Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

On spacelike immersions in locally symmetric semi-Riemannian spaces

por

Weiller Felipe Chaves Barboza

Campina Grande - PB
Junho/2022

On spacelike immersions in locally symmetric semi-Riemannian spaces

por

Weiller Felipe Chaves Barboza †

sob orientação do

Prof. Dr. Marco Antonio Lázaro Velásquez

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

Campina Grande - PB

Junho/2022

† Este trabalho contou com apoio financeiro da CAPES

B239s Barboza, Weiller Felipe Chaves.
On Spacelike immersions in locally symmetric semi-Riemannian spaces / Weiller Felipe Chaves Barboza. - Campina Grande, 2023.
133 f. : il. color.

Tese (Doutorado em Matemática) - Universidade Federal de Campina Grande, Centro de Ciências e Tecnologia, 2022.
"Orientação: Prof. Dr. Marco Antonio Lázaro Velásquez."
Referências.

1. Geometria Diferencial. 2. Espaços Localmente Simétricos. 3. State Space. 4. Anti-de Sitter Space. 5. Subvariedades Tipo-Espaço. Princípio do Máximo. I. Velásquez, Marco Antonio. II. Título.

CDU 514.7(043)

FICHA CATALOGRÁFICA ELABORADA PELA BIBLIOTECÁRIA ITAPUANA SOARES DIAS GONÇALVES CRB-15/093

Universidade Federal da Paraíba

Universidade Federal de Campina Grande

Programa Associado de Pós-Graduação em Matemática

Doutorado em Matemática

Área de Concentração: Geometria Diferencial

Aprovada em: 21/06/2022

Prof. Dr. Márcio Henrique Batista da Silva (UFAL)

Examinador Externo

Prof. Dr. Fábio Reis dos Santos (UFPE)

Examinador Externo

Prof. Dr. Henrique Fernandes de Lima (UFCG)

Examinador Interno

Prof. Dr. Marco Antonio Lázaro Velásquez (UFCG)

Orientador

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

Junho/2022

Resumo

Na primeira parte desta tese estudamos a geometria de imersões de hipersuperfícies tipo-espaço em espaços de curvatura seccional constante, mais especificamente nos ambientes do Steady State space $\mathcal{H}^{n+1}$ e no Anti-de Sitter $\mathbb{H}_1^{n+1}$. Nesses resultados, utilizamos condições adequadas sobre o comportamento das curvaturas médias de ordem superiores para provar alguns resultados de caracterizações de hipersuperfícies totalmente umbílicas no $\mathcal{H}^{n+1}$ e $\mathbb{H}_1^{n+1}$. Nesse processo também foi usado uma extensão adequada do princípio do máximo generalizado de Omori-Yau devido a Alías, Impera e Rigoli em [10]. Na segunda parte estudamos a geometria de subvariedades tipo-espaço com vetor curvatura média normalizado paralelo em ambientes de curvatura seccional constantes, onde utilizamos técnicas de crescimento de volume polinomial e um princípio do máximo no infinito devido a Alías, Caminha e Nascimento [7]. Também abordamos estruturas que possuem hipóteses de serem estocasticamente completa, $\mathcal{L}-$parabólicas e L^1-Lioville para garantir que determinada subvariedade seja totalmente umbílica. Na terceira e última parte, estudamos a geometria de subvariedades Weingarten linear tipo-espaço completa imersa com vetor curvatura média normalizado paralelo e fibrado normal flat em espaços semi-Riemannianos localmente simétrico L_p^{n+p} com index p. Nesse sentido, nosso objetivo foi estabelecer condições suficientes para garantir que uma dada subvariedade M^n seja totalmente umbílica ou isométrica a uma hypersuperfície isoparamétrica de uma subvariedade totalmente geodésica $L_1^{n+1} \hookrightarrow L_p^{n+p}$.

Palavras-chave: Espaços localmente simétricos, Steady State space, Anti-de Sitter space, Subvairiedades tipo-espaço, Princípio do máximo.

Abstract

In the first part of this these we study the geometry of immersions of the spacelike hypersurfaces in constant sectional curvature space, more specifically into the Steady State Space $\mathcal{H}_1^{n+1}$ and Anti-De Sitter space $\mathbb{H}_1^{n+1}$. In these results, we use suitable conditions on the behavior of higher order mean curvatures H_r to prove some results of characterizations of totally umbilical hypersurfaces in the $\mathcal{H}^{n+1}$ and $\mathbb{H}_1^{n+1}$, also in this process was use an suitable extension of the Omori-Yau's generalized maximum principle due to Alías, Impera and Rigoli in [10]. In the second part we study the geometry of spacelike submanifolds with parallel normalized mean curvature vector in constant sectional curvature spaces, where we use polynomial volume growth techniques and a maximum principle at infinity established by Alías, Caminha and Nascimento [7], our objects have hypotheses like: stochastically completeness, $\mathcal{L}-$parabolicity and L^1-Lioville to ensure that a given submanifold is totally umbilical. In the third and last part, we study the geometry of linear Weingarten spacelike complete submanifolds immersed with parallel normalized mean curvature vector and flat normal bundle in locally symmetric semi-Riemannian spaces L_p^{n+p} with index p. In this sense, our objective was to establish sufficient conditions to guarantee that a given submanifold M^n is totally umbilical or isometric to an isoparametric hypersurface of a totally geodesic submanifold $L_1^{n+1} \hookrightarrow L_p^{n+p}$.

Keywords: Locally symmetric spaces, Steady State space, Anti-de Sitter space, Spaceike submanifold, maximum principle.

Acknowledgment

A Deus, por tudo.

A minha mãe, Maria de Fátima por todo incentivo e apoio.

Ao meu pai, José Alberis (In memorian), por todo amor e proteção.

A minha esposa Rayzza, por todo amor e compreensão nos momentos difíceis...

Ao meu filho William, por trazer todo o amor e sentido para minha vida...

Ao Professor Marco Antonio Lázaro Velásquez, por ter me concedido a oportunidade de estudar e aprender durante todo o tempo do doutorado, pela excelente orientação, disponibilidade, ensinamentos, palavras de sabedoria, incentivos constantes durante todo o meu doutorado, só tenho que dizer MUITO OBRIGADO...

Ao Professor Henrique Fernandes de Lima, por ter me acompanhado durante todo o tempo de estudo, me oferecendo muita atenção, orientação, disponibilidade, ensinamentos, e principalmente muitas ideias de problemas para serem solucionados. Agradeço profundamente a sua presença na minha trajetória de estudo do doutorado...

Aos professores Fábio Reis e Márcio Batista, por oferecerem seu precioso tempo para avaliar e ler meu trabalho trazendo muita relevância no caráter científico e também por compor a minha banca de defesa...

A todos os professores da UAMAT que contribuíram com a minha formação de mestrado e doutorado, pelas amizades, ensinamentos e principalmente pelo conhecimento ofertado...

A todos os meus amigos da unidade acadêmica de Matemática - UAMAT por todos os momentos compartilhados, pela união, amizade e por contribuírem para a realização deste trabalho...

A CAPES, pelo incentivo financeiro ofertado durante todo o meu Doutorado...

"Se o conhecimento pode criar problemas, não é através da ignorância que podemos solucioná-los."

Isaac Asimov

Dedication

Aos meus pais, minha esposa e filho.

Sumário

Introduction

The investigation of the geometric behavior of spacelike hypersurfaces immersed in a Lorentzian space is an important thematic, from both the physical and mathematical points of view. For example, Marsden and Tipler [95] and Stumbles [115] pointed out that spacelike hypersurfaces immersed with constant mean curvature in a Lorentz manifold play an important role in the general relativity, in that they serve as convenient initial data for the Cauchy problem for Einstein's equations. From a mathematical viewpoint, a basic question related to this topic is the existence and uniqueness of spacelike hypersurfaces in Lorentz manifolds, under the assumption of some reasonable geometric properties, like the constancy of the mean or scalar curvature, for instance. A first relevant result in this direction was the proof of the famous conjecture due to Calabi [46] for maximal hypersurfaces (that is, hypersurfaces with vanishing mean curvature) in the Lorentz-Minkowski space, given by Cheng and Yau [55]. As for the case of the de Sitter space, Goddard [72] conjectured that every complete spacelike hypersurface with constant mean curvature should be totally umbilical. Although the conjecture turned out to be false in its original form, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses (see, for example, [22, 98]).

Here, initially we deal with complete spacelike hypersurfaces immersed in a special Lorentz space form with negative constant sectional curvature equal to 1. This space is known as an open region of de Sitter space, is the so-called *steady state space* $\mathcal{H}^{n+1}$ and is defined as been the hyperquadric

$$\mathcal{H}^{n+1} = \left\{ p \in \mathbb{S}_1^{n+1} : \langle p, a \rangle > 0 \right\}.$$

The importance of considering $\mathcal{H}^{n+1}$ comes from the fact that, in Cosmology, $\mathcal{H}^4$ is the steady state model of the universe proposed by Bondi-Gold [35] and Hoyle [84], when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous), but also at all times. For more details, we recommend for the readers to see Section 5.2 of [83] or Section 14.8 of [121]. From a mathematical point of view, the interest in the study of spacelike hypersurfaces immersed in a Lorentzian space is motivated by their nice Bernstein-type properties. In this direction, several authors have approached the problem of to characterizing spacelike hyperplanes of $\mathcal{H}^{n+1}$, which are totally umbilical spacelike hypersurfaces isometric to the Euclidean space $\mathbb{R}^n$ and give a complete foliation of $\mathcal{H}^{n+1}$. We refer to readers, for instance, the works [1, 13, 44, 45, 101].

Inspired by this construction and importance, we developed the following results below that can be found and seen in more detail in Chapter 2.

Theorem A.1 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$. Suppose that the mean curvature H of Σ^n is positive, bounded and satisfies*

$$H \leq H_2. \tag{1}$$

If

$$|a^\top| \leq C \inf_\Sigma (H_2 - H), \tag{2}$$

for some positive constant C, then Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$.

Theorem A.2 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$, with sectional curvature $K_\Sigma \leq 1$ and bounded from below. Suppose that, for some $1 \leq r \leq n-1$, H_{r+1} is bounded and satisfies*

$$\beta \leq H_r \leq H_{r+1}, \tag{3}$$

where β is a positive constant. If

$$|a^\top| \leq C \inf_\Sigma (H_{r+1} - H_r), \tag{4}$$

for some positive constant C, then Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$.

Theorem A.3 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$*

orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$*, and locally tangent from above to a spacelike hyperplane* $\mathcal{E}_{\tilde{\tau}}$*, with* $\tilde{\tau} \leq \tau$*. Suppose that* H *is bounded and, for some* $1 \leq r \leq n-1$*,* H_{r+1} *is positive and such that*

$$H_r \leq H_{r+1}. \tag{5}$$

If

$$|a^{\top}| \leq C \inf_{\Sigma}(H_{r+1} - H_r), \tag{6}$$

for some positive constant C*, then* Σ^n *must be the spacelike hyperplane* $\mathcal{E}_{\tilde{\tau}}$*.*

Theorem A.4 *Let* $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ *be a complete spacelike hypersurface of* $\mathcal{H}^{n+1}$ *contained in the closure of the interior domain enclosed by a spacelike hyperplane* $\mathcal{E}_\tau$ *orthogonal to a nonzero null vector* $a \in \mathbb{L}^{n+2}$*. Suppose that* $l_a = \lambda f_a$ *for some positive constant* $\lambda \in \mathbb{R}$*, the mean curvature* H *of* Σ^n *is bounded and that*

$$H_2 \geq 1. \tag{7}$$

Then Σ^n *is a spacelike hyperplane* $\mathcal{E}_{\tilde{\tau}}$ *with* $\tilde{\tau} \leq \tau$*.*

Theorem A.5 *Let* $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ *be a complete spacelike hypersurface of* $\mathcal{H}^{n+1}$ *contained in the closure of the interior domain enclosed by a spacelike hyperplane* $\mathcal{E}_\tau$ *orthogonal to a nonzero null vector* $a \in \mathbb{L}^{n+2}$*, and locally tangent from above to a spacelike hyperplane* $\mathcal{E}_{\tilde{\tau}}$*, with* $\tilde{\tau} \leq \tau$*. Suppose that* $l_a = \lambda f_a$ *for some positive constant* $\lambda \in \mathbb{R}$*, and that, for some* $1 \leq r \leq n-2$*, the* r*-th mean curvature* H_r *of* Σ^n *is bounded and such that*

$$\beta \leq H_r \leq H_{r+2}, \tag{8}$$

where β *is a positive constant. Then,* Σ^n *must be the spacelike hyperplane* $\mathcal{E}_{\tilde{\tau}}$*.*

In the sequence of this work, we deal with complete hypersurfaces immersed in a special Lorentz space form with negative constant sectional curvature equal to -1. Such manifold is known as an *anti-de Sitter space* and is defined as been the hyperquadric

$$\mathbb{H}_1^{n+1} = \{x \in \mathbb{R}_2^{n+2} : ds^2(x,x) = -1\},$$

where $\mathbb{R}_2^{n+2}$ is the $(n+2)$-dimensional semi-Euclidean space with index 2. We observe that an interesting feature of the four-dimensional anti-de Sitter space $\mathbb{H}_1^4$ is that, as a cosmological model, this spacetime is a *maximally symmetric universe* with constant negative curvature, which is conformally related to the half of the Einstein static universe. Consequently, $\mathbb{H}_1^4$ represents (locally) an unique solution to Einstein's equation in the absence of any ordinary matter or gravitational radiation. So, this spacetime may be thought of as a ground state of General Relativity (see, for instance, Chapter 6 of [122] and Chapter 14 of [123]).

Concerning the study of complete spacelike hypersurfaces immersed in the anti-de Sitter space, Choi, Ki and Kim [57] used the generalized maximum principle of Omori [105] and Yau [131] in order to show that if the height function with respect to a timelike vector of a complete maximal spacelike hypersurface Σ^n of $\mathbb{H}_1^{n+1}$ obeys a certain boundedness, then Σ^n must be totally geodesic. Later on, by extending a technique due to Yau [132], the second author jointly with Camargo [48] obtained another rigidity results to complete maximal spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$, imposing suitable conditions on both the norm of the second fundamental form and a certain height function naturally attached to the hypersurface. They also characterized complete maximal spacelike graphs satisfying a certain assumption on the gradient of the function which determines the graph. Afterwards, working with a suitable warped product model of $\mathbb{H}_1^{n+1}$, these same authors jointly with Caminha and Parente [47] extended the main result of [48] showing that if Σ^n is a complete spacelike hypersurface with constant mean curvature and bounded scalar curvature in $\mathbb{H}_1^{n+1}$, such that the gradient of its height function with respect to a timelike vector has integrable norm, then Σ^n must be totally umbilical. Next, the second author jointly with Aquino [14] obtained another characterizations theorems concerning complete constant mean curvature spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$, under suitable constraints on the behavior of the Gauss mapping. Furthermore, these same authors jointly with the fourth author [15] obtained similar results related to complete spacelike hypersurfaces with constant scalar curvature in $\mathbb{H}_1^{n+1}$.

Related to higher codimension, Ishihara [85] proved that a n-dimensional complete maximal spacelike submanifold immersed in the anti-de Sitter space $\mathbb{H}_p^{n+p}$ of index p must have the squared norm of the second fundamental form bounded from above by np. Moreover, the only ones that attain this estimate are the maximal hyperbolic cylinders $\mathbb{H}^{k_1}\left(-\frac{n}{k_1}\right) \times \cdots \times \mathbb{H}^{k_{p+1}}\left(-\frac{n}{k_{p+1}}\right)$, where $k_1 + \cdots + k_{p+1} = n$. Later on, Cao and Wei [50] showed that, if $n \geq 3$, then every n-dimensional complete maximal spacelike hypersurface in $\mathbb{H}_1^{n+1}$ with exactly two principal curvatures everywhere is isometric to some hyperbolic cylinder under an additional condition on these curvatures. Afterwards, Perdomo [107] studied the 2-dimensional case and constructed new examples of complete maximal surfaces in $\mathbb{H}_1^3$. More recently, Chaves, Sousa and Valério [52] studied complete maximal spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$ with either constant scalar curvature or constant non-zero Gauss-Kronecker curvature. In this setting, they characterized the hyperbolic cylinders as the only such hypersurfaces with $(n-1)$ principal curvatures with the same sign everywhere.

Motivated by these works, we were able to establish the following results, which can be found in more detail in the Chapter 2

Theorem B.1 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface such that $f_a^2 \leq \frac{1}{2}$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that the mean curvature H is positive, bounded and that the second mean curvature satisfies*

$$0 \leq H_2 \leq 1.$$

If

$$|a^\top| \leq C \inf_\Sigma (H - H_2),$$

for some positive constant C, then Σ^n is a totally umbilical spacelike hypersurface M_τ, with $\tau^2 = \frac{1}{2}$.

Theorem B.2 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below satisfying $K_\Sigma \leq -1$, and such that $f_a^2 \leq \frac{1}{2}$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that, for some $1 \leq r \leq n-1$, H_r is bounded and such that*

$$0 \leq H_{r+1} \leq H_r.$$

If

$$|a^\top| \leq C \inf_\Sigma (H_r - H_{r+1}),$$

for some positive constant C, then Σ^n is a totally umbilical spacelike hypersurface M_τ with $\tau^2 = \frac{1}{2}$.

Theorem B.3 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface such that $f_a^2 \leq \frac{1}{2}$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that H is bounded, for some $1 \leq r \leq n-1$, H_{r+1} is positive and such that*

$$H_{r+1} \leq H_r.$$

Let us assume in addition that

$$|a^\top| \leq C \inf_\Sigma (H_r - H_{r+1}),$$

for some positive constant C.

(i) If Σ^n is contained in $\Omega^+(a,\rho)$, for some $\frac{1}{\sqrt{2}} < \rho \leq 1$, and it is locally tangent from bellow to a totally umbilical spacelike hypersurface M_τ, with $0 < \tau < \rho$, then Σ^n is isometric to M_τ and $\tau = \frac{1}{\sqrt{2}}$;

(ii) If Σ^n is contained in $\Omega^-(a,\rho)$, for some $-1 \leq \rho < -\frac{1}{\sqrt{2}}$, and it is locally tangent from above to a totally umbilical spacelike hypersurface M_τ, with $\rho < \tau < 0$, then Σ^n is isometric to M_τ and $\tau = -\frac{1}{\sqrt{2}}$.

Theorem B.4 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete r-maximal $(2 \leq r \leq n-1)$ spacelike hypersurface with sectional curvature bounded from below satisfying $K_\Sigma \leq -1$, and such that $f_a^2 \leq \frac{1}{2}$ and $|a^\top|$ is bounded for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. If H_r is a nonnegative constant, then the index of minimum relative nullity ν_0 of Σ^n is at least $n-r+1$. Moreover, if H_{r-1} does not vanish on Σ^n, then through every point of Σ^n there passes an $(n-r+1)$-dimensional hyperbolic space $\mathbb{H}^{n-r+1} \hookrightarrow \mathbb{H}_1^{n+1}$ totally contained in Σ^n.*

Theorem B.5 *There does not exist complete 1-maximal spacelike hypersurface $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ with nonnegative constant mean curvature and such that $f_a^2 \leq \frac{1}{2}$ and $|a^\top|$ is bounded for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$.*

Theorem B.6 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If Σ^n is contained either in $\Omega^-(a,\rho)$ or in $\Omega^+(a,\rho)$, for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0 < \rho < 1$, then*

$$\sup_\Sigma H_r \geq \left(\frac{\sup_\Sigma u^\pm}{\sqrt{1 - (\sup_\Sigma u^\pm)^2}} \right)^r, \quad \textit{for all } r = 1, \ldots, n,$$

where $u^\pm \in C^\infty(\Sigma)$ is defined by $u^\pm = \pm l_a$ as we have $\Sigma^n \subset \Omega^-(a,\rho)$ or $\Sigma^n \subset \Omega^+(a,\rho)$.

Theorem B.7 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If Σ^n is contained either in $\Omega^-(a,\rho)$ or in $\Omega^+(a,\rho)$, for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0 < \rho < 1$, then its Ricci curvature satisfies*

$$\inf_\Sigma \mathrm{Ric} = \inf_{\substack{p \in \Sigma \\ v \in T_p\Sigma \\ |v|=1}} \mathrm{Ric}_p(v,v) \leq \frac{n-1}{(\sup_\Sigma u^\pm)^2 - 1},$$

where $u^\pm \in C^\infty(\Sigma)$ is defined by $u^\pm = \pm l_a$ as we have $\Sigma^n \subset \Omega^-(a,\rho)$ or $\Sigma^n \subset \Omega^+(a,\rho)$.

Theorem B.8 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface of $\mathbb{H}_1^{n+1}$ with bounded second fundamental form, such that $f_a^2 \leq 1/2$ for some fixed unit timelike* *vector $a \in \mathbb{R}^{n+2}_2$. Suppose that, for some $1 \le r \le n-1$, the r-th mean curvature H_r of Σ^n is positive and satisfies*

$$0 \le H_{r+1} \le H_r.$$

If $|a^\top| \in \mathcal{L}^1(\Sigma)$ and Σ^n is contained in the open region Ω^+_a (respect. Ω^-_a), then Σ^n is the totally umbilical spacelike hypersurface M_τ of $\mathbb{H}^{n+1}_1$ with $\tau = \sqrt{2}/2$ (respect. $\tau = -\sqrt{2}/2$).

Theorem B.9 *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}_1$ be a complete r-maximal spacelike hypersurface, $2 \le r \le n-1$, with bounded second fundamental form, such that $f_a^2 \le 1/2$ and $|a^\top| \in \mathcal{L}^1(\Sigma)$ for some fixed unit timelike vector $a \in \mathbb{R}^{n+2}_2$. If H_r is a nonnegative constant and Σ^n is contained either in Ω^+_a or Ω^-_a, then the index of minimum relative nullity ν_0 of Σ^n is at least $n-r+1$. Moreover, if H_{r-1} does not vanish on Σ^n, then through every point of Σ^n there passes an $(n-r+1)$-dimensional hyperbolic space $\mathbb{H}^{n-r+1} \hookrightarrow \mathbb{H}^{n+1}_1$ which is totally contained in Σ^n.*

Theorem B.10 *There does not exist complete 1-maximal spacelike hypersurface $\psi : \Sigma^n \to \mathbb{H}^{n+1}_1$ with nonnegative constant mean curvature, which is contained either in Ω^+_a or Ω^-_a, for some fixed unit timelike vector $a \in \mathbb{R}^{n+2}_2$, and such that $f_a^2 \le \frac{1}{2}$ and $|a^\top| \in \mathcal{L}^1(\Sigma)$.*

Theorem B.11 *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}_1$ be a complete spacelike hypersurface of $\mathbb{H}^{n+1}_1$ with bounded second fundamental form such that $f_a^2 \le 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}^{n+2}_2$. Suppose, for some $1 \le r \le n$, that the r-th mean curvature H_r of Σ^n satisfies*

$$0 < H_r \le 1.$$

If $|a^\top| \in \mathcal{L}^1(\Sigma)$, $l_a \ge -f_a$, Σ^n is contained in the open region Ω^+_a (respect. Ω^-_a) and it is locally tangent from below (respect. above) to a totally umbilical spacelike hypersurface M_τ, then $\tau = \sqrt{2}/2$ (respect. $\tau = -\sqrt{2}/2$) and Σ^n is the totally umbilical spacelike hypersurface M_τ.

Theorem B.12 *Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}_1$ be a complete spacelike hypersurface of $\mathbb{H}^{n+1}_1$ with bounded second fundamental form such that $f_a^2 \le 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}^{n+2}_2$. Suppose that the second mean curvature satisfies*

$$0 < H_2 \le 1.$$

If $|a^\top| \in \mathcal{L}^1(\Sigma)$ and $l_a \geq -f_a$, then Σ^n is a totally umbilical spacelike hypersurface M_τ, with $\tau^2 = 1/2$.

The second part of this thesis is related to geometry of spacelike submanifolds immersed in a Lorentzian space form. This investigation constitutes a classical and still fruitful thematic into the scop of Differential Geometry. Working in this branch, Ishihara [85] proved that the only n-dimensional complete maximal submanifold immersed in the pseudo-Euclidean space $\mathbb{R}^{n+p}_p$ of index p must be the totally geodesic ones. When the ambient spacetime is the anti-de Sitter space $\mathbb{H}^{n+p}_p$ of index p, Ishihara also proved that an n-dimensional complete maximal spacelike submanifold immersed in $\mathbb{H}^{n+p}_p$ must have the squared norm of the second fundamental form bounded from above by np. Moreover, the only ones that attain this estimate are generalized hyperbolic cylinders.

In [60], Cheng extended previous results due to Akutagawa [22] and Ramanathan [111] showing that an n-dimensional complete spacelike submanifold with parallel mean curvature vector h (that is, h is parallel as a section of the normal bundle) in the de Sitter space $\mathbb{S}^{n+p}_p$ of index p, such that $H^2 \leq 1$, when $n = 2$, or $H^2 \leq 4(n-1)/n^2$, when $n \geq 3$, must be totally umbilical. Here, $H = ||h||$ stands for the mean curvature function. Afterwards, Aiyama [20] studied compact spacelike submanifolds in $\mathbb{S}^{n+p}_p$ with parallel mean curvature vector and proved that if the normal connection is flat, then these spacelike submanifolds must be totally umbilical. Furthermore, she proved that a compact spacelike submanifold in $\mathbb{S}^{n+p}_p$ with parallel mean curvature vector and nonnegative sectional curvature must be totally umbilical. Next, Cheng [58] obtained a refinement of Ishihara's result [85] for the case of complete maximal spacelike surfaces immersed in $\mathbb{H}^{2+p}_p$.

In [12], Alías and Romero developed some integral formulas for compact spacelike submanifolds in $\mathbb{S}^{n+p}_q$, with index $1 \leq q \leq p$, which have a very clear geometric meaning and, as application, they obtained a Bernstein type result for complete maximal submanifolds, extending a previous result due to Ishihara [85]. Moreover, they extended Ramanathan's result [111] showing that the only compact spacelike surfaces in $\mathbb{S}^{2+p}_p$ with parallel mean curvature vector are the totally umbilical ones. Afterwards, Cheng and Ishikawa [61] also investigated complete maximal spacelike submanifolds immersed in $\mathbb{S}_q^{n+p}$, with index $1 \leq q \leq p$, obtaining characterizations results for totally geodesic spacelike submanifolds under pinching conditions on scalar curvature, Ricci curvature and sectional curvature, respectively.

Later on, Brasil, Chaves and Colares [32] considered n-dimensional complete spacelike submanifolds immersed in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector. In this setting, they used a Simons type inequality to obtain some rigidity results characterizing umbilical submanifolds and hyperbolic cylinders in $\mathbb{S}_p^{n+p}$. In [96], also applied a Simons type inequality in order to obtain sharp estimates for the supremum of the scalar curvature of complete spacelike submanifolds with parallel mean curvature vector in an indefinite space form.

More recently, Yang and Li [129] applied the Omori-Yau maximum principle [105, 131] in order to get characterization results concerning complete spacelike submanifolds with parallel mean curvature vector h in $\mathbb{S}_q^{n+p}$, for $1 \leq q \leq p$, where h is supposed to be either spacelike or timelike. Afterwards, the second and third authors jointly with dos Santos [62, 63] used the technique due to Alías and Romero [12] and, under appropriate restrictions on the Ricci curvature and second fundamental form, they showed that an n-dimensional complete maximal spacelike submanifold of either $\mathbb{R}_q^{n+p}$ or $\mathbb{H}_q^{n+p}$ must be totally geodesic.

The study of spacelike submanifolds immersed in a Lorentzian space is motivated by their nice Bernstein type properties. For instance, it was proved by Calabi [46] (for $n = 4$) and by Cheng and Yau [55] (for all n) that the only complete maximal spacelike hypersurfaces (that is, with mean curvature identically zero) of the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$ are the spacelike hyperplanes. In [102], Nishikawa proved that a complete maximal spacelike hypersurface in the de Sitter space $\mathbb{S}_1^{n+1}$ must be totally geodesic. In [72], Goddard conjectured that the complete spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ with constant mean curvature H must be totally umbilical. Ramanathan [111] proved Goddard's conjecture in $\mathbb{S}_1^3$ for $0 \leq H \leq 1$. Moreover, for $H > 1$, he showed that the conjecture is false, as can be seen from an example due to Dajczer and Nomizu in [68]. Independently, Akutagawa [22] proved that Goddard's conjecture is true when either $n = 2$ and $H^2 \leq 1$ or $n \geq 3$ and $H^2 < 4(n-1)/n^2$. He also constructed complete spacelike rotation surfaces in $\mathbb{S}_1^3$ having constant mean curvature $H > 1$ and which are not totally umbilical. Next, Montiel [98] showed that Goddard's conjecture is true for closed (that is, compact without boundary) spacelike hypersurfaces. Furthermore, he exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature $H^2 \geq 4(n-1)/n^2$ and being non totally umbilical, the so-called hyperbolic cylinders.

In higher codimension, Ishihara [85] obtained an extension of Cheng-Yau's result showing that the only n-dimensional complete maximal submanifold immersed in the pseudo-Euclidean space $\mathbb{R}_p^{n+p}$ of index p must be the totally geodesic ones. However, in the case that the mean curvature is a positive constant, Treibergs [119] surprisingly showed that there are many entire solutions of the corresponding constant mean curvature equation in $\mathbb{R}_1^{n+1}$, which he was able to classify by their projective boundary values at infinity. In [60], Cheng extended Akutagawa's result for complete spacelike submanifolds with parallel mean curvature vector h (that is, h is parallel as a section of the normal bundle) in the de Sitter space $\mathbb{S}_p^{n+p}$ of index p. Afterwards, Aiyama [20] studied compact spacelike submanifolds in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and proved that if the normal connection is flat, then these spacelike submanifolds must be totally umbilical. Furthermore, she proved that a compact spacelike submanifold in $\mathbb{S}_p^{n+p}$ with parallel mean curvature vector and nonnegative sectional curvature must be totally umbilical.

Meanwhile, Alías and Romero [12] developed some integral formulas for compact spacelike submanifolds in $\mathbb{S}_q^{n+p}$ $(1 \leq q \leq p)$ which have a very clear geometric meaning and, as application, they obtained a Bernstein type result for complete maximal submanifolds, extending a previous result due to Ishihara [85]. Moreover, they extended Ramanathan's result [111] showing that the only compact spacelike surfaces in $\mathbb{S}_p^{2+p}$ with parallel mean curvature vector are the totally umbilical ones and, in particular, they also reproved Cheng's result [60] establishing that every complete spacelike surface in $\mathbb{S}_p^{2+p}$ with parallel mean curvature vector h such that $H^2 < 1$ is totally umbilical, where $H = ||h||$ stands for the mean curvature function. Next, Li [88] showed that Montiel's result [98] still holds for higher codimensional spacelike submanifolds in $\mathbb{S}_p^{n+p}$. In [61], Cheng and Ishikawa investigated complete maximal spacelike submanifolds immersed in $\mathbb{S}_q^{n+p}$, with index $1 \leq q \leq p$, obtaining characterizations results for totally geodesic spacelike submanifolds under pinching conditions on scalar curvature, Ricci curvature and sectional curvature, respectively.

When the ambient spacetime is the anti-de Sitter space $\mathbb{H}_p^{n+p}$ of index p, in [85] Ishihara also proved that an n-dimensional complete maximal spacelike submanifold immersed in $\mathbb{H}_p^{n+p}$ must have the squared norm of the second fundamental form bounded from above by np. Moreover, the only ones that attain this estimate are generalized hyperbolic cylinders. Later on, Cheng [58] obtained a refinement of Ishihara's result [85] for the case of complete maximal spacelike surfaces immersed in $\mathbb{H}_p^{2+p}$.

More recently, Yang and Li [129] applied the Omori-Yau maximum principle [105, 131] in order to get characterization results concerning complete spacelike submanifolds with parallel mean curvature vector in $\mathbb{S}_q^{n+p}$, for $1 \leq q \leq p$. Afterwards, the second and third authors jointly with dos Santos [62, 63] used the technique due to Alías and Romero [12] and, under appropriate constraints on the Ricci curvature and second fundamental form, they showed that an n-dimensional complete maximal spacelike submanifold of either $\mathbb{R}_q^{n+p}$ or $\mathbb{H}_q^{n+p}$ must be totally geodesic. Moreover, they established sufficient conditions to guarantee that a complete spacelike submanifold with nonzero parallel mean curvature vector h in these ambient spaces must be pseudo-umbilical, which means that h is an umbilical direction. Their approach was based on a generalized form of a maximum principle at the infinity due to Yau [133].

Motivated by all these works, here we establish the following rigidity results with respect to spacelike submanifolds in Lorentzian spacial forms, which can be found in more detail in Chapter 3.

Theorem C.1 *Let M^n be a complete spacelike submanifold immersed in $\mathbb{L}_q^{n+p}(c)$, with $c \in \{0, -1, 1\}$ and $1 \leq q < p-1$, having spacelike and parallel mean curvature vector. When $c = -1$, suppose in addition that $H > 1$. If M has polynomial volume growth, $|\nabla\Phi|$ is bounded and assuming that there is a constant α suth that $\sup_M|\Phi| \leq \alpha < \alpha^*$, where α^* is the positive root of the function*

$$P_H(x) := -5x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}}Hx + 2n(c+H^2). \tag{9}$$

Then, $\sup_M|\Phi| = 0$ and M^n is a totally umbilical submanifold.

Theorem C.2 *Let M^n be a complete spacelike submanifold immersed in de Sitter space $\mathbb{S}_q^{n+p}$, with $1 < q < p-1$, having timelike and parallel mean curvature vector. Suppose that $H < 1$. If M has polynomial volume growth, $|\nabla\Phi|$ is bounded and assuming that there is a constant β suth that $\sup_M|\Phi| \leq \beta < \beta^*$, where β^* is the positive root of the* *function*

$$Q_H(x) := -\frac{4(2q-1)}{q-1}x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}}Hx + 2n(1-H^2). \tag{10}$$

Then, $\sup_M|\Phi|= 0$ *and* M^n *is a totally umbilical submanifold.*

Theorem C.3 *Let* M^n *be an* n*-dimensional complete noncompact spacelike submanifold immersed with spacelike and parallel mean curvature vector in an* $(n+p)$*-dimensional pseudo-Riemannian space form* $\mathbb{L}_q^{n+p}(c)$*, with constant sectional curvature* $c \in \{0, -1, 1\}$ *and index* $1 \leq q < p-1$*. When* $c = -1$*, suppose in addition that the mean curvature satisfies* $H > 1$*. If* $|\Phi|$ *converges to zero at infinity with* $\sup_M |\Phi| \leq \alpha^*$*, where* α^* *is the positive root of the polynomial function*

$$P_H(x) := -5x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}}Hx + 2n(c+H^2), \tag{11}$$

then $\sup_M |\Phi| = 0$ *and* M^n *is a totally umbilical submanifold of* $\mathbb{L}_q^{n+p}(c)$*.*

Theorem C.4 *Let* M^n *be an* n*-dimensional complete noncompact spacelike submanifold immersed with timelike and parallel mean curvature vector in the* $(n+p)$*- dimensional de Sitter space* $\mathbb{S}_q^{n+p}$*, with index* $1 < q < p-1$*. Suppose in addition that the mean curvature satisfies* $H < 1$*. If* $|\Phi|$ *converges to zero at infinity with* $\sup_M |\Phi| \leq \beta^*$*, where* β^* *is the positive root of the polynomial function*

$$Q_H(x) := -\frac{4(2q-1)}{q-1}x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}}Hx + 2n(1-H^2), \tag{12}$$

then $\sup_M |\Phi| = 0$ *and* M^n *is a totally umbilical submanifold of* $\mathbb{S}_q^{n+p}$*.*

Theorem C.5 *Let* M^n *be a stochastically complete spacelike submanifold immersed in* $\mathbb{L}_q^{n+p}(c)$*, with* $c \in \{0, -1, 1\}$ *and* $1 \leq q < p-1$*, having spacelike and parallel mean curvature vector. When* $c = -1$*, suppose in addition that* $H > 1$*. Then, either*

(i) $\sup_M|\Phi|= 0$ *and* M^n *is a totally umbilical submanifold, or*

(ii) $\sup_M|\Phi|\geq \alpha^*(n, c, H)$*, where* $\alpha^*(n, c, H)$ *is the positive root of the function*

$$P_H(x) := -\frac{5}{2}x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}Hx + n(c+H^2). \tag{13}$$

Moreover, if the equality holds and this supremum is attained at some point of M^n*, then* M^n *is a pseudo-umbilical submanifold of* $\mathbb{L}_q^{n+p}(c)$ *such that its principal curvatures are constant.*

Theorem C.6 *Let* M^n *be a stochastically complete spacelike submanifold immersed in de Sitter space* $\mathbb{S}_q^{n+p}$*, with* $1 < q < p-1$*, having timelike and parallel mean curvature vector. Suppose that* $H < 1$*. Then, either* *(i)* $\sup_M|\Phi|=0$ *and* M^n *is a totally umbilical submanifold, or*

(ii) $\sup_M|\Phi|\geq \beta^*(n,q,H)$*, where* $\beta^*(n,q,H)$ *is the positive root of the function*

$$Q_H(x) := -\frac{2(2q-1)}{q-1}x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}Hx + n(1-H^2). \tag{14}$$

Moreover, if the equality holds and this supremum is attained at some point of M^n*, then* M^n *is a pseudo-umbilical submanifold of* $\mathbb{S}_q^{n+p}$ *such that its principal curvatures are constant.*

Theorem C.7 *Let* M^n *be a parabolic spacelike submanifold immersed in* $\mathbb{L}_q^{n+p}(c)$*, with* $c\in\{0,-1,1\}$ *and* $1\leq q<p-1$*, having spacelike and parallel mean curvature vector. When* $c=-1$*, suppose in addition that* $H>1$*. Then either* $\sup_M|\Phi|=0$ *and* M^n *is a totally umbilical submanifold or* $\sup_M|\Phi|\geq \alpha^*(n,c,H)$*, where* $\alpha^*(n,c,H)$ *is the positive root of Teorema C.5. Moreover, when* $\sup_M|\Phi|=\alpha^*(n,c,H)$*,* M^n *is a pseudo-umbilical submanifold of* $\mathbb{L}_q^{n+p}(c)$ *such that its principal curvatures are constant.*

Theorem C.8 *Let* M^n *be a parabolic spacelike submanifold immersed in de Sitter space* $\mathbb{S}_q^{n+p}$*, with* $1<q<p-1$*, having timelike and parallel mean curvature vector, suppose in addition that* $H<1$*. Then either* $\sup_M|\Phi|=0$ *and* M^n *is a totally umbilical submanifold or* $\sup_M|\Phi|\geq \beta^*(n,q,H)$*, where* $\beta^*(n,q,H)$ *is the positive root of Teorema C.6. Moreover, when* $\sup_M|\Phi|=\beta^*(n,q,H)$*,* M^n *is a pseudo-umbilical submanifold of* $\mathbb{S}_q^{n+p}$ *such that its principal curvatures are constant.*

Theorem C.9 *Let* M^n *be a* L^1*-Liouville spacelike submanifold immersed in* $\mathbb{L}_q^{n+p}(c)$*, with* $c\in\{0,-1,1\}$ *and* $1\leq q<p-1$*, having spacelike and parallel mean curvature vector. When* $c=-1$*, suppose in addition that* $H>1$*. If* $\sup_M|\Phi|\leq\alpha^*(n,c,H)$ *and* $\varphi := (\alpha^*(n,c,H))^2-|\Phi|^2\in L^1(M)$*, where* $\alpha^*(n,c,H)$ *is the positive root of Teorema C.5, then either* $|\Phi|\equiv 0$ *and* M^n *is a totally umbilical submanifold or* $|\Phi|\equiv\alpha^*(n,c,H)$ *and* M^n *is a pseudo-umbilical submanifold of* $\mathbb{L}_q^{n+p}(c)$ *such that its principal curvatures are constant.*

Theorem C.10 *Let* M^n *be a* L^1*-Liouville spacelike submanifold immersed in de Sitter space* $\mathbb{S}_q^{n+p}$*, with* $1<q<p-1$*, having timelike and parallel mean curvature vector, suppose in addition that* $H<1$*. If* $\sup_M|\Phi|\leq\beta^*(n,q,H)$ *and* $\zeta := (\beta^*(n,q,H))^2 - |\Phi|^2\in L^1(M)$*, where* $\beta^*(n,q,H)$ *is the positive root of Teorema C.6, then either* $|\Phi|\equiv 0$

and M^n is a totally umbilical submanifold or $|\Phi| \equiv \beta^*(n, q, H)$ and M^n is a pseudo-umbilical submanifold of $\mathbb{S}_q^{n+p}$ such that its principal curvatures are constant.

Theorem C.11 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in the de Sitter space $\mathbb{S}_p^{n+p}$ with index $p > 1$, such that $R = aH + b$ with $a \geq 0$ and $0 < b \leq 1$. Then*

(i) either $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold,

(ii) or

$$\sup_M |\Phi| \geq \alpha(n, p, a, b) > 0, \tag{15}$$

where $\alpha(n, p, a, b)$ is a positive constant that depends only on n, p, a, b. Moreover, if M^n has nonnegative sectional curvature, $b < 1$, the equality $\sup_M |\Phi| = \alpha(n, p, a, b)$ holds and this supremum is attained at some point of M^n, then M^n is isometric to a product $M_1 \times M_2 \times \ldots \times M_k$, where the factors M_i are totally umbilical submanifolds of $\mathbb{S}_p^{n+p}$ which are mutually perpendicular along their intersections.

Theorem C.12 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in the de Sitter space $\mathbb{S}_p^{n+p}$ with index $p > 1$, such that $R = aH + b$ with $a \geq 0$ and $0 < b \leq 1$. If M^n is a $\mathcal{L}$-parabolic submanifold with nonnegative sectional curvature and such that $\sup_M |\Phi| \leq \alpha(n, p, a, b)$, where $\alpha(n, p, a, b)$ is the positive constant depending only on n, p, a, b which was obtained in Theorem C.11, then either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $\sup_M |\Phi| = \alpha(n, p, a, b)$ and M^n is isometric to a product $M_1 \times M_2 \times \ldots \times M_k$, where the factors M_i are totally umbilical submanifolds of $\mathbb{S}_p^{n+p}$ which are mutually perpendicular along their intersections.*

Theorem C.13 *Let M^n be a complete linear Weingarten spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector, such that $R = aH + b$ with $a \geq 0$ and $0 < b \leq 1$. If $\sup_M |\Phi|^2 < +\infty$ and, for some reference point $o \in M^n$,*

$$\int_0^{+\infty} \frac{dr}{\mathrm{vol}(\partial B_r)} = +\infty, \tag{16}$$

then M^n is $\mathcal{L}$-parabolic. Here B_r denotes the geodesic ball of radius r in M^n centered at the origin o.

