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The Gravitoelectromagnetic Analogy in the Context of Brans-Dicke Theory

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Abstract

We discuss the gravitoelectromagnetic analogy in the Brans-Dicke theory framework, exhibiting the field equations in a similar structure to Maxwell's equations. Moreover, in this formalism, we find the expression to the gravitoelectromagnetic force law. We compare the results obtained with those predicted by General Relativity.

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1 Introduction

In the framework of the General Relativity theory is possible, when one considers weak field approximation and low rotation velocity of the source, the definition of similar potentials to the electromagnetic potentials [10, 11]. In this sense, mass and mass currents generate fields called gravitoelectric and gravitomagnetic, respectively. Really, gravitational effects associated with the rotation of massive bodies, as the Lense-Thirring effect of frame dragging, can be understood in terms of gravitomagnetism [5]; it is interesting to mention that the Lense-Thirring effect was verified by the GP-B experiment with an accuracy of 19% [6]. The gravitoelectromagnetic analogy is established more deeply when the gravitational field equations are written in a similar form to Maxwell's equations, while a gravitoelectromagnetic force law is defined too [3].

In turn, taking the Brans-Dicke theory [4] as the fundamental theory for the description of gravitational phenomena, we will develop the formalism of the gravitoelectromagnetism to obtain the field equations in a similar form to Maxwell's equations and the gravitoelectromagnetic force law. The studies involving the scalar-tensorial theories of gravity, as Brans-Dicke theory, evaluate the contribution of the scalar field ϕ in the gravitational activity, so that several aspects are currently investigated [7, 1].

The paper is organized as follows. In Section 2, the gravitoelectric and gravitomagnetic fields are defined and the field equations for gravitoelectromagnetism are presented. After, in Section 3, we obtain the gravitoelectromagnetic force law, comparing the expression with the General Relativity prediction. Lastly, in Section 4, our conclusions are exposed.

2 The Field Equations for Gravitoelectromagnetism

The Brans-Dicke weak field equations are given by [13]

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\left(\frac{2\omega+3}{2\omega+4}\right)T_{\mu\nu}, \tag{1}$$

$$\Box\varepsilon = \frac{8\pi T}{c^4(2\omega+3)}, \tag{2}$$

where ω is the scalar field coupling constant, $T = T^{\alpha}{}_{\alpha}$ and G is the Newton's gravitational constant. We consider that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = diag(-1,1,1,1)$ denotes Minkowski metric tensor and $h_{\mu\nu}$ is a small perturbation term, so that we keep only first-order terms in $h_{\mu\nu}$. Besides, we take the scalar field $\phi = \phi_0 + \varepsilon$, where $\phi_0^{-1} = \left(\frac{2\omega+3}{2\omega+4}\right)G$ and $\varepsilon = \varepsilon(x)$ is also a small perturbation term with $|\varepsilon/\phi_0| \ll 1$. In this approach, the Brans-Dicke gauge $\overline{h}^{\mu\nu}{}_{,\mu} = 0$ is valid and still

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \varepsilon\phi_0^{-1}\eta_{\mu\nu}, \tag{3}$$

being $h = h^{\alpha}{}_{\alpha}$.

Assuming a localized matter distribution with density ρ and velocity field $\overrightarrow{v}$, with the condition $|\overrightarrow{v}| << c$, the relevant components of $\bar{h}_{\mu\nu}$ are $\overline{h}_{00}$ and $\overline{h}_{0i}$ [13]. Now, we define [10]:

$$\overline{h}_{00} = \frac{4\Phi}{c^2}, \tag{4}$$

$$\overline{h}_{0i} = -\frac{2A_i}{c^2}, \tag{5}$$

being Φ the gravitoelectric potential and $\vec{A}$ the gravitomagnetic vector potential. In the stationary case, if the matter distribution is confined around the origin of spatial coordinates, so far from the source we will have the solutions [13]

$$\Phi = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{GM}{r}, \tag{6}$$

$$\vec{A} = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{G(\vec{J}\times\vec{r})}{cr^3}, \tag{7}$$

where $r = |\vec{r}|$, M and $\vec{J}$ are the mass and angular momentum of the source.

From gauge expression $\overline{h}^{\mu\nu}{}_{,\mu} = 0$ and equations (4) and (5), we obtain