In the third and last part of this work we study immersions of spacelike submanifolds in locally symmetric semi-Riemannian spaces. Let us denote by L_p^{n+p} an $(n+p)$-dimensional connected semi-Riemannian space with index p. We recall that L_p^{n+p} is said to be *locally symmetric* when its curvature tensor $\bar{R}$ is parallel in the sense that $\overline{\nabla}\bar{R} = 0$, where $\overline{\nabla}$ denotes the Levi-Civita connection of L_p^{n+p}. In 1984, Nishikawa [102] introduced an important class of locally symmetric Lorentz spaces satisfying certain curvature constraints. In this setting, he extended the classical results of Calabi [46] and Cheng-Yau [55] showing that the only complete maximal spacelike hypersurfaces immersed in this ambient space having nonnegative sectional curvature are the totally geodesic ones. This seminal Nishikawa's paper induced the appearing of several works approaching the problem of characterizing complete spacelike hypersurfaces immersed in such a locally symmetric Lorentz space (see, for instance, [39, 90, 64, 65, 93]).

We also recall that a spacelike submanifold M^n of L_p^{n+p} is called *linear Weingarten* if its mean curvature H and its normalized scalar curvature R satisfy a linear relation of the type $R = aH + b$, for some real constants a and b. When the ambient space is the de Sitter space $\mathbb{S}_1^{n+1}$, Cheng [59], studying the case $b = 0$, proved that if M^n is a complete linear Weingarten spacelike hypersurface with nonnegative sectional curvature such that H attains its maximum, then M^n must be totally umbilical. For higher codimension, considering again the particular case $b = 0$, Liu [92] showed that the totally umbilical round spheres are the only n-dimensional compact linear Weingarten spacelike submanifolds of $\mathbb{S}_p^{n+p}$ with nonnegative sectional curvature and flat normal bundle. Generalizing the ideas of a previous work [127], Yang and Hou [128] showed that a linear Weingarten spacelike submanifold in $\mathbb{S}_p^{n+p}$, with $a > 0$, $b < 1$, having parallel normalized mean curvature vector field (that is, the mean curvature function is positive and that the corresponding normalized mean curvature vector field is parallel as a section of the normal bundle) and such that the squared norm of its second fundamental form satisfies a suitable boundedness, must be either totally umbilical or isometric to a certain hyperbolic cylinder. More recently, Araújo, De Lima, Velásquez and dos Santos [17] studied n-dimensional complete spacelike submanifolds M^n with flat normal bundle and parallel normalized mean curvature vector immersed in an $(n+p)$-dimensional locally symmetric semi-Riemannian manifold L_p^{n+p} of index p obeying some standard curvature conditions which are naturally satisfied when the ambient space is a semi-Riemannian space form. In this setting, they obtained sufficient conditions to guarantee that, in fact, $p = 1$ and M^n is isometric to an isoparametric hypersurface of L_1^{n+1} having two distinct principal curvatures, one of which is simple.

Proceeding with this picture, in this work we also consider complete linear Weingarten spacelike submanifolds with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} obeying certain curvature conditions, which are inspired in those ones considered in Nishikawa's paper [102]. Extending the techniques developed in [17] and [128], our purpose is establish sufficient conditions to guarantee that such a spacelike submanifold M^n be either totally umbilical or isometric to an isoparametric hypersurface of a totally geodesic submanifold $L_1^{n+1} \hookrightarrow L_p^{n+p}$, with two distinct principal curvatures, one of which is simple (see Theorem 4.1.2). Before, in Section 3.6 we recall some basic facts concerning spacelike submanifolds immersed in a semi-Riemannian space. Afterwards, in Chapter 4 we present our set up, jointly with an example of a semi-Riemannian space which has no constant sectional curvature but obeys ours curvature constraints (see Example 1.2.1), and some key lemmas which are used to prove our main result.

Motivated by these last works described above, here we establish some rigidity results with respect to spacelike submanifolds in locally symmetric semi-Riemannian spaces, which can be found in more detail in Chapter 4

Theorem D.1 *Let M^n be an n-dimensional spacelike submanifold immersed with flat normal bundle and parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying curvature conditions (1.36), (1.37) and (1.38). Then, we have*

$$\frac{1}{2}\Delta \mathrm{tr}(h^{n+1})^2 \geq \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} + cn\mathrm{tr}(h^{n+1})^2 - cn^2H^2 \qquad (17)$$
$$-nH\mathrm{tr}(h^{n+1})^3 + (\mathrm{tr}(h^{n+1})^2)^2 + \sum_{\beta>n+1}(\mathrm{tr}(h^{n+1}h^{\beta}))^2,$$

and

$$\frac{1}{2}\Delta \parallel \tau \parallel^2 \geq \sum_{i,j,k,\alpha>n+1}(h_{ijk}^{\alpha})^2 + cn \parallel \tau \parallel^2 - nH \sum_{\alpha>n+1} \mathrm{tr}((h^{\alpha})^2 h^{n+1}) \qquad (18)$$
$$+ \sum_{\alpha>n+1}(\mathrm{tr}(h^{n+1}h^{\alpha}))^2 + \sum_{\alpha,\beta>n+1}(\mathrm{tr}(h^{\alpha}h^{\beta}))^2,$$

where $c = \frac{c_1}{n} + 2c_2$.

Theorem D.2 *Let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying curvature conditions (1.36), (1.37), (1.38) and (1.39), with paral-*

lel normalized mean curvature vector field and such that $R = aH + b$ *for some* $a, b \in \mathbb{R}$, *with* $(n-1)a^2 + 4n(\overline{\mathcal{R}} - b) \geq 0$. *If* $c = \frac{c_1}{n} + 2c_2 > 0$ *and* $S \leq 2\sqrt{n-1}\,c$, *then either*

(i) M^n *is totally umbilical, or*

(ii) $\sup_M S = 2\sqrt{n-1}\,c$. *Moreover, if* L_p^{n+p} *is conformally flat,* $\sup_M S$ *is attained at some point in* M^n *and* $\overline{\mathcal{R}} > b$, *then* M^n *is isometric to an isoparametric hypersurface of a totally geodesic submanifold* $L_1^{n+1} \hookrightarrow L_p^{n+p}$, *with two distinct principal curvatures, one of which is simple.*

Theorem D.3 *Let* M^n *be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space* L_p^{n+p} *with* $p > 1$ *and satisfying conditions* (1.36), (1.37), (1.38) *and* (1.39), *such that* $R = aH + b$, *with* $a \geq 0$ *and* $b \leq \overline{\mathcal{R}} < b + c$, *where* $c = \frac{c_1}{n} + 2c_2$. *Suppose that there exists an orthogonal basis for* TM *that diagonalizes simultaneously all* $A_\xi, \xi \in TM^\perp$. *Then,*

(i) either $|\Phi| \equiv 0$ *and* M^n *is a totally umbilical submanifold,*

(ii) or

$$\sup_M |\Phi| \geq \alpha(n, p, a, b, c, \overline{\mathcal{R}}) > 0,$$

where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ *is a positive constant that depends only on* $n, p, a, b, c, \overline{\mathcal{R}}$. *Moreover, if* $b < \overline{\mathcal{R}}$, *the equality* $\sup_M |\Phi| = \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ *holds and this supremum is attained at some point of* M^n, *then* M^n *is an isoparametric submanifold, in the sense that their principal curvatures are constant.*

Theorem D.4 *Let* M^n *be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space* L_p^{n+p} *with* $p > 1$ *and satisfying conditions* (1.36), (1.38) *and* (1.39), *such that* $R = aH + b$, *with* $a \geq 0$ *and* $b \leq \overline{\mathcal{R}} < b + c$, *where* $c = \frac{c_1}{n} + 2c_2$. *Suppose that there exists an orthogonal basis for* TM *that diagonalizes simultaneously all* $A_\xi, \xi \in TM^\perp$. *Assume in addition that* $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, *where* $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ *is the positive constant which was obtained in Theorem 4.2.2. If* M^n *is a* $\mathcal{L}$*-parabolic submanifold, then either* $|\Phi| \equiv 0$ *and* M^n *is totally umbilical, or* $|\Phi| \equiv \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ *and* M^n *is an isoparametric submanifold.*

Theorem D.5 *Let* M^n *be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field in a locally symmetric Einstein semi-Riemannian space* L_p^{n+p} *satisfying conditions* (1.36), (1.38) *and* (1.39), *such that*

$R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = \dfrac{c_1}{n} + 2c_2$. If $\sup_M |\Phi|^2 < +\infty$ and, for some reference point $o \in M^n$,

$$\int_0^{+\infty} \frac{dr}{\mathrm{vol}(\partial B_r)} = +\infty, \tag{19}$$

then M^n is $\mathcal{L}$-parabolic. Here B_r denotes the geodesic ball of radius r in M^n centered at the origin o.

Theorem D.6 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in locally symmetric Einstein semi-Riemannian space L_p^{n+p} with $p > 1$ and satisfying conditions (1.36), (1.37), (1.38) and (1.39), such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = \dfrac{c_1}{n} + 2c_2$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all $A_\xi, \xi \in TM^\perp$. Assume in addition that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ is the positive constant which was obtained in Theorem D.3. If $|\nabla H| \in \mathcal{L}^1(M)$, then either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $|\Phi| \equiv \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ and M^n is an isoparametric submanifold.*

This work is presented with the following organization. In Chapter 1 we establish the notations and preliminary facts that will be used throughout the text. In Chapter 2 we establish some rigidity results for complete spacelike hypersurface immersed into the steady state space $\mathcal{H}^{n+1}$ and the anti-de Sitter space $\mathbb{H}_1^{n+1}$. In the case of $\mathbb{H}_1^{n+1}$, we also show some curvature estimation and nonexistence results, and we estimate the nullity index for r-maximal spacelike hypersurfaces. In Chapter 3 we study the geometry of spacelike submanifolds immersed into pseudo-Riemannian space form. Here, via the application of new maximum principles in Riemannian manifolds, we establish some results for stochastically, parabolic, L^1-Liouville and linear Weingarten complete spacelike submanifolds. Finally, in Chapter 4 we provide some results for complete spacelike submanifolds in locally symmetric semi-Riemannian spaces, ambient spaces that extend the pseudo-Riemannian space forms.

Capítulo 1

Preliminary

In this first chapter we aim to establish the notations that will be used in the other chapters of this work, as well as the basic facts of the theory of isometric immersions which we will make use of later. For more details, we indicate as references [103], [67], [49] and [41].

Initially, if M^n is a smooth manifold then $C^\infty(M)$ will always denote the ring of real functions of class C^∞ on M^n and $\mathfrak{X}(M)$ the $C^\infty(M)$-module of vector fields of class C^∞ on M^n. Next, we describe in two sections the elements that we must consider about the theory of ambient spaces and also about the isometric immersions in these spaces, objects that are of great interest in our study.

1.1 Spacelike hypersurfaces in Lorentz manifolds with constant sectional curvature

Let $\mathbb{R}^{n+2}_\nu$ be the $(n+2)$-dimensional semi-Euclidean space endowed with the metric tensor $\langle\, , \rangle$ of index $\nu \in \{1, 2\}$, given by

$$\langle v, w\rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2}$$

if $\nu = 1$, or

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i - v_{n+1} w_{n+1} - v_{n+2} w_{n+2},$$

when $\nu = 2$. As is common in the current literature, $\mathbb{R}_1^{n+2}$ is denoted by $\mathbb{L}^{n+2}$, and is called the $(n+2)$-dimensional *Lorentz-Minkowski space*. The $(n+1)$-dimensional *de Sitter space* is defined as the following hyperquadric of $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{L}^{n+2}; \langle x, x\rangle = 1\},$$

while the $(n+1)$-dimensional *anti-de Sitter space* corresponds to the following hyperquadric of $\mathbb{R}_2^{n+2}$

$$\mathbb{H}_1^{n+1} = \{x \in \mathbb{R}_2^{n+2}; \langle x, x\rangle = -1\}.$$

As it is well known, $\mathbb{L}^{n+1}$, $\mathbb{S}_1^{n+1}$ and $\widetilde{\mathbb{H}}_1^{n+1}$ are the standard simply connected Lorentzian space forms of constant sectional curvature 0, 1 and -1, respectively, where $\widetilde{\mathbb{H}}_1^{n+1}$ denotes the universal covering of $\mathbb{H}_1^{n+1}$ (see, for instance, Section 5.3 of [36] or Section 8.6 of [103]). To describe another Lorentz manifold of constant sectional curvature equal to 1, let $a \in \mathbb{L}^{n+2} \setminus \{0\}$ be a past-pointing null vector, that is, $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$. Then, the open region of the de Sitter space $\mathbb{S}_1^{n+1}$, given by

$$\mathcal{H}^{n+1} = \left\{p \in \mathbb{S}_1^{n+1} : \langle p, a\rangle > 0\right\}$$

is the so-called *steady state space*.

In order to simplify our notation, throughout this work we will denote these $(n+1)$-dimensional spaces by $\mathbb{L}_1^{n+1}(c)$ according to $c \in \{-1, 0, 1\}$. More specifically, $\mathbb{L}_1^{n+1}(c) = \mathbb{H}_1^{n+1}$ when $c = -1$, $\mathbb{L}_1^{n+1}(c) = \mathbb{L}^{n+1}$ if $c = 0$ and $\mathbb{L}_1^{n+1}(c) = \mathbb{S}_1^{n+1}$ or $\mathbb{L}_1^{n+1}(c) = \mathcal{H}^{n+1}$ when $c = 1$.

In this setting, let $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ be a connected spacelike hypersurface immersed into $\mathbb{L}_1^{n+1}(c)$, which means that the induced metric via ψ is a Riemannian metric on Σ^n. In order to set up our notation, we will denote by ∇^0,$\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{R}_2^{n+2}$, $\mathbb{L}_1^{n+1}(c)$ and Σ^n, respectively. Then, the Gauss and Weingarten formulas corresponding to Σ^n are given, respectively, by

$$\nabla^0{}_X Y = \nabla_X Y - \langle A(X), Y\rangle N - c\langle X, Y\rangle\psi \tag{1.1}$$

and

$$A(X) = -\overline{\nabla}_X N = -\nabla^0{}_X N, \tag{1.2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where A stands for the Weingarten operator of $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ with respect to a choice of timelike orientation N for Σ^n.

As in [103], the curvature tensor R of the spacelike hypersurface $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$. So, the Gauss equation is given by

$$R(X,Y)Z = c\,\{\langle X,Z\rangle Y - \langle Y,Z\rangle X\} + \langle A(Y),Z\rangle A(X) - \langle A(X),Z\rangle A(Y), \qquad (1.3)$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$.

On the other hand, at each $p \in \Sigma^n$, the Weingarten operator A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. For $0 \le r \le n$, let $S_r(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p. Thus, one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by convention. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$. This allows us to define the r-th mean curvature H_r of $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$, $0 \le r \le n$, by

$$\binom{n}{r} H_r = (-1)^r S_r. \qquad (1.4)$$

We observe that $H_0 = 1$, while $H_1 = -(1/n)S_1$ is the usual mean curvature H of $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$. It also follows from Gauss equation that H_2 is, up to a constant, the normalized scalar curvature R of $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$. Indeed, from (1.3) we have that the Ricci curvature of $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ is given by

$$\mathrm{Ric}(X,Y) = c(n-1)\langle X,Y\rangle - \mathrm{tr}(A)\langle A(X),Y\rangle + \langle A(X),A(Y)\rangle, \qquad (1.5)$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. Hence, we obtain the following relation

$$|A|^2 = n^2H^2 - n(n-1)H_2 = n^2H^2 + n(n-1)(R+1). \qquad (1.6)$$

For $0 \leq r \leq n$, one defines the r-th Newton transformation P_r on Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = \binom{n}{r} H_r I + A P_{r-1}. \tag{1.7}$$

With a trivial induction, from (1.7) we verify that

$$P_r = \binom{n}{r} H_r I + \binom{n}{r-1} H_{r-1} A + \binom{n}{r-2} H_{r-2} A^2 + \cdots + A^r, \tag{1.8}$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p\Sigma$ diagonalizing A at $p \in \Sigma^n$ also diagonalize all of the P_r at p. So, let $\{e_1, \dots, e_n\}$ be an orthonormal frame on $T_p\Sigma$ which diagonalizes A_p, $A_p(e_i) = \lambda_i(p) e_i$, then from (1.8) we have that

$$(P_r)_p e_i = (-1)^r \sum_{i_1 < \ldots < i_r, i_j \neq i} \lambda_{i_1}(p) \ldots \lambda_{i_r}(p) e_i. \tag{1.9}$$

Moreover, it is not difficult to check that $P_r e_i = (-1)^r S_r(A_i) e_i$ and, consequently, we obtain the following lemma (see Lemma 2.1 of [38]).

Lemma 1.1.1 *With the above notations, the following formulas hold:*

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;

(b) $\operatorname{tr}(P_r) = (-1)^r \sum_{i=1}^{n} S_r(A_i) = (-1)^r (n-r) S_r = c_r H_r$;

(c) $\operatorname{tr}(A P_r) = (-1)^r \sum_{i=1}^{n} \lambda_i S_r(A_i) = (-1)^r (r+1) S_{r+1} = -c_r H_{r+1}$;

(d) $\operatorname{tr}(A^2 P_r) = (-1)^r \sum_{i=1}^{n} \lambda_i^2 S_r(A_i) = \binom{n}{r+1} (nHH_{r+1} - (n-r-1) H_{r+2})$,

where $c_r = (n-r)\binom{n}{r}$.

Associated to each Newton transformation P_r one has the second-order linear differential operator L_r, defined by

$$L_r f = \operatorname{tr}(P_r \nabla^2 f), \tag{1.10}$$

where $\nabla^2 f : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, which is given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X \nabla f, Y \rangle = Hess f(X, Y),$$

for all $X, Y \in \mathfrak{X}(\Sigma)$.

For a smooth function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ and $f \in C^\infty(\Sigma)$, it follows from the properties of the Hessian of functions that

$$L_r(\varphi \circ f) = \varphi'(f) L_r(f) + \varphi''(f) \langle P_r \nabla f, \nabla f \rangle. \tag{1.11}$$

In particular, for $r = 0$, we get the well known Laplacian operator $L_0 = \Delta$, which is always elliptic. The next lemma gives a geometric condition which guarantees the ellipticity of L_1 (cf. Lemma 3.2 of [5]).

Lemma 1.1.2 *Let $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ be a spacelike hypersurface. If $H_2 > 0$ on Σ^n, then L_1 is elliptic or, equivalently, P_1 is positive definite (for an appropriate choice of orientation N).*

When $r \geq 2$, the following lemma establishes sufficient conditions to guarantee the ellipticity of L_r (cf. Lemma 3.3 of [5]).

Lemma 1.1.3 *Let $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ be a spacelike hypersurface. If there exists an elliptic point of Σ^n, with respect to an appropriate choice of orientation N, and $H_{r+1} > 0$ on Σ^n, for $2 \leq r \leq n-1$, then for all $1 \leq k \leq r$ the operator L_k is elliptic or, equivalently, P_k is positive definite (for an appropriate choice of orientation N, if k is odd).*

Here, by an *elliptic point* in a spacelike hypersurface $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ we mean a point $p_0 \in \Sigma^n$ where all principal curvatures $\lambda(p_0)$ are negative.

The next lemma was done by Alías, Brasil Jr. and Colares [4] in a more general setting, when they studied spacelike hypersurfaces in conformally stationary spacetime (see Lemma 5.4 of [4]). Taking into account our purposes, we rewrote it as follows.

Lemma 1.1.4 *Let V be a complete closed conformal timelike vector field globally defined on the Lorentzian manifold $\mathbb{L}_1^{n+1}(c)$, and let $\psi : \Sigma^n \to \mathbb{L}_1^{n+1}(c)$ be a complete spacelike hypersurface. Suppose that the divergence of V on $\mathcal{H}^{n+1}$, $\mathrm{Div} V$, does not vanish at a point of Σ^n where the restriction $|V|_\Sigma = \sqrt{-\langle V, V \rangle}|_\Sigma$ of $|V|$ to Σ^n attains a local minimum. Then, there exists an elliptic point $p_0 \in \Sigma^n$.*

In [132] Yau, generalizing a previous result due to Gaffney [79], established the following version of Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold Σ^n: *If $\omega \in \Omega^{n-1}(\Sigma)$ is an integrable $(n-1)$-differential form on Σ^n,* *then there exists a sequence* B_i *of domains on* Σ^n *such that* $B_i \subset B_{i+1}$*,* $\Sigma^n = \bigcup_{i\geq 1} B_i$ *and* $\lim_{i\to+\infty} \int_{B_i} d\omega = 0$.

Supposing that Σ^n is oriented by the volume element $d\Sigma$, denoting by $\mathcal{L}^1(\Sigma)$ the space of Lebesgue integrable functions on Σ^n and considering $\omega = \iota_X d\Sigma$ the contraction of $d\Sigma$ in the direction of a smooth vector field X on Σ^n, Caminha obtained the following consequence of Yau's result (cf. Proposition 2.1 of [43]).

Lemma 1.1.5 *Let* X *be a smooth vector field on the* n*-dimensional complete oriented Riemannian manifold* Σ^n*, such that* $\mathrm{div}X$ *does not change sign on* Σ^n*. If* $|X| \in \mathcal{L}^1(\Sigma)$*, then* $\mathrm{div}X = 0$.

For a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ and $f \in C^\infty(\Sigma)$, it follows from the properties of the Hessian of functions that

$$L_r(\varphi \circ f) = \varphi'(f)L_r(f) + \varphi''(f)\langle P_r\nabla f, \nabla f\rangle. \tag{1.12}$$

Furthermore, according to [112], we observe that

$$\begin{aligned} \mathrm{div}(P_r(\nabla f)) &= \sum_{i=1}^{n}\langle(\nabla_{e_i}P_r)(\nabla f), e_i\rangle + \sum_{i=1}^{n}\langle P_r(\nabla_{e_i}\nabla f), e_i\rangle \\ &= \langle \mathrm{div}P_r, \nabla f\rangle + L_r f, \end{aligned} \tag{1.13}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on Σ^n and the divergence of P_r on Σ^n is given by

$$\mathrm{div}P_r = \mathrm{tr}(\nabla P_r) = \sum_{i=1}^{n}(\nabla_{e_i}P_r)(e_i).$$

Consequently, since Corollary 3.2 of [4] guarantees that P_r has divergence free when the ambient space has constant sectional curvature, what happens to $\mathbb{L}_1^{n+1}(c)$, from (1.13) we get that

$$L_r f = \mathrm{div}(P_r\nabla f). \tag{1.14}$$

We close this section recalling the description of the totally umbilical spacelike hypersurfaces of the anti-de Sitter space $\mathbb{H}_1^{n+1}$ (see, for instance, Section 4 of [21] or Example 2 of [94]). For this, we fix an unit timelike vector $a \in \mathbb{R}_2^{n+2}$ (that is, $\langle a, a\rangle = -1$) and consider the smooth function $h_a : \mathbb{H}_1^{n+1} \to \mathbb{R}$ defined by $h_a(p) = \langle p, a\rangle$. A straightforward computation allows us to conclude that for every real number τ, such that $|\tau| < 1$, the level set

$$M_\tau = h_a^{-1}(\tau) = \{p \in \mathbb{H}_1^{n+1} : \langle p, a\rangle = \tau\}$$

is a totally umbilical hypersurface in $\mathbb{H}_1^{n+1}$, with the Gauss mapping

$$N_\tau(p) = \frac{1}{\sqrt{|\langle a, a\rangle + \tau^2|}}\,(a + \tau p). \tag{1.15}$$

Hence, the shape operator A_τ of M_τ is given by

$$A_\tau(X) = -\frac{\tau}{\sqrt{|\langle a, a\rangle + \tau^2|}}X, \tag{1.16}$$

for all smooth vector field X tangent to M_τ. Consequently, we have the following possibilities:

(1) if a is a unit spacelike vector, then M_τ is isometric to the anti-de Sitter space $\mathbb{H}_1^n(-\sqrt{1+\tau^2})$ of constant sectional curvature $-\dfrac{1}{1+\tau^2}$;

(2) if a is a nonzero null vector, then $\tau \neq 0$ and M_τ is isometric to the Lorentz-Minkowski space $\mathbb{L}^n$;

(3) if a is a unit timelike vector, then either $|\tau| > 1$ and M_τ is isometric to a de Sitter space $\mathbb{S}_1^n(\sqrt{\tau^2-1})$ of constant sectional curvature $\dfrac{1}{\tau^2-1}$, or $|\tau| < 1$ and M_τ is isometric to a hyperbolic space $\mathbb{H}^n(-\sqrt{1-\tau^2})$ of constant sectional curvature $-\dfrac{1}{1-\tau^2}$.

1.2 Spacelike submanifolds imersed in locally symetric semi-Riemannian spaces

Let us denote by L_p^{n+p} an $(n+p)$-dimensional connected *semi-Riemannian space with index p*, which means that in every tangent space of L_p^{n+p} there is a subspace of dimension p in which a the metric tensor is negative. We say that L_p^{n+p} is *locally symmetric* when its curvature tensor $\bar{R}$ is parallel in the sense that $\overline{\nabla}\bar{R} = 0$, where $\overline{\nabla}$ denotes the Levi-Civita connection of L_p^{n+p}. For the moment, we can record that if L_p^{n+p} has constant sectional curvature then L_p^{n+p} is locally symmetric.

Let M^n be a spacelike submanifold immersed in a locally symmetric semi-Riemannian space L_p^{n+p}, which means that the induced metric of L_p^{n+p} is a Riemannian metric on M^n. In this context, we choose a local field of semi-Riemannian orthonormal frames

$e_1, \ldots, e_{n+p}$ in L_p^{n+p}, with dual coframes $\omega_1, \ldots, \omega_{n+p}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n. We will use the following convention of indices

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

In this setting, the semi-Riemannian metric of L_p^{n+p} is given by

$$d\overline{s}^2 = \sum_A \epsilon_A \, \omega_A^2,$$

where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$. Denoting by $\{\omega_{AB}\}$ the connection forms of L_p^{n+p}, we have that the structure equations of L_p^{n+p} are given by:

$$d\omega_A = \sum_B \epsilon_B \, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{1.17}$$

$$d\omega_{AB} = \sum_C \epsilon_C \, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \epsilon_C \epsilon_D K_{ABCD} \, \omega_C \wedge \omega_D, \tag{1.18}$$

where, $\overline{R}_{ABCD}$, $\overline{R}_{CD}$ and $\overline{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Lorentz space L_p^{n+p}. In this setting, we have

$$\overline{R}_{CD} = \sum_B \varepsilon_B \overline{R}_{CBDB} \quad \text{and} \quad \overline{R} = \sum_A \varepsilon_A \overline{R}_{AA}. \tag{1.19}$$

Moreover, the components $\overline{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor L_p^{n+p} are defined by

$$\sum_E \varepsilon_E \overline{R}_{ABCD;E} \omega_E = d\overline{R}_{ABCD} - \sum_E \varepsilon_E (\overline{R}_{EBCD} \omega_{EA} + \overline{R}_{AECD} \omega_{EB} + \overline{R}_{ABED} \omega_{EC} + \overline{R}_{ABCE} \omega_{ED}).$$

Next, we restrict all the tensors to M^n. First of all,

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p.$$

Consequently, the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. Since

$$\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha = 0,$$

from Cartan's Lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{1.20}$$

This gives the second fundamental form of M^n, $B = \sum_{\alpha,i,j} h_{ij}^{\alpha}\omega_i \otimes \omega_j e_{\alpha}$, and its square length from second fundamental form is $S = |B|^2 = \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2$. Furthermore, we define the mean curvature vector field $\mathbf{H}$ and the mean curvature function H of M^n respectively by

$$\mathbf{H} = \frac{1}{n}\sum_{\alpha}\left(\sum_{i} h_{ii}^{\alpha}\right)e_{\alpha} \quad \text{and} \quad H = |\mathbf{H}| = \frac{1}{n}\sqrt{\sum_{\alpha}\left(\sum_{i} h_{ii}^{\alpha}\right)^2}.$$

The structure equations of M^n are given by

$$d\omega_i = -\sum_j \omega_{ij}\wedge\omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ik}\wedge\omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l,$$

where R_{ijkl} are the components of the curvature tensor of M^n. Using the previous structure equations, we obtain Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} - \sum_{\beta}(h_{ik}^{\beta}h_{jl}^{\beta} - h_{il}^{\beta}h_{jk}^{\beta}). \tag{1.21}$$

and

$$n(n-1)R = \sum_{i,j}\overline{R}_{ijij} - n^2H^2 + S. \tag{1.22}$$

We also state the structure equations of the normal bundle of M^n

$$d\omega_{\alpha} = -\sum_{\beta}\omega_{\alpha\beta}\wedge\omega_{\beta}, \quad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0,$$

$$d\omega_{\alpha\beta} = -\sum_{\gamma}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} - \frac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\omega_k\wedge\omega_l.$$

Supposing that M^n has normal bundle flat, that is, $R^{\perp} = 0$ (equivalently $R_{\alpha\beta jk} = 0$), we get the following Ricci equation

$$\overline{R}_{\alpha\beta ij} = \sum_k (h_{ik}^{\alpha}h_{kj}^{\beta} - h_{kj}^{\alpha}h_{ik}^{\beta}). \tag{1.23}$$

The components h_{ijk}^{α} of the covariant derivative ∇B satisfy

$$\sum_k h_{ijk}^{\alpha}\omega_k = dh_{ij}^{\alpha} + \sum_k h_{ik}^{\alpha}\omega_{kj} + \sum_k h_{jk}^{\alpha}\omega_{ki} - \sum_{\beta} h_{ij}^{\beta}\omega_{\beta\alpha}. \tag{1.24}$$

In this setting, from (1.20) and (1.24) we get Codazzi equation

$$\overline{R}_{\alpha ijk} = h^{\alpha}_{ijk} - h^{\alpha}_{ikj}. \tag{1.25}$$

The first and the second covariant derivatives of h^{α}_{ij} are denoted by h^{α}_{ijk} and h^{α}_{ijkl}, respectively, which satisfy

$$\sum_l h^{\alpha}_{ijkl}\omega_l = dh^{\alpha}_{ijk} + \sum_l h^{\alpha}_{ljk}\omega_{li} + \sum_l h^{\alpha}_{ilk}\omega_{lj} + \sum_l h^{\alpha}_{ijl}\omega_{lk} - \sum_\beta h^{\beta}_{ijk}\omega_{\beta\alpha}.$$

Thus, taking the exterior derivative in (1.24), we obtain the following Ricci identity

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum_m h^{\alpha}_{im} R_{mjkl} + \sum_m h^{\alpha}_{mj} R_{mikl}. \tag{1.26}$$

Restricting the covariant derivative $\overline{R}_{ABCD;E}$ of $\overline{R}_{ABCD}$ on M^n, then $\overline{R}_{\alpha ijk;l}$ is given by

$$\overline{R}_{\alpha ijkl} = \overline{R}_{\alpha ijk;l} + \sum_\beta \overline{R}_{\alpha\beta jk} h^{\beta}_{il} + \sum_\beta \overline{R}_{\alpha i\beta k} h^{\beta}_{jl} + \sum_\beta \overline{R}_{\alpha ij\beta} h^{\beta}_{kl} + \sum_{m,k} \overline{R}_{mijk} h^{\alpha}_{lm}, \tag{1.27}$$

where $\overline{R}_{\alpha ijkl}$ denotes the covariant derivative of $\overline{R}_{\alpha ijk}$ as a tensor on M^n.

For our purposes, we will consider that the mean curvature function H is positive, so that in the local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ we take $e_{n+1} = \frac{h}{H}$. Thus, we deal with the traceless second fundamental form Φ, which is defined as been the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi^{\alpha}_{ij} \omega_i \otimes \omega_j e_\alpha,$$

where $\Phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$. Here, H^{α} denotes the mean curvature function of M^n in the direction of e_α, that is,

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} = \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \geq n+2, \tag{1.28}$$

where $h^{\alpha} = (h^{\alpha}_{ij})$ denotes the second fundamental form of M^n in direction e_α for every $n+1 \leq \alpha \leq n+p$. From here it is not difficult to verify that Φ is a traceless tensor, that is, $\mathrm{tr}(\Phi) = 0$ and that holds the following relation

$$|\Phi|^2 = S - nH^2. \tag{1.29}$$

Moreover, $|\Phi|$ vanishes identically on M^n if and only if M^n is a totally umbilical spacelike submanifold. For this reason, Φ is also called the total umbilicity tensor of M^n. We also note that, by (1.22), the following relation is trivially satisfied:

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijij} - n(n-1)H^2 + |\Phi|^2. \tag{1.30}$$

At this point, we will assume that M^n is a *linear Weingarten* spacelike submanifold, which means that the normalized scalar curvature and mean curvature functions are linearly related in the following way: there exist real constants $a, b \in \mathbb{R}$ such that

$$R = aH + b.$$

Related to the geometry of linear Weingarten spacelike submanifolds there exists a Cheng-Yau type differential operator, which recently has been considered by many authors. More precisely, let us introduce the second order linear differential operator $\mathcal{L} : C^\infty(M) \to C^\infty(M)$ defined by

$$\mathcal{L} = L + \frac{n-1}{2} a\Delta, \tag{1.31}$$

where Δ is the Laplacian operator on M^n and $L : C^\infty(M) \to C^\infty(M)$ denotes the standard Cheng-Yau's operator [56], which is given by

$$Lu = \operatorname{tr}(P \circ \nabla^2 u), \tag{1.32}$$

for every $u \in C^\infty(M)$. Here, $\nabla^2 u$ is the self-adjoint linear tensor metrically equivalent to the Hessian of u and $P : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the first Newton transformation of M^n, that is, the tensor

$$P = nHI - h^{n+1}. \tag{1.33}$$

Thus, from (1.31) and (1.32) we get

$$\mathcal{L}u = \operatorname{tr}(\mathcal{P} \circ \nabla^2 u), \tag{1.34}$$

where

$$\mathcal{P} = \left(nH + \frac{n-1}{2} a\right) I - h^{n+1}. \tag{1.35}$$

Proceeding, inspired by the configuration assumed in [39], here we will suppose that there exist constants c_1, c_2 and c_3 such that the sectional curvature $\overline{K}$ and the curvature tensor $\overline{R}$ of the ambient space L_p^{n+p} satisfy the following constraints:

$$\overline{K}(u, \eta) = \frac{c_1}{n}, \tag{1.36}$$

for any spacelike vector u and any timelike vector η; when $p > 1$, suppose that

$$\langle \overline{R}(\xi, u)\eta, u \rangle = 0. \tag{1.37}$$

for any spacelike vector u and timelike vectors ξ, η, with $\langle \xi, \eta \rangle = 0$.

$$\overline{K}(u, v) \geq c_2, \tag{1.38}$$

for any spacelike vectors u, v;

$$\overline{K}(\eta, \xi) = \frac{c_3}{p}, \tag{1.39}$$

for timelike vectors η, ξ.

The curvature conditions (1.36) and (1.38), are natural extensions for higher codimension of conditions assumed by Nishikawa [102] in context of hypersurfaces. When the ambient manifold L_p^{n+p} has constant sectional curvature c, then it satisfies conditions (1.36), (1.37), (1.38) and (1.39). On the other hand, the next example gives us a situation where the curvature conditions (1.36), (1.37), (1.38) and (1.39) are satisfied but the ambient space has not constant sectional curvature.

Example 1.2.1 *Let $L_p^{n+p} = \mathbb{R}_p^p \times \mathbb{S}^n$ be a semi-Riemannian space, where $\mathbb{R}_p^p$ stands for the p-dimensional semi-Euclidean space of index p and $\mathbb{S}^n$ is the n-dimensional unit Euclidean sphere. We consider the spacelike submanifold $M^n = \{0\} \times \mathbb{S}^n$ of L_p^{n+p}. Taking into account that the normal bundle of M^n is equipped with p linearly independent timelike vector fields $\xi^1, \xi^2, \ldots, \xi^p$, it is not difficult to verify that the sectional curvature $\overline{K}$ of L_p^{n+p} satisfies*

$$\overline{K}\,(\xi_i, X) \;\; = \;\; \langle R_{\mathbb{R}_p^p}(\xi^i, X)\xi^i, X\rangle_{\mathbb{R}_p^p} + \langle R_{\mathbb{S}^n}(0, U)0, U\rangle_{\mathbb{S}^n} = 0, \tag{1.40}$$

for each $i \in \{1, \ldots, p\}$, where $R_{\mathbb{R}_p^p}$ and $R_{\mathbb{S}^n}$ denote the curvature tensors of $\mathbb{R}_p^p$ and $\mathbb{S}^n$, respectively, $\xi_i = (\xi^i, 0) \in T^{\perp}M$ and $X = (0, X_2) \in TM$ with $\langle \xi_i, \xi_i \rangle = \langle X, X \rangle = 1$. On the other hand, by a direct computation we obtain

$$\overline{K}(X, Y) = \langle R_{\mathbb{R}_p^p}(0, 0)0, 0\rangle_{\mathbb{R}_p^p} + \langle R_{\mathbb{S}^n}(X_2, Y_2)X_2, Y_2\rangle_{\mathbb{S}^n}, \tag{1.41}$$

for every $X = (0, X_2), Y = (0, Y_2) \in TM$ such that $\langle X, Y \rangle = 0, \langle X, X \rangle = \langle Y, Y \rangle = 1$. Consequently, since

$$\langle X_2, Y_2 \rangle = 0, \langle X_2, X_2 \rangle = \langle Y_2, Y_2 \rangle = 1,$$

from (3.137) we get

$$\overline{K}(X, Y) = |X_2|^2 |Y_2|^2 - \langle X_2, Y_2 \rangle^2 = 1. \tag{1.42}$$

Moreover, we have that

$$\overline{K}(\xi_i, \xi_j) = 0, \quad \textit{for all} \quad i, j \in \{1, \ldots, p\} \tag{1.43}$$

and

$$\langle \overline{R}(\xi_i, X)\xi_j, X\rangle = 0, \quad \textit{for all} \quad i, j \in \{1, \ldots, p\}. \tag{1.44}$$

We observe from (1.40), (1.42), (1.43) and (1.44) that the curvature constraints (1.36), (1.37), (1.38) and (1.39) are satisfied with $c_1 = c_3 = 0$, $c_2 = 1$ and $c = \frac{c_1}{n} + 2c_2 = 2$. Furthermore, we also note that the ambient space $L_p^{n+p} = \mathbb{R}_p^p \times \mathbb{S}^n$ is conformally flat (see, for instance, Chapter 7 of [67]). This property will be assumed in the last part of our main result (cf. Theorem 4.2.2 in Section 4.1).

Now, we denote by $\overline{R}_{CD}$ the components of the Ricci tensor of L_p^{n+p}. So, its scalar curvature $\overline{R}$ is given by

$$\overline{R} = \sum_A \varepsilon_A \overline{R}_{AA} = \sum_{i,j} \overline{R}_{ijij} - 2\sum_{i,\alpha} \overline{R}_{i\alpha i\alpha} + \sum_{\alpha,\beta} \overline{R}_{\alpha\beta\alpha\beta}.$$

Furthermore, if L_p^{n+p} satisfies conditions (1.36) and (1.39), then

$$\overline{R} = \sum_{i,j} \overline{R}_{ijij} - 2pc_1 + (p-1)c_3. \tag{1.45}$$

But, it is well known that the scalar curvature of a locally symmetric Lorentz space is constant (see Proposition 8.10 of [103]). Consequently, $\frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$ is a constant naturally attached to a locally symmetric Lorentz space satisfying conditions (1.36) and (1.39), which will be denoted by $\overline{\mathcal{R}}$.

Considering the previous digression, we obtain the following lemma whose proof can be found in [17].

Lemma 1.2.2 *Let M^n be a linear Weingarten spacelike submanifold immersed in locally symmetric space L_p^{n+p} satisfying conditions (1.36) and (1.39), such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \geq 0. \tag{1.46}$$

Then,

$$|\nabla A|^2 \geq n^2 |\nabla H|^2. \tag{1.47}$$

Moreover, if the equality holds in (4.2) on M^n, then H is constant on M^n.

Capítulo 2

Results for spacelike hypersurface in the $\mathcal{H}^{n+1}$ and $\mathbb{H}_1^{n+1}$

In this chapter, we study the complete spacelike hypersurfaces immersed in an open region of the de Sitter space $\mathbb{S}_1^{n+1}$ which is known as the *steady state space* $\mathcal{H}^{n+1}$ and also the geometry of complete spacelike hypersurfaces immersed in the Anti-de Sitter space $\mathbb{H}_1^{n+1}$. In this setting, under suitable constraints on the behavior of the higher order mean curvatures of these hypersurfaces, we prove that they must be totally umbilical spacelike hypersurfaces of $\mathcal{H}^{n+1}$ or $\mathbb{H}_1^{n+1}$. For more details, you can look at the works [26], [27] and [29].

2.1 Rigidity of complete spacelike hypersurfaces in the $\mathcal{H}^{n+1}$

As introduced before, the steady state space $\mathcal{H}^{n+1}$ is the hyperquadric

$$\mathcal{H}^{n+1} = \{x \in \mathbb{S}_1^{n+1}; \langle x, a\rangle > 0\},$$

with $a \in \mathbb{L}^{n+2}$ be a past-pointing null vector, that is, $\langle a, a\rangle = 0$ and $\langle a, e_{n+2}\rangle > 0$, where $e_{n+2} = (0, \ldots, 0, 1)$.

According to Example 2 in Section 4 of [100], the timelike vector field

$$V = \langle x, a\rangle x + a$$

is such that

$$\overline{\nabla}_W V = \langle x, a\rangle W, \tag{2.1}$$

for all $W \in \mathfrak{X}(\mathcal{H}^{n+1})$, where $\overline{\nabla}$ stands for the Levi-Civita connection of $\mathcal{H}^{n+1}$. Thus, V is a complete closed conformal timelike vector field globally defined on $\mathcal{H}^{n+1}$. Proposition 1 of [100] guarantees that the n-dimensional distribution $\mathcal{D}(V)$ defined on $\mathcal{H}^{n+1}$ by

$$x \in \mathcal{H}^{n+1} \longmapsto \mathcal{D}(x) = \{v \in T_x\mathcal{H}^{n+1} : \langle V(x), v\rangle = 0\}$$

determines a codimension one spacelike foliation $\mathcal{F}(V)$ which is oriented by V. Furthermore, the leaves of $\mathcal{D}(V)$ are given by

$$\mathcal{E}_\tau = \{x \in \mathcal{H}^{n+1} : \langle x, a\rangle = \tau\}, \ \text{ with } \ \tau > 0,$$

which are totally umbilical hypersurfaces of $\mathcal{H}^{n+1}$ isometric to the n-dimensional Euclidean space $\mathbb{R}^n$, having constant r-th mean curvature $H_r = 1$ (with respect to the unit normal fields $N_\tau = x - \dfrac{1}{\tau}a, \quad x \in \mathcal{L}_\tau$, when r is odd).

Figure 1: Foliating $\mathcal{H}^{n+1}$ via spacelike hyperplanes $\mathcal{E}_\tau$.

In this setting, we will consider two particular functions naturally attached to a spacelike hypersurface Σ^n immersed on $\mathcal{H}^{n+1}$, namely, the height and angle functions with respect to a previously fixed nonzero null vector $a \in \mathbb{L}^{n+2}$, which are defined, respectively, by $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$. It is not difficult to check that $\nabla l_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ stands for the orthogonal projection of a onto the tangent bundle $T\Sigma$. Moreover, using Gauss and Weingarten formulas, we obtain

$$\nabla_X \nabla l_a = -f_a AX - l_a X, \tag{2.2}$$

for all $X \in \mathfrak{X}(\Sigma)$. From Lemma 1.1.1 jointly with (2.2), we can deduce the formula for the operator L_r acting on the height function, that is,

$$L_r l_a = c_r(-l_a H_r + f_a H_{r+1}), \tag{2.3}$$

where $c_r = (n-r)\binom{n}{r}$.

In what follows, we say that a spacelike hypersurface Σ^n immersed in $\mathcal{H}^{n+1}$ is contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$, with $\tau > 0$, when its height function satisfies $l_a \leq \tau$ (see Figure 2).

Figure 2: Σ^n is contained in the closure of the interior domain enclosed by $\mathcal{E}_\tau$.

Lemma 2.1.1 *Let Σ^n be a complete Riemannian manifold with sectional curvature bounded from below, and $f \in C^\infty(\Sigma)$ be a function which is bounded from above on Σ^n. If $\mathcal{P}$ is positive semi-definite and $\mathrm{tr}(\mathcal{P})$ is bounded from above on Σ^n, then there exists a sequence $(p_k)_{k\geq 1}$ in Σ^n such that*

$$\lim_k f(p_k) = \sup_\Sigma f, \quad \lim_k |\nabla f(p_k)| = 0 \quad \text{and} \quad \limsup_k \mathcal{L}f(p_k) \leq 0,$$

where the operator $\mathcal{L}$ is given by (4.4).

Now, we are in position to state and prove our first rigidity result for complete spacelike hypersurfaces $\psi : \Sigma^n \to \mathcal{H}^{n+1}$. Fixed a past-pointing nonzero null vector $a \in \mathbb{L}^{n+2}$ as before and taking a future-pointing orientation N for such a ψ, along this chapter we will assume that its angle function f_a is positive.

Theorem 2.1.2 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$. Suppose that the mean curvature H of Σ^n is positive, bounded and satisfies*

$$H \leq H_2. \tag{2.4}$$

If

$$|a^\top| \leq C \inf_\Sigma (H_2 - H), \tag{2.5}$$

for some positive constant C, then Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$.

Proof. Regarding that $a^\top$ stands for the orthogonal projection of a onto the tangent bundle $T\Sigma$, we have that $a^\top = a + f_a N - l_a \psi$. Consequently,

$$|a^\top|^2 = f_a^2 - l_a^2. \tag{2.6}$$

In particular, we obtain that $f_a \geq l_a > 0$. From (2.3) we can see that

$$L_1(\, l_a\,) = -c_1 H l_a + c_1 H_2 f_a \geq c_1 (H_2 - H) l_a, \tag{2.7}$$

where $c_1 = n(n-1)$.

From Cauchy-Schwarz inequality we have that $H_2 \leq H^2$ and, since we are assuming that H is bounded, we get that H_2 is also bounded. Thus, taking into the algebraic relation

$$\sum_{i=1}^{n} \lambda_i^2 = |A|^2 = n^2H^2 - n(n-1)H_2, \tag{2.8}$$

we have that all the principal curvatures λ_i of Σ^n are bounded. Consequently, from Gauss equation

$$K_\Sigma(e_i, e_j) = 1 - \lambda_i\lambda_j, \tag{2.9}$$

we conclude that the sectional curvature K_Σ of Σ^n is bounded from below.

We note that our assumption that Σ^n is contained in the closure of the interior domain enclosed by $\mathcal{E}_\tau$ guarantees that l_a is bounded. On the other hand, using hypothesis (2.4), it follows from Lemma 3.10 of [70] that L_1 is elliptic and, in particular, P_1 is positive semi-definite and $\mathrm{tr}(P_1) = c_1H$ is bounded.

Hence, since (2.7) gives

$$L_1(\, l_a\,) \geq n(n-1)\,(H_2 - H)\, l_a \geq 0.$$

we can apply Lemma 2.1.1 to obtain a sequence $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k l_a(p_k) = \sup_\Sigma l_a, \quad \lim_k |\nabla l_a(p_k)| = 0 \quad \text{and} \quad \limsup_k L_1(\, l_a\,)(p_k) \leq 0. \tag{2.10}$$

Consequently, from (2.7) and (2.10) we have that

$$0 \geq \limsup_k L_1(\, l_a\,)(p_k) \geq n(n-1)(\sup_\Sigma l_a) \limsup_k\, (H_2 - H)\,(p_k) \geq 0. \tag{2.11}$$

Thus, since $\sup_\Sigma l_a > 0$, from (2.11) we get that

$$\limsup_k\, (H_2 - H)\,(p_k) = 0$$

and, in particular,

$$\inf_\Sigma\, (H_2 - H) = 0. \tag{2.12}$$

Therefore, hypothesis (2.5) jointly with (2.12) guarantee that $a^\top = \nabla l_a$ vanishes identically on Σ^n, that is, l_a is constant on Σ^n, which implies that Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde\tau}$ with $\tilde\tau \leq \tau$. ■

Taking into account that $H_2 = 1 - R$, where R is the normalized scalar curvature of the hypersurface, from Theorem (2.1.2) we obtain the following consequence:

Corollary 2.1.3 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$. Suppose that the mean curvature H of Σ^n is positive, bounded and satisfies*

$$H + R \leq 1,$$

where R is the normalized scalar curvature of Σ^n. If

$$|a^\top| \leq C\{1 - \sup_\Sigma (H + R)\},$$

for some positive constant C, then Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$.

In the next result, we consider the rigidity via suitable constraints on the higher order mean curvatures.

Theorem 2.1.4 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$, with sectional curvature $K_\Sigma \leq 1$ and bounded from below. Suppose that, for some $1 \leq r \leq n-1$, H_{r+1} is bounded and satisfies*

$$\beta \leq H_r \leq H_{r+1}, \tag{2.13}$$

where β is a positive constant. If

$$|a^\top| \leq C \inf_\Sigma (H_{r+1} - H_r), \tag{2.14}$$

for some positive constant C, then Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$.

Proof. Since (2.6) guarantees that $f_a \geq l_a$, from (2.3) we get that

$$L_r(\, l_a\,) = -c_r H_r l_a + c_r H_{r+1} f_a \geq c_r (H_{r+1} - H_r) l_a, \tag{2.15}$$

where $c_r = (n-r)\binom{n}{r}$.

We define on Σ^n the following self-adjoint operator $\mathcal{P}_r : \mathfrak{X}(\Sigma^n) \longrightarrow \mathfrak{X}(\Sigma^n)$ by

$$\mathcal{P}_r := H_r P_r. \tag{2.16}$$

Taking a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ such that $Ae_i = \lambda_i e_i$, from (2.16) and (1.9) we have that

$$\langle \mathcal{P}_r e_i, e_i \rangle = \binom{n}{r}^{-1} \sum_{i_1 < \ldots < i_r, i_j \neq i, j_1 < \ldots < j_r} (\lambda_{i_1}\lambda_{j_1}) \ldots (\lambda_{i_r}\lambda_{j_r}). \tag{2.17}$$

But, since we are assuming that $K_\Sigma \leq 1$, from Gauss equation we obtain

$$\lambda_i \lambda_j = 1 - K_\Sigma(e_i, e_j) \geq 0, \tag{2.18}$$

for all $i, j \in \{1, \ldots, n\}$, with $i \neq j$. Thus, from (2.17) and (2.18) we get

$$\langle \mathcal{P}_r e_i, e_i \rangle \geq 0.$$

Consequently, $\mathcal{P}_r$ is positive semi-definite. In addition, since we are also assuming that H_r is bounded on Σ^n, we have that the same happens for $\mathrm{tr}(\mathcal{P}_r) = c_r H_r^2$.

Extending the idea of the proof of Theorem 2.1.4, we consider the operator $\mathcal{L}_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$\mathcal{L}_r f = \mathrm{tr}(\mathcal{P}_r \nabla^2 f). \tag{2.19}$$

Since $\mathcal{P}_r$ is positive semi-definite, from (2.13), (3.137) and (2.19) we get

$$\mathcal{L}_r(\, l_a \,) \geq c_r \left(H_{r+1} - H_r \right) l_a H_r \geq 0. \tag{2.20}$$

So, taking into account that our assumption that Σ^n is contained in the closure of the interior domain enclosed by $\mathcal{E}_\tau$ implies that l_a is bounded, we can apply Lemma 2.1.1 to obtain a sequence $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k l_a(p_k) = \sup_\Sigma l_a, \quad \lim_k |\nabla l_a(p_k)| = 0 \quad \text{and} \quad \limsup_k \mathcal{L}_r(\, l_a \,)(p_k) \leq 0. \tag{2.21}$$

Consequently, from (2.20) and (2.21) we have that

$$0 \geq \limsup_k \mathcal{L}_r(\, l_a \,)(p_k) \geq c_r \beta(\sup_\Sigma l_a) \limsup_k \left(H_{r+1} - H_r \right)(p_k) \geq 0. \tag{2.22}$$

Hence, since $\sup_\Sigma l_a > 0$, from (2.22) we get that

$$\limsup_k \left(H_{r+1} - H_r \right)(p_k) = 0$$

and, in particular,

$$\inf_\Sigma \left(H_{r+1} - H_r \right) = 0. \tag{2.23}$$

Therefore, hypothesis (2.14) guarantees that $\nabla l_a = a^\top$ vanishes identically on Σ^n, that is, l_a is constant on Σ^n, which implies that Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$. ■

From the proof of Theorem (2.1.4), we obtain the following nonexistence result:

Corollary 2.1.5 *There do not exist complete spacelike hypersurfaces $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$, with sectional curvature $K_\Sigma \leq 1$ and bounded from below such that, for some $1 \leq r \leq n-1$, H_r and H_{r+1} are positive constant satisfying $H_r < H_{r+1}$.*

Proof. Suppose by contradiction that there is such a spacelike hypersurface Σ^n. But, from the proof of Theorem (2.1.4) we obtain that

$$\inf_\Sigma (H_{r+1} - H_r) = 0, \tag{2.24}$$

which implies that $H_r = H_{r+1}$, contradicting our hypothesis that $H_{r+1} > H_r$. ■

Before presenting our next result, we will need to establish the following definition. We say that an immersed hypersurface Σ^n in $\mathcal{H}^{n+1}$ is *locally tangent from above* to a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ of $\mathcal{H}^{n+1}$, when there exist a point $p \in \Sigma^n$ and a neighborhood $\mathcal{U} \subset \Sigma^n$ of p such that $l_a(p) = \widetilde{\tau}$ and $l_a(q) \geq \widetilde{\tau}$ for all $q \in \mathcal{U}$ (see Figure 3).

Figure 3: Σ^n is locally tangent from above to a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$.

In this setting, we obtain the following rigidity result:

Theorem 2.1.6 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$, and locally tangent from above to a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$, with $\tilde{\tau} \leq \tau$. Suppose that H is bounded and, for some $1 \leq r \leq n-1$, H_{r+1} is positive and such that*

$$H_r \leq H_{r+1}. \tag{2.25}$$

If

$$|a^\top| \leq C \inf_\Sigma (H_{r+1} - H_r), \tag{2.26}$$

for some positive constant C, then Σ^n must be the spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$.

Proof. Let consider the vector field on $\mathcal{H}^{n+1}$ defined in the beginning of this section, namely $V(p) = -\langle p, a\rangle p + a$. It is not difficult to verify that $|V|_\Sigma = l_a$ and $\mathrm{Div}V(p) = (n+1)\langle p, a\rangle$. Thus, since we are supposing that Σ^n is locally tangent from above to a spacelike hyperplane $\mathcal{E}_\tau$, we have that $|V|_\Sigma$ attains a minimum local on Σ^n.

Consequently, we can apply Lemma 1.1.4 to guarantee the existence of an elliptic point on Σ^n. Hence, since we are also supposing that $H_{r+1} > 0$, it follows from Lemma 1.1.3 that P_j is positive definite and, since $\mathrm{tr}(P_j) = c_j H_j$, H_j is positive for all $1 \leq j \leq r$. Moreover, taking into account once more (2.55) and that $H_2 > 0$, we get that

$$\sum_i \lambda_i^2 \leq n^2 H^2.$$

Consequently, the boundedness of H implies in the boundedness of all principal curvatures of Σ^n. In particular, we have that H_r is bounded and, from Gauss equation (2.56), K_Σ is bounded from bellow. Thus, from (2.3) we have that

$$L_r(\, l_a \,) \geq c_r \, (H_{r+1} - H_r) \, l_a \geq 0. \tag{2.27}$$

Hence, we can apply Lemma 2.1.1 to obtain a sequence $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k l_a(p_k) = \sup_\Sigma l_a, \quad \lim_k |\nabla l_a(p_k)| = 0 \quad \text{and} \quad \limsup_k L_r(\, l_a \,)(p_k) \leq 0. \tag{2.28}$$

Consequently, from (2.7) and (2.28) we have that

$$0 \geq \limsup_k L_r(\, l_a \,)(p_k) \geq c_r(\sup_\Sigma l_a) \limsup_k \, (H_{r+1} - H_r) \, (p_k) \geq 0. \tag{2.29}$$

Hence, since $\sup_\Sigma l_a > 0$, from (2.29) we get that

$$\limsup_k \, (H_{r+1} - H_r) \, (p_k) = 0$$

and, in particular,

$$\inf_\Sigma \, (H_{r+1} - H_r) = 0. \tag{2.30}$$

Therefore, hypothesis (2.26) jointly with (2.30) guarantee that $\nabla l_a = a^\top$ vanishes identically on Σ^n, that is, l_a is constant on Σ^n, which implies that Σ^n is the spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$. ■

From the proof of Theorem 2.1.6 we obtain the following nonexistence result:

Corollary 2.1.7 *There do not exist complete spacelike hypersurfaces $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$, locally tangent from above to a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$, with $\tilde{\tau} \leq \tau$, having bounded mean curvature and such that, for some $1 \leq r \leq n-1$, H_r and H_{r+1} are positive constant satisfying $H_r < H_{r+1}$.*

Rigidity results in $\mathcal{H}^{n+1}$

Motivated by the fact that the spacelike hyperplanes $\mathcal{E}_\tau$ of $\mathcal{H}^{n+1}$ satisfy the condition $l_a = f_a$ (considering the unit normal fields $N_\tau = x - \frac{1}{\tau}a$), in the last section of this chapter we present further rigidity results supposing that the height and angle functions of the spacelike hypersurfaces are linearly related. We start proving the following:

Theorem 2.1.8 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$. Suppose that $l_a = \lambda f_a$ for some positive constant $\lambda \in \mathbb{R}$, the mean curvature H of Σ^n is bounded and that*

$$H_2 \geq 1. \tag{2.31}$$

Then Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$.

Proof. Since $l_a = \lambda f_a$, we observe that

$$|\nabla l_a|^2 = f_a^2 - l_a^2 = (\lambda^{-2} - 1) l_a^2. \tag{2.32}$$

In particular, from (2.32) we have that $\lambda^{-2} - 1 \geq 0$. Now, we define on Σ^n the following operator $\mathcal{L} : C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma)$ by

$$\mathcal{L}f = \frac{\lambda^{-1}}{n(n-1)} L_1 f + \frac{1}{n}\Delta f, \tag{2.33}$$

From equation (2.3) and the hypothesis (2.31), we obtain that

$$\begin{aligned} \mathcal{L}(l_a) &= \frac{\lambda^{-1}}{n(n-1)}\{n(n-1)(-l_a H_1 + f_a H_2)\} + \frac{1}{n}\{n(-l_a + f_a H_1)\} \\ &= \lambda^{-2} H_2 l_a - l_a \geq (\lambda^{-2} - 1) l_a \geq 0. \end{aligned} \tag{2.34}$$

Hence, since we are also supponsing that $H_2 > 0$, it follows from Lemma 1.1.3 that P_1 is positive definite, consequently, H is positive. Moreover, we know that

$$\mathrm{tr}(P_1) = c_1 H. \tag{2.35}$$

Thus, taking into account once more that $H_2 \geq 1$ and $H_2 < H^2$, from (2.8) we get that

$$\sum_i \lambda_i^2 \leq n^2 H^2.$$

Consequently, the boundedness of H implies in the boundedness of all principal curvatures of Σ^n. From Gauss equation (2.9), K_Σ is bounded from bellow. Since our assumption that Σ^n is contained in the closure of the interior domain enclosed by $\mathcal{E}_\tau$ implies that l_a is bounded, we can apply Lemma 2.1.1 to get a sequence $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k l_a(p_k) = \sup_\Sigma l_a, \quad \lim_k |\nabla l_a(p_k)| = 0 \quad \text{and} \quad \limsup_k \mathcal{L}(\, l_a\,)(p_k) \leq 0. \tag{2.36}$$

Hence, from (2.42) and (2.34), we obtain that

$$0 \geq \limsup_k \mathcal{L}(\, l_a\,)(p_k) \geq (\lambda^2 - 1)(\sup_\Sigma l_a) \geq 0. \tag{2.37}$$

So, as $\sup_\Sigma l_a > 0$, then

$$\lambda^{-2} - 1 = 0.$$

Therefore, returning to (2.34), we obtain that $\lambda = 1$. From (2.39), we have that $\nabla l_a = 0$ and the height function l_a is constant on Σ^n, consequently, Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ for some $\tilde{\tau} \leq \tau$. ■

From Theorem 2.1.8 and using once more the relation between H_2 and the normalized scalar curvature, we obtain the following consequence:

Corollary 2.1.9 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$. Suppose that $l_a = \lambda f_a$, for some positive constant $\lambda \in \mathbb{R}$, the mean curvature is bounded and the normalized scalar curvature is nonpositive. Then, Σ^n is a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$ with $\tilde{\tau} \leq \tau$.*

We close this paper extending Theorem 2.1.8 for the context of higher order mean curvatures.

Theorem 2.1.10 *Let $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete spacelike hypersurface of $\mathcal{H}^{n+1}$ contained in the closure of the interior domain enclosed by a spacelike hyperplane $\mathcal{E}_\tau$ orthogonal to a nonzero null vector $a \in \mathbb{L}^{n+2}$, and locally tangent from above to a spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$, with $\tilde{\tau} \leq \tau$. Suppose that $l_a = \lambda f_a$ for some positive constant $\lambda \in \mathbb{R}$, and that, for some $1 \leq r \leq n-2$, the r-th mean curvature H_r of Σ^n is bounded and such that*

$$\beta \leq H_r \leq H_{r+2}, \tag{2.38}$$

where β is a positive constant. Then, Σ^n must be the spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$.

Proof. Since $l_a = \lambda f_a$, we observe that

$$|\nabla l_a|^2 = f_a^2 - l_a^2 = (\lambda^{-2} - 1) l_a^2. \tag{2.39}$$

In particular, from (2.39) we have that $\lambda^{-2} - 1 \geq 0$. Now, we define on Σ^n the following operator $\mathcal{L} : C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma)$ by

$$\mathcal{L}f = \frac{\lambda^{-1}}{c_{r+1}} L_{r+1} f + \frac{1}{c_r} L_r f, \tag{2.40}$$

where $c_i = (i+1)\binom{n}{i+1}$. From equation (2.3) and the hypothesis (2.38) , we obtain that

$$\begin{aligned}
\mathcal{L}(l_a) &= \frac{\lambda^{-1}}{c_{r+1}} L_{r+1} l_a + \frac{1}{c_r} L_r l_a \\
&= \frac{\lambda^{-1}}{c_{r+1}} \{c_{r+1}(-l_a h_{r+1} + f_a H_{r+2})\} + \frac{1}{c_r}\{c_r(-l_a H_r + f_a H_{r+1})\} \\
&= \lambda^{-2} H_{r+2} l_a - l_a H_r \geq (\lambda^{-2} - 1) H_r l_a \geq 0
\end{aligned} \tag{2.41}$$

On the other hand, we can reason as in the beginning of the proof of Theorem 2.1.6 to guarantee the existence of an elliptic point on Σ^n. Hence, since we are also supposing that $H_r > 0$, it follows from Lemma 1.1.3 that P_j is positive definite, consequently, H_j is positive for all $1 \leq j \leq r-1$. Moreover, taking into account once more that $H_2 > 0$ and $H_2 < H^2$, from (2.8) we get once more that

$$\sum_i \lambda_i^2 \leq n^2 H^2.$$

Consequently, the boundedness of H implies in the boundedness of all principal curvatures of Σ^n and, hence, $\mathrm{tr}(P_j) = c_j H_j$ is also bounded. From Gauss equation (2.9) we also have that K_Σ is bounded from bellow. So, since our assumption that Σ^n is contained in the closure of the interior domain enclosed by $\mathcal{E}_\tau$ implies that l_a is bounded, we can apply Lemma 2.1.1 to get a sequence $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k l_a(p_k) = \sup_\Sigma l_a, \quad \lim_k |\nabla l_a(p_k)| = 0 \quad \text{and} \quad \limsup_k \mathcal{L}(\, l_a\,)(p_k) \leq 0. \tag{2.42}$$

Hence, from (2.41) and (2.42) we obtain that

$$0 \geq \limsup_k \mathcal{L}(\, l_a\,)(p_k) \geq (\lambda^2 - 1)\beta(\sup_\Sigma l_a) \geq 0. \tag{2.43}$$

Thus, since $\sup_\Sigma l_a > 0$, we get $\lambda^{-2} - 1 = 0$. Consequently, returning to (2.41) we conclude that $\lambda = 1$. Therefore, from (2.39), we have that $\nabla l_a = 0$ and the height function is constant on Σ^n, which implies that Σ^n must be the spacelike hyperplane $\mathcal{E}_{\tilde{\tau}}$.

■

2.2 Rigidity of complete spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$

For a fixed vector $a \in \mathbb{R}_2^{n+2}$, let us consider the *height* and *angle* functions attached to a spacelike hypersurface $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$, which are defined, respectively, by $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$. A direct computation allows us to conclude that the gradient of such functions are given, respectively, by

$$\nabla l_a = a^\top \tag{2.44}$$

and

$$\nabla f_a = -A(a^\top), \tag{2.45}$$

where $a^\top$ is the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is,

$$a^\top = a + f_a N + l_a \psi. \tag{2.46}$$

Using Gauss and Weingarten formulas (1.1) and (1.2), from (2.44) it is not difficult to verify that

$$\nabla_X \nabla l_a = -f_a AX + l_a X, \tag{2.47}$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus, it follows from Lemma (1.1.1), (1.10) and (2.47) that

$$L_r l_a = c_r \left(H_{r+1} f_a + H_r l_a\right). \tag{2.48}$$

When $a \in \mathbb{R}_2^{n+2}$ is a fixed unit timelike vector, we obtain the following suitable formula.

2.2.1 Main results for $\mathbb{H}_1^{n+1}$

Proposition 2.2.1 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a spacelike hypersurface such that H_r is positive on Σ^n, and let $a \in \mathbb{R}_2^{n+2}$ be a fixed unit timelike vector. Then,*

$$\begin{aligned} L_r(l_a^2) = c_r \left(\sqrt{H_r} f_a + \frac{H_{r+1}}{\sqrt{H_r}} l_a\right)^2 + c_r \left(H_r - \frac{H_{r+1}^2}{H_r}\right) l_a^2 \\ + \; c_r H_r |\nabla l_a|^2 + c_r H_r (1 - 2f_a^2) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle. \end{aligned} \tag{2.49}$$

Proof. From (1.12) and (2.48), we have that

$$\begin{aligned} L_r(l_a^2) &= 2l_a L_r(l_a) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle \\ &= 2l_a\{c_r H_{r+1} f_a + c_r H_r l_a\} + 2\langle P_r(\nabla l_a), \nabla l_a\rangle \\ &= 2c_r H_{r+1} l_a f_a + 2c_r H_r l_a^2 + 2\langle P_r(\nabla l_a), \nabla l_a\rangle. \end{aligned} \tag{2.50}$$

Thus, by adding and subtracting the terms $c_r \frac{H_{r+1}^2}{H_r} l_a^2$ and $c_r H_r f_a^2$ in (2.50), we obtain

$$\begin{aligned} L_r(l_a^2) = c_r \left(\sqrt{H_r} f_a + \frac{H_{r+1}}{\sqrt{H_r}} l_a\right)^2 + c_r \left(H_r - \frac{H_{r+1}^2}{H_r}\right) l_a^2 \\ + c_r H_r (l_a^2 - f_a^2) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle. \end{aligned} \tag{2.51}$$

On the other hand, taking into account that $a \in \mathbb{R}_2^{n+2}$ is a unit timelike vector, from (2.46) we get

$$l_a^2 = |\nabla l_a|^2 + 1 - f_a^2. \tag{2.52}$$

Therefore, inserting (2.52) in (2.51), we conclude the proof of (2.49). ■

Motivated by the description of the totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$, in our next results we infer the rigidity of spacelike hypersurfaces Σ^n immersed in $\mathbb{H}^{n+1}$. For this, we will assume that the orientation N of such a spacelike hypersurface is in the same time-orientation of a certain fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$, which means that $f_a < 0$.

Theorem 2.2.2 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface such that $f_a^2 \leq \frac{1}{2}$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that the mean curvature H is positive, bounded and that the second mean curvature satisfies*

$$0 \leq H_2 \leq 1. \tag{2.53}$$

If

$$|a^\top| \leq C \inf_\Sigma (H - H_2), \tag{2.54}$$

for some positive constant C, then Σ^n is a totally umbilical spacelike hypersurface M_τ, with $\tau^2 = \frac{1}{2}$.

Proof. First of all, we observe that straightforward computations show that the totally umbilical spacelike hypersurfaces $M_{-\sqrt{2}/2}$ and $M_{\sqrt{2}/2}$ satisfy all the hypotheses of the theorem. Now, we shall see that these are the only such hypersurfaces. To do so, let Σ^n

be a complete spacelike hypersurface satisfying the hypotheses of the theorem. Taking into account the algebraic relation

$$\sum_{i=1}^{n} \lambda_i^2 = |A|^2 = n^2 H^2 - n(n-1)H_2, \tag{2.55}$$

since H is bounded and (2.53) guarantees that H_2 is also bounded, we have that all the principal curvatures λ_i of Σ^n are also bounded. Consequently, from Gauss equation

$$K_\Sigma(e_i, e_j) = -1 - \lambda_i \lambda_j, \tag{2.56}$$

we conclude that the sectional curvature K_Σ of Σ^n is bounded from below.

On the other hand, using again hypothesis (2.53), Lemma 3.10 in [70] gives that $\mathcal{P}_1 := HP_1$ is positive semi-definite, with $\operatorname{tr}(\mathcal{P}) = c_1 H^2$ been bounded. So, we consider the operator $\mathcal{L}_1 : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$\mathcal{L}_1 f = \operatorname{tr}(\mathcal{P}_1 \nabla^2 f). \tag{2.57}$$

From (2.53), we also have

$$H_2^2 \le H_2 \le H^2.$$

Thus, since we are assuming that the angle function satisfies $f_a^2 \le \frac{1}{2}$, from Proposition 2.2.1 we get

$$L_r(l_a) \ge c_1 \left(H - \frac{H_2^2}{H} \right) l_a^2. \tag{2.58}$$

Hence, from (2.57) and (2.58) we obtain

$$\mathcal{L}_1(\, l_a^2\,) \ge n(n-1) \left(H^2 - H_2^2 \right) l_a^2 \ge 0. \tag{2.59}$$

In view of (2.52), our hypothesis (2.54) implies that the function l_a is bounded. Thus, we can apply Lemma 2.1.1 to obtain a sequence of points $\{p_k\}_{k\ge1}$ in Σ^n such that

$$\lim_k l_a^2(p_k) = \sup_\Sigma l_a^2, \quad \lim_k |\nabla l_a^2(p_k)| = 0 \quad \text{and} \quad \limsup_k \mathcal{L}_1(\, l_a^2\,)(p_k) \le 0. \tag{2.60}$$

Consequently, from (2.59) and (2.60) we have that

$$0 \ge \limsup_k \mathcal{L}_1(\, l_a^2\,)(p_k) \ge n(n-1)(\sup_\Sigma l_a^2) \limsup_k \left(H^2 - H_2^2 \right)(p_k) \ge 0. \tag{2.61}$$

Hence, since equation (2.52) jointly with our constraint on f_a imply in particular that $\sup_\Sigma l_a^2 > 0$, from (2.61) we get that

$$\limsup_k \left(H^2 - H_2^2\right)(p_k) = 0,$$

and, consequently,

$$\inf_\Sigma (H - H_2) = 0. \tag{2.62}$$

Therefore, from (2.54) and (2.62), we have that $a^\top$ vanishes identically on Σ^n, which means that l_a is constant on Σ^n and, hence, Σ^n is a totally umbilical spacelike hypersurface M_τ of $\mathbb{H}_1^{n+1}$. Since we must have $H = H_2$, we also get

$$\frac{\tau}{\sqrt{1-\tau^2}} = \left(\frac{\tau}{\sqrt{1-\tau^2}}\right)^2,$$

which allows us to conclude that $\tau^2 = \frac{1}{2}$. ∎

Proceeding, under a suitable control on the sectional curvature of the spacelike hypersurface, we obtain an extension of Theorem 2.2.2 for the case of higher order mean curvatures.

Theorem 2.2.3 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below satisfying $K_\Sigma \leq -1$, and such that $f_a^2 \leq \frac{1}{2}$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that, for some $1 \leq r \leq n-1$, H_r is bounded and such that*

$$0 \leq H_{r+1} \leq H_r. \tag{2.63}$$

If

$$|a^\top| \leq C \inf_\Sigma (H_r - H_{r+1}), \tag{2.64}$$

for some positive constant C, then Σ^n is a totally umbilical spacelike hypersurface M_τ with $\tau^2 = \frac{1}{2}$.

Proof. We define a self-adjoint operator $\mathcal{P}_r : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ by

$$\mathcal{P}_r = H_r P_r. \tag{2.65}$$

For each $p \in \Sigma^n$, we take a local orthonormal frame $\{e_1, \ldots, e_n\}$ such that $Ae_i = \lambda_i e_i$. So, from (1.9) we have that

$$P_r e_i = (-1)^r \sum_{i_1 < \ldots < i_r, i_j \neq i} \lambda_{i_1} \ldots \lambda_{i_r} e_i.$$

Thus, for any $i \in \{1, \ldots, n\}$, we get

$$\langle \mathcal{P}_r e_i, e_i \rangle = \binom{n}{r}^{-1} \sum_{\substack{i_1 < \ldots < i_r, i_j \neq i \\ j_1 < \ldots < j_r}} (\lambda_{i_1} \lambda_{j_1}) \ldots (\lambda_{i_r} \lambda_{j_r}). \tag{2.66}$$

Moreover, from Gauss equation (2.56) and taking into account our constraint on the sectional curvature K_Σ of Σ^n, we have that

$$\lambda_i \lambda_j = -1 - K_\Sigma(e_i, e_j) \geq 0,$$

for all $i, j \in \{1, \cdots, n\}$, with $i \neq j$. Hence, from (2.66) we get that

$$\langle \mathcal{P}_r e_i, e_i \rangle \geq 0,$$

for any $i \in \{1, \ldots, n\}$, which implies that $\mathcal{P}_r$ is positive semi-definite. In addition, since we are assuming that H_r is bounded on Σ^n, from Lemma 1.1.1 and (2.65) we have that the same is true for $\operatorname{tr}(\mathcal{P}_r) = c_r H_r^2$.

Now, extending the idea of the proof of Theorem 2.2.2, we consider the operator $\mathcal{L}_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$\mathcal{L}_r f = \operatorname{tr}(\mathcal{P}_r \nabla^2 f). \tag{2.67}$$

Since $\mathcal{P}_r$ is positive semi-definite and $f_a^2 \leq \frac{1}{2}$, from Proposition 2.2.1, (2.63) and (2.67) we get

$$\mathcal{L}_r(\, l_a^2\,) \geq c_r \left(H_r^2 - H_{r+1}^2 \right) l_a^2 \geq 0. \tag{2.68}$$

Furthermore, taking into account once more relation (2.52), hypothesis (2.64) implies that l_a is bounded. Thus, we can apply Lemma 2.1.1 to obtain a sequence of points $\{p_k\}_{k \geq 1}$ in Σ^n such that

$$\lim_k l_a^2(p_k) = \sup_\Sigma l_a^2, \quad \lim_k |\nabla l_a^2(p_k)| = 0 \quad \text{and} \quad \limsup_k \mathcal{L}_r(\, l_a^2\,)(p_k) \leq 0. \tag{2.69}$$

Consequently, from (2.68) and (2.69) we have that

$$0 \geq \limsup_k \mathcal{L}_r(\, l_a^2\,)(p_k) \geq c_r (\sup_\Sigma l_a^2) \limsup_k \left(H_r^2 - H_{r+1}^2 \right)(p_k) \geq 0. \tag{2.70}$$

Hence, since $\sup_\Sigma l_a^2 > 0$, from (2.70) we get that

$$\limsup_k \left(H_r^2 - H_{r+1}^2 \right)(p_k) = 0$$

and, in particular,

$$\inf_{\Sigma} (H_r - H_{r+1}) = 0. \tag{2.71}$$

Therefore, from (2.64) and (2.71) we get that $a^\top$ vanishes identically on Σ^n, which means that l_a is constant on Σ^n and, hence, Σ^n is a totally umbilical spacelike hypersurface M_τ of $\mathbb{H}_1^{n+1}$. Furthermore, since in this case $H_r = H_{r+1}$, we must have

$$\left(\frac{\tau}{\sqrt{1-\tau^2}}\right)^r = \left(\frac{\tau}{\sqrt{1-\tau^2}}\right)^{r+1},$$

which implies that we must have $\tau^2 = \frac{1}{2}$. ■

Remark 2.2.4 *We point out that the assumption $K_\Sigma \leq -1$ in Theorem 2.2.3 is compatible with our previous conclusion. Indeed, from item (3) of the description of the totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$ quoted in the beginning of this section, for $\tau^2 = \frac{1}{2}$ we have $K_\Sigma = K_{M_\tau} = -\dfrac{1}{1-\tau^2} = -2$.*

We say that a spacelike hypersurface Σ^n is *locally tangent from below* to a totally umbilical hypersurface M_τ of $\mathbb{H}_1^{n+1}$, when there exists a point $p \in \Sigma^n$ and a neighborhood $\mathcal{U} \subset \Sigma^n$ of p such that $l_a(p) = \tau$ and $l_a(q) \leq \tau$ for all $q \in \mathcal{U}$ (see Figure 1).

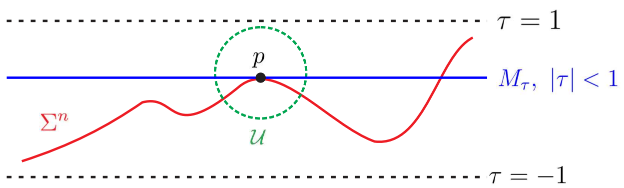

Figure 1: Σ^n is locally tangent from bellow to a hypersurface M_τ.

On the other hand, we say that Σ^n is *locally tangent from above* to M_τ, when there exists a point $p \in \Sigma^n$ and a neighborhood $\mathcal{U} \subset \Sigma^n$ of p such that $l_a(p) = \tau$ and $l_a(q) \leq \tau$ for all $q \in \mathcal{U}$ (see Figure 2).

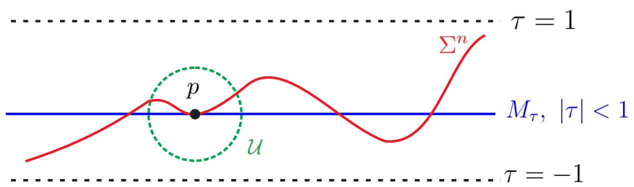

Figure 2: Σ^n is locally tangent from above to a hypersurface M_τ.

In our next result, for a constant $0 < \rho \leq 1$, we will also consider the open regions

$$\Omega^+(a, \rho) = \{p \in \mathbb{H}_1^{n+1} : 0 < \langle p, a \rangle < \rho\}$$

of the chronological past, and

$$\Omega^-(a, \rho) = \{p \in \mathbb{H}_1^{n+1} : -\rho < \langle p, a \rangle < 0\}$$

of the chronological future of $\mathbb{H}_1^{n+1}$, with respect to a fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$ (see Figure 3).

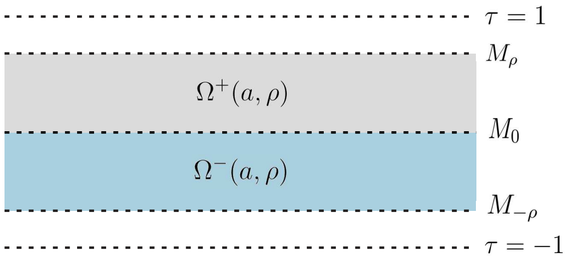

Figure 3: Open regions of $\mathbb{H}_1^{n+1}$ determined by a fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$.

Considering this previous setting, we obtain the following:

Theorem 2.2.5 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface such that $f_a^2 \leq \frac{1}{2}$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that H is bounded, for some $1 \leq r \leq n-1$, H_{r+1} is positive and such that*

$$H_{r+1} \leq H_r. \tag{2.72}$$

Let us assume in addition that

$$|a^\top| \leq C \inf_\Sigma (H_r - H_{r+1}), \tag{2.73}$$

for some positive constant C.

(i) If Σ^n is contained in $\Omega^+(a, \rho)$, for some $\frac{1}{\sqrt{2}} < \rho \leq 1$, and it is locally tangent from bellow to a totally umbilical spacelike hypersurface M_τ, with $0 < \tau < \rho$, then Σ^n is isometric to M_τ and $\tau = \frac{1}{\sqrt{2}}$;

(ii) If Σ^n is contained in $\Omega^-(a, \rho)$, for some $-1 \leq \rho < -\frac{1}{\sqrt{2}}$, and it is locally tangent from above to a totally umbilical spacelike hypersurface M_τ, with $\rho < \tau < 0$, then Σ^n is isometric to M_τ and $\tau = -\frac{1}{\sqrt{2}}$.

Proof. Let us consider the vector field X defined on $\mathbb{H}_1^{n+1}$ by

$$X(p) = \langle p, a \rangle p + a.$$

From Example 3 of [99] we have that X is a complete closed conformal vector field, with

$$\mathrm{div} X(p) = (n+1)\langle p, a \rangle, \tag{2.74}$$

and

$$|X|_\Sigma = \sqrt{-\langle X, X \rangle} = \sqrt{1 - l_a^2}. \tag{2.75}$$

Thus, supposing for instance that Σ^n is contained in $\Omega^+(a, \rho)$ and that it is locally tangent from below to M_τ, with $0 < \tau < \rho$, from (2.75) we have that $|X|_\Sigma$ attains a local minimum on Σ^n. Consequently, we can apply Lemma 1.1.4 to guarantee the existence of an elliptic point on Σ^n. Hence, since we are also supposing that $H_{r+1} > 0$, it follows from Lemma 1.1.3 that P_j is positive definite and, consequently, H_j is positive for all $1 \leq j \leq r$.

Moreover, taking into account once more (2.55) and that $H_2 > 0$, we get that

$$\sum_i \lambda_i^2 \leq n^2 H^2.$$

Consequently, the boundedness of H implies in the boundedness of all principal curvatures of Σ^n. So, from Gauss equation (2.56) we conclude that K_Σ is bounded from below. Moreover, we also have that H_r is bounded and, hence, $\mathrm{tr}(P_r) = c_r H_r$ is bounded.

From Proposition 2.2.1 we get that

$$L_r(l_a^2) \geq c_r \left(\frac{H_r^2 - H_{r+1}^2}{H_r} \right) l_a^2 \geq 0. \tag{2.76}$$

Thus, we can apply Lemma 2.1.1 to obtain a sequence of points $\{p_k\}_{k \geq 1}$ in Σ^n such that

$$\lim_k l_a^2(p_k) = \sup_\Sigma l_a^2, \quad \lim_k |\nabla l_a^2(p_k)| = 0 \quad \text{and} \quad \limsup_k L_r(\, l_a^2\,)(p_k) \leq 0. \tag{2.77}$$

Consequently, from (2.76) and (2.77) we have that

$$0 \geq \limsup_k L_r(l_a^2)(p_k) \geq c_r (\sup_\Sigma l_a^2) \limsup_k \left(\frac{H_r^2 - H_{r+1}^2}{H_r} \right) \geq 0. \tag{2.78}$$

Since $\sup_\Sigma l_a^2 > 0$, from (2.78) we obtain

$$\limsup_k \left(\frac{H_r^2 - H_{r+1}^2}{H_r}\right) = 0. \tag{2.79}$$

In particular,

$$\inf_\Sigma \left(\frac{H_r - H_{r+1}}{H_r}\right) = 0, \tag{2.80}$$

and, since $H_r > 0$, we conclude that

$$\inf_\Sigma (H_r - H_{r+1}) = 0. \tag{2.81}$$

At this point, we finish the proof of item (i) reasoning as in the last part of the proof of Theorem 2.2.3. The proof of item (ii) is similar. ∎

2.2.2 Nullity of r-maximal spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$

Let $\psi : \Sigma^n \to \mathbb{H}^{n+1}$ be a spacelike hypersurface immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}$, with second fundamental form A. According to [67], for $p \in \Sigma^n$, we define the *space of relative nullity* $\mathcal{N}(p)$ of Σ^n at p by

$$\mathcal{N}(p) = \{v \in T_p\Sigma; v \in \ker(A_p)\},$$

where $\ker(A_p)$ denotes the kernel of A_p. The *index of relative nullity* $\nu(p)$ of Σ^n at p is the dimension of $\mathcal{N}(p)$, that is,

$$\nu(p) = \dim\left(\mathcal{N}(p)\right),$$

and the *index of minimum relative nullity* ν_0 of Σ^n is defined by

$$\nu_0 = \min_{p\in\Sigma} \nu(p).$$

We also recall that a spacelike hypersurface Σ^n immersed in $\mathbb{H}_1^{n+1}$ is said to be *r-maximal* if H_{r+1} vanishes identically on Σ^n. In this setting, we obtain the following result:

Theorem 2.2.6 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete r-maximal ($2 \leq r \leq n-1$) spacelike hypersurface with sectional curvature bounded from below satisfying $K_\Sigma \leq -1$, and such that $f_a^2 \leq \frac{1}{2}$ and $|a^\top|$ is bounded for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. If H_r is a nonnegative constant, then the index of minimum relative nullity ν_0 of Σ^n is at least $n - r + 1$. Moreover, if H_{r-1} does not vanish on Σ^n, then through every point of Σ^n there passes an $(n - r + 1)$-dimensional hyperbolic space $\mathbb{H}^{n-r+1} \hookrightarrow \mathbb{H}_1^{n+1}$ totally contained in Σ^n.*

Proof. Let us suppose, by contradiction, that $H_r > 0$. Reasoning as in the proof of Theorem 2.2.3, we conclude that $\inf_\Sigma(H_r - H_{r+1}) = 0$. Thus, since Σ^n is supposed to be r-maximal, we get that $H_r = 0$. Hence, from Proposition 2.3(c) of [42], we see that $H_j = 0$ for all $j \geq r$ and, hence, $\nu_0 \geq n - r + 1$.

Now, let as assume that H_{r-1} does not vanish on Σ^n. From Theorem 5.3 of [67] (see also [71]), the distribution $p \mapsto \mathcal{N}(p)$ of minimal relative nullity of Σ^n is smooth and integrable with complete leaves, totally geodesic in Σ^n and in $\mathbb{H}_1^{n+1}$. Therefore, the result follows from the characterization of complete totally geodesic submanifolds of $\mathbb{H}_1^{n+1}$ as hyperbolic spaces of suitable dimension. ■

We close this section with the following nonexistence result.

Theorem 2.2.7 *There does not exist complete* 1*-maximal spacelike hypersurface* $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ *with nonnegative constant mean curvature and such that* $f_a^2 \leq \frac{1}{2}$ *and* $|a^\top|$ *is bounded for some fixed unit timelike vector* $a \in \mathbb{R}_2^{n+2}$*.*

Proof. Let us assume the existence of such a complete 1-maximal spacelike hypersurface Σ^n. Consequently, we have two cases to infer:

(i) if $H = 0$, from (1.6) we get that $|A| = 0$ and, consequently, we conclude that Σ^n must be a totally geodesic spacelike hypersurface M_0. But, from (1.16) and (2.52) we get $f_a^2 = 1$, contradicting the fact that $f_a^2 \leq \frac{1}{2}$.

(ii) If $H > 0$, we can reason as in the proof of Theorem 2.2.2 concluding that $H = 0$, allowing us to a contradiction. ■

2.2.3 Curvature estimates and further nonexistence results

In order to prove our next results, we recall the classical Omori's generalized maximum principle [105].

Lemma 2.2.8 *Let* Σ^n *be a complete Riemannian manifold with sectional curvature bounded from below and let* $u : \Sigma^n \to \mathbb{R}$ *be a smooth function bounded from above. Then, for each* $\epsilon > 0$ *there exists a point* $p_\epsilon \in \Sigma^n$ *such that*

(i) $|\nabla u(p_\epsilon)| < \epsilon$*;*

(ii) $\operatorname{Hess} u(v, v) < \epsilon$ *for all unit tangent vector* $v \in T_p\Sigma$*;*

(iii) $\sup_\Sigma u - \epsilon < u(p_\epsilon) \leq \sup_\Sigma u$*.*

Now we are in position to present the following result.

Theorem 2.2.9 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If Σ^n is contained either in $\Omega^-(a,\rho)$ or in $\Omega^+(a,\rho)$, for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0<\rho<1$, then*

$$\sup_\Sigma H_r \geq \left(\frac{\sup_\Sigma u^\pm}{\sqrt{1-(\sup_\Sigma u^\pm)^2}}\right)^r, \quad \text{for all } r = 1,\ldots,n,$$

where $u^\pm \in C^\infty(\Sigma)$ is defined by $u^\pm = \pm l_a$ as we have $\Sigma^n \subset \Omega^-(a,\rho)$ or $\Sigma^n \subset \Omega^+(a,\rho)$.

Proof. We will first assume that $\Sigma^n \subset \Omega^+(a,\rho)$ for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0<\rho<1$. In such a case, we will choose the orientation N of Σ^n in the same time-orientation of a, so that $f_a < 0$. On Σ^n, we define the smooth function $u = l_a$ (where, for simplicity, we wrote u instead of u^+). From (2.47) we obtain that the Hessian of u is given by

$$\operatorname{Hess} u(X,X) = -f_a\langle A(X),X\rangle + u\langle X,X\rangle. \tag{2.82}$$

Since u is a smooth function on Σ^n bounded from above, we know from Lemma 2.2.8 that for each $j \in \mathbb{N}$ there exists a point $p_j \in \Sigma^n$ such that

$$|\nabla u(p_j)| < \frac{1}{j}, \tag{2.83}$$

$$\operatorname{Hess} u(p_j)(v,v) < \frac{1}{j}, \tag{2.84}$$

for each tangent vector $v \in T_{p_j}\Sigma$ with $|v| = 1$, and

$$\sup_\Sigma u - \frac{1}{j} < u(p_j) \leq \sup_\Sigma u. \tag{2.85}$$

Let $\{e_i^j\}_{i=1}^n$ be an orthonormal basis of principal directions at p_j satisfying $A_{p_j}(e_i^j) = \lambda_i(p_j)e_i^j$. From (2.82) and (2.84), we achieve at

$$\operatorname{Hess} u(p_j)(e_i^j,e_i^j) = -f_a(p_j)\lambda_i(p_j) + u(p_j) < \frac{1}{j}.$$

Thus, since N was chosen to be in the same time-orientation of a, it follows that

$$\lambda_i(p_j) < -\frac{1/j - u(p_j)}{f_a(p_j)}. \tag{2.86}$$

On the other hand, it follows from (2.85) that $1/j - u(p_j) \to -\sup_\Sigma u < 0$ as $j \to \infty$, so that by (2.86) we infer that $\lambda_i(p_j)$ is negative for all j large enough. We will assume from now on that j is large enough, so that

$$\lambda_i(p_j) < -\frac{1/j - u(p_j)}{f_a(p_j)} < 0. \tag{2.87}$$

Thus, from (2.87), we have that

$$\binom{n}{r} H_r(p_j) > \binom{n}{r} \left(\frac{u(p_j) - 1/j}{-f_a(p_j)} \right)^r. \tag{2.88}$$

From (2.52) we can deduce that $|\nabla u|^2 = -1 + f_a^2 + u^2$ and, using (2.83) and (2.85), we get that

$$\lim_{j\to\infty} -f_a(p_j) = \sqrt{1 - \left(\sup_\Sigma u\right)^2}. \tag{2.89}$$

Letting $j \to \infty$ and using (2.85) and (2.89), we obtain from (2.88)

$$\sup_\Sigma H_r \geq \left(\frac{\sup_\Sigma u}{\sqrt{1 - (\sup_\Sigma u)^2}} \right)^r.$$

If Σ^n is contained in the region $\Omega^-(a, \rho)$ of the chronological future determined by a unit timelike vector $a \in \mathbb{R}_2^{n+2}$, for some $0 < \rho < 1$, we will choose N in the opposite time-orientation of a. Then, by performing computations very similar to those in the first part of the proof with u^- instead of u, we will achieve at

$$\sup_\Sigma H_r \geq \left(\frac{\sup_\Sigma u^-}{\sqrt{1 - (\sup_\Sigma u^-)^2}} \right)^r,$$

for all $r = 1, \ldots, n$. The proof is now complete. ■

From Theorem 2.2.9 we obtain the following consequence

Corollary 2.2.10 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below.*

(i) If one of the intrinsic r-th mean curvatures (that is, when r is even) $H_r \leq 0$, then Σ^n cannot be contained in any $\Omega^-(a, \rho)$ or in any $\Omega^+(a, \rho)$, for every unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and for every $0 < \rho < 1$.

(ii) If one of the extrinsic r-th mean curvatures (that is, when r is odd) $H_r \leq 0$ (for an appropriate choice of orientation N), then Σ^n cannot be contained in any $\Omega^+(a, \rho)$, for every unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and for every $0 < \rho < 1$.

As it was mentioned in the introduction of this paper, Ishihara [85] proved that a complete maximal spacelike hypersurface immersed in $\mathbb{H}_1^{n+1}$ must have the squared norm of the second fundamental form bounded from above by n, and that this bounded is reached only by the maximal hyperbolic cylinders $\mathbb{H}^m(-\frac{n}{m}) \times \mathbb{H}^{n-m}(-\frac{n}{n-m})$, with $1 \leq m \leq n-1$ (see Theorems 1.2 and 1.3 of [85]).

We observe that this Ishihara's result jointly with Gauss equation guarantee that a complete maximal spacelike hypersurface of $\mathbb{H}_1^{n+1}$ must have, in particular, sectional curvature bounded from below. Thus, taking into account (1.6), it is not difficult to verify that item (i) of Corollary 2.2.10 allows us to obtain the following nonexistence result concerning complete maximal spacelike hypersurfaces of $\mathbb{H}_1^{n+1}$:

Corollary 2.2.11 *There do not exist complete maximal spacelike hypersurfaces $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ contained in any $\Omega^-(a,\rho)$ or in any $\Omega^+(a,\rho)$, for every unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and for every $0 < \rho < 1$.*

As a consequence of (1.6), we get that the normalized scalar curvature of Σ^n satisfies $R = -1 - H_2$. Hence, under the assumptions of Theorem 2.2.9, we obtain the following estimate

$$\inf_{\Sigma} R \leq \frac{1}{(\sup u^{\pm})^2 - 1}. \tag{2.90}$$

In view of the relation (2.90), from item (i) of Corollary 2.2.10 we also get the following result:

Corollary 2.2.12 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If the normalized scalar curvature satisfies $R > -1$, then Σ^n cannot be contained in any $\Omega^-(a,\rho)$ or in any $\Omega^+(a,\rho)$, for every unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and for every $0 < \rho < 1$.*

Proceeding, we obtain the following estimate for the Ricci curvature of a complete spacelike hypersurface in $\mathbb{H}_1^{n+1}$:

Theorem 2.2.13 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If Σ^n is contained either in $\Omega^-(a,\rho)$ or in $\Omega^+(a,\rho)$, for some unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and some $0 < \rho < 1$, then its Ricci curvature satisfies*

$$\inf_{\Sigma} \mathrm{Ric} = \inf_{\substack{p \in \Sigma \\ v \in T_p\Sigma \\ |v|=1}} \mathrm{Ric}_p(v,v) \leq \frac{n-1}{(\sup_{\Sigma} u^{\pm})^2 - 1},$$

where $u^{\pm} \in C^{\infty}(\Sigma)$ is defined by $u^{\pm} = \pm l_a$ as we have $\Sigma^n \subset \Omega^-(a,\rho)$ or $\Sigma^n \subset \Omega^+(a,\rho)$.

Proof. As in the proof of Theorem 2.2.9, we will first assume that $\Sigma^n \subset \Omega^+(a,\rho)$ for some unit timelike vector $a \in \mathbb{R}^{n+2}_2$ and for some $0<\rho<1$. In such a case, we will choose the orientation N of Σ^n in the same time-orientation of a. It follows from (1.5) and (2.87) that

$$\begin{aligned}\operatorname{Ric}(e_j^k,e_j^k) &= -(n-1)-\sum_{i=1}^{n}\lambda_i(p_k)\lambda_j(p_k)+\lambda_j^2(p_k)\\ &= -(n-1)-\sum_{i\neq j}\lambda_i(p_k)\lambda_j(p_k) \qquad (2.91)\\ &\leq -(n-1)-(n-1)\left(\frac{1/k-u(p_k)}{-f_a(p_k)}\right)^2,\end{aligned}$$

where $u=l_a$ (here, for simplicity, we also wrote u instead of u^+). Letting $k\to\infty$ and using (2.85) and (2.89), from (2.91) we get that

$$\inf_{\substack{p\in\Sigma\\ v\in T_p\Sigma\\ |v|=1}} \operatorname{Ric}_p(v,v) \leq -(n-1)\left[1+\left(\frac{-\sup_\Sigma u}{\sqrt{1-(\sup_\Sigma u)^2}}\right)^2\right] = \frac{n-1}{(\sup_\Sigma u)^2-1}.$$

In the case where $\Sigma^n\subset\Omega^-(a,\rho)$, we consider the function $u^-\in C^\infty(\Sigma)$ defined by $u^-=-l_a$, which is smooth and bounded from above. A straightforward computation shows that the Hessian of u^- is given by

$$\operatorname{Hess} u^-(X,X) = f_a\langle A(X),X\rangle + u^-\langle X,X\rangle,$$

for all tangent vector field $X\in\mathfrak{X}(\Sigma)$. Let $(q_k)\subset\Sigma^n$ be a maximizing sequence for u^- in the sense of Lemma 2.2.8. For each $k\in\mathbb{N}$, let $\{e_j^k\}_{j=1}^n$ be an orthonormal basis of principal directions at p_k satisfying $A_{p_k}(e_j^k)=\lambda_j(p_k)e_j^k$. All this allows us to obtain that

$$\lambda_j(p_k) < \frac{1/k-u^-(p_k)}{f_a(p_k)} < 0, \qquad (2.92)$$

where the last inequality holds for all k large enough, since $1/k-u^-(p_k)\to-\sup_\Sigma u^-<0$ as $k\to\infty$, and $f_a>0$ on Σ^n. Moreover, it can be easily seen that

$$\lim_{k\to\infty} f_a(p_k) = 1-\left(\sup_\Sigma u^-\right)^2. \qquad (2.93)$$

On the other hand, from (1.5) and (2.92) we get that

$$\operatorname{Ric}(e_j^k,e_j^k) \leq -(n-1)-(n-1)\left(\frac{1/k-u^-(p_k)}{f_a(p_k)}\right)^2. \qquad (2.94)$$

Therefore, letting $k \to \infty$ and using (2.93), from (2.94) we obtain

$$\inf_{\substack{p\in\Sigma\\ v\in T_p\Sigma\\ |v|=1}} \mathrm{Ric}_p(v,v) \leq -(n-1)\left[1+\left(\frac{-\sup_\Sigma u^-}{\sqrt{1-(\sup_\Sigma u^-)^2}}\right)^2\right] = \frac{n-1}{(\sup_\Sigma u^-)^2-1}.$$

This finishes the proof of Theorem 2.2.13. ■

We close our paper with the following consequence of Theorem 2.2.13:

Corollary 2.2.14 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface with sectional curvature bounded from below. If its Ricci curvature satisfies* $\mathrm{Ric} > -(n-1)$, *then Σ^n cannot be contained in any $\Omega^-(a,\rho)$ or in any $\Omega^+(a,\rho)$, for every unit timelike vector $a \in \mathbb{R}_2^{n+2}$ and for every $0 < \rho < 1$.*

2.2.4 More results of umbilicity for spacelike hypersurfaces in the $\mathbb{H}_1^{n+1}$

For a fixed vector $a \in \mathbb{R}_2^{n+2}$, let us consider the *height* and *angle* functions attached to a spacelike hypersurface $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$, which are defined, respectively, by $l_a = \langle \psi, a\rangle$ and $f_a = \langle N, a\rangle$. A direct computation allows us to conclude that the gradient of such functions are given, respectively, by

$$\nabla l_a = a^\top \tag{2.95}$$

and

$$\nabla f_a = -A(a^\top), \tag{2.96}$$

where $a^\top$ is the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is,

$$a^\top = a + f_a N + l_a \psi. \tag{2.97}$$

Using Gauss and Weingarten formulas (1.1) and (1.2), from (2.95) it is not difficult to verify that

$$\nabla_X \nabla l_a = -f_a AX + l_a X, \tag{2.98}$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus, it follows from (1.1.1), (1.10) and (2.98) that

$$L_r l_a = c_r \left(H_{r+1} f_a + H_r l_a\right). \tag{2.99}$$

Taking into account the description of the totally umbilical hypersurfaces of $\mathbb{H}_1^{n+1}$ recalled in the end of the previous section, we will consider two suitable open regions of $\mathbb{H}_1^{n+1}$ determined by a fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$; more precisely,

$$\Omega_a^+ := \{p \in \mathbb{H}_1^{n+1} : 0 < \langle a, p \rangle < 1\}$$

and

$$\Omega_a^- := \{p \in \mathbb{H}_1^{n+1} : -1 < \langle a, p \rangle < 0\},$$

which are illustrated in Figure 1.

Figure 1: Open regions of $\mathbb{H}_1^{n+1}$ determined by a fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$.

Now, we are in position to present our first uniqueness result.

Theorem 2.2.15 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface of $\mathbb{H}_1^{n+1}$ with bounded second fundamental form, such that $f_a^2 \leq 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that, for some $1 \leq r \leq n-1$, the r-th mean curvature H_r of Σ^n is positive and satisfies*

$$0 \leq H_{r+1} \leq H_r.$$

If $|a^\top| \in \mathcal{L}^1(\Sigma)$ and Σ^n is contained in the open region Ω_a^+ (respect. Ω_a^-), then Σ^n is the totally umbilical spacelike hypersurface M_τ of $\mathbb{H}_1^{n+1}$ with $\tau = \sqrt{2}/2$ (respect. $\tau = -\sqrt{2}/2$).

Proof. Initially, from (1.12) and (2.3) we have that

$$\begin{aligned} L_r(l_a^2) &= 2l_a L_r(l_a) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle \\ &= 2l_a\{c_r H_{r+1} f_a + c_r H_r l_a\} + 2\langle P_r(\nabla l_a), \nabla l_a\rangle \\ &= 2c_r H_{r+1} l_a f_a + 2c_r H_r l_a^2 + 2\langle P_r(\nabla l_a), \nabla l_a\rangle. \end{aligned} \tag{2.100}$$

Thus, by adding and subtracting the terms $c_r \dfrac{H_{r+1}^2}{H_r} l_a^2$ and $c_r H_r f_a^2$ in (2.100), we obtain

$$\begin{aligned} L_r(l_a^2) = c_r \left(\sqrt{H_r} f_a + \frac{H_{r+1}}{\sqrt{H_r}} l_a\right)^2 &+ c_r \left(H_r - \frac{H_{r+1}^2}{H_r}\right) l_a^2 \\ &+ c_r H_r (l_a^2 - f_a^2) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle. \end{aligned} \tag{2.101}$$

On the other hand, taking into account that $a \in \mathbb{R}_2^{n+2}$ is a unit timelike vector, from (2.97) we get

$$l_a^2 = |\nabla l_a|^2 + 1 - f_a^2. \tag{2.102}$$

Hence, inserting (2.102) in (2.101) we conclude that

$$L_r(l_a^2) = c_r\left(\sqrt{H_r}f_a + \frac{H_{r+1}}{\sqrt{H_r}}l_a\right)^2 + c_r\left(H_r - \frac{H_{r+1}^2}{H_r}\right)l_a^2 + c_rH_r|\nabla l_a|^2 + c_rH_r(1-2f_a^2) + 2\langle P_r(\nabla l_a), \nabla l_a\rangle, \tag{2.103}$$

where $c_r = (n-r)\binom{n}{r}$. Thus from (1.12) and (2.103) we obtain

$$\begin{aligned} 2l_aL_r(l_a) &= c_r\left(\sqrt{H_r}f_a + \frac{H_{r+1}}{\sqrt{H_r}}l_a\right)^2 + c_r\left(H_r - \frac{H_{r+1}^2}{H_r}\right)l_a^2 \\ &+ c_rH_r|\nabla l_a|^2 + c_rH_r(1-2f_a^2). \end{aligned} \tag{2.104}$$

But, supposing that either $\Sigma^n \subset \Omega_a^+$ or $\Sigma^n \subset \Omega_a^-$, we get that l_a has strict sign on Σ^n. Consequently, we can rewrite (2.104) as follows

$$\begin{aligned} L_r(l_a) &= \frac{1}{2l_a}\Big\{c_r\left(\sqrt{H_r}f_a + \frac{H_{r+1}}{\sqrt{H_r}}l_a\right)^2 + c_r\left(H_r - \frac{H_{r+1}^2}{H_r}\right)l_a^2 \\ &+ c_rH_r|\nabla l_a|^2 + c_rH_r(1-2f_a^2)\Big\}. \end{aligned} \tag{2.105}$$

Since we are also assuming that $0 \leq H_{r+1} \leq H_r$, from (2.105) we conclude that $L_r(l_a)$ does not change sign on Σ^n. Consequently, since we are supposing that the second fundamental form is bounded and that $|\,a^\top| \in \mathcal{L}^1(\Sigma)$, from (1.7) and (1.14) it is not difficult to see that we can apply Lemma 1.1.5 to get that $L_r(l_a) = 0$ on Σ^n. Hence, returning to equation (2.105), we get that $H_r|\nabla l_a|^2 = 0$ and, since $H_r > 0$, it follows that $\nabla l_a = 0$. Therefore, l_a is constant on Σ^n, which means that Σ^n is a totally umbilical spacelike hypersurface M_τ of $\mathbb{H}_1^{n+1}$. Moreover, returning once more to (2.105), we also obtain that $f_a^2 = 1/2$, and (2.102) gives that $\tau^2 = 1/2$. ∎

Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a spacelike hypersurface immersed in the anti-de Sitter space $\mathbb{H}_1^{n+1}$, with second fundamental form A. According to [67], for $p \in \Sigma^n$, we define the *space of relative nullity* $\mathcal{N}(p)$ of Σ^n at p by

$$\mathcal{N}(p) = \{v \in T_p\Sigma; v \in \ker(A_p)\},$$

where $\ker(A_p)$ denotes the kernel of A_p. The *index of relative nullity* $\nu(p)$ of Σ^n at p is the dimension of $\mathcal{N}(p)$, that is,

$$\nu(p) = \dim\left(\mathcal{N}(p)\right),$$

and the *index of minimum relative nullity* ν_0 of Σ^n is defined by

$$\nu_0 = \min_{p\in\Sigma} \nu(p).$$

We also recall that a spacelike hypersurface Σ^n immersed in $\mathbb{H}_1^{n+1}$ is said to be *r-maximal* if H_{r+1} vanishes identically on Σ^n. In this setting, we obtain the following result:

Theorem 2.2.16 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete r-maximal spacelike hypersurface, $2 \leq r \leq n-1$, with bounded second fundamental form, such that $f_a^2 \leq 1/2$ and $|a^\top| \in \mathcal{L}^1(\Sigma)$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. If H_r is a nonnegative constant and Σ^n is contained either in Ω_a^+ or Ω_a^-, then the index of minimum relative nullity ν_0 of Σ^n is at least $n-r+1$. Moreover, if H_{r-1} does not vanish on Σ^n, then through every point of Σ^n there passes an $(n-r+1)$-dimensional hyperbolic space $\mathbb{H}^{n-r+1} \hookrightarrow \mathbb{H}_1^{n+1}$ which is totally contained in Σ^n.*

Proof. Let us suppose, by contradiction, that $H_r > 0$. Reasoning as in the proof of Theorem 2.2.15, we get that $L_r(l_a) = 0$ in Σ^n. Consequently, since H_{r+1} is identically zero, from (2.3) we conclude that $H_r l_a = 0$, which cannot occur since we are also assuming that either $\Sigma^n \subset \Omega_a^+$ or $\Sigma^n \subset \Omega_a^-$. Thus, we get that H_r must be zero. Hence, from Proposition 2.3(c) of [42], we see that $H_j = 0$ for all $j \geq r$ and, hence, $\nu_0 \geq n-r+1$.

Furthermore, supposing in addition that H_{r-1} does not vanish on Σ^n, from Theorem 5.3 of [67] (see also [71]) we have that the distribution $p \mapsto \mathcal{N}(p)$ of minimal relative nullity of Σ^n is smooth and integrable with complete leaves, totally geodesic in Σ^n and in $\mathbb{H}_1^{n+1}$. Therefore, the result follows from the characterization of complete totally geodesic submanifolds of $\mathbb{H}_1^{n+1}$ as hyperbolic spaces of suitable dimension. ∎

Our next result is related to the nonexistence of complete 1-maximal spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$.

Theorem 2.2.17 *There does not exist complete 1-maximal spacelike hypersurface $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ with nonnegative constant mean curvature, which is contained either in Ω_a^+ or Ω_a^-, for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$, and such that $f_a^2 \leq \frac{1}{2}$ and $|a^\top| \in \mathcal{L}^1(\Sigma)$.*

Proof. Let us assume the existence of such a complete 1-maximal spacelike hypersurface Σ^n. if $H = 0$, from (1.6) we get that $|A| = 0$ and, consequently, we conclude that Σ^n must be a totally geodesic spacelike hypersurface M_0. But, from (1.16) and (2.102) we get $f_a^2 = 1$, contradicting the fact that $f_a^2 \leq \frac{1}{2}$. Otherwise, if $H > 0$, we can reason once more as in the proof of Theorem 2.2.15 to get that $L_1(l_a) = 0$ in Σ^n.

Hence, using again (2.3), we obtain that $Hl_a = 0$, allowing us to a contradiction with the assumption that Σ^n is contained either in Ω_a^+ or Ω_a^-. ■

Remark 2.2.18 *Wu [124] and Yang [126] investigated complete r-maximal spacelike hypersurfaces with two principal curvature in $\mathbb{H}_1^{n+1}$, $n \geq 3$, obtaining characterization results concerning the hyperbolic cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = -1$, which extend a previous one due to Cao and Wei [50]. It is also worth to mention that Perdomo [107] treated the 2-dimensional case and constructed new examples of complete maximal surfaces in $\mathbb{H}_1^3$. Moreover, Chaves, Sousa and Valério [51] studied complete maximal spacelike hypersurfaces in $\mathbb{H}_1^{n+1}$ with either constant scalar curvature or constant non-zero Gauss-Kronecker curvature, characterizing the hyperbolic cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{H}^1(c_2)$ as the only such a maximal spacelike hypersurface with $(n-1)$ principal curvatures with the same sign everywhere.*

In order to state our next result, we need to establish the following definition: We say that a spacelike hypersurface Σ^n is *locally tangent from below* to a totally umbilical hypersurface M_τ of $\mathbb{H}_1^{n+1}$, when there exist a point $p \in \Sigma^n$ and a neighborhood $\mathcal{U} \subset \Sigma^n$ of p such that $l_a(p) = \tau$ and $l_a(q) \leq \tau$ for all $q \in \mathcal{U}$ (see Figure 2).

Figure 2: Σ^n is locally tangent from bellow to M_τ.

On the other hand, we say that Σ^n is *locally tangent from above* to M_τ, when there exists a point $p \in \Sigma^n$ and a neighborhood $\mathcal{U} \subset \Sigma^n$ of p such that $l_a(p) = \tau$ and $l_a(q) \geq \tau$ for all $q \in \mathcal{U}$ (see Figure 3).

Figure 3: Σ^n is locally tangent from above to M_τ.

Considering this previous setting, we obtain the following:

Theorem 2.2.19 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface of $\mathbb{H}_1^{n+1}$ with bounded second fundamental form such that $f_a^2 \leq 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose, for some $1 \leq r \leq n$, that the r-th mean curvature H_r of Σ^n satisfies*

$$0 < H_r \leq 1.$$

If $|a^\top| \in \mathcal{L}^1(\Sigma)$, $l_a \geq -f_a$, Σ^n is contained in the open region Ω_a^+ (respect. Ω_a^-) and it is locally tangent from below (respect. above) to a totally umbilical spacelike hypersurface M_τ, then $\tau = \sqrt{2}/2$ (respect. $\tau = -\sqrt{2}/2$) and Σ^n is the totally umbilical spacelike hypersurface M_τ.

Proof. Let us prove the case that Σ^n is contained in the open region Ω_a^+ and it is locally tangent from below to a totally umbilical spacelike hypersurface M_τ (the proof of the other case is similar).

For this, we consider $V \in \mathfrak{X}(\mathbb{H}_1^{n+1})$ given by

$$V(p) = \langle p, a\rangle p + a.$$

According to Example 4.3 of [99], we have that V is a complete closed conformal vector field, with

$$\mathrm{Div}V(p) = (n+1)\langle p, a\rangle,$$

and

$$|V|_\Sigma = \sqrt{-\langle V, V\rangle} = \sqrt{1 - l_a^2}.$$

Thus, since we are supposing that Σ^n is locally tangent from below to a totally umbilical spacelike hypersurface M_τ, we have that $|V|_\Sigma$ attains a local minimum on Σ^n. Consequently, since $\Sigma^n \subset \Omega_a^+$ implies that $\mathrm{Div}V|_\Sigma = (n+1)l_a > 0$, we can apply Lemma 1.1.4 to guarantee the existence of an elliptic point on Σ^n. Hence, since we are also supposing that $H_r > 0$, it follows from Lemma 1.1.3 that the Newton transformation P_j is positive definite and, consequently, H_j is positive for all $1 \le j \le r-1$.

Now, we define the following vector field tangent to Σ^n

$$X = \sum_{i=0}^{r-1} \frac{1}{c_i} P_i(\nabla l_a), \tag{2.106}$$

where $c_i = (i+1)\binom{n}{i+1}$. From (2.3) we obtain

$$\begin{aligned} \mathrm{div}X &= \sum_{i=0}^{r-1} \frac{1}{c_i} L_i(l_a) = (H_r f_a + H_{r-1} l_a) + \cdots + (H f_a + l_a) \\ &= H_r f_a + H_{r-1}(f_a + l_a) + \cdots + H(f_a + l_a) + l_a. \end{aligned} \tag{2.107}$$

Consequently, since we are assuming that $H_r \le 1$ and $l_a \ge -f_a$ on Σ^n, from (2.107) we conclude that

$$\begin{aligned} \mathrm{div}X &= H_r f_a + H_{r-1}(f_a + l_a) + \cdots + H(f_a + l_a) + l_a \\ &\ge H_r f_a + H_{r-1}(f_a + l_a) + \cdots + H(f_a + l_a) + H_r l_a \\ &= (H_r + \cdots + H)(f_a + l_a) \ge 0. \end{aligned} \tag{2.108}$$

Thus, using that H_j is positive, for all $1 \leq j \leq r$, joint with the equation (2.108) we deduce that $\mathrm{div}X \geq 0$. Moreover, since we are supposing that the second fundamental form of Σ^n is bounded and that $|\nabla l_a| = |a^\top| \in \mathcal{L}^1(\Sigma)$, from (1.7) and (2.106) we have that $|X| \in \mathcal{L}^1(\Sigma)$. Hence, we can apply Lemma 1.1.5 to get that $\mathrm{div}X = 0$ on Σ^n and, returning to (2.108), $l_a = -f_a$ on Σ^n. Now, we observe from (2.102) and the fact that $f_a^2 \leq 1/2$, we conclude

$$0 \leq |\nabla l_a|^2 = l_a^2 + f_a^2 - 1 = 2f_a^2 - 1 \leq 0, \tag{2.109}$$

that is, $|\nabla l_a|^2 = 0$. Hence, l_a is constant on Σ^n, from (2.109) we get that $f_a^2 = 1/2$. Therefore Σ^n is the totally umbilical spacelike hypersurface M_τ, with $\tau = \sqrt{2}/2$. ■

When $r = 2$, we can reason as in the proof of Theorem 2.2.17 but using Lemma 1.1.2 instead of Lemma 1.1.3. For this reason, in this case there is no need to assume that Σ^n is contained in the open region Ω_a^+ (respect. Ω_a^-) and neither that it is locally tangent from below (respect. above) to a totally umbilical spacelike hypersurface M_τ. So, we obtain the following result.

Theorem 2.2.20 *Let $\psi : \Sigma^n \to \mathbb{H}_1^{n+1}$ be a complete spacelike hypersurface of $\mathbb{H}_1^{n+1}$ with bounded second fundamental form such that $f_a^2 \leq 1/2$ for some fixed unit timelike vector $a \in \mathbb{R}_2^{n+2}$. Suppose that the second mean curvature satisfies*

$$0 < H_2 \leq 1.$$

If $|a^\top| \in \mathcal{L}^1(\Sigma)$ and $l_a \geq -f_a$, then Σ^n is a totally umbilical spacelike hypersurface M_τ, with $\tau^2 = 1/2$.

Capítulo 3

Results for spacelike submanifolds in pseudo-Riemannian space forms

In this chapter, we deal with n-dimensional spacelike submanifolds immersed with parallel mean curvature vector (which is supposed to be either spacelike or timelike) in a pseudo-Riemannian space form $\mathbb{L}_q^{n+p}(c)$ of index $1 \leq q \leq p$ and constant sectional curvature $c \in \{-1, 0, 1\}$. Under suitable constraints on the traceless second fundamental form, we apply a maximum principle for complete noncompact Riemannian manifolds having polynomial volume growth, recently established by Alías, Caminha and Nascimento [7], to prove that such a spacelike submanifold must be totally umbilical. For more details you can look at the works [24], [25], [30] and [31].

3.1 Set up and key lemmas

Let M^n be an n-dimensional connected spacelike submanifold immersed in an $(n+p)$-dimensional pseudo-Riemannian space form $\mathbb{L}_q^{n+p}(c)$ of index q, with $1 \leq q \leq p$, and constant curvature $c \in \{-1, 0, 1\}$. We choose a local field of pseudo-Riemannian orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $\mathbb{L}_q^{n+p}(c)$, with dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n and $e_{n+1}, \dots, e_{n+p}$ are normal to M^n. We have that the pseudo-Riemannian metric $d\overline{s}^2$ of $\mathbb{L}_q^{n+p}(c)$ can be written as

$$d\overline{s}^2 = \sum_A \epsilon_A \omega_A^2, \tag{3.1}$$

where

$\epsilon_i = 1,\ 1 \le i \le n;\ \ \epsilon_\beta = 1,\ n+1 \le \beta \le n+p-q;\ \ \epsilon_\gamma = -1,\ n+p-q+1 \le \gamma \le n+p.$

We will use the following convention for indices

$$1 \le A, B, C, \ldots \le n+p; \quad 1 \le i, j, k, \ldots \le n; \quad n+1 \le \alpha \le n+p.$$

$$n+1 \le \beta, \beta' \le n+p-q; \quad n+p-q+1 \le \gamma, \gamma' \le n+p$$

Denoting by $\{\omega_{AB}\}$ the connection forms of $\mathbb{L}_q^{n+p}(c)$, we have that the structure equations of $\mathbb{L}_q^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \epsilon_B\, \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{3.2}$$

$$d\omega_{AB} = \sum_C \epsilon_C\, \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} \epsilon_C \epsilon_D K_{ABCD}\, \omega_C \wedge \omega_D, \tag{3.3}$$

and

$$K_{ABCD} = c\epsilon_A\epsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}),$$

where $\epsilon_A = 1$ for $1 \le A \le n+p-q$, $\epsilon_A = -1$ for $n+p-q+1 \le A \le n+p$, and K_{ABCD} denote the components of indefinite curvature tensor of $\mathbb{L}_q^{n+p}(c)$.

We restrict forms to M^n, so that we have

$$\omega_\alpha = 0, \quad \alpha = n+1, \cdots, n+p,$$

and the induced metric ds^2 of M^n is written as $ds^2 = \sum_i w_i^2$. Since $\sum_i \omega_{\alpha i} \wedge \omega_i = d\omega_\alpha$ and by Cartan's Lemma we can write

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha. \tag{3.4}$$

The quadratic form

$$A = \sum_{i,j,\alpha} \epsilon_\alpha h_{ij}^\alpha \omega_i \omega_j e_\alpha,$$

is the second fundamental form of M^n. Denote

$$H^\alpha = \frac{1}{n}\sum_i h_{ii}^\alpha \qquad \alpha = n+1, \cdots, n+p.$$

Then the mean curvature vector h is expressed as $h = \sum_\alpha \epsilon_\alpha H^\alpha e_\alpha$ and denote by H the length of h and by S the square of the length of the second fundamental form, i.e.

$$H = ||h|| = \sqrt{\sum_\alpha (H^\alpha)^2} \quad \text{and} \quad S = \sum_{\alpha,i,j} (h^\alpha_{ij})^2.$$

We can write out the structure equations of M

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ji} + \omega_{ij} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l.$$

where R_{ijkl} are the components of the curvature tensor of M^n. Using the previous structure equations, we obtain the Gauss equation

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\beta (h^\beta_{ik} h^\beta_{jl} - h^\beta_{il} h^\beta_{jk}) - \sum_\gamma (h^\gamma_{ik} h^\gamma_{jl} - h^\gamma_{il} h^\gamma_{jk}). \tag{3.5}$$

In particular, the components of the Ricci tensor R_{ik} and the normalized scalar curvature R are given, respectively, by

$$R_{ik} = c(n-1)\delta_{ik} + \sum_\beta \sum_j (h^\beta_{ik} h^\beta_{jl} - h^\beta_{il} h^\beta_{jk}) - \sum_\gamma \sum_j (h^\gamma_{ik} h^\gamma_{jl} - h^\gamma_{il} h^\gamma_{jk}),$$

and

$$R = cn(n-1) + n \sum_\beta (H^\beta)^2 - n \sum_\gamma (H^\gamma)^2 - S_1 + S_2, \tag{3.6}$$

where $H^\beta = \frac{1}{n} \sum_j h^\beta_{jj}$, $H^\gamma = \frac{1}{n} \sum_j h^\gamma_{jj}$ and

$$S_1 = \sum_{\beta,i,j} (h^\beta_{ij})^2, \qquad S_2 = \sum_{\gamma,i,j} (h^\gamma_{ij})^2;$$

we now put $S = S_1 + S_2$. Moreover, the normal curvature tensor $\{R_{\alpha\beta kl}\}$ is expressed as

$$R_{\alpha\beta kl} = \sum_{m=1}^{n} (h^\alpha_{km} h^\beta_{lm} - h^\alpha_{lm} h^\beta_{km}). \tag{3.7}$$

Define the first and the second covariant derivarives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$ by

$$\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} + \sum_k (h^\alpha_{kj} \omega_{ki} + h^\alpha_{ik} \omega_{\beta\alpha}) + \sum_\beta h^\beta_{ij} \omega_{\beta\alpha} - \sum_\gamma h^\gamma_{ij} \omega_{\gamma\alpha}, \tag{3.8}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega_{li} + \sum_l h^\alpha_{ilk}\omega_{lj} + \sum_l h^\alpha_{ijl}\omega_{lk} + \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha} - \sum_\gamma h^\gamma_{ijk}\omega_{\gamma\alpha}. \tag{3.9}$$

Then, by exterior differentiation of (3.4), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik}. \tag{3.10}$$

By exterior differentiation of (1.18), we have the following Ricci identity

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{mi}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl} - \sum_\gamma h^\gamma_{ij}R_{\gamma\alpha kl}. \tag{3.11}$$

In order to prove our results, we will also need the following algebraic lemmas, whose proofs can be found in [116] and [87], respectively.

Lemma 3.1.1 *Let B_1 and B_2 be symmetric $n \times n$ matrices such that $[B_1, B_2] = 0$ and $\mathrm{tr}B_1 = \mathrm{tr}B_2 = 0$. Then*

$$|\mathrm{tr}B_1^2B_2| \leq \frac{n-2}{\sqrt{n(n-1)}}(\mathrm{trB}_1^2)\sqrt{\mathrm{tr}B_2^2},$$

and the equality holds if and only if $n-1$ of the eigenvalues x_i of B_1 and the corresponding eigenvalues y_i of B_2 satisfy

$$|x_i| = \frac{(\mathrm{tr}B_1^2)^{1/2}}{\sqrt{n(n-1)}}, \qquad y_i = \frac{(\mathrm{tr}B_2^2)^{1/2}}{\sqrt{n(n-1)}} \quad \left(resp., y_i = -\frac{(\mathrm{tr}B_2^2)^{1/2}}{\sqrt{n(n-1)}}\right).$$

Lemma 3.1.2 *Let $B_1, \ldots, B_p, p \geq 2$ be symmetric $n \times n$ matrices. Then*

$$\sum_{\alpha,\beta=1}^{p} (\mathrm{tr}[B_\alpha, B_\beta]^2 - (\mathrm{tr}B_\alpha B_\beta)^2) \geq -\frac{3}{2}\left(\sum_{\alpha=1}^{p}\mathrm{tr}B_\alpha^2\right)^2.$$

To close this section, we will quote the maximum principle that will be used to prove our main results. For this, let M be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p,t)$ the geodesic ball centered at p and with radius t.

Given a polynomial function $\sigma : (0, +\infty) \longrightarrow (0, +\infty)$, we say that M has polynomial volume growth like $\sigma(t)$ if there exists $p \in M$ such that

$$\mathrm{vol}(B(p,t)) = \mathcal{O}(\sigma(t)),$$

as $t \longrightarrow +\infty$, where vol denotes the Riemannian volume.

If $p, q \in M$ are at distance d from each other, it is straightforward to check that

$$\frac{\mathrm{vol}(B(p,t))}{\sigma(t)} \geq \frac{\mathrm{vol}(B(q,t-d))}{\sigma(t-d)} \cdot \frac{\sigma(t-d)}{\sigma(t)}.$$

Hence, the choice of p in the notion of volume growth is immaterial, so that, henceforth, we shall simply say that M has polynomial volume growth.

On the other hand, Alías, Caminha and Nascimento deduct a new form of maximum principle for smooth function on a complete noncompact Riemannian manifold M (See Theorem 2.1 of [7]). According to this new result, we can obtain the following Lemma:

Lemma 3.1.3 *Let M be a connected, oriented, complete noncompact Riemannian manifold, and let $f \in \mathcal{C}^\infty(M)$ be nonnegative and such that $\Delta f \geq af$ on M, for some $a > 0$. If M has polynomial volume growth and $|\nabla f|$ is bounded on M, then $f \equiv 0$ on M.*

Gap type results

We denote by ∇ and Δ the gradient and the Laplacian operator in the metric of the spacelike submanifold M^n. Then the Laplacian of the second fundamental form h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_{k=1}^n h_{ijkk}^\alpha$. From (3.10) and (3.11), we obtain

$$\Delta h_{ij}^\alpha = \sum_k h_{kkij}^\alpha + \sum_{m,k} h_{mk}^\alpha R_{mijk} + \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{k,\beta} h_{ik}^\beta R_{\beta\alpha jk} - \sum_{k,\gamma} h_{ik}^\gamma R_{\gamma\alpha jk}. \quad (3.12)$$

Since

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha, \quad (3.13)$$

from (3.13) and (3.12), we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha,k} h_{ij}^\alpha h_{kkij}^\alpha + \sum_{i,j,\alpha,m,k} h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum_{i,j,\alpha,m,k} h_{ij}^\alpha h_{im}^\alpha R_{mkjk} \quad (3.14)$$
$$+ \sum_{i,j,\alpha,k,\beta} h_{ij}^\alpha h_{ik}^\beta R_{\beta\alpha jk} - \sum_{i,j,\alpha,k,\gamma} h_{ij}^\alpha h_{ik}^\gamma R_{\gamma\alpha jk},$$

by using (1.20) and (3.7), it is straightforward to verify that

$$\begin{aligned}\frac{1}{2}\Delta S = &\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{\alpha,i,j} n h_{ij}^\alpha h_{kkij}^\alpha + cnS - cn^2H^2 \\ &+ \sum_{\alpha,\beta} \mathrm{tr}(h_{ij}^\beta)\mathrm{tr}((h_{ij}^\alpha)^2 h_{ij}^\beta) - \sum_{\alpha,\beta}(N(h_{ij}^\alpha h_{ij}^\beta - h_{ij}^\beta h_{ij}^\alpha) + (\mathrm{tr} h_{ij}^\alpha h_{ij}^\beta)^2) \\ &- \sum_{\alpha,\gamma} \mathrm{tr}(h_{ij}^\gamma)\mathrm{tr}((h_{ij}^\alpha)^2 h_{ij}^\gamma) + \sum_{\alpha,\gamma}(N(h_{ij}^\alpha h_{ij}^\gamma - h_{ij}^\gamma h_{ij}^\alpha) + (\mathrm{tr} h_{ij}^\alpha h_{ij}^\gamma)^2),\end{aligned} \quad (3.15)$$

where $N(h_{ij}^{\alpha}) = \mathrm{tr}((h_{ij}^{\alpha})^T h_{ij}^{\alpha})$. From (3.15) we that following result.

Proposition 3.1.4 *Considering all the previous notation, it holds the following Simons type formula:*

$$\begin{aligned}\frac{1}{2}\Delta S = &\sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 + n\sum_{\alpha,i,j} h_{ij}^{\alpha} h_{kkij}^{\alpha} + nc(S - nH^2)\\ &+ \sum_{\alpha,\beta}\mathrm{tr}(h_{ij}^{\beta})\mathrm{tr}((h_{ij}^{\alpha})^2 h_{ij}^{\beta}) - \sum_{\beta,\beta'}(N(h_{ij}^{\alpha}h_{ij}^{\beta} - h_{ij}^{\beta}h_{ij}^{\alpha}) + (\mathrm{tr} h_{ij}^{\alpha}h_{ij}^{\beta})^2)\\ &- \sum_{\alpha,\gamma}\mathrm{tr}(h_{ij}^{\gamma})\mathrm{tr}((h_{ij}^{\alpha})^2 h_{ij}^{\gamma}) + \sum_{\gamma,\gamma'}(N(h_{ij}^{\alpha}h_{ij}^{\gamma} - h_{ij}^{\gamma}h_{ij}^{\alpha}) + (\mathrm{tr} h_{ij}^{\alpha}h_{ij}^{\gamma})^2).\end{aligned}$$

Now, we will recall the definition of the traceless second fundamental form. For this,

$$\phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij},$$

we consider the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j}\phi_{ij}^{\alpha}\omega_i \otimes \omega_j \otimes e_{\alpha}. \tag{3.16}$$

It is easy to check that Φ is traceless and

$$|\Phi|^2 = \sum_{\alpha}\mathrm{tr}\Phi_{\alpha}^2 = S - nH^2, \tag{3.17}$$

where Φ_{α} denote the matrix (ϕ_{ij}^{α}). Moreover, we observe that $|\Phi|^2 = 0$ if and only if M^n is a totally umbilical submanifold of $\mathbb{L}_q^{n+p}(c)$.

The next key lemma is due to Barros et al. (see Lemma 1 of [23]).

Lemma 3.1.5 *Let M^n be a Riemannian manifold isomatrically immersed into a Riemannian manifold N^{n+p}. Consider $\Psi = \sum_{\alpha,i,j}\Psi_{ij}^{\alpha}\omega_i \otimes \omega_j \otimes e_{\alpha}$ a traceless symmetric tensor satisfying Codazzi equation. Then the following inequality holds*

$$|\nabla|\Psi|^2|^2 \leq \frac{4n}{n+2}|\Psi|^2|\nabla\Psi|^2,$$

where $|\Psi|^2 = \sum_{\alpha,i,j}(\Psi_{ij}^{\alpha})^2$ and $|\nabla\Psi|^2 = \sum_{\alpha,i,j,k}(\Psi_{ijk}^{\alpha})^2$. In particular the conclusion holds for the tensor Φ defined in (3.55).

Now, we are in position to establish our first gap result.

Theorem 3.1.6 *Let M^n be a complete spacelike submanifold immersed in $\mathbb{L}_q^{n+p}(c)$, with $c \in \{0, -1, 1\}$ and $1 \le q < p-1$, having spacelike and parallel mean curvature vector. When $c = -1$, suppose in addition that $H > 1$. If M has polynomial volume growth, $|\nabla \Phi|$ is bounded and assuming that there is a constant α suth that $\sup_M |\Phi| \le \alpha < \alpha^*$, where α^* is the positive root of the function*

$$P_H(x) := -5x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}} Hx + 2n(c + H^2). \tag{3.18}$$

Then, $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold.

Proof. Taking into account that the mean curvature vector is spacelike, we choose e_{n+1} to have the same direction an h, so that $h = He_{n+1}$; Then we have

$$H^{n+1} = H; \qquad H^\alpha = 0, \quad \alpha = n+2, \cdots, n+p. \tag{3.19}$$

Since h in nonzero and parallel, we see that H is a nonzero constant and e_{n+1} is parallel. It follows that $h_{ij}^{n+1} h_{ij}^\alpha = h_{ij}^\alpha h_{ij}^{n+1}$ and

$$\sum_k h_{kki}^\alpha = 0, \qquad \sum_k h_{kkij}^\alpha = 0. \tag{3.20}$$

From (3.19) and (3.17) we have

$$\phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij}, \qquad \text{tr}(\Phi_{n+1}^2) = \text{tr}(h_{ij}^{n+1})^2 - nH^2, \tag{3.21}$$

$$\text{tr}(h_{ij}^{n+1})^3 = \text{tr}\Phi_{n+1}^3 + 3H\text{tr}\Phi_{n+1}^2 + nH^3, \tag{3.22}$$

$$\phi_{ij}^\alpha = h_{ij}^\alpha, \qquad \text{tr}\Phi_\alpha^2 = \text{tr}(h_\alpha^2), \quad \alpha \ge n+2. \tag{3.23}$$

Substituting (3.19) - (3.23) into Proposition 3.1.4 we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 = & \sum_\alpha |\nabla \Phi_\alpha|^2 + n(c+H^2)|\Phi|^2 + n\sum_\alpha H\text{tr}(\Phi_\alpha^2 \Phi_{n+1}) \\ & - \sum_{\beta,\beta'} (N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) + (\text{tr}\Phi_{\beta'}\Phi_\beta)^2) \\ & + \sum_{\gamma,\gamma'} (N(\Phi_{\gamma'}\Phi_\gamma - \Phi_\gamma\Phi_{\gamma'}) + (\text{tr}\Phi_{\gamma'}\Phi_\gamma)^2). \end{aligned} \tag{3.24}$$

Now we shall estimate separately each term of the right-hand side of (3.24). First, we define

$$|\Phi_1|^2 = \sum_\beta \sum_{i,j} (\Phi_{ij}^\beta)^2 = S_1 - nH^2, \qquad |\Phi_2|^2 = \sum_\gamma \sum_{i,j} (\Phi_{ij}^\gamma)^2 = S_2, \tag{3.25}$$

then $|\Phi|^2 = |\Phi_1|^2 + |\Phi_2|^2$. Since

$$[\Phi_{n+1}, \Phi_\alpha] = [h_{ij}^{n+1}, h_{ij}^\alpha] = 0, \qquad \mathrm{tr}\Phi_\alpha = 0, \quad \alpha = n+1, \cdots, n+p,$$

we may apply Lemma (3.1.1) to the third term of (3.24), obtaining

$$\sum_\alpha \mathrm{tr}\Phi_{n+1}\Phi_\alpha^2 \geq -\frac{n-2}{\sqrt{n(n-1)}}\left(\sum_\alpha \mathrm{tr}\Phi_\alpha^2\right)\sqrt{\mathrm{tr}\Phi_{n+1}^2} = -\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^2|\Phi_{n+1}|. \tag{3.26}$$

The fourth term of (3.24) can be rewritten as follows:

$$\sum_{\beta,\beta' \neq n+1}(-N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2) - |\Phi_{n+1}|^4 - 2\sum_{\beta \neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2,$$

it follows from Lemma (3.1.2) that, for $p - q \geq 3$,

$$\begin{aligned}\sum_{\beta,\beta' \neq n+1}(-N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2) &\geq -\frac{3}{2}\left(\sum_{\beta \neq n+1}|\Phi_\beta|^2\right)^2 \\ &\geq -\frac{3}{2}(|\Phi|^2 - |\Phi_{n+1}|^2)^2,\end{aligned} \tag{3.27}$$

and the second equality holds if and only if $S_2 = 0$; When $p - q = 2$, (3.27) becomes

$$-(\mathrm{tr}\Phi_{n+2}\Phi_{n+2})^2 \geq -\frac{3}{2}(\mathrm{tr}\Phi_{n+2}^2)^2,$$

which, of course, hold, and we really obtain (3.27) for $p - q \geq 2$. On the other hand, by the Cauchy-Schwarz inequality, we have

$$\sum_{\beta \neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2 \leq |\Phi_{n+1}|^2\sum_{\beta \neq n+1}|\Phi_\beta|^2 \leq |\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}|^2), \tag{3.28}$$

and the second equality holds if and only if $S_2 = 0$. It follows from (3.27) and (3.28) that

$$\begin{aligned}&\sum_{\beta,\beta' \neq n+1}(-N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2) - |\Phi_{n+1}|^4 - 2\sum_{\beta \neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2, \\ &\geq -\frac{3}{2}(|\Phi|^2 - |\Phi_{n+1}|^2)^2 - |\Phi_{n+1}|^4 - 2|\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}^2|).\end{aligned} \tag{3.29}$$

For the last term of (3.24), we have

$$\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_\gamma - \Phi_\gamma\Phi_{\gamma'}) + (\mathrm{tr}\Phi_{\gamma'}\Phi_\gamma)^2) \geq 0, \tag{3.30}$$

and the equality holds if and only if $S_2 = 0$. Substituting (3.26), (3.29) and (3.30) into (3.24), we obtain

$$\frac{1}{2}\Delta|\Phi|^2 \geq \sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2+\frac{1}{2}|\Phi_{n+1}|^2\left(|\Phi|^2-|\Phi_{n+1}|^2\right) \tag{3.31}$$
$$+\,|\Phi|^2\left(-\frac{3}{2}\left(|\Phi|^2-|\Phi_{n+1}|^2\right)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+1}|-|\Phi_{n+1}|^2+n(c+H^2)\right).$$

Since $|\Phi_{n+1}|\leq|\Phi|$ and $H>0$, we can rewritten as follows

$$\Delta|\Phi|^2 \geq\ |\Phi|^2 P_H(|\Phi|), \tag{3.32}$$

where $P_H(x)$ is the function defined by (3.18). Knowing that $\sup_M|\Phi|\leq \alpha < \alpha^*$, then for the behavior of $P_H(x)$, we obtain

$$\Delta|\Phi|^2 \geq\ a|\Phi|^2,$$

where $a = P_H(\alpha) > 0$.

On the order hand, as $\sup_M|\Phi|\leq \alpha < \alpha^*$ and $|\nabla\Phi|$ is bounded, from Lemma 3.1.5 we can guarantee that $|\nabla|\Phi|^2|$ is bounded, Therefore we can apply Lemma 3.4.1 to conclude $\sup_M|\Phi|= 0$ and M^n is a totally umbilical submanifold.

■

When the mean curvature vector is timelike and the ambient space is de Sitter space $\mathbb{S}_q^{n+p}$, we also get the following result

Theorem 3.1.7 *Let M^n be a complete spacelike submanifold immersed in de Sitter space $\mathbb{S}_q^{n+p}$, with $1 < q < p-1$, having timelike and parallel mean curvature vector. Suppose that $H < 1$. If M has polynomial volume growth, $|\nabla\Phi|$ is bounded and assuming that there is a constant β suth that $\sup_M|\Phi|\leq \beta < \beta^*$, where β^* is the positive root of the function*

$$Q_H(x) := -\frac{4(2q-1)}{q-1}x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}}Hx + 2n(1-H^2). \tag{3.33}$$

Then, $\sup_M|\Phi|= 0$ and M^n is a totally umbilical submanifold.

Proof. Taking into account that the mean curvature vector is timelike, we choose e_{n+p} to have the same direction an h, so that $h = He_{n+p}$; Then we have

$$H^{n+p} = H; \qquad H^\alpha = 0, \quad \alpha = n+1,\cdots,n+p-1. \tag{3.34}$$

Since h in nonzero and parallel, we see that H is a nonzero constant and e_{n+p} is parallel. It follows that $h_{ij}^{n+p}h_{ij}^{\alpha} = h_{ij}^{\alpha}h_{ij}^{n+p}$ and

$$\sum_k h_{kki}^{\alpha} = 0, \qquad \sum_k h_{kkij}^{\alpha} = 0. \tag{3.35}$$

From (3.34) and (3.17) we have

$$\phi_{ij}^{n+p} = h_{ij}^{n+p} - H\delta_{ij}, \qquad \mathrm{tr}(\Phi_{n+p}^2) = \mathrm{tr}(h_{ij}^{n+p})^2 - nH^2, \tag{3.36}$$

$$\mathrm{tr}(h_{ij}^{n+p})^3 = \mathrm{tr}\Phi_{n+p}^3 + 3H\mathrm{tr}\Phi_{n+p}^2 + nH^3, \tag{3.37}$$

$$\phi_{ij}^{\alpha} = h_{ij}^{\alpha}, \qquad \mathrm{tr}\Phi_{\alpha}^2 = \mathrm{tr}(h_{\alpha}^2), \quad n+1 \leq \alpha \leq n+p-1. \tag{3.38}$$

Substituting (3.34) - (3.38) into Proposition 3.1.4 we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 = & \sum_{\alpha}|\nabla\Phi_{\alpha}|^2 + n(1-H^2)|\Phi|^2 - n\sum_{\alpha}H\mathrm{tr}(\Phi_{\alpha}^2\Phi_{n+1}) \\ & - \sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_{\beta} - \Phi_{\beta}\Phi_{\beta'}) + (\mathrm{tr}\Phi_{\beta'}\Phi_{\beta})^2) \\ & + \sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_{\gamma} - \Phi_{\gamma}\Phi_{\gamma'}) + (\mathrm{tr}\Phi_{\gamma'}\Phi_{\gamma})^2). \end{aligned} \tag{3.39}$$

Now we shall estimate separately each term of the right-hand side of (3.39). First, we define

$$|\Phi_1|^2 = S_1 = \sum_{\beta}\sum_{i,j}(\Phi_{ij}^{\beta})^2, \qquad |\Phi_2|^2 = \sum_{\gamma}\sum_{i,j}(\Phi_{ij}^{\gamma})^2 = S_2 - nH^2; \tag{3.40}$$

then $|\Phi|^2 = |\Phi_1|^2 + |\Phi_2|^2$. Since

$$\mathrm{tr}\Phi_{\alpha} = 0, \qquad [\Phi_{n+p}, \Phi_{\alpha}] = [h_{ij}^{n+p}, h_{ij}^{\alpha}] = 0, \qquad \alpha = n+1, \cdots, n+p,$$

we may apply Lemma (3.1.1) to the third term of (3.39), obtaining

$$\sum_{\alpha}\mathrm{tr}\Phi_{n+p}\Phi_{\alpha}^2 \leq \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{\alpha}\mathrm{tr}\Phi_{\alpha}^2\right)\sqrt{\mathrm{tr}\Phi_{n+p}^2} = \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^2|\Phi_{n+p}|. \tag{3.41}$$

Using Lemma (3.1.2) in the fourth term of (3.39), we can write

$$\begin{aligned} -\sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_{\beta} - \Phi_{\beta}\Phi_{\beta'}) + (\mathrm{tr}\Phi_{\beta'}\Phi_{\beta})^2) & \geq -\frac{3}{2}(|\Phi|^2 - |\Phi_2|^2)^2 \\ & = -\frac{3}{2}(|\Phi|^4 - 2|\Phi_2|^2|\Phi|^2 + |\Phi_2|^4) \\ & \geq -\frac{3}{2}(|\Phi|^4 - 2|\Phi_{n+p}|^2|\Phi|^2 + |\Phi|^4) \\ & = -3|\Phi|^2(|\Phi|^2 - |\Phi_{n+p}|^2), \end{aligned} \tag{3.42}$$

and the equalities hold if and only if $|\Phi_1|^2 = S_1 = 0$ and $|\Phi_2|^2 = |\Phi_{n+p}|^2$.

For the last term of (3.39), we have

$$\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_\gamma - \Phi_\gamma\Phi_{\gamma'}) \geq 0, \tag{3.43}$$

and

$$\begin{aligned}\sum_{\gamma,\gamma'}(\mathrm{tr}\Phi_{\gamma'}\Phi_\gamma)^2 &\geq |\Phi_{n+p}|^4 + \frac{1}{q-1}(|\Phi_2|^2 - |\Phi_{n+p}|^2)^2 \\ &= |\Phi_{n+p}|^4 + \frac{1}{q-1}(|\Phi_2|^4 - 2|\Phi_2|^2|\Phi_{n+p}|^2 + |\Phi_{n+p}|^4) \\ &\geq |\Phi_{n+p}|^4 + \frac{1}{q-1}(|\Phi_{n+p}|^4 - 2|\Phi|^2|\Phi_{n+p}|^2 + |\Phi_{n+p}|^4) \\ &= |\Phi_{n+p}|^4 + \frac{2}{q-1}|\Phi_{n+p}|^2(|\Phi_{n+p}|^2 - |\Phi|^2),\end{aligned} \tag{3.44}$$

where the equalities hold if and only if $|\Phi_1|^2 = S_1 = 0$ and $|\Phi_2|^2 = |\Phi_{n+p}|^2$.

Substituting (3.41)-(3.44) into (3.39), we obtain

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 \geq& \sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2 - \frac{q+1}{q-1}|\Phi_{n+p}|^2(|\Phi|^2 - |\Phi_{n+p}|^2) \\ +& \ |\Phi|^2\left(-3(|\Phi|^2 - |\Phi_{n+p}|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}| + |\Phi_{n+p}|^2 + n(1-H^2)\right).\end{aligned} \tag{3.45}$$

Since $|\Phi_{n+p}| \leq |\Phi|$ e $H > 0$, we can rewritten as follows

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 \geq& \sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2 - \frac{q+1}{q-1}|\Phi|^2(|\Phi|^2 - |\Phi_{n+p}|^2) \\ +& \ |\Phi|^2\left(-3(|\Phi|^2 - |\Phi_{n+p}|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}| + \right. \\ +& \left. |\Phi_{n+p}|^2 + n(1-H^2)\right) \\ \geq& \ |\Phi|^2\left(-\frac{2(2q-1)}{q-1}(|\Phi|^2 - |\Phi_{n+p}|^2) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}| + \right. \\ +& \left. |\Phi_{n+p}|^2 + n(1-H^2)\right) \\ \geq& \ \frac{2(2q-1)}{q-1}|\Phi|^2|\Phi_{n+p}|^2 + |\Phi|^2\frac{Q_H(|\Phi|)}{2},\end{aligned} \tag{3.46}$$

or yet, knowing that $\sup_M|\Phi| \leq \beta < \beta^*$, then for the behavior of $Q_H(x)$, we obtain

$$\Delta|\Phi|^2 \ \geq \ |\Phi|^2 Q_H(|\Phi|) \ \geq \ b|\Phi|^2, \tag{3.47}$$

where $b = Q_H(\beta) > 0$ and $Q_H(x)$ is the function defined in (3.33).

Therefore, we can reason as in the last part of the proof of Theorem 3.1.7 to conclude that M^n must be totally umbilical.

■

3.2 Umbilicity of spacelike submanifolds with parallel mean vector via a maximum principle at infinity

Our approach is based on a suitable maximum principle at infinity for complete noncompact Riemannian manifolds due to Alías, Caminha and Nascimento [9]. To quote it, we need to recall the following concept established in the beginning of [9, Section 2]: Let M^n be a complete noncompact Riemannian manifold and let $d(\,\cdot\,, o) : M^n \to [0, +\infty)$ denote the Riemannian distance of M^n, measured from a fixed point $o \in M^n$. We say that a smooth function $f \in C^\infty(M)$ *converges to zero at infinity*, when it satisfies the following condition

$$\lim_{d(x,o)\to+\infty} f(x) = 0. \tag{3.48}$$

Keeping in mind this concept, the following maximum principle at infinite corresponds to item (a) of [9, Theorem 2.2].

Lemma 3.2.1 *Let M^n be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(M)$ be a vector field on M^n. Assume that there exists a nonnegative, non-identically vanishing function $f \in C^\infty(M)$ which converges to zero at infinity and such that $\langle \nabla f, X\rangle \geq 0$. If $\operatorname{div}X \geq 0$ on M^n, then $\langle \nabla f, X\rangle \equiv 0$ on M^n.*

So, our purpose is to apply Lemma 3.2.1 jointly with a suitable Simons type formula (see Proposition 3.1.4)in order to obtain our characterization results of totally umbilical spacelike submanifolds of a pseudo-Riemannian space form. Denoting by $|\Phi|$ the Hilbert-Schmidt norm of the traceless second fundamental form Φ and assuming that the mean curvature vector is spacelike and parallel, we state our first characterization result.

Theorem 3.2.2 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with spacelike and parallel mean curvature vector in an $(n+p)$-dimensional pseudo-Riemannian space form $\mathbb{L}^{n+p}_q(c)$, with constant sectional curvature $c \in \{0, -1, 1\}$ and index $1 \leq q < p-1$. When $c = -1$, suppose in addition that the* *mean curvature satisfies $H > 1$. If $|\Phi|$ converges to zero at infinity with $\sup_M |\Phi| \leq \alpha^*$, where α^* is the positive root of the polynomial function*

$$P_H(x) := -5x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}} Hx + 2n(c + H^2), \tag{3.49}$$

then $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold of $\mathbb{L}_q^{n+p}(c)$.

In the case that the mean curvature vector is timelike and the ambient space is de Sitter space $\mathbb{S}_q^{n+p}$, we obtain our second characterization result of totally umbilical spacelike submanifolds.

Theorem 3.2.3 *Let M^n be an n-dimensional complete noncompact spacelike submanifold immersed with timelike and parallel mean curvature vector in the $(n+p)$- dimensional de Sitter space $\mathbb{S}_q^{n+p}$, with index $1 < q < p-1$. Suppose in addition that the mean curvature satisfies $H < 1$. If $|\Phi|$ converges to zero at infinity with $\sup_M |\Phi| \leq \beta^*$, where β^* is the positive root of the polynomial function*

$$Q_H(x) := -\frac{4(2q-1)}{q-1} x^2 - \frac{2n(n-2)}{\sqrt{n(n-1)}} Hx + 2n(1 - H^2), \tag{3.50}$$

then $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.

In order to prove our results in the next section, we will also need the following algebraic lemmas, whose proofs can be found in [116] and [87], respectively.

Lemma 3.2.4 *Let B_1 and B_2 be symmetric $n \times n$ matrices such that $[B_1, B_2] = 0$ and $\mathrm{tr} B_1 = \mathrm{tr} B_2 = 0$. Then*

$$|\mathrm{tr} B_1^2 B_2| \leq \frac{n-2}{\sqrt{n(n-1)}} (\mathrm{tr} B_1^2) \sqrt{\mathrm{tr} B_2^2},$$

and the equality holds if and only if $n-1$ of the eigenvalues x_i of B_1 and the corresponding eigenvalues y_i of B_2 satisfy

$$|x_i| = \frac{(\mathrm{tr} B_1^2)^{1/2}}{\sqrt{n(n-1)}}, \qquad y_i = \frac{(\mathrm{tr} B_2^2)^{1/2}}{\sqrt{n(n-1)}} \quad \left(resp., y_i = -\frac{(\mathrm{tr} B_2^2)^{1/2}}{\sqrt{n(n-1)}} \right).$$

Lemma 3.2.5 *Let $B_1, \cdots, B_p$, $p \geq 2$, be symmetric $n \times n$ matrices. Then*

$$\sum_{\alpha,\beta=1}^{p} (\mathrm{tr}[B_\alpha, B_\beta]^2 - (\mathrm{tr} B_\alpha B_\beta)^2) \geq -\frac{3}{2} \left(\sum_{\alpha=1}^{p} \mathrm{tr} B_\alpha^2 \right)^2.$$

A Simons type formula and the proofs of Theorems 3.2.2 and 3.2.3

Our aim in this section is to present the proofs of our characterization results for n-dimensional totally umbilical spacelike submanifolds of a pseudo-Riemannian space form $\mathbb{L}_q^{n+p}(c)$ of index $1 \le q \le p$ and constant sectional curvature $c \in \{-1, 0, 1\}$. To achieve our goal, we will adapt the technique developed by Yang and Li in [129].

In what follows, we denote by ∇ and Δ the gradient and the Laplacian operator in the metric of such a spacelike submanifold M^n immersed in $\mathbb{L}_q^{n+p}(c)$. Then the Laplacian of the second fundamental form h_{ij}^α is defined by

$$\Delta h_{ij}^\alpha = \sum_{k=1}^n h_{ijkk}^\alpha.$$

So, from (1.2) and (3.11) we obtain

$$\begin{aligned}\Delta h_{ij}^\alpha &= \sum_k h_{kkij}^\alpha + \sum_{m,k} h_{mk}^\alpha R_{mijk} + \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{k,\beta} h_{ik}^\beta R_{\beta\alpha jk} \\ &\quad - \sum_{k,\gamma} h_{ik}^\gamma R_{\gamma\alpha jk}.\end{aligned} \tag{3.51}$$

Since

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha, \tag{3.52}$$

from (3.52) and (3.51) we have

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha,k} h_{ij}^\alpha h_{kkij}^\alpha + \sum_{i,j,\alpha,m,k} h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum_{i,j,\alpha,m,k} h_{ij}^\alpha h_{im}^\alpha R_{mkjk} \\ &\quad + \sum_{i,j,\alpha,k,\beta} h_{ij}^\alpha h_{ik}^\beta R_{\beta\alpha jk} - \sum_{i,j,\alpha,k,\gamma} h_{ij}^\alpha h_{ik}^\gamma R_{\gamma\alpha jk}.\end{aligned} \tag{3.53}$$

Hence, by using (1.20), (3.7) and (3.53), we reach at

$$\begin{aligned}\frac{1}{2}\Delta S =& \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{\alpha,i,j} n h_{ij}^\alpha h_{kkij}^\alpha + cnS - cn^2H^2 \\ &+ \sum_{\alpha,\beta} \mathrm{tr}(h_{ij}^\beta)\mathrm{tr}((h_{ij}^\alpha)^2 h_{ij}^\beta) - \sum_{\alpha,\beta}(N(h_{ij}^\alpha h_{ij}^\beta - h_{ij}^\beta h_{ij}^\alpha) + (\mathrm{tr} h_{ij}^\alpha h_{ij}^\beta)^2) \\ &- \sum_{\alpha,\gamma} \mathrm{tr}(h_{ij}^\gamma)\mathrm{tr}((h_{ij}^\alpha)^2 h_{ij}^\gamma) + \sum_{\alpha,\gamma}(N(h_{ij}^\alpha h_{ij}^\gamma - h_{ij}^\gamma h_{ij}^\alpha) + (\mathrm{tr} h_{ij}^\alpha h_{ij}^\gamma)^2),\end{aligned} \tag{3.54}$$

where $N(h_{ij}^\alpha) = \mathrm{tr}((h_{ij}^\alpha)^T h_{ij}^\alpha)$. Therefore, from (4.3) we obtain the following Simons type formula:

Proposition 3.2.6 *Considering all the previous notations, it holds*

$$\begin{aligned}\frac{1}{2}\Delta S = & \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + n\sum_{\alpha,i,j}h^\alpha_{ij}h^\alpha_{kkij} + nc(S-nH^2)\\ & + \sum_{\alpha,\beta}\mathrm{tr}(h^\beta_{ij})\mathrm{tr}((h^\alpha_{ij})^2h^\beta_{ij}) - \sum_{\beta,\beta'}(N(h^\alpha_{ij}h^\beta_{ij}-h^\beta_{ij}h^\alpha_{ij}) + (\mathrm{tr}h^\alpha_{ij}h^\beta_{ij})^2)\\ & - \sum_{\alpha,\gamma}\mathrm{tr}(h^\gamma_{ij})\mathrm{tr}((h^\alpha_{ij})^2h^\gamma_{ij}) + \sum_{\gamma,\gamma'}(N(h^\alpha_{ij}h^\gamma_{ij}-h^\gamma_{ij}h^\alpha_{ij}) + (\mathrm{tr}h^\alpha_{ij}h^\gamma_{ij})^2).\end{aligned}$$

Now, we recall that the traceless second fundamental form Φ is defined as been the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j}\phi^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha, \tag{3.55}$$

where $\phi^\alpha_{ij} = h^\alpha_{ij} - H^\alpha\delta_{ij}$. It is not difficult to check that Φ is, indeed, traceless and that its squared norm satisfies the following algebraic relation

$$|\Phi|^2 = \sum_\alpha \mathrm{tr}\Phi^2_\alpha = S - nH^2, \tag{3.56}$$

where Φ_α denotes the matrix (ϕ^α_{ij}). Moreover, we observe that $|\Phi|$ vanishes identically on M^n if and only if M^n is a totally umbilical submanifold of $\mathbb{L}^{n+p}_q(c)$.

Proof of Theorem 3.2.2

Taking into account that the mean curvature vector is spacelike, we choose e_{n+1} to have the same direction an h, so that $h = He_{n+1}$; Then we have

$$H^{n+1} = H; \qquad H^\alpha = 0, \quad \alpha = n+2,\cdots,n+p. \tag{3.57}$$

Since h in nonzero and parallel, we see that H is a nonzero constant and e_{n+1} is parallel. It follows that $h^{n+1}_{ij}h^\alpha_{ij} = h^\alpha_{ij}h^{n+1}_{ij}$ and

$$\sum_k h^\alpha_{kki} = 0, \qquad \sum_k h^\alpha_{kkij} = 0. \tag{3.58}$$

From (3.57) and (3.56) we have

$$\phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij}, \qquad \mathrm{tr}(\Phi^2_{n+1}) = \mathrm{tr}(h^{n+1}_{ij})^2 - nH^2, \tag{3.59}$$

$$\mathrm{tr}(h^{n+1}_{ij})^3 = \mathrm{tr}\Phi^3_{n+1} + 3H\mathrm{tr}\Phi^2_{n+1} + nH^3, \tag{3.60}$$

$$\phi^\alpha_{ij} = h^\alpha_{ij}, \qquad \mathrm{tr}\Phi^2_\alpha = \mathrm{tr}(h^2_\alpha), \quad \alpha \ge n+2. \tag{3.61}$$

Substituting (3.57) - (3.61) into Proposition 3.1.4 we obtain

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 &= \sum_\alpha |\nabla\Phi_\alpha|^2 + n(c+H^2)|\Phi|^2 + n\sum_\alpha H\mathrm{tr}(\Phi_\alpha^2\Phi_{n+1}) \\ &\quad -\sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) + (\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2) \\ &\quad +\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_\gamma - \Phi_\gamma\Phi_{\gamma'}) + (\mathrm{tr}\Phi_{\gamma'}\Phi_\gamma)^2).\end{aligned} \tag{3.62}$$

Now, we will estimate separately each term of the right-hand side of (3.62). First, we define

$$|\Phi_1|^2 = \sum_\beta\sum_{i,j}(\Phi_{ij}^\beta)^2 = S_1 - nH^2, \qquad |\Phi_2|^2 = \sum_\gamma\sum_{i,j}(\Phi_{ij}^\gamma)^2 = S_2, \tag{3.63}$$

then $|\Phi|^2 = |\Phi_1|^2 + |\Phi_2|^2$. Since

$$[\Phi_{n+1}, \Phi_\alpha] = [h_{ij}^{n+1}, h_{ij}^\alpha] = 0, \qquad \mathrm{tr}\Phi_\alpha = 0, \quad \alpha = n+1, \cdots, n+p,$$

we may apply Lemma 3.2.4 to the third term of (3.24), obtaining

$$\sum_\alpha \mathrm{tr}\Phi_{n+1}\Phi_\alpha^2 \geq -\frac{n-2}{\sqrt{n(n-1)}}\left(\sum_\alpha \mathrm{tr}\Phi_\alpha^2\right)\sqrt{\mathrm{tr}\Phi_{n+1}^2} = -\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^2|\Phi_{n+1}|. \tag{3.64}$$

The fourth term of (3.62) can be rewritten as follows:

$$\sum_{\beta,\beta'\neq n+1}(-N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2) - |\Phi_{n+1}|^4 - 2\sum_{\beta\neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2,$$

it follows from Lemma 3.2.5 that, for $p-q \geq 3$,

$$\begin{aligned}\sum_{\beta,\beta'\neq n+1}(-N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2) &\geq -\frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi_\beta|^2\right)^2 \\ &\geq -\frac{3}{2}(|\Phi|^2 - |\Phi_{n+1}|^2)^2,\end{aligned} \tag{3.65}$$

and the second equality holds if and only if $S_2 = 0$; When $p - q = 2$, (3.65) becomes

$$-(\mathrm{tr}\Phi_{n+2}\Phi_{n+2})^2 \geq -\frac{3}{2}(\mathrm{tr}\Phi_{n+2}^2)^2,$$

which, of course, hold, and we really obtain (3.65) for $p-q \geq 2$. On the other hand, by the Cauchy-Schwarz inequality, we have

$$\sum_{\beta\neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2 \leq |\Phi_{n+1}|^2\sum_{\beta\neq n+1}|\Phi_\beta|^2 \leq |\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}|^2), \tag{3.66}$$

and the second equality holds if and only if $S_2 = 0$. It follows from (3.65) and (3.66) that

$$\sum_{\beta,\beta'\neq n+1}(-N(\Phi_{\beta'}\Phi_\beta - \Phi_\beta\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2) - |\Phi_{n+1}|^4 - 2\sum_{\beta\neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2$$
$$\geq -\frac{3}{2}(|\Phi|^2 - |\Phi_{n+1}|^2)^2 - |\Phi_{n+1}|^4 - 2|\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}^2|). \tag{3.67}$$

For the last term of (3.24), we have

$$\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_\gamma - \Phi_\gamma\Phi_{\gamma'}) + (\mathrm{tr}\Phi_{\gamma'}\Phi_\gamma)^2) \geq 0, \tag{3.68}$$

and the equality holds if and only if $S_2 = 0$. Substituting (3.64), (3.67) and (3.68) into (3.24), we obtain

$$\frac{1}{2}\Delta|\Phi|^2 \geq \sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2 + \frac{1}{2}|\Phi_{n+1}|^2\left(|\Phi|^2 - |\Phi_{n+1}|^2\right) \tag{3.69}$$
$$+|\Phi|^2\left(-\frac{3}{2}\left(|\Phi|^2 - |\Phi_{n+1}|^2\right) - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+1}| - |\Phi_{n+1}|^2 + n(c+H^2)\right).$$

Since $|\Phi_{n+1}| \leq |\Phi|$ and $H > 0$, (3.31) can be rewritten as follows

$$\Delta|\Phi|^2 \;\geq\; |\Phi|^2 P_H(|\Phi|), \tag{3.70}$$

where $P_H(x)$ is the function defined by (3.49).

Let us suppose by the contradiction that M^n is not totally umbilical or, equivalently, that $f = |\Phi|^2$ is a non-identically vanishing smooth function on M^n. So, considering on M^n the tangent vector field $X = \nabla|\Phi|^2$, we have that

$$\langle\nabla f, X\rangle = |\nabla|\Phi|^2|^2 \geq 0.$$

Moreover, since $\sup_M|\Phi| \leq \alpha^*$, from (3.70) we obtain

$$\mathrm{div}X = \Delta|\Phi|^2 \geq 0.$$

Hence, since we are assuming that $|\Phi|$ converges to zero at infinity, we can apply Lemma 3.2.1 to conclude that $|\nabla|\Phi|^2|^2 \equiv 0$, that is, $|\Phi|$ is constant on M^n. But, taking into account once more that $|\Phi|$ converges to zero at infinity, we have that $|\Phi|$ must be identically zero on M^n and we reach at a contradiction.

Proof of Theorem 3.2.3

Taking into account that the mean curvature vector is timelike, we choose e_{n+p} to have the same direction an h, so that $h = He_{n+p}$; Then we have

$$H^{n+p} = H; \qquad H^{\alpha} = 0, \quad \alpha = n+1, \cdots, n+p-1. \tag{3.71}$$

Since h in nonzero and parallel, we see that H is a nonzero constant and e_{n+p} is parallel. It follows that $h_{ij}^{n+p} h_{ij}^{\alpha} = h_{ij}^{\alpha} h_{ij}^{n+p}$ and

$$\sum_k h_{kki}^{\alpha} = 0, \qquad \sum_k h_{kkij}^{\alpha} = 0. \tag{3.72}$$

From (3.108) and (3.90) we have

$$\phi_{ij}^{n+p} = h_{ij}^{n+p} - H\delta_{ij}, \qquad \mathrm{tr}(\Phi_{n+p}^2) = \mathrm{tr}(h_{ij}^{n+p})^2 - nH^2, \tag{3.73}$$

$$\mathrm{tr}(h_{ij}^{n+p})^3 = \mathrm{tr}\Phi_{n+p}^3 + 3H\mathrm{tr}\Phi_{n+p}^2 + nH^3, \tag{3.74}$$

$$\phi_{ij}^{\alpha} = h_{ij}^{\alpha}, \qquad \mathrm{tr}\Phi_{\alpha}^2 = \mathrm{tr}(h_{\alpha}^2), \quad n+1 \le \alpha \le n+p-1. \tag{3.75}$$

Substituting (3.71) - (3.75) into Proposition 3.1.4 we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 \;\; = \;\; & \sum_{\alpha} |\nabla\Phi_{\alpha}|^2 + n(1-H^2)|\Phi|^2 - n\sum_{\alpha} H\mathrm{tr}(\Phi_{\alpha}^2\Phi_{n+1}) \\ & -\sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_{\beta} - \Phi_{\beta}\Phi_{\beta'}) + (\mathrm{tr}\Phi_{\beta'}\Phi_{\beta})^2) \\ & +\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_{\gamma} - \Phi_{\gamma}\Phi_{\gamma'}) + (\mathrm{tr}\Phi_{\gamma'}\Phi_{\gamma})^2). \end{aligned} \tag{3.76}$$

Now, we will estimate separately each term of the right-hand side of (3.76). First, we define

$$|\Phi_1|^2 = S_1 = \sum_{\beta}\sum_{i,j}(\Phi_{ij}^{\beta})^2, \qquad |\Phi_2|^2 = \sum_{\gamma}\sum_{i,j}(\Phi_{ij}^{\gamma})^2 = S_2 - nH^2; \tag{3.77}$$

then $|\Phi|^2 = |\Phi_1|^2 + |\Phi_2|^2$. Since

$$\mathrm{tr}\Phi_{\alpha} = 0, \qquad [\Phi_{n+p}, \Phi_{\alpha}] = [h_{ij}^{n+p}, h_{ij}^{\alpha}] = 0, \qquad \alpha = n+1, \cdots, n+p,$$

we may apply Lemma 3.2.4 to the third term of (3.76), obtaining

$$\sum_{\alpha} \mathrm{tr}\Phi_{n+p}\Phi_{\alpha}^2 \le \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{\alpha}\mathrm{tr}\Phi_{\alpha}^2\right)\sqrt{\mathrm{tr}\Phi_{n+p}^2} = \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^2|\Phi_{n+p}|. \tag{3.78}$$

Using Lemma 3.2.5 in the fourth term of (3.76), we can write

$$-\sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_\beta-\Phi_\beta\Phi_{\beta'})+(\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2)\geq -\frac{3}{2}(|\Phi|^2-|\Phi_2|^2)^2 \tag{3.79}$$
$$=-\frac{3}{2}(|\Phi|^4-2|\Phi_2|^2|\Phi|^2+|\Phi_2|^4)$$
$$\geq -\frac{3}{2}(|\Phi|^4-2|\Phi_{n+p}|^2|\Phi|^2+|\Phi|^4)$$
$$=-3|\Phi|^2(|\Phi|^2-|\Phi_{n+p}|^2),$$

and the equalities hold if and only if $|\Phi_1|^2=S_1=0$ and $|\Phi_2|^2=|\Phi_{n+p}|^2$.

For the last term of (3.76), we have

$$\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_\gamma-\Phi_\gamma\Phi_{\gamma'})\geq 0, \tag{3.80}$$

and

$$\sum_{\gamma,\gamma'}(\mathrm{tr}\Phi_{\gamma'}\Phi_\gamma)^2\geq|\Phi_{n+p}|^4+\frac{1}{q-1}(|\Phi_2|^2-|\Phi_{n+p}|^2)^2 \tag{3.81}$$
$$=|\Phi_{n+p}|^4+\frac{1}{q-1}(|\Phi_2|^4-2|\Phi_2|^2|\Phi_{n+p}|^2+|\Phi_{n+p}|^4)$$
$$\geq|\Phi_{n+p}|^4+\frac{1}{q-1}(|\Phi_{n+p}|^4-2|\Phi|^2|\Phi_{n+p}|^2+|\Phi_{n+p}|^4)$$
$$=|\Phi_{n+p}|^4+\frac{2}{q-1}|\Phi_{n+p}|^2(|\Phi_{n+p}|^2-|\Phi|^2),$$

where the equalities hold if and only if $|\Phi_1|^2=S_1=0$ and $|\Phi_2|^2=|\Phi_{n+p}|^2$.

Substituting (3.78)-(3.81) into (3.76), we obtain

$$\frac{1}{2}\Delta|\Phi|^2\geq\sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2-\frac{q+1}{q-1}|\Phi_{n+p}|^2(|\Phi|^2-|\Phi_{n+p}|^2) \tag{3.82}$$
$$+\ \ |\Phi|^2\left(-3(|\Phi|^2-|\Phi_{n+p}|^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}|+|\Phi_{n+p}|^2+n(1-H^2)\right).$$

Since $|\Phi_{n+p}|\leq|\Phi|$ and $H>0$, we can rewritten as follows

$$\frac{1}{2}\Delta|\Phi|^2\geq\sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2-\frac{q+1}{q-1}|\Phi|^2(|\Phi|^2-|\Phi_{n+p}|^2)$$
$$+|\Phi|^2\left(-3(|\Phi|^2-|\Phi_{n+p}|^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}|+|\Phi_{n+p}|^2+n(1-H^2)\right)$$
$$\geq|\Phi|^2\Big(-\frac{2(2q-1)}{q-1}(|\Phi|^2-|\Phi_{n+p}|^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}|+$$
$$+\ \ |\Phi_{n+p}|^2+n(1-H^2)\Big)$$
$$\geq\frac{2(2q-1)}{q-1}|\Phi|^2|\Phi_{n+p}|^2+\frac{1}{2}|\Phi|^2Q_H(|\Phi|), \tag{3.83}$$

where $Q_H(x)$ is the function defined in (3.50).

Since $\sup_M |\Phi| \leq \beta^*$, from the behavior of $Q_H(x)$ jointly with (3.83) we obtain

$$\Delta|\Phi|^2 \geq \ |\Phi|^2 Q_H(|\Phi|) \geq 0.$$

At this point, we can reason as in the last part of the proof of Theorem 3.2.2 to conclude that M^n must be a totally umbilical submanifold of $\mathbb{S}_q^{n+p}$.

3.3 A Simons type formula for spacelike submanifolds

In what follows, we denote by ∇ and Δ the gradient and the Laplacian operator in the metric of the spacelike submanifold M^n. Then, the Laplacian of the second fundamental form h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_{k=1}^n h_{ijkk}^\alpha$. From (3.10) and (3.11), we obtain

$$\Delta h_{ij}^\alpha = \sum_k h_{kkij}^\alpha + \sum_{m,k} h_{mk}^\alpha R_{mijk} + \sum_{m,k} h_{im}^\alpha R_{mkjk} + \sum_{k,\beta} h_{ik}^\beta R_{\beta\alpha jk} - \sum_{k,\gamma} h_{ik}^\gamma R_{\gamma\alpha jk}. \quad (3.84)$$

On the other hand, we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha. \quad (3.85)$$

Thus, inserting (3.84) into (3.85) we get

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha,k} h_{ij}^\alpha h_{kkij}^\alpha + \sum_{i,j,\alpha,m,k} h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum_{i,j,\alpha,m,k} h_{ij}^\alpha h_{im}^\alpha R_{mkjk} \quad (3.86)$$
$$+ \sum_{i,j,\alpha,k,\beta} h_{ij}^\alpha h_{ik}^\beta R_{\beta\alpha jk} - \sum_{i,j,\alpha,k,\gamma} h_{ij}^\alpha h_{ik}^\gamma R_{\gamma\alpha jk}.$$

Hence, using (1.20) and (3.7), from (3.86) we reach at

$$\begin{aligned}\frac{1}{2}\Delta S = &\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{\alpha,i,j} n h_{ij}^\alpha h_{kkij}^\alpha + cnS - cn^2H^2 \quad (3.87)\\ &+ \sum_{\alpha,\beta} \mathrm{tr}(h_{ij}^\beta)\mathrm{tr}((h_{ij}^\alpha)^2 h_{ij}^\beta) - \sum_{\alpha,\beta}(N(h_{ij}^\alpha h_{ij}^\beta - h_{ij}^\beta h_{ij}^\alpha) + (\mathrm{tr} h_{ij}^\alpha h_{ij}^\beta)^2)\\ &- \sum_{\alpha,\gamma} \mathrm{tr}(h_{ij}^\gamma)\mathrm{tr}((h_{ij}^\alpha)^2 h_{ij}^\gamma) + \sum_{\alpha,\gamma}(N(h_{ij}^\alpha h_{ij}^\gamma - h_{ij}^\gamma h_{ij}^\alpha) + (\mathrm{tr} h_{ij}^\alpha h_{ij}^\gamma)^2),\end{aligned}$$

where $N(h_{ij}^\alpha) = \mathrm{tr}((h_{ij}^\alpha)^T h_{ij}^\alpha)$. Therefore, from (3.87) we obtain the following Simons type formula

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2 + n\sum_{\alpha,i,j} h^{\alpha}_{ij}h^{\alpha}_{kkij} + nc(S - nH^2) \tag{3.88}$$
$$+ \sum_{\alpha,\beta}\mathrm{tr}(h^{\beta}_{ij})\mathrm{tr}((h^{\alpha}_{ij})^2 h^{\beta}_{ij}) - \sum_{\beta,\beta'}(N(h^{\alpha}_{ij}h^{\beta}_{ij} - h^{\beta}_{ij}h^{\alpha}_{ij}) + (\mathrm{tr}h^{\alpha}_{ij}h^{\beta}_{ij})^2)$$
$$- \sum_{\alpha,\gamma}\mathrm{tr}(h^{\gamma}_{ij})\mathrm{tr}((h^{\alpha}_{ij})^2 h^{\gamma}_{ij}) + \sum_{\gamma,\gamma'}(N(h^{\alpha}_{ij}h^{\gamma}_{ij} - h^{\gamma}_{ij}h^{\alpha}_{ij}) + (\mathrm{tr}h^{\alpha}_{ij}h^{\gamma}_{ij})^2).$$

3.4 Stochastically complete spacelike submanifolds

A (non necessarily complete) Riemannian manifold M^n is said to be *stochastically complete* if, for some (and, hence, for any) $(x,t) \in M^n \times (0,+\infty)$, the heat kernel $p(x,y,t)$ of the Laplace-Beltrami operator Δ satisfies the conservation property

$$\int_M p(x,y,t)d\mu(y) = 1. \tag{3.89}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (3.89) means that the total probability of the particle to be found in the state space is constantly equal to one (see [69, 75, 76, 113]).

On the other hand, Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori-Yau maximum principle (See Theorem 1.1 of [108] and Theorem 3.1 of [109]), as is expressed below.

Lemma 3.4.1 *A Riemannian manifold M^n is stochastically complete if, and only if, for every $u \in C^2(M)$ satisfying $\sup_M u < +\infty$ there exists a sequence of points $\{p_k\} \subset M^n$ such that*

$$\lim_{k\to\infty} u(p_k) = \sup_M u \qquad \textit{and} \qquad \limsup_{k\to\infty} \Delta u(p_k) \leq 0.$$

We also note that stochastic completeness of Riemannian manifold M^n is equivalent (among other conditions) to the fact that for every $\lambda > 0$, the only nonnegative bounded smooth solution u of $\Delta u \geq \lambda u$ on M^n is the constant $u = 0$. Moreover, it is a direct consequence of Lemma 3.4.1 jointly with the Omori-Yau maximum principle [105, 131] that complete Riemannian manifolds having Ricci curvature bounded from below are stochastically complete.

Our aim in this section is to present some gap results concerning stochastically complete spacelike submanifolds M^n with parallel mean curvature vector in the indefinite space form $\mathbb{L}_q^{n+p}(c)$. For this, setting

$$\phi_{ij}^\alpha = h_{ij}^\alpha - H^\alpha \delta_{ij},$$

we consider the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \phi_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha.$$

It is easy to check that Φ is traceless and

$$|\Phi|^2 = \sum_\alpha \mathrm{tr}\Phi_\alpha^2 = S - nH^2, \tag{3.90}$$

where Φ_α denote the matrix (ϕ_{ij}^α). Moreover, we observe that $|\Phi|^2 = 0$ if and only if M^n is a totally umbilical submanifold of $\mathbb{L}_q^{n+p}(c)$.

Considering stochastically complete spacelike submanifolds having spacelike and parallel mean curvature vector, we establish our first main result.

Theorem 3.4.2 *Let M^n be a stochastically complete spacelike submanifold immersed in $\mathbb{L}_q^{n+p}(c)$, with $c \in \{0, -1, 1\}$ and $1 \leq q < p - 1$, having spacelike and parallel mean curvature vector. When $c = -1$, suppose in addition that $H > 1$. Then, either*

(i) $\sup_M |\Phi| = 0$ *and M^n is a totally umbilical submanifold, or*

(ii) $\sup_M |\Phi| \geq \alpha^*(n, c, H)$, *where $\alpha^*(n, c, H)$ is the positive root of the function*

$$P_H(x) := -\frac{5}{2}x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} Hx + n(c + H^2). \tag{3.91}$$

Moreover, if the equality holds and this supremum is attained at some point of M^n, then M^n is a pseudo-umbilical submanifold of $\mathbb{L}_q^{n+p}(c)$ such that its principal curvatures are constant.

Proof. Taking into account that the mean curvature vector is spacelike, we choose e_{n+1} to have the same direction of h, so that $h = He_{n+1}$. Then we have

$$H^{n+1} = H; \qquad H^\alpha = 0, \quad \alpha = n+2, \cdots, n+p. \tag{3.92}$$

Since h is nonzero and parallel, we see that H is a nonzero constant and e_{n+1} is parallel. It follows that $h_{ij}^{n+1} h_{ij}^\alpha = h_{ij}^\alpha h_{ij}^{n+1}$ and

$$\sum_k h_{kki}^\alpha = 0, \qquad \sum_k h_{kkij}^\alpha = 0. \tag{3.93}$$

From (3.92) and (3.90) we have

$$\phi_{ij}^{n+1} = h_{ij}^{n+1} - H\delta_{ij}, \qquad \mathrm{tr}(\Phi_{n+1}^2) = \mathrm{tr}(h_{ij}^{n+1})^2 - nH^2, \tag{3.94}$$

$$\mathrm{tr}(h_{ij}^{n+1})^3 = \mathrm{tr}\Phi_{n+1}^3 + 3H\mathrm{tr}\Phi_{n+1}^2 + nH^3, \tag{3.95}$$

$$\phi_{ij}^{\alpha} = h_{ij}^{\alpha}, \qquad \mathrm{tr}\Phi_{\alpha}^2 = \mathrm{tr}(h_{\alpha}^2), \quad \alpha \geq n+2. \tag{3.96}$$

Substituting (3.92) - (3.96) into (3.2.6) we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 =& \sum_{\alpha}|\nabla\Phi_{\alpha}|^2 + n(c+H^2)|\Phi|^2 + n\sum_{\alpha}H\mathrm{tr}(\Phi_{\alpha}^2\Phi_{n+1}) \\ &- \sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_{\beta} - \Phi_{\beta}\Phi_{\beta'}) + (\mathrm{tr}\Phi_{\beta'}\Phi_{\beta})^2) \\ &+ \sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_{\gamma} - \Phi_{\gamma}\Phi_{\gamma'}) + (\mathrm{tr}\Phi_{\gamma'}\Phi_{\gamma})^2). \end{aligned} \tag{3.97}$$

Now we shall estimate separately each term of the right-hand side of (3.97). Defining

$$|\Phi_1|^2 = \sum_{\beta}\sum_{i,j}(\Phi_{ij}^{\beta})^2 = S_1 - nH^2, \qquad |\Phi_2|^2 = \sum_{\gamma}\sum_{i,j}(\Phi_{ij}^{\gamma})^2 = S_2, \tag{3.98}$$

we get $|\Phi|^2 = |\Phi_1|^2 + |\Phi_2|^2$. Since

$$[\Phi_{n+1}, \Phi_{\alpha}] = [h_{ij}^{n+1}, h_{ij}^{\alpha}] = 0, \qquad \mathrm{tr}\Phi_{\alpha} = 0, \quad \alpha = n+1, \cdots, n+p,$$

we can apply Lemma 3.1.1 to the third term of (3.97), obtaining

$$\sum_{\alpha}\mathrm{tr}\Phi_{n+1}\Phi_{\alpha}^2 \geq -\frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{\alpha}\mathrm{tr}\Phi_{\alpha}^2\right)\sqrt{\mathrm{tr}\Phi_{n+1}^2} = -\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^2|\Phi_{n+1}|. \tag{3.99}$$

The fourth term of (3.97) can be rewritten as follows:

$$\sum_{\beta,\beta'\neq n+1}(-N(\Phi_{\beta'}\Phi_{\beta} - \Phi_{\beta}\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_{\beta})^2) - |\Phi_{n+1}|^4 - 2\sum_{\beta\neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_{\beta})^2,$$

it follows from Lemma 3.1.2 that, for $p - q \geq 3$,

$$\begin{aligned} \sum_{\beta,\beta'\neq n+1}(-N(\Phi_{\beta'}\Phi_{\beta} - \Phi_{\beta}\Phi_{\beta'}) - (\mathrm{tr}\Phi_{\beta'}\Phi_{\beta})^2) &\geq -\frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi_{\beta}|^2\right)^2 \\ &\geq -\frac{3}{2}(|\Phi|^2 - |\Phi_{n+1}|^2)^2, \end{aligned} \tag{3.100}$$

and the second equality holds if and only if $S_2 = 0$. When $p - q = 2$, (3.100) becomes

$$-(\mathrm{tr}\Phi_{n+2}\Phi_{n+2})^2 \geq -\frac{3}{2}(\mathrm{tr}\Phi_{n+2}^2)^2,$$

which, of course, holds, and we really obtain (3.100) for $p-q\geq 2$. On the other hand, by the Cauchy-Schwarz inequality, we have

$$\sum_{\beta\neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2\leq|\Phi_{n+1}|^2\sum_{\beta\neq n+1}|\Phi_\beta|^2\leq|\Phi_{n+1}|^2(|\Phi|^2-|\Phi_{n+1}|^2),\tag{3.101}$$

and the second equality holds if and only if $S_2=0$. It follows from (3.100) and (3.101) that

$$\begin{aligned}\sum_{\beta,\beta'\neq n+1}&(-N(\Phi_{\beta'}\Phi_\beta-\Phi_\beta\Phi_{\beta'})-(\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2)-|\Phi_{n+1}|^4-2\sum_{\beta\neq n+1}(\mathrm{tr}\Phi_{n+1}\Phi_\beta)^2,\\ &\geq-\frac{3}{2}(|\Phi|^2-|\Phi_{n+1}|^2)^2-|\Phi_{n+1}|^4-2|\Phi_{n+1}|^2(|\Phi|^2-|\Phi_{n+1}^2|).\end{aligned}\tag{3.102}$$

For the last term of (3.97), we have

$$\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_\gamma-\Phi_\gamma\Phi_{\gamma'})+(\mathrm{tr}\Phi_{\gamma'}\Phi_\gamma)^2)\geq 0,\tag{3.103}$$

and the equality holds if and only if $S_2=0$. Substituting (3.99), (3.102) and (3.103) into (3.97), we obtain

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2\geq&\sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2+\frac{1}{2}|\Phi_{n+1}|^2\big(|\Phi|^2-|\Phi_{n+1}|^2\big)\\ &+|\Phi|^2\left(-\frac{3}{2}\big(|\Phi|^2-|\Phi_{n+1}|^2\big)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+1}|-|\Phi_{n+1}|^2+n(c+H^2)\right).\end{aligned}\tag{3.104}$$

Since $|\Phi_{n+1}|\leq|\Phi|$ and $H>0$, we can rewrite (3.104) as follows

$$\frac{1}{2}\Delta|\Phi|^2\ \geq\ \sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2+\frac{3}{2}|\Phi|^2|\Phi_{n+1}|^2+|\Phi|^2P_H(|\Phi|)\geq|\Phi|^2P_H(|\Phi|),\tag{3.105}$$

where $P_H(x)$ is the function defined by (3.91).

To conclude the proof, we can apply Lemma 3.1.3 to the Laplacian operator acting on the function $|\Phi|^2$. Indeed, if $\sup_M|\Phi|=+\infty$, then (ii) is trivially satisfied. So, let us suppose that $\sup_M|\Phi|<+\infty$. Thus, Lemma 3.4.1 guarantees that there exists a sequence of points $\{p_k\}\subset M^n$ such that

$$\lim_{k\to\infty}|\Phi|^2(p_k)=\sup_M|\Phi|^2\qquad\text{and}\qquad\limsup_{k\to\infty}\Delta|\Phi|^2(p_k)\leq 0.$$

Consequently, taking into account the continuity of the function $P_H(x)$, from (3.105) we get

$$\begin{aligned}0\geq\frac{1}{2}\limsup_{k\to\infty}\Delta|\Phi|^2(p_k)\ &\geq\ \limsup_{k\to\infty}\big(|\Phi|^2P_H(|\Phi|)\big)(p_k)=\lim_{k\to\infty}\big(|\Phi|^2P_H(|\Phi|)\big)(p_k)\\ &=\ \lim_{k\to\infty}|\Phi|^2(p_k)P_H(\lim_{k\to\infty}|\Phi|(p_k))=\sup_M|\Phi|^2P_H(\sup_M|\Phi|).\end{aligned}$$

Hence, we obtain

$$\sup_M |\Phi|^2 P_H(\sup_M |\Phi|) \leq 0. \tag{3.106}$$

It follows from here that either $\sup_M |\Phi| = 0$, which means that $|\Phi| \equiv 0$ and the spacelike submanifold is totally umbilical, or $\sup_M |\Phi| > 0$ and then (3.106) gives

$$P_H(\sup_M |\Phi|) \leq 0,$$

which implies that $\sup_M |\Phi| \geq \alpha^*(n, c, H)$, where $\alpha^*(n, c, H)$ is the positive root of (3.91). We note that it was used the fact that $P_H(0) = n(c + H^2) > 0$.

Finally, let us assume that $\sup_M |\Phi| = \alpha^*(n, c, H)$ and the $\sup_M |\Phi|$ is attained at some point of M^n, then as Laplacian operator is elliptic we have from Hopf maximum principle that $|\Phi|$ is constant. Hence, returning to (3.105), we obtain that $|\Phi_{n+1}| = 0$, which means to say that M^n is a pseudo-umbilical submanifold of $\mathbb{L}_q^{n+p}(c)$. Furthermore, since we also have that $\sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 = 0$, we conclude that the principal curvatures of M^n are constant. ■

Remark 3.4.3 As it was observed by the referee of this manuscript, an interesting open problem is to know if the conclusion of Theorem 3.4.2 is sharp in the sense that it is no more true for a stochastically *incomplete* spacelike submanifold immersed in $\mathbb{L}_q^{n+p}(c)$; in particular, for the case that such a submanifold is rotationally symmetric.

When the mean curvature vector is timelike and the ambient space is de Sitter space $\mathbb{S}_q^{n+p}$, we also get the following result.

Theorem 3.4.4 *Let M^n be a stochastically complete spacelike submanifold immersed in de Sitter space $\mathbb{S}_q^{n+p}$, with $1 < q < p-1$, having timelike and parallel mean curvature vector. Suppose that $H < 1$. Then, either*

(i) $\sup_M |\Phi| = 0$ *and M^n is a totally umbilical submanifold, or*

(ii) $\sup_M |\Phi| \geq \beta^*(n, q, H)$, *where $\beta^*(n, q, H)$ is the positive root of the function*

$$Q_H(x) := -\frac{2(2q-1)}{q-1} x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H x + n(1 - H^2). \tag{3.107}$$

Moreover, if the equality holds and this supremum is attained at some point of M^n, then M^n is a pseudo-umbilical submanifold of $\mathbb{S}_q^{n+p}$ such that its principal curvatures are constant.

Proof. Taking into account that the mean curvature vector is timelike, we choose e_{n+p} to have the same direction of h, so that $h = He_{n+p}$. Then we have

$$H^{n+p} = H; \qquad H^{\alpha} = 0, \quad \alpha = n+1, \cdots, n+p-1. \tag{3.108}$$

Since h is nonzero and parallel, we see that H is a nonzero constant and e_{n+p} is parallel. It follows that $h_{ij}^{n+p}h_{ij}^{\alpha} = h_{ij}^{\alpha}h_{ij}^{n+p}$ and

$$\sum_k h_{kki}^{\alpha} = 0, \qquad \sum_k h_{kkij}^{\alpha} = 0. \tag{3.109}$$

From (3.108) and (3.90) we have

$$\phi_{ij}^{n+p} = h_{ij}^{n+p} - H\delta_{ij}, \qquad \mathrm{tr}(\Phi_{n+p}^2) = \mathrm{tr}(h_{ij}^{n+p})^2 - nH^2, \tag{3.110}$$

$$\mathrm{tr}(h_{ij}^{n+p})^3 = \mathrm{tr}\Phi_{n+p}^3 + 3H\mathrm{tr}\Phi_{n+p}^2 + nH^3, \tag{3.111}$$

$$\phi_{ij}^{\alpha} = h_{ij}^{\alpha}, \qquad \mathrm{tr}\Phi_{\alpha}^2 = \mathrm{tr}(h_{\alpha}^2), \quad n+1 \le \alpha \le n+p-1. \tag{3.112}$$

Substituting (3.108) - (3.112) into (3.2.6) we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 = &\sum_{\alpha}|\nabla\Phi_{\alpha}|^2 + n(1-H^2)|\Phi|^2 - n\sum_{\alpha} H\mathrm{tr}(\Phi_{\alpha}^2\Phi_{n+1}) \\ &- \sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_{\beta} - \Phi_{\beta}\Phi_{\beta'}) + (\mathrm{tr}\Phi_{\beta'}\Phi_{\beta})^2) \\ &+ \sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_{\gamma} - \Phi_{\gamma}\Phi_{\gamma'}) + (\mathrm{tr}\Phi_{\gamma'}\Phi_{\gamma})^2). \end{aligned} \tag{3.113}$$

Now we shall estimate separately each term of the right-hand side of (3.113). Defining

$$|\Phi_1|^2 = S_1 = \sum_{\beta}\sum_{i,j}(\Phi_{ij}^{\beta})^2, \qquad |\Phi_2|^2 = \sum_{\gamma}\sum_{i,j}(\Phi_{ij}^{\gamma})^2 = S_2 - nH^2, \tag{3.114}$$

we get $|\Phi|^2 = |\Phi_1|^2 + |\Phi_2|^2$. Since

$$\mathrm{tr}\Phi_{\alpha} = 0, \qquad [\Phi_{n+p}, \Phi_{\alpha}] = [h_{ij}^{n+p}, h_{ij}^{\alpha}] = 0, \qquad \alpha = n+1, \cdots, n+p,$$

we can apply Lemma 3.1.1 to the third term of (3.113), obtaining

$$\sum_{\alpha}\mathrm{tr}\Phi_{n+p}\Phi_{\alpha}^2 \le \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{\alpha}\mathrm{tr}\Phi_{\alpha}^2\right)\sqrt{\mathrm{tr}\Phi_{n+p}^2} = \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^2|\Phi_{n+p}|. \tag{3.115}$$

Using Lemma 3.1.2 in the fourth term of (3.113), we can write

$$-\sum_{\beta,\beta'}(N(\Phi_{\beta'}\Phi_\beta-\Phi_\beta\Phi_{\beta'})+(\mathrm{tr}\Phi_{\beta'}\Phi_\beta)^2)\geq -\frac{3}{2}(|\Phi|^2-|\Phi_2|^2)^2 \tag{3.116}$$
$$=-\frac{3}{2}(|\Phi|^4-2|\Phi_2|^2|\Phi|^2+|\Phi_2|^4)$$
$$\geq -\frac{3}{2}(|\Phi|^4-2|\Phi_{n+p}|^2|\Phi|^2+|\Phi|^4)$$
$$=-3|\Phi|^2(|\Phi|^2-|\Phi_{n+p}|^2),$$

and the equalities hold if and only if $|\Phi_1|^2=S_1=0$ and $|\Phi_2|^2=|\Phi_{n+p}|^2$.

For the last term of (3.113), we have

$$\sum_{\gamma,\gamma'}(N(\Phi_{\gamma'}\Phi_\gamma-\Phi_\gamma\Phi_{\gamma'})\geq 0, \tag{3.117}$$

and

$$\sum_{\gamma,\gamma'}(\mathrm{tr}\Phi_{\gamma'}\Phi_\gamma)^2\geq |\Phi_{n+p}|^4+\frac{1}{q-1}(|\Phi_2|^2-|\Phi_{n+p}|^2)^2 \tag{3.118}$$
$$=|\Phi_{n+p}|^4+\frac{1}{q-1}(|\Phi_2|^4-2|\Phi_2|^2|\Phi_{n+p}|^2+|\Phi_{n+p}|^4)$$
$$\geq |\Phi_{n+p}|^4+\frac{1}{q-1}(|\Phi_{n+p}|^4-2|\Phi|^2|\Phi_{n+p}|^2+|\Phi_{n+p}|^4)$$
$$=|\Phi_{n+p}|^4+\frac{2}{q-1}|\Phi_{n+p}|^2(|\Phi_{n+p}|^2-|\Phi|^2),$$

where the equalities hold if and only if $|\Phi_1|^2=S_1=0$ and $|\Phi_2|^2=|\Phi_{n+p}|^2$.

Substituting (3.115)-(3.118) into (3.113), we obtain

$$\frac{1}{2}\Delta|\Phi|^2\geq\sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2-\frac{q+1}{q-1}|\Phi_{n+p}|^2(|\Phi|^2-|\Phi_{n+p}|^2) \tag{3.119}$$
$$+\quad |\Phi|^2\left(-3(|\Phi|^2-|\Phi_{n+p}|^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}|+|\Phi_{n+p}|^2+n(1-H^2)\right).$$

Since $|\Phi_{n+p}|\leq|\Phi|$ and $H>0$, we can rewrite (3.119) as follows

$$\frac{1}{2}\Delta|\Phi|^2\geq\sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2-\frac{q+1}{q-1}|\Phi|^2(|\Phi|^2-|\Phi_{n+p}|^2) \tag{3.120}$$
$$+\quad |\Phi|^2\left(-3(|\Phi|^2-|\Phi_{n+p}|^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}|+|\Phi_{n+p}|^2+n(1-H^2)\right)$$
$$\geq\quad |\Phi|^2\left(-\frac{2(2q-1)}{q-1}(|\Phi|^2-|\Phi_{n+p}|^2)-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi_{n+p}|\right.$$
$$\left.+\quad |\Phi_{n+p}|^2+n(1-H^2)\right)$$
$$\geq\quad \frac{2(2q-1)}{q-1}|\Phi|^2|\Phi_{n+p}|^2+|\Phi|^2Q_H(|\Phi|)\geq|\Phi|^2Q_H(|\Phi|),$$

where $Q_H(x)$ is the function defined in (3.107),

On the other hand, reasoning as in the deduction of inequality (3.106), we can apply once more Lemma 3.4.1 to the Laplacian operator acting on the function $|\Phi|^2$ and, from (3.120), we obtain

$$(\sup_M|\Phi|)^2 Q_H(\sup_M|\Phi|) \leq 0. \tag{3.121}$$

It follows from here that either $\sup_M|\Phi|= 0$, which means that $|\Phi|\equiv 0$ and M^n is totally umbilical, or $\sup_M|\Phi|> 0$ and then (3.121) gives

$$Q_H(\sup_M|\Phi|) \leq 0,$$

which implies that $\sup_M|\Phi|\geq \beta^*(n,q,H)$, where $\beta^*(n,q,H)$ is the positive root of (3.107). We note that it was used the fact that $Q_H(0) = n(1-H^2) > 0$.

Now, let us assume that $\sup_M|\Phi|= \beta^*(n,q,H)$ and the $\sup_M|\Phi|$ is attained at some point of M^n, then as Laplacian operator is elliptic we have from Hopf maximum principle that $|\Phi|$ is constant. Hence, from (3.120), we obtain that

$$\frac{2(2q-1)}{q-1}|\Phi|^2|\Phi_{n+p}|^2 = 0 \qquad \text{and} \qquad \sum_{\alpha=n+1}^{n+p}|\nabla\Phi_\alpha|^2 = 0. \tag{3.122}$$

Therefore, from (3.122) we conclude that M^n is pseudo-umbilical submanifold of $\mathbb{S}_q^{n+p}$ and its principal curvatures are constant. ■

3.5 Parabolic and L^1-Liouville spacelike submanifolds

We recall that a (non necessarily complete) Riemannian manifold M^n is said to be *parabolic* (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above, that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

We observe that every parabolic Riemannian manifold is stochastically complete. As a consequence, the weak maximum principle holds on every parabolic Riemannian manifold (see Corollary 6.4 of [76]). Obviously, every closed Riemannian manifold M^n is parabolic, where by closed we mean compact and without boundary. Moreover, there are several interesting geometric conditions which imply the parabolicity of a Riemannian manifold M^n. For instance, in dimension $n = 2$ parabolicity is strongly related to the behaviour of the Gaussian curvature; for instance, from a classical result by Ahlfors [16] and Blanc et al. [81] it is well known that every complete Riemannian surface with nonnegative Gaussian curvature is parabolic. More generally, every complete Riemannian surface with finite total curvature is parabolic (see Section 10 of [89]).

As it was observed in [77], when M^n $(n \geq 2)$ is a complete Riemannian manifold, we can state sufficient conditions for parabolicity and stochastic completeness in terms of the volume function $V(r) = V(B(x_0, r))$, where $B(x_0, r)$ is the geodesic ball of radius r centered at a fixed point $x_0 \in M^n$. Namely, the following implications are true:

$$\int_{r_0}^{\infty} \frac{rdr}{V(r)} = \infty \quad \Rightarrow \quad M^n \text{ is parabolic,} \tag{3.123}$$

$$\int_{r_0}^{\infty} \frac{rdr}{\log V(r)} = \infty \quad \Rightarrow \quad M^n \text{ is stochastically complete.} \tag{3.124}$$

For example, $V(r) \leq Cr^2$ and $V(r) \leq \exp(Cr^2)$ will imply the volume conditions in (3.123) and (3.124), respectively. Cheng and Yau in [54] proved that $V(r) \leq Cr^2$ is a sufficient condition for parabolicity. The sharp sufficient condition (3.123) for parabolicity was proved by several authors in [73], [74], [86] and [120]. Several authors [66], [82], [117] showed that $V(r) \leq \exp(Cr^2)$ is a sufficient condition for stochastic completeness (see also an earlier result [80]), and the sharp result (3.124) was obtained in [74] (see [78] and [114] for its extensions). For a model manifold with pole at x_0, both the parabolicity and stochastic completeness can be characterized solely in terms of the function $V(r)$ and its derivative (see [76] and [33]).

Considering the context of spacelike submanifolds immersed in a pseudo-Riemannian space form, we obtain the following gap result:

Theorem 3.5.1 *Let M^n be a parabolic spacelike submanifold immersed in $\mathbb{L}_q^{n+p}(c)$, with $c \in \{0, -1, 1\}$ and $1 \leq q < p - 1$, having spacelike and parallel mean curvature vector. When $c = -1$, suppose in addition that $H > 1$. Then either $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold or $\sup_M |\Phi| \geq \alpha^*(n, c, H)$, where $\alpha^*(n, c, H)$ is the positive root of (3.91). Moreover, when $\sup_M |\Phi| = \alpha^*(n, c, H)$, M^n is a pseudo-umbilical submanifold of $\mathbb{L}_q^{n+p}(c)$ such that its principal curvatures are constant.*

Proof. First all recall that the weak Omori-Yau maximum principle holds on every parabolic Riemannian manifold. Then, if $\sup_M |\Phi|^2 < +\infty$, there is nothing to prove.

On the other hand, in the case that $0 < \sup_M |\Phi|^2 \leq +\infty$, reasoning as in the first part of the proof of Theorem 3.4.4, we guarantee, the $\sup_M |\Phi|^2 \geq \alpha^*(n, c, H)$. Moreover, if $\sup_M |\Phi|^2 = \alpha^*(n, c, H)$, then $P_H(\sup_M \Phi) \leq 0$ and, consequently, the function $|\Phi|^2$ is subharmonic on M^n. Therefore, from the parabolicity of M^n we conclude that the function $|\Phi|^2$ must be constant and equal to $\alpha^*(n, c, H)$. To close the proof, we can reason as in the proof of Theorem 3.4.4. ■

When the mean curvature vector is timelike, we get.

Theorem 3.5.2 *Let M^n be a parabolic spacelike submanifold immersed in de Sitter space $\mathbb{S}_q^{n+p}$, with $1 < q < p - 1$, having timelike and parallel mean curvature vector, suppose in addition that $H < 1$. Then either $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold or $\sup_M |\Phi| \geq \beta^*(n, q, H)$, where $\beta^*(n, q, H)$ is the positive root of (3.107). Moreover, when $\sup_M |\Phi| = \beta^*(n, q, H)$, M^n is a pseudo-umbilical submanifold of $\mathbb{S}_q^{n+p}$ such that its principal curvatures are constant.*

Proof. We note that, if $\sup_M |\Phi|^2 = +\infty$, there is nothing to prove. for this reason, in the case that $0 < \sup_M |\Phi|^2 \leq +\infty$, reasoning as in the first part of the proof of Theorem 3.5.1, we guarantee, the $\sup_M |\Phi|^2 \geq \beta^*(n, q, H)$. Moreover, if $\sup_M |\Phi|^2 = \beta^*(n, q, H)$, then $Q_H(\sup_M \Phi) \leq 0$ and, consequently, the function $|\Phi|^2$ is subharmonic on M^n. Therefore, from the parabolicity of M^n we conclude that the function $|\Phi|^2$ must be constant and equal to $\beta^*(n, q, H)$. To close the proof, we can reason as in the proof of Theorem 3.5.1. ■

According to [34], a Riemannian manifold M^n is said be L^1-*Liouville* when every nonnegative superharmonic function $u \in L^1(M) := \{f : M^n \to \mathbb{R} : \int_M |f| dM < +\infty\}$ must be constant. Taking into account Corollary 3 of [34] which ensures that a stochastically complete manifold is always L^1-Liouville, we see that $\mathbb{R}^n$ $(n > 2)$ and $\mathbb{H}^n$ constitute examples of L^1-Liouville Riemannian manifolds which are not parabolic. On the other hand, we also observe that in Section 2 of [34] the authors constructed nontrivial examples of stochastically incomplete (and, in particular, nonparabolic) L^1-Liouville manifolds.

Considering a L^1-Liouville spacelike submanifold immersed in $\mathbb{L}_q^{n+p}(c)$, we obtain the following results.

Theorem 3.5.3 *Let M^n be a L^1-Liouville spacelike submanifold immersed in $\mathbb{L}_q^{n+p}(c)$, with $c \in \{0, -1, 1\}$ and $1 \leq q < p - 1$, having spacelike and parallel mean curvature* *vector. When $c = -1$, suppose in addition that $H > 1$. If $\sup_M |\Phi| \leq \alpha^*(n, c, H)$ and $\varphi := (\alpha^*(n, c, H))^2 - |\Phi|^2 \in L^1(M)$, where $\alpha^*(n, c, H)$ is the positive root of (3.91), then either $|\Phi| \equiv 0$ and M^n is a totally umbilical submanifold or $|\Phi| \equiv \alpha^*(n, c, H)$ and M^n is a pseudo-umbilical submanifold of $\mathbb{L}_q^{n+p}(c)$ such that its principal curvatures are constant.*

Proof. Since we are assuming that $\sup_M |\Phi| \leq \alpha^*(n, c, H)$, from (3.105) we get $\Delta\varphi \leq 0$. Thus, since φ is a nonnegative superharmonic function with $\varphi \in L^1(M)$, we have that φ must be constant on M^n, which implies that $|\Phi|$ is constant on M^n. Consequently, we can reason as in the last part of the proof of Theorem 3.4.4 to conclude the proof. ■

Taking into account the proof of Theorem 3.4.4, we see that the proof of the next result is quite similar to that of Theorem 3.5.3.

Theorem 3.5.4 *Let M^n be a L^1-Liouville spacelike submanifold immersed in de Sitter space $\mathbb{S}_q^{n+p}$, with $1 < q < p - 1$, having timelike and parallel mean curvature vector, suppose in addition that $H < 1$. If $\sup_M |\Phi| \leq \beta^*(n, q, H)$ and $\zeta := (\beta^*(n, q, H))^2 - |\Phi|^2 \in L^1(M)$, where $\beta^*(n, q, H)$ is the positive root of (3.107), then either $|\Phi| \equiv 0$ and M^n is a totally umbilical submanifold or $|\Phi| \equiv \beta^*(n, q, H)$ and M^n is a pseudo-umbilical submanifold of $\mathbb{S}_q^{n+p}$ such that its principal curvatures are constant.*

Now, we quote the following lemma which corresponds to Theorem 7 of [133].

Lemma 3.5.5 *Every complete noncompact Riemannian manifold, whose Ricci curvature is nonnegative, has infinite volume.*

To close this paper, it is not difficult to verify that from Lemma 3.5.5 jointly with Theorems 3.5.2 and 3.5.3 we obtain the following nonexistence results

Corollary 3.5.6 *There does not exist a complete noncompact L^1-Liouville spacelike submanifold M^n, whose Ricci curvature is nonnegative, immersed in $\mathbb{L}_q^{n+p}(c)$, with $c \in \{0, -1, 1\}$ and $1 \leq q < p - 1$, having spacelike and parallel mean curvature vector (when $c = -1$, assume in addition that $H > 1$), such that $\sup_M |\Phi| < \alpha^*(n, c, H)$ and $\varphi := (\alpha^*(n, c, H))^2 - |\Phi|^2 \in L^1(M)$, where $\alpha^*(n, c, H)$ is the positive root of (3.91).*

Corollary 3.5.7 *There does not exist a complete noncompact L^1-Liouville spacelike submanifold M^n, whose Ricci curvature is nonnegative, immersed in de Sitter space $\mathbb{S}_q^{n+p}$, with $1 < q < p - 1$, having timelike and parallel mean curvature vector, with $H < 1$, such that $\sup_M |\Phi| < \beta^*(n, q, H)$ and $\zeta := (\beta^*(n, q, H))^2 - |\Phi|^2 \in L^1(M)$, where $\beta^*(n, q, H)$ is the positive root of (3.107).*

3.6 Spacelike Submanifolds immersed in the De Sitter space

Let us denote by $\mathbb{R}_p^{n+p+1}$ the $(n+p+1)$-dimensional Lorentz-Minkowski space of index p, that is, the Euclidean space $\mathbb{R}^{n+p+1}$ endowed with the semi-Riemanannian metric

$$\langle \ , \ \rangle_p = -dx_1^2 - \ldots - dx_p^2 + dx_{p+1}^2 + \ldots + dx_{n+p+1}^2. \tag{3.125}$$

The $(n+p)$-dimensional *de Sitter space* $\mathbb{S}_p^{n+p}$ of index p is the semi-Riemannian manifold of constant sectional curvature 1 given by the following hyperquadric of $\mathbb{R}_p^{n+p+1}$:

$$\mathbb{S}_p^{n+p} = \{x \in \mathbb{R}_p^{n+p+1} \ ; \ \langle x, x\rangle_p = 1\}. \tag{3.126}$$

Let M^n be an n-dimensional connected spacelike submanifold isometrically immersed into the de Sitter space $\mathbb{S}_p^{n+p}$, meaning that the induced metric on M^n via immersion is a Riemannian metric. We choose a local field of semi-Riemannian orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{S}_p^{n+p}$, with dual coframe $\{\omega_1, \ldots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

It is well known that the second fundamental form A of M^n is defined by

$$A = \sum_{\alpha,i,j} h_{ij}^{\alpha} \omega_i \otimes \omega_j e_\alpha, \tag{3.127}$$

where the functions h_{ij}^α are given by the Cartan's Lemma and satisfy $h_{ij}^\alpha = h_{ji}^\alpha$. Then the square of the norm of the second fundamental form is $|A|^2 = \sum_{\alpha,i,j}(h_{ij}^\alpha)^2$. We also define the mean curvature vector h and the mean curvature function H of M^n, respectively, by

$$h = \frac{1}{n}\sum_\alpha \left(\sum_i h_{ii}^\alpha\right) e_\alpha \quad \text{and} \quad H = |h| = \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha\right)^2}. \tag{3.128}$$

In the case when $H > 0$, we define the normalized mean curvature vector by $\frac{h}{H}$. We also recall that $\frac{h}{H}$ is said to be parallel if it is parallel as a section of the normal bundle of M^n.

From this and by the Gauss equation, it is not difficult to check that the normalized scalar curvature R of M^n is given by

$$n(n-1)R = n(n-1) - n^2H^2 + |A|^2. \tag{3.129}$$

For our purposes, in what follows we will consider the case $H > 0$, so that in the local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ we take $e_{n+1} = \frac{h}{H}$. Thus, we consider the traceless second fundamental form of the hypersurface Φ, which is defined as the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi_{ij}^{\alpha} \omega_i \otimes \omega_j e_{\alpha},$$

where $\Phi_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}$. Here, H^{α} denotes the mean curvature function of M^n in the direction of e_{α}, that is,

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} = \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \geq n+2, \tag{3.130}$$

where $h^{\alpha} = (h_{ij}^{\alpha})$ denotes the second fundamental form of M^n in direction e_{α} for every $n+1 \leq \alpha \leq n+p$. From here it is not difficult to verify that Φ is a traceless tensor, that is, $\mathrm{tr}(\Phi) = 0$ and that holds the following relation,

$$|\Phi|^2 = |A|^2 - nH^2. \tag{3.131}$$

Moreover, $|\Phi|$ vanishes identically on M^n if and only if M^n is a totally umbilical hypersurface. For this reason, Φ is also called the total umbilicity tensor of M^n. We also note that, by equation (3.129), the following relation is trivially satisfied:

$$n(n-1)R = n(n-1)(1-H^2) + |\Phi|^2. \tag{3.132}$$

At this point, we will assume that M^n is a *linear Weingarten submanifold*, which means that the normalized scalar curvature and mean curvature functions are linearly related in the following way: there exist real constants $a, b \in \mathbb{R}$ such that

$$R = aH + b. \tag{3.133}$$

Related with the geometry of linear Weingarten spacelike hypersurfaces there exists an interesting Cheng-Yau type differential operator, which recently has been considered by many authors. To be more precise, let us introduce the second order linear differential operator $\mathcal{L} : C^\infty(M) \to C^\infty(M)$ defined by

$$\mathcal{L} = L + \frac{n-1}{2} a\Delta, \tag{3.134}$$

where Δ is the Laplacian operator on M^n and $L : C^\infty(M) \to C^\infty(M)$ denotes the standard Cheng-Yau's operator, which is given by

$$Lu = \operatorname{tr}(P \circ \operatorname{hess} u).$$

for every $u \in C^\infty(M)$. Here, $P : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the first Newton transformation of M^n, that is, the tensor

$$P = nHI - h^{n+1}. \tag{3.135}$$

Thus,

$$\mathcal{L}u = \operatorname{tr}(\mathcal{P} \circ \operatorname{hess} u), \tag{3.136}$$

where hess u is the self-adjoint linear tensor metrically equivalent to the Hessian of u and

$$\mathcal{P} = \left(nH + \frac{n-1}{2} a \right) I - h^{n+1}.$$

3.7 Main result of umbilicity of linear Weingarten spacelike submanifold in the $\mathbb{S}_p^{n+p}$

This section is dedicate to state and prove our main results concerning linear Weingarten spacelike submanifolds immersed into de Sitter space $\mathbb{S}_p^{n+p}$ having parallel normalized mean curvature vector. For this, we need of the next lemma which collects two important properties of the operator $\mathcal{L}$, namely: a sufficient conditions for the ellipticity property of $\mathcal{L}$ and the validity of a generalized version of the Omori-Yau's maximum principle on M^n, meaning that for any function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $\{p_j\} \subset M^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) < \frac{1}{j},$$

for every $j \in \mathbb{N}$.

Lemma 3.7.1 *Let M^n be a complete linear Weingarten spacelike hypersurface immersed into the de Sitter space $\mathbb{S}_p^{n+p}$, such that $R = aH + b$ with $b < 1$ (resp. $b \leq 1$). The following holds:*

(i) The operator $\mathcal{L}$ is elliptic (resp. semi-elliptic) or, equivalently, $\mathcal{P}$ is positive definite (resp. semi-definite);

(ii) If $\sup_M |\Phi|^2 < +\infty$, then the Omori-Yau's maximum principle holds on M^n for the operator $\mathcal{L}$.

Proof. The proof of (i) can be found in Lemma 3.1 of [91]. Then, let us proof item (ii). By equation (1.30) we find

$$|\Phi|^2 = n(n-1)(H^2 + aH) - n(n-1)(b-1), \tag{3.137}$$

which assures that $\sup_M H < +\infty$ because of our assumption on $|\Phi|^2$. From here and of equation (3.129), for every α, i, j, it holds that

$$(h_{ij}^\alpha)^2 \leq |A|^2 = n\left(nH^2 + (n-1)aH\right) + n(n-1)(b-1),$$

so that $\sup_M h_{ij}^\alpha < +\infty$. Thus, it follows from the Gauss equation that

$$R_{ijij} = 1 - \sum_\alpha \left(h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2\right) \geq 1 - \sum_\alpha h_{ii}^\alpha h_{jj}^\alpha > -\infty, \tag{3.138}$$

that is, the sectional curvatures of M^n are bounded from below.

Besides, one verifies that

$$\operatorname{tr}(\mathcal{P}) = n(n-1)H + \frac{n(n-1)a}{2}. \tag{3.139}$$

In particular, $\sup_M \operatorname{tr}(\mathcal{P}) < +\infty$. Therefore, taking into account (4.41) we can apply Theorem 6.13 of [8] to conclude the desired result. ■

Now, we are in position to state and prove our first main result.

Theorem 3.7.2 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in the de Sitter space $\mathbb{S}_p^{n+p}$ with index $p > 1$, such that $R = aH + b$ with $a \geq 0$ and $0 < b \leq 1$. Then*

(i) either $\sup_M |\Phi| = 0$ and M^n is a totally umbilical submanifold,

(ii) or

$$\sup_M |\Phi| \geq \alpha(n,p,a,b) > 0, \tag{3.140}$$

where $\alpha(n,p,a,b)$ is a positive constant that depends only on n,p,a,b. Moreover, if M^n has nonnegative sectional curvature, $b<1$, the equality $\sup_M |\Phi| = \alpha(n,p,a,b)$ holds and this supremum is attained at some point of M^n, then M^n is isometric to a product $M_1 \times M_2 \times \ldots \times M_k$, where the factors M_i are totally umbilical submanifolds of $\mathbb{S}_p^{n+p}$ which are mutually perpendicular along their intersections.

Proof. Initially we must to obtain a suitable lower boundedness for the operator $\mathcal{L}$ acting on the squared norm of the total umbilicity tensor Φ of M^n. To get it, let us begin observing that, since M^n is a linear Weingarten, by equation (3.132) we get

$$\begin{aligned} \frac{n}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(n^2H^2) + \frac{an}{2}\mathcal{L}(nH) \\ &= nH\mathcal{L}(nH) + n^2\langle \mathcal{P}\nabla H, \nabla H\rangle + \frac{an}{2}\mathcal{L}(nH). \end{aligned} \tag{3.141}$$

By using Lemma 3.7.1 (ii), we have that $\mathcal{P}$ is positive definite. In particular, from (3.141) we find

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq \left(H + \frac{a}{2}\right)\mathcal{L}(nH). \tag{3.142}$$

Then, Proposition 1 of [18] (which also holds for $b=1$) gives

$$\mathcal{L}(nH) \geq |\Phi|^2 \left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}|\Phi|H - n\left(H^2-1\right)\right). \tag{3.143}$$

Besides, from (3.132) we have

$$H + \frac{a}{2} = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4}+1-b\right)}. \tag{3.144}$$

This jointly with equations (3.142) and (3.143) enables us to deduce that

$$\frac{1}{2}\mathcal{L}(|\Phi|^2) \geq (n-1)|\Phi|^2 Q_{n,p,a,b}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + 1 - b}, \tag{3.145}$$

where the function $Q_{n,p,a,b}(x)$ is given by

$$\begin{aligned} Q_{n,p,a,b}(x) &= \frac{n-p-1}{p(n-1)}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\right)\sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} + 1 - b} \\ &+ \frac{n(n-2)a}{2\sqrt{n(n-1)}}x - n\left(\frac{a^2}{2} - b\right). \end{aligned} \tag{3.146}$$

At this point, we will make a brief analysis of the behavior of the function $Q_{n,p,a,b}(x)$, considering $p > 1$, $a \geq 0$ and $0 < b \leq 1$. Let us observe that when $x > 0$, isolating x^2 we get

$$\lim_{x\to\infty} x^2 \left\{ \frac{n-p-1}{p(n-1)} + \left(\frac{na}{x} - \frac{n(n-2)}{\sqrt{n(n-1)}} \right) \sqrt{\frac{1}{n(n-1)} + \frac{a^2}{4x^2} + \frac{1-b}{x^2}} + \right.$$

$$\left. + \frac{n(n-2)a}{2x\sqrt{n(n-1)}} - \frac{n}{x^2}\left(\frac{a^2}{4} - b\right) \right\}.$$

Thus, when $x \to +\infty$, we have

$$\lim_{x\to\infty} x^2 \left\{ \frac{n-p-1}{p(n-1)} - \frac{n-2}{n-1} \right\}. \tag{3.147}$$

Hence, considering $p > 1$ in (3.147), we obtain

$$\lim_{x\to+\infty} Q_{n,p,a,b}(x) = -\infty.$$

Since we are also assuming that $0 < b \leq 1$ and $a \geq 0$, we also obtain that

$$Q_{n,p,a,b}(0) = na\sqrt{\frac{a^2}{4} + 1 - b} - n\left(\frac{a^2}{2} - b\right) \geq nb > 0.$$

According to these facts, we will define $\alpha(n,p,a,b)$ as being the first positive root of the function $Q_{n,p,a,b}(x)$.

We are now going to finish the proof by applying the Omori-Yau maximum principle to the operator $\mathcal{L}$ acting on the function $|\Phi|^2$. Indeed, if $\sup_M |\Phi| = +\infty$, then the claim (ii) of Theorem 3.7.2 trivially holds and there is nothing to prove.

So, let us assume without loss of generality that $\sup_M |\Phi| < +\infty$. In this case, from Lemma 3.7.1 we obtain a sequence $\{p_j\}$ in M^n satisfying

$$\lim |\Phi|(p_j) = \sup_M |\Phi| \quad \text{and} \quad \mathcal{L}(|\Phi|^2)(p_j) < \frac{1}{j},$$

which jointly with estimate (3.145) gives

$$\frac{1}{j} > \mathcal{L}(|\Phi|^2)(p_j) \geq (n-1)|\Phi|^2(p_j) Q_{n,p,a,b}(|\Phi|(p_j)) \sqrt{\frac{|\Phi|^2(p_j)}{n(n-1)} + \frac{a^2}{4} + 1 - b}.$$

Taking the limit as $j \to +\infty$, we infer

$$\left(\sup_M |\Phi|\right)^2 Q_{n,p,a,b}(\sup_M |\Phi|) \sqrt{\frac{(\sup_M |\Phi|)^2}{n(n-1)} + \frac{a^2}{4} + 1 - b} \leq 0.$$

Since we are assuming $b \leq 1$ it follows from here that either $\sup_M |\Phi| = 0$, which means that $|\Phi| \equiv 0$ and the hypersurface is totally umbilical, or $\sup_M |\Phi| > 0$ and then

$$Q_{n,p,a,b}(\sup_M |\Phi|) \leq 0.$$

Therefore, from the behavior of the function $Q_{n,p,a,b}(x)$ and according to our choice of the positive constant $\alpha(n,p,a,b)$, we deduce the lower estimate (3.140).

Now, let us assume that $\sup_M |\Phi| = \alpha(n,p,a,b)$. In this case, from (3.145) and taking into account once more the behavior of $Q_{n,p,a,b}(x)$, we get that $\mathcal{L}(|\Phi|^2) \geq 0$. But, since we are assuming that $b < 1$, item (i) of Lemma 3.7.1 guarantees that $\mathcal{L}$ is elliptic. Consequently, since we are also supposing that $\sup_M |\Phi|$ is attained at some point of M^n, we conclude that $|\Phi|$ is constant on M^n and, from (3.137), the same holds for H and R. Thus, M^n has, in fact, parallel mean curvature vector and constant normalized scalar curvature. Therefore, since M^n has nonnegative sectional curvature, the result follows applying Theorem 1.11 of [53]. ■

We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

Extending this previous concept for the operator $\mathcal{L}$ defined in (3.136), M^n is said to be $\mathcal{L}$-parabolic (or parabolic with respect to the operator $\mathcal{L}$) if the constant functions are the only functions $u \in \mathcal{C}^2(M)$ which are bounded from above and satisfying $\mathcal{L}u \geq 0$. That is, for a function $u \in \mathcal{C}^2(M)$

$$\mathcal{L}u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

In this setting, we obtain the following gap result:

Theorem 3.7.3 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in the de Sitter space $\mathbb{S}_p^{n+p}$ with index $p > 1$, such that $R = aH + b$ with $a \geq 0$ and $0 < b \leq 1$. If M^n is a $\mathcal{L}$-parabolic submanifold with nonnegative sectional curvature and such that $\sup_M |\Phi| \leq \alpha(n,p,a,b)$, where $\alpha(n,p,a,b)$ is the positive constant depending only on n,p,a,b which was obtained in Theorem 3.7.2, then either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $\sup_M |\Phi| = \alpha(n,p,a,b)$ and M^n is isometric to a product $M_1 \times M_2 \times \ldots \times M_k$, where the factors*

M_i are totally umbilical submanifolds of $\mathbb{S}_p^{n+p}$ which are mutually perpendicular along their intersections.

Proof. Suppose that $0 < \sup_M |\Phi|^2 \leq \alpha(n,p,a,b)$. In this case, from item (ii) of Theorem 3.7.2 we get that $\sup_M |\Phi|^2 = \alpha(n,p,a,b)$. Moreover, $\mathcal{L}(|\Phi|^2) \geq 0$ on M^n. Hence, from the $\mathcal{L}$-parabolicity of M^n we conclude that $|\Phi|$ must be constant and equal to $\alpha(n,p,a,b)$. Therefore, we can reason as in the last part of the proof of Theorem 3.7.2 to conclude the result. ■

By a standard tensor computation, it is not difficult to see that

$$\mathcal{L}(u) = \mathrm{div}(\mathcal{P}(\nabla u)) - \langle \mathrm{div}\mathcal{P}, \nabla u\rangle \tag{3.148}$$

for every function $u \in \mathcal{C}^2(M)$, where $\mathcal{P}$ is defined in (3.135) and

$$\mathrm{div}\mathcal{P} = \mathrm{tr}(\nabla\mathcal{P}) = \sum_{i=1}^{n} \nabla\mathcal{P}(e_i, e_i)$$

with

$$\nabla\mathcal{P}(X,Y) = (\nabla_Y\mathcal{P})X = \nabla_Y(\mathcal{P}X) - \mathcal{P}(\nabla_Y X)$$

for every $X, Y \in TM$. Thus, being M^n a linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector in $\mathbb{S}_p^{n+p}$, we can apply Lemma 5 of [6] to guarantee that $\mathrm{div}\mathcal{P} = 0$ and, consequently,

$$\mathcal{L}(u) = \mathrm{tr}(\mathcal{P} \circ \nabla^2 u) = \mathrm{div}(\mathcal{P}(\nabla u)) \tag{3.149}$$

is a divergence type operator. Taking into account this digression, we close our paper showing the following $\mathcal{L}$-parabolicity criterium, which extends Proposition 3 of [6]:

Proposition 3.7.4 *Let M^n be a complete linear Weingarten spacelike submanifold immersed in $\mathbb{S}_p^{n+p}$ with parallel normalized mean curvature vector, such that $R = aH + b$ with $a \geq 0$ and $0 < b \leq 1$. If $\sup_M |\Phi|^2 < +\infty$ and, for some reference point $o \in M^n$,*

$$\int_0^{+\infty} \frac{dr}{\mathrm{vol}(\partial B_r)} = +\infty, \tag{3.150}$$

then M^n is $\mathcal{L}$-parabolic. Here B_r denotes the geodesic ball of radius r in M^n centered at the origin o.

Proof. We consider on M^n the symmetric $(0,2)$ tensor field ξ given by $\xi(X,Y) = \langle \mathcal{P}X, Y\rangle$, or, equivalently, $\xi(\nabla u,\cdot)^\sharp = \mathcal{P}(\nabla u)$, where $\mathcal{P}$ is defined in (3.135) and $^\sharp : T^*M \to TM$ denotes the musical isomorphism. Thus, from (3.149) we get

$$\mathcal{L}(u) = \operatorname{div}\left(\xi(\nabla u,\cdot)^\sharp\right).$$

On the other hand, since we are assuming that $\sup_M |\Phi|^2 < +\infty$ and $a \geq 0$, from (3.135) we get that $\sup_M H < +\infty$. So, we can define a positive continuous function ξ_+ on $[0,+\infty)$, by

$$\xi_+(r) = 2n \sup_{\partial B_r} H. \tag{3.151}$$

Thus, from (3.151) we have

$$\xi_+(r) = 2n \sup_{\partial B_r} H \leq 2n \sup_M H < +\infty. \tag{3.152}$$

Hence, from (4.55) and (3.152) we get

$$\int_0^{+\infty} \frac{dr}{\xi_+(r)\mathrm{vol}(\partial \mathrm{B_r})} = +\infty.$$

Therefore, we can apply Theorem 2.6 of [110] to conclude the proof. ∎

Capítulo 4

Results for spacelike submanifolds in locally symmetric semi-Riemannian spaces

In this chapter, let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} of index p, which obeys standard curvature constraints (such an ambient space can be regarded as an extension of a semi-Riemannian space form). In this setting, our purpose is to establish sufficient conditions guaranteeing that such a spacelike submanifold M^n be either totally umbilical or isometric to an isoparametric hypersurface of a totally geodesic submanifold $L_1^{n+1} \hookrightarrow L_p^{n+p}$, with two distinct principal curvatures, one of which is simple. Our approach is based on a suitable Simons type formula jointly with a version of the Omori-Yau's generalized maximum principle for a Cheng-Yau's modified operator. For more details, you can look at the works [19] and [28].

Lemma 4.0.1 *Let M^n be an n-dimensional linear Weingarten spacelike submanifold immersed in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying curvature conditions (1.36) and (1.39), and such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \geq 0. \tag{4.1}$$

Then,

$$|\nabla B|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 \geq n^2 |\nabla H|^2. \tag{4.2}$$

Moreover, if the equality holds in (4.2)*, then* H *is constant on* M^n*.*

At this point, we will deal with spacelike submanifolds M^n of L_p^{n+p} having parallel normalized mean curvature vector field, which means that the mean curvature function H is positive and that the corresponding normalized mean curvature vector field $\frac{\mathbf{H}}{H}$ is parallel as a section of the normal bundle. In this setting, we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $e_{n+1} = \frac{\mathbf{H}}{H}$, we have that

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^\alpha = \frac{1}{n}\mathrm{tr}(h^\alpha) = 0, \quad for \quad \alpha \geq n+2.$$

The following Simons type formula for locally symmetric spaces was obtained in Lemma 2 of [17].

Lemma 4.0.2 *Let* M^n *be an* n*-dimensional spacelike submanifold immersed with flat normal bundle and parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space* L_p^{n+p}*. Then, we have*

$$\begin{aligned}\frac{1}{2}\Delta S = & \sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 + 2\left(\sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha \overline{R}_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik}\right) + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha i\beta k} \\ & - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha k\beta i} + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta \overline{R}_{\alpha k\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{kk}^\beta \overline{R}_{\alpha i\beta j} + n\sum_{i,j} h_{ij}^{n+1} H_{ij} \\ & - nH\sum_{i,j,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1} + \sum_{\alpha,\beta}[\mathrm{tr}(h^\alpha h^\beta)]^2 + \frac{3}{2}\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha), \qquad (4.3)\end{aligned}$$

where $N(A) = \mathrm{tr}(AA^t)$*, for all matrix* $A = (a_{ij})$*, and* $h^\alpha = (h_{ij}^\alpha)$*.*

In order to study linear Weingarten submanifolds, we will consider, for each $a \in \mathbb{R}$, an appropriated Cheng-Yau's modified operator, given by

$$L = \Box + \frac{n-1}{2}a\Delta, \qquad (4.4)$$

where, according to [56], the square operator is defined by

$$\Box f = \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})f_{ij}, \qquad (4.5)$$

for each $f \in C^\infty(M)$, and the normal vector field e_{n+1} is taken in the direction of the mean curvature vector field, that is, $e_{n+1} = \dfrac{\mathbf{H}}{H}$.

The next lemma gives sufficient conditions to guarantee the elipticity of the operator L, and it is an extension of Lemma 3.2 of [64].

Lemma 4.0.3 *Let M^n be an n-dimensional linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying curvature condition (1.39), and such that $R = aH + b$ for some $a, b \in \mathbb{R}$, with $b < \overline{\mathcal{R}}$. Then, L is elliptic.*

Proof. Let us consider the case that $a = 0$. Since $R = b < \overline{\mathcal{R}}$, from Equation (1.22) if we choose a (local) orthonormal frame $\{e_i\}$ on M^n such that $h_{ij}^{n+1} = \lambda_i \delta_{ij}$, we have that $\sum_{i<j} \lambda_i \lambda_j > 0$. Consequently,

$$n^2 H^2 = \sum_i \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j > \lambda_i^2 \tag{4.6}$$

for every $i = 1, \dots, n$ and, hence, we have that $nH - |\lambda_i| > 0$ for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From Equation (1.22) we get that

$$a = \frac{1}{n(n-1)H} \left(S - n^2 H^2 + n(n-1)\overline{\mathcal{R}} - n(n-1)b \right). \tag{4.7}$$

For any i, from (4.37) we have

$$\begin{aligned} nH - \lambda_i^{n+1} + \frac{n-1}{2} a &= nH - \lambda_i^{n+1} + \frac{1}{2nH} \left(S - n^2 H^2 + n(n-1)(\overline{\mathcal{R}} - b) \right) \\ &= \left(\frac{1}{2}(nH)^2 - nH\lambda_i^{n+1} + \frac{1}{2} S + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b) \right)(nH)^{-1}. \end{aligned} \tag{4.8}$$

Since $\sum_j \lambda_j^{n+1} = nH$ and $S \geq \sum_j (\lambda_j^{n+1})^2$, from (4.38) we have

$$\begin{aligned} nH - \lambda_i^{n+1} + \frac{n-1}{2} a \geq & \left\{ \frac{1}{2} \left(\sum_j \lambda_j^{n+1} \right)^2 - \lambda_i^{n+1} \sum_j \lambda_j^{n+1} + \frac{1}{2} \sum_j (\lambda_j^{n+1})^2 \right\} (nH)^{-1} \\ & + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b)(nH)^{-1} \\ = & \left\{ \sum_j (\lambda_j^{n+1})^2 + \frac{1}{2} \sum_{l \neq j} \lambda_l^{n+1} \lambda_j^{n+1} - \lambda_i^{n+1} \sum_j \lambda_j^{n+1} \right\} (nH)^{-1} \\ & + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b)(nH)^{-1} \\ = & \left\{ \sum_{i \neq j} (\lambda_j^{n+1})^2 + \frac{1}{2} \sum_{l \neq j, l, j \neq i} \lambda_l^{n+1} \lambda_j^{n+1} + \frac{1}{2} n(n-1)(\overline{\mathcal{R}} - b) \right\} (nH)^{-1} \\ = & \frac{1}{2} \left\{ \sum_{i \neq j} (\lambda_j^{n+1})^2 + \left(\sum_{j \neq i} \lambda_j^{n+1} \right)^2 + n(n-1)(\overline{\mathcal{R}} - b) \right\} (nH)^{-1}. \end{aligned} \tag{4.9}$$

Therefore, taking into account our assumption $b < \overline{\mathcal{R}}$, we conclude that $nH - \lambda_i^{n+1} + \frac{n-1}{2} a > 0$, which implies that L is an elliptic operator. ■

The next lemma guarantees us the existence of an Omori-type sequence related to the operator L, and it corresponds to Lemma 3 of [17].

Lemma 4.0.4 *Let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying conditions* (1.36), (1.38) *and* (1.39), *such that $R = aH + b$, with $a \geq 0$ and $(n-1)a^2 + 4n\left(\overline{\mathcal{R}} - b\right) \geq 0$. If H is positive and bounded on M^n, then there is a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^n$ such that*

$$\lim_k nH(q_k) = \sup_M nH, \quad \lim_k |\nabla nH(q_k)| = 0 \quad \textit{and} \quad \limsup_k L(nH(q_k)) \leq 0.$$

We will also need of the following two algebraic lemmas, whose proofs can be founded in [106] and [134], respectively.

Lemma 4.0.5 *Let μ_i $(1 \leq i \leq n)$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta$, where β is a nonnegative constant. Then,*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3.$$

Moreover, the equality holds if and only if at least $(n-1)$ of the μ_i are equal.

Lemma 4.0.6 *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers satisfying $\sum_i b_i = 0$. Then,*

$$\sum_{i,j} a_i a_j (b_i - b_j)^2 \leq \frac{n}{\sqrt{n-1}} \sum_i a_i^2 \sum_j b_j^2.$$

4.1 Umbilicity of submanifold in a locally symmetric semi-Riemannian space L_p^{n+p}

Proceeding with the same set up of the previous section and defining

$$\mu_{ij} = h_{ij}^{n+1} - H\delta_{ij}, \quad \tau_{ij}^{\beta} = h_{ij}^{\beta}, \quad \beta > n+1, \tag{4.10}$$

we have

$$\| \mu \|^2 = \operatorname{tr}(h^{n+1})^2 - nH^2, \quad \| \tau \|^2 = \sum_{i,j,\alpha > n+1} (h_{ij}^{\alpha})^2, \tag{4.11}$$

and

$$S = \| \mu \|^2 + \| \tau \|^2 + nH^2. \tag{4.12}$$

It is not difficult to see that $\| \mu \|^2$ and $\| \tau \|^2$ are functions globally defined on M^n. Moreover, they are independent of the choice of the frame field. Thus, we obtain the following auxiliary result:

Proposition 4.1.1 *Let M^n be an n-dimensional spacelike submanifold immersed with flat normal bundle and parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying curvature conditions* (1.36), (1.37) *and* (1.38)*. Then, we have*

$$\begin{aligned}\frac{1}{2}\Delta \mathrm{tr}(h^{n+1})^2 \geq & \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} + cn\mathrm{tr}(h^{n+1})^2 - cn^2H^2 \\ & -nH\mathrm{tr}(h^{n+1})^3 + (\mathrm{tr}(h^{n+1})^2)^2 + \sum_{\beta>n+1}(\mathrm{tr}(h^{n+1}h^\beta))^2,\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\frac{1}{2}\Delta \parallel \tau \parallel^2 \geq & \sum_{i,j,k,\alpha>n+1}(h_{ijk}^{\alpha})^2 + cn \parallel \tau \parallel^2 - nH \sum_{\alpha>n+1}\mathrm{tr}((h^\alpha)^2 h^{n+1}) \\ & + \sum_{\alpha>n+1}(\mathrm{tr}(h^{n+1}h^\alpha))^2 + \sum_{\alpha,\beta>n+1}(\mathrm{tr}(h^\alpha h^\beta))^2,\end{aligned} \tag{4.14}$$

where $c = \dfrac{c_1}{n} + 2c_2$.

Proof. Let us consider $\{e_1, \dots, e_n\}$ a local orthonormal frame on M^n such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, for all $\alpha \in \{n+1, \dots, n+p\}$. From (4.3), we obtain

$$\begin{aligned}2\left(\sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha \overline{R}_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik}\right) &= 2\sum_{i,k,\alpha}\left((\lambda_i^\alpha)^2 \overline{R}_{ikik} + \lambda_i^\alpha \lambda_k^\alpha \overline{R}_{kiik}\right) \\ &= \sum_{i,k,\alpha} \overline{R}_{ikik}(\lambda_i^\alpha - \lambda_k^\alpha)^2.\end{aligned}$$

Since L_p^{n+p} satisfies condition (1.38) we have

$$\begin{aligned}2\left(\sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha \overline{R}_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{jm}^\alpha \overline{R}_{mkik}\right) &\geq c_2 \sum_{i,k,\alpha}(\lambda_i^\alpha - \lambda_k^\alpha)^2 \\ &= 2nc_2(S - nH^2).\end{aligned} \tag{4.15}$$

Moreover, since L_p^{n+p} also satisfies conditions (1.36) and (1.37), it is not difficult to verify that we also get

$$\begin{aligned}&\sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha i\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{jk}^\beta \overline{R}_{\alpha k\beta i} + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta \overline{R}_{\alpha k\beta k} - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{kk}^\beta \overline{R}_{\alpha i\beta j} \\ &\quad = c_1(S - nH^2).\end{aligned} \tag{4.16}$$

On the other hand, we have that

$$\sum_{\alpha,\beta} N(h^\alpha h^\beta - h^\beta h^\alpha) \geq 0. \tag{4.17}$$

Hence, from (4.3) and using (4.15), (4.16) and (4.17) we conclude that

$$\frac{1}{2}\Delta S \geq \sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2 + cn(S - nH^2) + n\sum_{i,j} h^{n+1}_{ij}H_{ij} \\ -nH\sum_{i,j,m,\alpha} h^{\alpha}_{ij}h^{\alpha}_{mi}h^{n+1}_{mj} + \sum_{\alpha,\beta}(\sum_{i,j} h^{\alpha}_{ij}h^{\beta}_{ij})^2. \tag{4.18}$$

Therefore, considering the cases $\alpha = n+1$ and $\alpha > n+1$ in (4.18) we obtain (4.13) and (4.14), respectively. ■

Finally, we are in position to state and prove our main result.

Theorem 4.1.2 *Let M^n be an n-dimensional complete linear Weingarten spacelike submanifold immersed with flat normal bundle in a locally symmetric semi-Riemannian space L^{n+p}_p satisfying curvature conditions (1.36), (1.37), (1.38) and (1.39), with parallel normalized mean curvature vector field and such that $R = aH + b$ for some $a, b \in \mathbb{R}$, with $(n-1)a^2 + 4n(\overline{\mathcal{R}} - b) \geq 0$. If $c = \frac{c_1}{n} + 2c_2 > 0$ and $S \leq 2\sqrt{n-1}\,c$, then either*

(i) M^n is totally umbilical, or

(ii) $\sup_M S = 2\sqrt{n-1}\,c$. Moreover, if L^{n+p}_p is conformally flat, $\sup_M S$ is attained at some point in M^n and $\overline{\mathcal{R}} > b$, then M^n is isometric to an isoparametric hypersurface of a totally geodesic submanifold $L^{n+1}_1 \hookrightarrow L^{n+p}_p$, with two distinct principal curvatures, one of which is simple.

Proof. We choose a local frame of orthonormal vector field $\{e_i\}$ such that $h^{\alpha}_{ij} = \lambda^{\alpha}_i\delta_{ij}$, for each $\alpha \geq n+1$. In particular, we consider $h^{n+1}_{ij} = \lambda_i\delta_{ij}$. We also consider $\mu_i = \lambda_i - H$ and $\| \mu \|^2 = \sum \mu_i^2 = \sum_i(\lambda_i - H)^2 = \sum_i \lambda_i^2 - nH^2 = \mathrm{tr}(h^{n+1})^2 - nH^2$.

By applying Lemma 4.0.5, we have

$$-nH\mathrm{tr}(h^{n+1})^3 = -3nH^2 \| \mu \|^2 - n^2H^4 - nH\sum_i \mu_i^3 \\ \geq -3nH^2 \| \mu \|^2 - n^2H^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H \| \mu \|^3 . \tag{4.19}$$

Substituting (4.19) into (4.13), we have

$$\frac{1}{2}\Delta\mathrm{tr}(h^{n+1})^2 \geq \sum_{i,j,k}(h^{n+1}_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij}(nH)_{ij} \\ + \| \mu \|^2 \left(\| \mu \|^2 + cn - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H \| \mu \| \right). \tag{4.20}$$

Let us define the quadratic form

$$Q(x,y) = x^2 - \frac{n-2}{\sqrt{n-1}}xy - y^2. \tag{4.21}$$

It is not difficult to see that, by the orthogonal transformation

$$u = \frac{1}{\sqrt{2n}}\{(1+\sqrt{n-1})x + (1-\sqrt{n-1})y\} \tag{4.22}$$
$$v = \frac{1}{\sqrt{2n}}\{(\sqrt{n-1}-1)x + (1+\sqrt{n-1})y\},$$

equation (4.21) becomes

$$Q(x,y) = \frac{n}{2\sqrt{n-1}}(u^2 - v^2). \tag{4.23}$$

Defining $x = \parallel \mu \parallel$ and $y = \sqrt{nH^2}$, we have that $u^2 + v^2 = x^2 + y^2 = \parallel \mu \parallel^2 + nH^2 = \mathrm{tr}(h^{n+1})^2$. Hence, we obtain

$$cn + Q(x,y) = cn - \frac{n}{2\sqrt{n-1}}(u^2+v^2) + \frac{n}{\sqrt{n-1}}u^2 \geq cn - \frac{n}{2\sqrt{n-1}}S. \tag{4.24}$$

It follows from (4.20) and (4.23) that

$$\frac{1}{2}\Delta\mathrm{tr}(h^{n+1})^2 \geq \sum_{i,j,k}(h^{n+1}_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij}(nH)_{ij} + \parallel \mu \parallel^2 \left(cn - \frac{n}{2\sqrt{n-1}}S\right). \tag{4.25}$$

On the other hand, since $\sum_i h^{\alpha}_{ii} = 0$ for $\alpha > n+1$, by applying Lemma 4.0.6 we have

$$-nH\mathrm{tr}((h^{\alpha})^2 h^{n+1}) + (\mathrm{tr}(h^{n+1}h^{\alpha}))^2 = -\frac{1}{2}\sum_{i,j} h^{n+1}_{ii} h^{n+1}_{jj}(h^{\alpha}_{ii} - h^{\alpha}_{jj})^2 \tag{4.26}$$
$$\geq -\frac{n}{2\sqrt{n-1}}\sum_j (h^{\alpha}_{jj})^2 \sum_i (h^{n+1}_{ii})^2.$$

Consequently, from (4.26) we obtain that

$$-nH\sum_{\alpha>n+1}\mathrm{tr}((h^{\alpha})^2 h^{n+1}) + \sum_{\alpha>n+1}(\mathrm{tr}(h^{n+1}h^{\alpha}))^2 \geq -\frac{n}{2\sqrt{n-1}} \parallel \tau \parallel^2 S. \tag{4.27}$$

Substituting (4.27) into (4.14) we get

$$\frac{1}{2}\Delta \parallel \tau \parallel^2 \geq \sum_{i,j,k,\alpha>n+1}(h^{\alpha}_{ijk})^2 + \parallel \tau \parallel^2 \left(cn - \frac{n}{2\sqrt{n-1}}S\right). \tag{4.28}$$

It follows from (4.25) and (4.28) that

$$\frac{1}{2}\Delta S \geq \sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2 + \sum_{i,j} h^{n+1}_{ij}(nH)_{ij} + (S - nH^2)\left(cn - \frac{n}{2\sqrt{n-1}}S\right). \tag{4.29}$$

Using (1.22) and (4.29) we have

$$\begin{aligned}\Box(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})(nH)_{ij} \\ &= \sum_i (nH)(nH)_{ii} - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH_i)^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &= \frac{1}{2}\Delta S - \frac{1}{2}n(n-1)\Delta R - n^2|\nabla H|^2 - \sum_{i,j} h_{ij}^{n+1}(nH)_{ij} \\ &\geq \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 - \frac{1}{2}n(n-1)\Delta R \\ &\quad + (S - nH^2)\left(cn - \frac{n}{2\sqrt{n-1}}S\right).\end{aligned} \tag{4.30}$$

Consequently, since $R = aH + b$, from (4.30) we get

$$\begin{aligned}L(nH) &= \Box(nH) + \frac{n-1}{2}a\Delta(nH) \\ &= \Box(nH) + \frac{1}{2}n(n-1)\Delta R \\ &\geq \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2 + (S - nH^2)\left(cn - \frac{n}{2\sqrt{n-1}}S\right).\end{aligned} \tag{4.31}$$

Thus, using Lemma 4.0.1, from (4.31) we obtain

$$L(nH) \geq (S - nH^2)\left(cn - \frac{n}{2\sqrt{n-1}}S\right). \tag{4.32}$$

Since $S \leq 2\sqrt{n-1}\,c$, we have that the mean curvature H and λ_i^{n+1} are bounded. Hence, the Ricci curvature of M^n is bounded from below, $nH - \lambda_i + \frac{n-1}{2}a$ is bounded and $a > 0$, and we can apply Lemma 4.0.4 to guarantee that there exists a sequence of points $\{q_k\}_{k\in\mathbb{N}} \subset M^n$ such that

$$\lim_k (nH)(q_k) = \sup_M (nH), \quad \lim_k \parallel \nabla(nH)(q_k) \parallel = 0 \quad \text{and} \quad \limsup_k L(nH)(q_k) \leq 0. \tag{4.33}$$

It follows from of equation $S = n^2H^2 + n(n-1)(aH + b - \overline{\mathcal{R}})$ that $\lim_k S(q_k) = \sup_M S$. Evaluating (4.32) at points q_k, we have

$$0 \geq (\sup_M S - n\sup_M H^2)\left(cn - \frac{n}{2\sqrt{n-1}}\sup_M S\right). \tag{4.34}$$

Since $S \leq 2\sqrt{n-1}\,c$, we have

$$\sup_M(S - nH^2)\left(cn - \frac{n}{2\sqrt{n-1}}\sup_M S\right) = 0. \tag{4.35}$$

Hence, we have either $\sup_M(S - nH^2) = 0$ and M^n is totally umbilical, or $\sup_M S = 2\sqrt{n-1}\,c$.

Now, suppose that $\sup_M S = 2\sqrt{n-1}\,c$ and $\sup_M S$ is attained on M^n. Then, $\sup_M H$ is also attained on M^n and, since $L(nH) \geq 0$ and $\overline{\mathcal{R}} < b$, we can use Lemma 4.0.3 to obtain that H is constant. Consequently, all the inequalities previously obtained become equalities. Since the equality in (4.27) holds, we have $\| \tau \|^2 = 0$. On the other hand, our assumptions that e_{n+1} is parallel and H is constant force that the mean curvature vector field is parallel in the normal bundle $T^{\perp}(M^n)$. Hence, since it is also assumed that L_p^{n+p} is conformally flat, we are in position to apply Theorem 1 of [130] to conclude that M^n is, in fact, isometrically immersed in a $(n+1)$-dimensional totally geodesic submanifold L_1^{n+1} of L_p^{n+p}. Therefore, since the equality holds in Lemma 4.0.5, we conclude that M^n must be isoparametric with two distinct principal curvatures, one of which is simple. ■

4.2 Via Omori-Yau's maximum principle

In order to prove our first result, we will make use of a generalized version of the Omori-Yau's maximum principle for trace type differential operators proved in [8]. Let M^n be a Riemannian manifold and let $\mathcal{L} = \mathrm{tr}(\mathcal{P} \circ \nabla^2)$ be a semi-elliptic operator, where $\mathcal{P} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a positive semi-definite symmetric tensor. Following the terminology introduced by Pigola, Rigoli and Setti [109], we say that the Omori-Yau's maximum principle holds on M^n for the operator $\mathcal{L}$ if, for any function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence of points $\{p_j\} \subset M^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) < \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

Equivalently, for any function $u \in C^2(M)$ with $u_* = \inf_M u > -\infty$ there exists a sequence of points $\{p_j\} \subset M^n$ satisfying

$$u(p_j) < u_* + \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) > -\frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

The following proposition establishes a suitable version of the Omori-Yau's maximum principle for the Cheng-Yau type differential operator $\mathcal{L}$ defined in (1.31).

Proposition 4.2.1 *Let M^n be an n-dimensional linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field in a locally symmetric semi-Riemannian space L_p^{n+p} satisfying curvature conditions* (1.36), (1.38), (1.39), *and such that $R = aH + b$ for some $a, b \in \mathbb{R}$, with $b < \overline{\mathcal{R}}$ (resp. $b \leq \overline{\mathcal{R}}$). The following holds:*

(i) The operator $\mathcal{L}$ defined in (1.31) *is elliptic (resp. semi-elliptic) or, equivalently, $\mathcal{P}$ defined in* (1.35) *is positive definite (resp. semi-definite);*

(ii) If $\sup_M |\Phi|^2 < +\infty$, then the Omori-Yau's maximum principle holds on M^n for the operator $\mathcal{L}$ defined in (1.31).

Proof. We recall that conditions (1.36), (1.39) guarantee that $\overline{\mathcal{R}}$ is constant. For proof of (i), let us consider the case that $a = 0$. Since $R = b < \overline{\mathcal{R}}$, from (1.22) if we choose a (local) orthonormal frame $\{e_i\}$ on M^n such that $h_{ij}^{n+1} = \lambda_i \delta_{ij}$, we have that $\sum_{i<j} \lambda_i \lambda_j > 0$. Consequently,

$$n^2H^2 = \sum_i \lambda_i^2 + 2\sum_{i<j} \lambda_i\lambda_j > \lambda_i^2 \tag{4.36}$$

for every $i = 1, \ldots, n$ and, hence, we have that $nH - |\lambda_i| > 0$ for every i. Therefore, in this case, we conclude that L is elliptic.

Now, suppose that $a \neq 0$. From (1.22) we get that

$$a = \frac{1}{n(n-1)H}\left(S - n^2H^2 + n(n-1)\overline{\mathcal{R}} - n(n-1)b\right). \tag{4.37}$$

For any i, from (4.37) we have

$$\begin{aligned} nH - \lambda_i^{n+1} + \frac{n-1}{2}a &= nH - \lambda_i^{n+1} + \frac{1}{2nH}\left(S - n^2H^2 + n(n-1)(\overline{\mathcal{R}} - b)\right) \\ &= \left(\frac{1}{2}(nH)^2 - nH\lambda_i^{n+1} + \frac{1}{2}S + \frac{1}{2}n(n-1)(\overline{\mathcal{R}} - b)\right)(nH)^{-1}. \end{aligned} \tag{4.38}$$

Since $\sum_j \lambda_j^{n+1} = nH$ and $S \geq \sum_j (\lambda_j^{n+1})^2$, from (4.38) we have

$$\begin{aligned} nH - \lambda_i^{n+1} + \frac{n-1}{2}a &\geq \left\{ \frac{1}{2}\left(\sum_j \lambda_j^{n+1}\right)^2 - \lambda_i^{n+1}\sum_j \lambda_j^{n+1} + \frac{1}{2}\sum_j (\lambda_j^{n+1})^2 \right\}(nH)^{-1} \\ &\quad + \frac{1}{2}n(n-1)(\overline{\mathcal{R}} - b)(nH)^{-1} \\ &= \left\{ \sum_j (\lambda_j^{n+1})^2 + \frac{1}{2}\sum_{l\neq j}\lambda_l^{n+1}\lambda_j^{n+1} - \lambda_i^{n+1}\sum_j \lambda_j^{n+1} \right\}(nH)^{-1} \qquad (4.39) \\ &\quad + \frac{1}{2}n(n-1)(\overline{\mathcal{R}} - b)(nH)^{-1} \\ &= \left\{ \sum_{i\neq j} (\lambda_j^{n+1})^2 + \frac{1}{2}\sum_{l\neq j, l,j\neq i}\lambda_l^{n+1}\lambda_j^{n+1} + \frac{1}{2}n(n-1)(\overline{\mathcal{R}} - b) \right\}(nH)^{-1} \\ &= \frac{1}{2}\left\{ \sum_{i\neq j} (\lambda_j^{n+1})^2 + \left(\sum_{j\neq i}\lambda_j^{n+1}\right)^2 + n(n-1)(\overline{\mathcal{R}} - b) \right\}(nH)^{-1}. \end{aligned}$$

Therefore, considering $b < \overline{\mathcal{R}}$ ($b \leq \overline{R}$), we conclude that $\mathcal{L}$ is an elliptic (semi-ellíptic) operator.

Now, let us proof item (ii). By (1.30) we find

$$|\Phi|^2 = n(n-1)(H^2 + aH) + n(n-1)(b - \overline{\mathcal{R}}), \qquad (4.40)$$

which assures that $\sup_M H < +\infty$ because of our assumption on $|\Phi|^2$. From here and of (1.22), for every α, i, j, it holds that

$$(h_{ij}^\alpha)^2 \leq |A|^2 = n\left(nH^2 + (n-1)aH\right) + n(n-1)(b - \overline{\mathcal{R}}),$$

so that $\sup_M h_{ij}^\alpha < +\infty$. Thus, it follows from the Gauss equation, (1.38) and (1.45) that

$$R_{ijij} = \overline{R}_{ijij} - \sum_\alpha \left(h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2\right) \geq c_2 - \sum_\alpha h_{ii}^\alpha h_{jj}^\alpha > -\infty, \qquad (4.41)$$

that is, the sectional curvatures of M^n are bounded from below.

Besides, from (1.35) one verifies that

$$\mathrm{tr}(\mathcal{P}) = n(n-1)H + \frac{n(n-1)a}{2}. \qquad (4.42)$$

In particular, from (4.42) we get $\sup_M \mathrm{tr}(\mathcal{P}) < +\infty$. Therefore, taking into account (3.136) and (4.41), we can apply Theorem 6.13 of [8] to conclude the desired result. ■

So, we apply Proposition 4.2.1 to establish the following characterization result:

Theorem 4.2.2 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} with $p > 1$ and satisfying conditions (1.36), (1.37), (1.38) and (1.39), such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = \frac{c_1}{n} + 2c_2$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all $A_\xi, \xi \in TM^\perp$. Then,*

(i) either $|\Phi| \equiv 0$ and M^n is a totally umbilical submanifold,

(ii) or

$$\sup_M |\Phi| \geq \alpha(n, p, a, b, c, \overline{\mathcal{R}}) > 0,$$

where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ is a positive constant that depends only on $n, p, a, b, c, \overline{\mathcal{R}}$. Moreover, if $b < \overline{\mathcal{R}}$, the equality $\sup_M |\Phi| = \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ holds and this supremum is attained at some point of M^n, then M^n is an isoparametric submanifold, in the sense that their principal curvatures are constant.

Proof. Initially we must to obtain a suitable lower boundedness for the operator $\mathcal{L}$ acting on the squared norm of the total umbilicity tensor Φ of M^n. To get it, let us begin observing that, since M^n is a linear Weingarten, by (1.30) we get

$$\begin{aligned} \frac{n}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(n^2H^2) + \frac{an}{2}\mathcal{L}(nH) \\ &= nH\mathcal{L}(nH) + n^2\langle \mathcal{P}\nabla H, \nabla H\rangle + \frac{an}{2}\mathcal{L}(nH). \end{aligned} \tag{4.43}$$

By using item (i) of Proposition 4.2.1, we have that $\mathcal{P}$ is positive semi-definite. In particular, from (4.43) we find

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geq \left(H + \frac{a}{2}\right)\mathcal{L}(nH). \tag{4.44}$$

On the other hand, since we are supposing that M^n has parallel normalized mean curvature vector field, flat normal bundle and that there exists an orthogonal basis for TM that diagonalizes simultaneously all $A_\xi, \xi \in TM^\perp$, from the proof of Proposition 1 in [17] (see the bottom of page 75) we have the following

$$\begin{aligned} \mathcal{L}(nH) &= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - n\sum_{i,j} h_{ij}^{n+1}H_{ij} \\ &\geq |\nabla A|^2 - n^2|\nabla H|^2 + cn|\Phi|^2 - nH\sum_{i,j,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1} + \sum_{\alpha,\beta}[\mathrm{tr}(h^\alpha h^\beta)]^2. \end{aligned} \tag{4.45}$$

Moreover, we see that

$$-nH\sum_\alpha \mathrm{tr}[h^{n+1}(h^\alpha)^2] + \sum_{\alpha,\beta}[\mathrm{tr}(h^\alpha h^\beta)]^2 \geq \frac{-n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 - nH^2|\Phi|^2 + \frac{|\Phi|^4}{p}. \tag{4.46}$$

From (4.45) and (4.46), we have

$$\mathcal{L}(nH) \geq |\nabla A|^2 - n^2|\nabla H|^2 + |\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(H^2-c)\right) \quad (4.47)$$

Besides, from (1.30) we have

$$H + \frac{a}{2} = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + \overline{\mathcal{R}} - b\right)}. \quad (4.48)$$

This jointly with (4.44), (4.47) and Lemma 1.2.2 enables us to deduce that

$$\frac{1}{2}\mathcal{L}(|\Phi|^2) \geq (n-1)|\Phi|^2 Q_{n,p,a,b,c,\overline{\mathcal{R}}}(|\Phi|)\sqrt{\frac{|\Phi|^2}{n(n-1)} + \frac{a^2}{4} + \overline{\mathcal{R}} - b}, \quad (4.49)$$

where the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$ is given by

$$\begin{aligned} Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x) &= \frac{n-p-1}{p(n-1)}x^2 + \left(na - \frac{n(n-2)}{\sqrt{n(n-1)}}x\right)\sqrt{\frac{x^2}{n(n-1)} + \frac{a^2}{4} + \overline{\mathcal{R}} - b} \\ &+ \frac{n(n-2)a}{2\sqrt{n(n-1)}}x + n\left(-\frac{a^2}{2} + b + c - \overline{\mathcal{R}}\right). \end{aligned} \quad (4.50)$$

At this point, we will make a brief analysis of the behavior of the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$, considering $p > 1$, $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$. Let us observe that when $x > 0$, from (4.50) we get

$$\begin{aligned} \lim_{x\to+\infty} Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x) &= \lim_{x\to+\infty} x^2\Big\{\frac{n-p-1}{p(n-1)} \\ &+ \left(\frac{na}{x} - \frac{n(n-2)}{\sqrt{n(n-1)}}\right)\sqrt{\frac{1}{n(n-1)} + \frac{a^2}{4x^2} + \frac{\overline{\mathcal{R}} - b}{x^2}} \\ &+ \frac{n(n-2)a}{2x\sqrt{n(n-1)}} + \frac{n}{x^2}\big(-\frac{a^2}{2} + b + c - \overline{\mathcal{R}}\big)\Big\}. \end{aligned} \quad (4.51)$$

Thus, taking into account that $p > 1$, from (4.51) we obtain

$$\lim_{x\to+\infty} Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x) = \lim_{x\to\infty} x^2\left\{\frac{n-p-1}{p(n-1)} - \frac{n-2}{n-1}\right\} = -\infty. \quad (4.52)$$

Since we are also assuming that $b \leq \overline{\mathcal{R}} < b + c$ and $a \geq 0$, we also have that

$$Q_{n,p,a,b,c,\overline{\mathcal{R}}}(0) = n\left(a\sqrt{\frac{a^2}{4} + \overline{\mathcal{R}} - b} - \frac{a^2}{2}\right) + n\left(b + c - \overline{\mathcal{R}}\right) \geq n(b + c - \overline{\mathcal{R}}) > 0. \quad (4.53)$$

From (4.52) and (4.53), we can define $\alpha(n,p,a,b,c,\overline{\mathcal{R}})$ as being the first positive root of the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$.

Now, we are going to finish the proof by applying our version of the Omori-Yau's maximum principle to the operator $\mathcal{L}$ acting on the function $|\Phi|^2$. Before, we note that if $\sup_M |\Phi| = +\infty$, then the claim (ii) of Theorem 4.2.2 trivially holds and there is nothing to prove

So, let us assume without loss of generality that $\sup_M |\Phi| < +\infty$. In this case, from item (ii) of Proposition 4.2.1 we obtain a sequence $\{p_j\}$ in M^n satisfying

$$\lim_j |\Phi|(p_j) = \sup_M |\Phi| \quad \text{and} \quad \mathcal{L}(|\Phi|^2)(p_j) < \frac{1}{j}, \quad \forall j \in \mathbb{N},$$

which jointly with (4.49) gives

$$\frac{1}{j} > \mathcal{L}(|\Phi|^2)(p_j) \geq (n-1)|\Phi|^2(p_j) Q_{n,p,a,b,c,\overline{\mathcal{R}}}(|\Phi|(p_j)) \sqrt{\frac{|\Phi|^2(p_j)}{n(n-1)} + \frac{a^2}{4} + \overline{\mathcal{R}} - b}, \quad \forall j \in \mathbb{N}.$$

Taking the limit as $j \to +\infty$, we infer

$$\left(\sup_M |\Phi|\right)^2 Q_{n,p,a,b,c,\overline{\mathcal{R}}}(\sup_M |\Phi|) \sqrt{\frac{(\sup_M |\Phi|)^2}{n(n-1)} + \frac{a^2}{4} + \overline{\mathcal{R}} - b} \leq 0.$$

It follows from here that either $\sup_M |\Phi| = 0$, which means that $|\Phi| \equiv 0$ and the submanifold is totally umbilical, or $\sup_M |\Phi| > 0$ and then

$$Q_{n,p,a,b,c,\overline{\mathcal{R}}}(\sup_M |\Phi|) \leq 0.$$

Thus, from the behavior of the function $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$ and according to our choice of the positive constant $\alpha(n,p,a,b,c,\overline{\mathcal{R}})$, we deduce that $\sup_M |\Phi| \geq \alpha(n,p,a,b,c,\overline{\mathcal{R}})$.

Finally, let us assume that $\sup_M |\Phi| = \alpha(n,p,a,b,c,\overline{\mathcal{R}})$. In this case, from (4.49) and taking into account once more the behavior of $Q_{n,p,a,b,c,\overline{\mathcal{R}}}(x)$, we get that $\mathcal{L}(|\Phi|^2) \geq 0$. But, since we are assuming that $b < \overline{\mathcal{R}}$, item (i) of Proposition 4.2.1 guarantees that $\mathcal{L}$ is elliptic. Consequently, since we are also supposing that $\sup_M |\Phi|$ is attained at some point of M^n, we conclude that $|\Phi|$ is constant on M^n and, from (4.40), the same holds for H. Hence, returning to (4.47) we obtain

$$\sum_{i,j,k} (h_{ijk}^\alpha)^2 = |\nabla A|^2 = n^2 |\nabla H|^2 = 0,$$

that is, $h_{ijk}^\alpha = 0$, for all i, j. Therefore, we conclude that M^n is an isoparametric submanifold of L_p^{n+p}. ■

Remark 4.2.3 We note that the example mentioned in Remark 1, besides checking the assumptions (1.36), (1.37), (1.38) and (1.39), also verifies the hypothesis $b \leq \overline{\mathcal{R}} < b + c$ and it is such that there exists an orthogonal basis for TM that diagonalizes simultaneously all $A_\xi, \xi \in TM^\perp$. Indeed, we have that

$$\overline{\mathcal{R}} = \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij} = \frac{1}{n(n-1)} \sum_{i,j} \overline{K}(e_i, e_j) = 1.$$

Consequently, since $\overline{\mathcal{R}} = 1$, $b = 1$ and $c = 2$, we conclude that $b \leq \overline{\mathcal{R}} < b+c$. Moreover, since $M^n = \{0\} \times \mathbb{S}^n$ is totally geodesic, we get that $A_\xi \equiv 0$ for all $\xi \in TM^\perp$.

4.3 Via $\mathcal{L}$-parabolicity

We recall that a Riemannian manifold M^n is said to be parabolic (with respect to the Laplacian operator) if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

From a physical viewpoint, parabolicity is closed related to the recurrence of the Brownian motion. Roughly speaking, the parabolicity is equivalent to the property of that all particles will pass through any open set at an arbitrary large time (for more details, see [76]).

Extending this previous concept for the operator $\mathcal{L}$ defined in (3.136), M^n is said to be $\mathcal{L}$-parabolic if the constant functions are the only functions $u \in \mathcal{C}^2(M)$ which are bounded from above and satisfying $\mathcal{L}u \geq 0$, that is, for a function $u \in \mathcal{C}^2(M)$,

$$\mathcal{L}u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

In this setting, we obtain the following gap result:

Theorem 4.3.1 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in a locally symmetric semi-Riemannian space L_p^{n+p} with $p > 1$ and satisfying conditions (1.36), (1.38) and (1.39), such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = \frac{c_1}{n} + 2c_2$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all $A_\xi, \xi \in TM^\perp$. Assume in addition that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ is the positive constant which was obtained in Theorem 4.2.2. If M^n is a $\mathcal{L}$-parabolic submanifold, then either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $|\Phi| \equiv \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ and M^n is an isoparametric submanifold.*

Proof. Suppose that M^n is not totally umbilical. Since we are assuming that $0 \leq |\Phi| \leq \alpha(n,p,a,b,c,\overline{\mathcal{R}})$, we obtain $0 < \sup_M |\Phi|^2 \leq \alpha(n,p,a,b,c,\overline{\mathcal{R}})$. In this case, from item (ii) of Theorem 4.2.2 we get that $\sup_M |\Phi|^2 = \alpha(n,p,a,b,c,\overline{\mathcal{R}})$. Moreover, since estimate (4.49) jointly with our restriction on $|\Phi|$ imply $\mathcal{L}(|\Phi|^2) \geq 0$ on M^n, from the $\mathcal{L}$-parabolicity of M^n we conclude that $|\Phi|$ must be constant and identically equal to $\alpha(n,p,a,b,c,\overline{\mathcal{R}})$. Therefore, at this point we can proceed as in the last part of the proof of Theorem 4.2.2 to conclude the result. ■

When the ambient space L_p^{n+p} is supposed to be Einstein, reasoning as in the first part of the proof of Theorem 1.1 in [90], from (1.31) and (1.32) it is not difficult to verify that

$$\mathcal{L}(f) = \operatorname{div}(\mathcal{P}(\nabla f)), \tag{4.54}$$

where $\mathcal{P}$ is just the operator defined in (1.35). Taking account this fact, we obtain the following criterion for $\mathcal{L}$-parabolicity of complete linear Weingarten spacelike submanifolds:

Proposition 4.3.2 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field in a locally symmetric Einstein semi-Riemannian space L_p^{n+p} satisfying conditions (1.36), (1.38) and (1.39), such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = \frac{c_1}{n} + 2c_2$. If $\sup_M |\Phi|^2 < +\infty$ and, for some reference point $o \in M^n$,*

$$\int_0^{+\infty} \frac{dr}{\operatorname{vol}(\partial B_r)} = +\infty, \tag{4.55}$$

then M^n is $\mathcal{L}$-parabolic. Here B_r denotes the geodesic ball of radius r in M^n centered at the origin o.

Proof. We consider on M^n the symmetric $(0,2)$ tensor field ξ given by

$$\xi(X,Y) = \langle \mathcal{P}X, Y\rangle \quad \text{or, equivalently,} \quad \xi(\nabla u, \cdot)^\sharp = \mathcal{P}(\nabla u),$$

where $\mathcal{P}$ is defined in (3.135) and $^\sharp : T^*M \to TM$ denotes the musical isomorphism. Thus, from (4.54) we get

$$\mathcal{L}(u) = \operatorname{div}\left(\xi(\nabla u, \cdot)^\sharp\right).$$

On the other hand, since we are assuming that $\sup_M |\Phi|^2 < +\infty$ and $a \geq 0$, from (4.40) we get that $\sup_M H < +\infty$. So, we can define a positive continuous function ξ_+ on $[0, +\infty)$, by

$$\xi_+(r) = 2n \sup_{\partial B_r} H. \tag{4.56}$$

Thus, from (4.56) we have

$$\xi_+(r) = 2n \sup_{\partial B_r} H \leq 2n \sup_{M} H < +\infty. \tag{4.57}$$

Hence, from (4.55) and (4.57) we get

$$\int_0^{+\infty} \frac{dr}{\xi_+(r)\mathrm{vol}(\partial \mathrm{B_r})} = +\infty.$$

Therefore, we can apply Theorem 2.6 of [110] to conclude the proof. ■

Remark 4.3.3 Taking into account Proposition 4.3.2, it is natural to ask oneself about the existence of Einstein manifolds which are locally symmetric. In this direction, Tod [118] showed that four-dimensional Einstein manifolds which are also D'Atri spaces are necessarily locally symmetric. Later on, Brendle [40] proved that a compact Einstein manifold of dimension $n \geq 4$ having nonnegative isotropic curvature must be locally symmetric, extending a previous result of Micallef and Wang for $n = 4$ (see Theorem 4.4 of [97]). See also [125] for another sufficient conditions for an Einstein manifold to be locally symmetric.

4.4 Via integrability property

In [132], Yau established the following version of Stokes' Theorem on an n-dimensional complete noncompact Riemannian manifold M^n: *If $\omega \in \Omega^{n-1}(M)$ is an $(n-1)$-differential form on M^n, then there exists a sequence B_i of domains on M^n such that $B_i \subset B_{i+1}$, $M^n = \bigcup_{i\geq 1} B_i$ and $\lim_{i\to +\infty} \int_{B_i} d\omega = 0$.* Supposing that M^n is oriented by the volume element dM and considering the contraction of dM in the direction of a smooth vector field X on M^n, that is, $\omega = \iota_X dM$, Caminha obtained a suitable consequence of Yau's result, which is described below (see Proposition 2.1 of [43]). In what follows, $\mathcal{L}^1(M)$ stands for the space of Lebesgue integrable functions on M^n.

Lemma 4.4.1 *Let X be a smooth vector field on the n-dimensional complete oriented Riemannian manifold M^n, such that* $\mathrm{div}X$ *does not change sign on M^n. If $|X| \in \mathcal{L}^1(M)$, then* $\mathrm{div}X = 0$.

We close our paper applying Lemma 4.4.1 in order to obtain the following characterization result.

Theorem 4.4.2 *Let M^n be a complete linear Weingarten spacelike submanifold immersed with parallel normalized mean curvature vector field and flat normal bundle in locally symmetric Einstein semi-Riemannian space L_p^{n+p} with $p > 1$ and satisfying conditions (1.36), (1.37), (1.38) and (1.39), such that $R = aH + b$, with $a \geq 0$ and $b \leq \overline{\mathcal{R}} < b + c$, where $c = \dfrac{c_1}{n} + 2c_2$. Suppose that there exists an orthogonal basis for TM that diagonalizes simultaneously all $A_\xi, \xi \in TM^\perp$. Assume in addition that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, where $\alpha(n, p, a, b, c, \overline{\mathcal{R}})$ is the positive constant which was obtained in Theorem 4.2.2. If $|\nabla H| \in \mathcal{L}^1(M)$, then either $|\Phi| \equiv 0$ and M^n is totally umbilical, or $|\Phi| \equiv \alpha(n, p, a, b, c, \overline{\mathcal{R}})$ and M^n is an isoparametric submanifold.*

Proof. Since $R = aH + b$ and taking into account that (4.40) gives that H is bounded on M^n, from (1.22) we have that A is bounded on M^n. Consequently, from (1.35) we conclude that the operator $\mathcal{P}$ is bounded, that is, there exists a positive constant C_1 such that $|\mathcal{P}| \leq C_1$. Since we are also assuming that $|\nabla H| \in \mathcal{L}^1(M)$ and (4.40), we obtain that

$$|P(\nabla H)| \leq |P||\nabla H| \leq C_1|\nabla H| \in \mathcal{L}^1(M). \tag{4.58}$$

Thus, taking into account (4.54) and (4.58), we can apply Lemma 4.4.1 to obtain

$$\mathcal{L}(nH) = \operatorname{div}(\mathcal{P}(nH)) = 0. \tag{4.59}$$

Hence, using the fact that $0 \leq |\Phi| \leq \alpha(n, p, a, b, c, \overline{\mathcal{R}})$, from (4.47) and (4.59) we conclude that

$$0 = \mathcal{L}(nH) \geq |\nabla A|^2 - n^2|\nabla H|^2 + |\Phi|^2 Q_{n,p,a,b,c,\overline{\mathcal{R}}}(|\Phi|) \geq 0. \tag{4.60}$$

Thus, from (4.60) we get that $|\nabla A|^2 = n^2|\nabla H|^2$ and, consequently, Lemma (1.2.2) guarantees that H is constant. Hence,

$$\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 = |\nabla A|^2 = n^2|\nabla H|^2 = 0,$$

that is, $h_{ijk}^\alpha = 0$ for all i, j, and we obtain that M^n is isoparametric. Therefore, the result follows once more as in the last part of the proof of Theorem 4.2.2. ■

Referências Bibliográficas

[1] A.L. Albujer and L.J. Alías, *Spacelike hypersurfaces with constant mean curvature in the steady state space*, Proc. American Math. Soc. **137** (2009), 711–721.

[2] D. Achour, K. Saadi. A polynomial characterization of Hilbert spaces, Collectanea Mathematica, **61** (2010), 291–301.

[3] L.J. Alias, D. Impera and M. Rigoli, *Spacelike hypersurfaces of constant higher order mean curvature in generalized Robertson-Walker spacetimes*, Math. Proc. Camb. Philos. Soc. **152** (2012), 365–383.

[4] L.J. Alías, A. Brasil Jr. and A.G. Colares, *Integral formulae for spacelike hypersurfaces in conformally stationary spacetimes and applications*, Proc. Edinburgh Math. Soc. **46** (2003), 465–488.

[5] L.J. Alías and A.G. Colares, *Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetime*, Math. Proc. Cambridge Philos. Soc. **143** (2007), 703–729.

[6] L.J. Alías, H.F. de Lima and F.R. dos Santos, *Characterizations of spacelike submanifolds with constant scalar curvature in the de Sitter space*, Mediterr. J. Math. (2018), **15**:12.

[7] L.J. Alías, A. Caminha and F.Y. do Nascimento, *A maximum principle related to volume growth and applications*, Ann. Mat. Pura Appl. (2020), **14** 27-34.

[8] L.J. Alías, P. Mastrolia and M. Rigoli, *Maximum principles and geometric applications*, Springer Monographs in Mathematics, Springer, Cham, 2016.

[9] L.J. Alías, A. Caminha and F.Y. do Nascimento, *A maximum principle at infinity with applications to geometric vector fields*, J. Math. Anal. Appl. **474** (2019), 242–247.

[10] L.J. Alías, D. Impera and M. Rigoli, *Spacelike hypersurfaces of constant higher order mean curvature in generalized Robertson-Walker spacetimes*, Math. Proc. Camb. Philos. Soc. **152** (2012), 365–383.

[11] L.J. Alías, P. Mastrolia and M. Rigoli, *Maximum Principles and Geometric Applications*, Springer Monographs in Mathematics, New York, 2016.

[12] L.J. Alías and A. Romero, *Integral formulas for compact spacelike n-submanifolds in de Sitter spaces. Applications to the parallel mean curvature vector case*, Manuscripta Math. **87** (1995), 405–416.

[13] C.P. Aquino, H.F. de Lima, F.R. dos Santos and M.A.L. Velásquez, *Characterizations of spacelike hyperplanes in the steady state space via generalized maximum principles*, Milan J. Math. **83** (2015), 199–209.

[14] C.P. Aquino and H.F. de Lima, *On the umbilicity of complete constant mean curvature spacelike hypersurfaces*, Math. Ann. **360** (2014), 555–569.

[15] C.P. Aquino, H.F. de Lima and M.A.L. Velásquez, *On the geometry of complete spacelike hypersurfaces in the anti-de Sitter space*, Geom. Dedicata **174** (2015), 13–23.

[16] L.V. Ahlfors, *Sur le type d'une surface de Riemann*, C. R. Acad. Sci. Paris **201** (1935), 30–32.

[17] J.G. Araújo, H.F. de Lima, F.R. dos Santos and M.A.L. Velásquez, *Characterizations of complete linear Weingarten spacelike submanifolds in a locally symmetric semi-Riemannian manifold*, Extracta Math. **32** (2017), 55–81.

[18] J.G. Araújo, H.F. de Lima, F.R. dos Santos and M.A.L Velásquez, *Complete linear Weingarten spacelike submanifolds with higher codimension in the de Sitter space*, **16** (2019), International Journal of Geometric Methods in Modern Physics.

[19] J. G. Araújo, W.F.C. Barboza, H. de Lima, M.A.L. Velásquez, *On the linear Weingarten spacelike submanifolds immersed in a locally symmetric semi-Riemannian space*, CONTRIBUTIONS TO ALGEBRA AND GEOMETRY/BEITRAGE ZUR ALGEBRA UND GEOMETRIE **61** (2020), 267–282.

[20] R. Aiyama, *Compact spacelike m-submanifolds in a pseudo-Riemannian sphere* $\mathbb{S}_p^{m+p}(c)$, Tokyo J. Math. **18** (1995), 81–90.

[21] N. Abe, N. Koike and S. Yamaguchi, *Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form*, Yokohama Math. J. **35** (1987), 123–136.

[22] K. Akutagawa, *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. **196** (1987), 13–19.

[23] A. Barros, A. Brasil and L. A. M. Sousa JR, *A maximum A new characterization of submanifolds with parallel mean curvature vector in* $\mathbb{S}^{n+p}$, Kodai Math. J. **27** (2004), 45–56.

[24] W.F.C. Barboza, E. L. Lima, H. de Lima, Marco A. L. Velásquez, *On the umbilicity of linear Weingarten spacelike submanifolds immersed in the de Sitter space*, Bulletin of Mathematical Sciences **10** (2020), 1–12.

[25] W.F.C. Barboza, H. de Lima, Marco A. L. Velásquez, *Gap type results for spacelike submanifolds with parallel mean curvature vector*, to appear at MATHEMATICA SCANDINAVICA.

[26] W.F.C. Barboza, H. de Lima, Marco A. L. Velásquez, *Uniqueness and nullity of complete spacelike hypersurfaces immersed in the anti-de Sitter space*, to appear at ANNALI DELL'UNIVERSITA' DI FERRARA.

[27] W.F.C. Barboza, H. de Lima, A.M.S. Oliveira, Marco A. L. Velásquez, *Complete spacelike hypersurfaces in the anti-de Sitter space: Rigidity, nonexistence and curvature estimates*, to appear at COLLOQUIUM MATHEMATICUM. .

[28] W.F.C. Barboza, H. de Lima, Marco A. L. Velásquez, *Revisiting the linear Weingarten spacelike submanifolds*, preprint.

[29] W.F.C. Barboza, H. de Lima, Marco A. L. Velásquez, *Rigidity and nonexistence of complete spacelike hypersurfaces in the steady state space*, preprint.

[30] W.F.C. Barboza, H. de Lima, Marco A. L. Velásquez, *Stochastically complete, parabolic and L1-Liouville spacelike submanifolds*, preprint.

[31] W.F.C. Barboza, H. de Lima, Marco A. L. Velásquez, *Umbilicity of spacelike submanifolds with parallel mean curvature vector via a maximum principle at infinity*, preprint.

[32] A. Brasil, R.M.B. Chaves and A.G. Colares, *Rigidity results for submanifolds with parallel mean curvature vector in de Sitter space*, Glasgow Math. J. **48** (2006), 1–10.

[33] C. Bar and G.P. Bessa, *Stochastic completeness and volume growth*, Proc. Amer. Math. Soc. **138** (2010), 2629–2640.

[34] G.P. Bessa, S. Pigola and A.G. Setti, *On the L^1-Liouville property of stochastically incomplete manifolds*, Potential Anal. **39** (2013), 313–324.

[35] H. Bondi and T. Gold, *On the generation of magnetism by fluid motion*, Monthly Not. Roy. Astr. Soc. **108** (1948), 252–270.

[36] J.K. Beem, P.E. Ehrlich and K.L. Easley, *Global Lorentzian Geometry*, Second Edition, CRC Press, New York, 1996.

[37] J. A. Barbosa, G. Botelho, D. Diniz, D. Pellegrino. Spaces of absolutely summing polynomials, Mathematica Scandinavica, **101** (2007), 219–237.

[38] J.L.M. Barbosa and A.G. Colares, *Stability of hypersurfaces with constant r-mean curvature*, Ann. Global Anal. Geom. **15** (1997), 277–297.

[39] J.O. Baek, Q.M. Cheng and Y.J. Suh, *Complete spacelike hypersurface in locally symmetric Lorentz spaces*, J. Geom. Phys. **49** (2004), 231–247.

[40] S. Brendle, *Einstein manifolds with nonnegative isotropic curvature are locally symmetric*, Duke Math. J. **151** (2009), 1–21.

[41] A. Caminha, *Tópicos de Geometria Diferencial*, Rio de Janeiro: SBM (2014).

[42] A. Caminha, *A rigidity theorem for complete CMC hypersurfaces in Lorentz manifolds*, Diff. Geom. Appl. **24** (2006), 652–659.

[43] A. Caminha, *The geometry of closed conformal vector fields on Riemannian spaces*, Bull. Braz. Math. Soc. **42** (2011), 277–300.

[44] A.G. Colares and H.F. de Lima, *Spacelike hypersurfaces with constant mean curvature in the steady state space*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 287–302.

[45] A.G. Colares and H.F. de Lima, *On the rigidity of spacelike hypersurfaces immersed in the steady state space* $\mathcal{H}^{n+1}$, Publ. Math. Debrecen **81** (2012), 103–119.

[46] E. Calabi, *Examples of Bernstein problems for some nonlinear equations*, Proc. Sympos. Pure Math. **15** (1970), 223–230.

[47] F. Camargo, A. Caminha, H.F. de Lima and U. Parente, *Generalized maximum principles and the rigidity of complete spacelike hypersurfaces*, Math. Proc. Cambridge Philos. Soc. **153** (2012), 541–556.

[48] F. Camargo and H.F. de Lima, *New characterizations of totally geodesic hypersurfaces in anti-de Sitter space* $\mathbb{H}^{n+1}$, J. Geom. Phys. **60** (2010), 1326–1332.

[49] M. P. Carmo, *Geometria Riemanniana*, Rio de Janeiro: IMPA 5 Ed (2011).

[50] L. Cao and G. Wei, *A new characterization of hyperbolic cylinder in anti-de Sitter space* $\mathbb{H}_1^{n+1}(-1)$, J. Math. Anal. Appl. **329** (2007), 408–414.

[51] R.M.B. Chaves, L.A.M. Sousa Jr. and B.C. Valério, *New characterizations for hyperbolic cylinders in anti-de Sitter spaces*, J. Math. Anal. Appl. **393** (2012), 166–176.

[52] R.M.B. Chaves, L.A.M. Sousa Jr. and B.C. Valério, *New characterizations for hyperbolic cylinders in anti-de Sitter spaces*, J. Math. Anal. Appl. **393** (2012), 166–176.

[53] R.M.B. Chaves and L.A.M. Sousa Jr., *On complete spacelike submanifolds in the De Sitter space with parallel mean curvature vector*, Rev. Union Mat. Argentina **47** (2006), 85–98.

[54] S.Y. Cheng and S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354.

[55] S.Y. Cheng and S.T. Yau, *Maximal Spacelike Hypersurfaces in the Lorentz-Minkowski Space*, Ann. of Math. **104** (1976), 407–419.

[56] S.Y. Cheng and S.T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), 195–204.

[57] S.M. Choi, U-H. Ki and H-J. Kim, *Complete maximal spacelike hypersurfaces in an anti-de Sitter space*, Bull. Korean Math. Soc. **31** (1994), 85–92.

[58] Q.M. Cheng, *Space-like surfaces in an anti-de Sitter space*, Colloq. Math. **66** (1993), 201–208.

[59] Q.M. Cheng, *Complete spacelike hypersurface of a de Sitter space with* $r = \kappa H$, Mem. Fac. Sci. Kyushu Univ. **44** (1990), 67–77.

[60] Q.M. Cheng, *Complete space-like submanifolds with parallel mean curvature vector*, Math. Z. **206** (1991), 333–339.

[61] Q.M. Cheng and S. Ishikawa, *Complete maximal spacelike submanifolds*, Kodai Math. J. **20** (1997), 208–217.

[62] H.F. De Lima, F.R. dos Santos and M.A.L. Velásquez, *Complete spacelike submanifolds with parallel mean curvature vector in a semi-Euclidean space*, Acta Math. Hungar. **150** (2016), 217–227.

[63] H.F. De Lima, F.R. dos Santos and M.A.L. Velásquez, *Characterizations of complete spacelike submanifolds in the* $(n+p)$*-dimensional anti-de Sitter space of index* q, RACSAM **111** (2017), 921–930.

[64] H.F. De Lima and J.R. de Lima, *Complete linear Weingarten spacelike hypersurfaces immersed in a locally symmetric Lorentz space*, Res. Math. **63** (2013), 865–876.

[65] H.F. De Lima, F.R. Santos, J.N. Gomes and M.A.L. Velásquez, *On the complete spacelike hypersurfaces immersed with two distinct principal curvatures in a locally symmetric Lorentz space*, Collect. Math. **67** (2016), 379–397.

[66] E.B. Davies, *Heat kernel bounds, conservation of probability and the Feller property*, Festschrift on the occasion of the 70th birthday of Shmuel Agmon, J. Anal. Math. **58** (1992), 99–119.

[67] M. Dajczer et al., *Submanifolds and Isometric Immersions*, Publish or Perish, Houston, 1990.

[68] M. Dajczer and K. Nomizu, *On the flat surfaces in* $\mathbb{S}^3_1$ *and* $\mathbb{H}^3_1$, Manifolds and Lie Groups, Birkhäuser, Boston, 1981.

[69] M. Émery, *Stochastic Calculus on Manifolds*, Springer-Verlag, Berlin, 1989.

[70] M.F. Elbert, *Constant positive* 2*-mean curvature hypersurfaces*, Illinois. J. Math. **46** (2002), 247–267.

[71] D. Ferus, *On the completeness of nullity foliations*, Mich. Math. J. **18** (1971), 61–64.

[72] A.J. Goddard, *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Philos. Soc. **82** (1977), 489–495.

[73] A. Grigor'yan, *On the existence of a Green function on a manifold*, Uspekhi Mat. Nauk **38** (1) (1983), 161–262 (in Russian), Engl. transl.: Russian Math. Surveys **38** (1) (1983), 190–191.

[74] A. Grigor'yan, *On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds*, Mat. Sb. **128** (3) (1985), 354–363 (in Russian), Engl. transl.: Math. USSR Sb 56 (1987), 349–358.

[75] A. Grigor'yan, *Stochastically complete manifolds and summable harmonic functions*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), 1102–1108; translation in Math. USSR-Izv. **33** (1989), 425–532.

[76] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Am. Math. Soc. (N.S.) **36** (1999), 135–249.

[77] A. Grigor'yan and J. Masamune, *Parabolicity and stochastic completeness of manifolds in terms of the Green formula*, J. Math. Pures Appl. **100** (2013), 607–632.

[78] A. Grigor'yan, *Heat kernels on weighted manifolds and applications*, in: The Ubiquitous Heat Kernel, in: Contemp. Math., Vol. 398 Amer. Math. Soc., Providence, RI, 2006, pp. 93–191.

[79] M. Gaffney, *A special Stokes' theorem for complete Riemannian manifolds*, Ann. of Math. **60** (1954), 140–145.

[80] M.P. Gaffney, *The conservation property of the heat equation on Riemannian manifolds*, Comm. Pure Appl. Math. **12** (1959), 1–11.

[81] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

[82] E.P. Hsu, *Heat semigroup on a complete Riemannian manifold*, Ann. Probab. **17** (1989), 1248–1254.

[83] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge Univ. Press, Cambridge, 1973.

[84] F. Hoyle, *A new model for the expanding universe*, Monthly Not. Roy. Astr. Soc. **108** (1948), 372–382.

[85] T. Ishihara, *Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature*, Mich. Math. J. **35** (1988), 345–352.

[86] L. Karp, *Subharmonic functions, harmonic mapping and isometric immersions*, in: S.T. Yau (Ed.), Seminar on Differential Geometry, in: Ann. of Math. Stud., vol. 102, Princeton University Press, 1983.

[87] A.M. Li and J.M. Li, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math. **58** (1992), 582–594.

[88] H. Li, *Complete spacelike submanifolds in de Sitter space with parallel mean curvature vector satisfying* $H^2 = 4(n-1)/n^2$, Ann. Global Anal. Geom. **15** (1997), 335–345.

[89] P. Li, *Curvature and function theory on Riemannian manifolds*, Surveys in differential geometry, pp. 375–432, Surv. Differ. Geom., VII, Int. Press, Somerville, MA, 2000.

[90] H.F. de Lima and J.R. de Lima, *Characterizations of linear Weingarten spacelike hypersurfaces in Einstein spacetimes*, Glasgow Math. J. **55** (2013), 567–579.

[91] J. Liu and J. Zhang, *Complete spacelike submanifolds in de Sitter spaces with* $R = aH + b$, Bull. Aust. Math. Soc. **87(3)** (2013), 386–399.

[92] X.M. Liu, *Space-like submanifolds in de Sitter spaces*, J. Phys. A Math. Gen. **34** (2001), 5463–5468.

[93] J. Liu and Z. Sun, *On spacelike hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces*, J. Math. Anal. App. **364** (2010), 195–203.

[94] P. Lucas and H.F. Ramírez-Ospina, *Hypersurfaces in non-flat Lorentzian space forms satisfying* $L_k\psi = A\psi + b$, Taiwanese J. Math. **16** (2012), 1173–1203.

[95] J. Marsden and F. Tipler, *Maximal hypersurfaces and foliations of constant mean curvature in general relativity*, Bull. Am. Phys. Soc. **23** (1978), 84.

[96] M. Mariano, *On spacelike submanifolds with parallel mean curvature in an indefinite space form*, Monatsh. Math. **166** (2012), 107–120.

[97] M.J. Micallef and M.Y. Wang, *Metrics with nonnegative isotropic curvature*, Duke Math. J. **72** (1993), 649–672.

[98] S. Montiel, *An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. **37** (1988), 909–917.

[99] S. Montiel, *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*, Math. Ann. **314** (1999), 529–553.

[100] S. Montiel, *Unicity of constant mean curvature hypersurface in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), 711–748.

[101] S. Montiel, *Complete non-compact spacelike hypersurfaces of constant mean curvature in de Sitter spaces*, J. Math. Soc. Japan **55** (2003), 915–938.

[102] S. Nishikawa, *On spacelike hypersurfaces in a Lorentzian manifold*, Nagoya Math. J. **95** (1984), 117–124.

[103] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, 1983.

[104] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.

[105] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.

[106] M. Okumura, *Hypersurfaces and a pinching problem on the second fundamental tensor*, Am. J. Math. **96** (1974), 207–213.

[107] O. Perdomo, *New examples of maximal spacelike surfaces in the anti-de Sitter space*, J. Math. Anal. Appl. **353** (2009), 403–409.

[108] S. Pigola, M. Rigoli and A.G. Setti, *A remark on the maximum principle and stochastic completeness*, Proc. Amer. Math. Soc. **131** (2003), 1283–1288.

[109] S. Pigola, M. Rigoli and A.G. Setti, *Maximum principles on Riemannian manifolds and applications*, Mem. American Math. Soc. **822** (2005).

[110] S. Pigola, M. Rigoli and A.G. Setti, *A Liouville-type result for quasi-linear elliptic equations on complete Riemannian manifolds*, J. Funct. Anal. **219** (2005), 400–432.

[111] J. Ramanathan, *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space*, Indiana Univ. Math. J. **36** (1987), 349–359.

[112] H. Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Sc. Math. **117** (1993), 217–239.

[113] D. Stroock, *An Introduction to the Analysis of Paths on a Riemannian Manifold*, Math. Surveys and Monographs, volume 4, American Math. Soc (2000).

[114] K.-Th Sturm, *Analysis on local Dirichelet spaces I. Recurrence, conservativeness and Liouville properties*, J. Reine Angew. Math **456** (1994), 173–196.

[115] S. Stumbles, *Hypersurfaces of constant mean extrinsic curvature*, Ann. Phys. **133** (1980), 28–56.

[116] W. Santos, *Submanifolds with parallel mean curvature vector in spheres*, Tohoku Math. J. **46** (1994), 403–415.

[117] M. Takeda, *On a martingale method for symmetric diffusion processes and its applications*, Osaka J. Math. **26** (1989), 650–623.

[118] K.P. Tod, *Four-dimensional D'Atri Einstein spaces are locally symmetric*, Diff. Geom. Appl. **11** (1999), 55–67.

[119] A.E. Treibergs, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space*, Invent. Math. **66** (1982), 39–56.

[120] N.Th Varopoulos, *Potential theory and diffusion of Riemannian manifolds*, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund, vols. I, II, in: Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 821–837.

[121] S. Weinberg, *Gravitation and Cosmology: Principles ans Applications of the General Theory of Relativity*, John Wiley & Sons, New York, 1972.

[122] R. Wald, *General Relativity*, Univ. of Chicago Press, Chicago, 1984.

[123] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, New York, 1972.

[124] B.Y. Wu, *On complete spacelike hypersurfaces with constant m-th mean curvature in an anti-de Sitter space*, Int. J. Math. **21** (2010), 551–569.

[125] H.W. Xu and J. Gu, *Rigidity of Einstein manifolds with positive scalar curvature*, Math. Ann. **358** (2014), 169–193.

[126] B. Yang, *On complete spacelike $(r-1)$-maximal hypersurfaces in the anti-de Sitter space* $\mathbb{H}_1^{n+1}(-1)$, Bull. Korean Math. Soc. **47** (2010), 1067–1076.

[127] D. Yang and Z.H. Hou, *Linear Weingarten spacelike hypersurfaces in de Sitter space*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 769–780.

[128] D. Yang and Z.H. Hou, *Linear Weingarten spacelike submanifolds in de Sitter space*, J. Geom. **103** (2012), 177–190.

[129] D. Yang and L. Li, *Spacelike submanifolds with parallel mean curvature vector in* $S_q^{n+p}(1)$, Math. Notes **100** (2016), 298–308.

[130] S.T. Yau, *Submanifolds with constant mean curvature I*, Amer. J. Math. **96** (1974), 346–366.

[131] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

[132] S.T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), 659–670.

[133] S.T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), 659–670.

[134] J.F. Zhang, *Submanifolds with constant scalar curvature and the harmonic function of Finsler manifold*, Ph.D. Thesis, Zhejiang University, 2005.