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\cdot\vec{A} = 0, \tag{8}$$

that is analogous to the Lorenz gauge of electromagnetism [8]. Then, in close analogy with electrodynamics, let us define the gravitoelectric field $\vec{E}$ and the gravitomagnetic field $\vec{B}$ as [3]

$$\vec{E} = -\nabla\Phi - \frac{1}{2c}\frac{\partial\vec{A}}{\partial t}, \tag{9}$$

$$\vec{B} = \nabla\times\vec{A}. \tag{10}$$

From (9) and (10), one obtains immediately the equations

$$\nabla\times\vec{E} = -\frac{1}{2c}\frac{\partial\vec{B}}{\partial t}, \tag{11}$$

$$\nabla\cdot\vec{B} = 0. \tag{12}$$

Furthermore, considering the equations (8)-(10) and the field equation (1) with the energy-momentum tensor components $T_{00} = \rho c^2$ and $T_{0i} = -cj_i$, where $\vec{j} = \rho\vec{v}$ is the mass current, we get

$$\nabla \cdot \vec{E} = 4\pi G \left(\frac{2\omega + 3}{2\omega + 4}\right) \rho, \tag{13}$$

$$\nabla \times \vec{B} = \frac{8\pi G}{c} \left(\frac{2\omega + 3}{2\omega + 4}\right) \vec{j} + \frac{2}{c}\frac{\partial \vec{E}}{\partial t}. \tag{14}$$

These equations contain the continuity equation $\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$.

The equations (11)-(14) represent the analog of the Maxwell equations in the context of Brans-Dicke theory, when one regards the weak field approximation and a localized slowly rotating source. It is interesting to note that, in the limit $\omega \to \infty$, the equations (11)-(14) are reduced to the expressions obtained in General Relativity [3]. Indeed, it is well known that, in the weak field approximation, when $\omega \to \infty$ the Brans-Dicke solution goes over to the corresponding solution in Einstein's General Relativity, although this is not always true in the case of exact solutions [12, 2].

3 Gravitoelectromagnetic Force Law

Let us now to obtain the gravitoelectromagnetic force law. We start with the calculus of the spacetime metric from equations (3)-(5). Then, we will have

$$h_{00} = \frac{2\Phi}{c^2} + \left(\frac{2\omega + 3}{2\omega + 4}\right) G\varepsilon, \tag{15}$$

$$h_{0i} = -\frac{2A_i}{c^2}, \tag{16}$$

$$h_{ij} = 0 \quad (i \neq j), \tag{17}$$

$$h_{11} = h_{22} = h_{33} = \frac{2\Phi}{c^2} - \left(\frac{2\omega + 3}{2\omega + 4}\right) G\varepsilon. \tag{18}$$

Therefore, the line element has the form

$$\begin{aligned} ds^2 &= -c^2 \left[1 - \frac{2\Phi}{c^2} - \left(\frac{2\omega + 3}{2\omega + 4}\right) G\varepsilon\right] dt^2 - \frac{4}{c}\left(\vec{A} \cdot d\vec{r}\right) dt \\ &\quad + \left[1 + \frac{2\Phi}{c^2} - \left(\frac{2\omega + 3}{2\omega + 4}\right) G\varepsilon\right] \delta_{ij} dx^i dx^j. \end{aligned} \tag{19}$$

With the definitions

$$2\frac{\Lambda}{c^2} = \frac{2\Phi}{c^2} + \left(\frac{2\omega + 3}{2\omega + 4}\right) G\varepsilon, \tag{20}$$

$$2\frac{\Psi}{c^2} = \frac{2\Phi}{c^2} - \left(\frac{2\omega+3}{2\omega+4}\right) G\varepsilon, \tag{21}$$

the equation (19) can be expressed as

$$ds^2 = -c^2\left(1 - 2\frac{\Lambda}{c^2}\right) dt^2 - \frac{4}{c}\left(\vec{A}\cdot d\vec{r}\right) dt + \left(1 + 2\frac{\Psi}{c^2}\right)\delta_{ij} dx^i dx^j. \tag{22}$$

The Lagrangian for the motion of a test particle of mass m is $L = -mc\dfrac{ds}{dt}$. Considering first-order terms in Λ, Ψ and $\vec{A}$, we get

$$L = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\gamma\Lambda + m\gamma\frac{v^2}{c^2}\Psi - \frac{2m}{c}\gamma\vec{A}\cdot\vec{v}, \tag{23}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Now, in the weak gravitational field, we assume that the material particle has a small velocity [9] and terms until second-order in $\dfrac{v}{c}$ are maintained. Thus, taking into account all approximations, one obtains

$$L = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\Lambda - \frac{2m}{c}\vec{A}\cdot\vec{v}, \tag{24}$$

which is analogous to the electromagnetic Lagrangian [8]. Hence, being the equation of motion $\dfrac{d\vec{p}}{dt} = \vec{F}$, with the linear momentum $\vec{p} = \gamma m\vec{v}$, we will find the expression

$$\vec{F} = -m\left(-\nabla\Lambda - \frac{2}{c}\frac{\partial\vec{A}}{\partial t}\right) - \frac{2m}{c}\vec{v}\times(\nabla\times\vec{A}). \tag{25}$$

From equations (10) and (20) it follows that

$$\vec{F} = -m\left[-\nabla\left(\Phi + \frac{c^2}{2}\left(\frac{2\omega+3}{2\omega+4}\right)G\varepsilon\right) - \frac{2}{c}\frac{\partial\vec{A}}{\partial t}\right] - \frac{2m}{c}\vec{v}\times\vec{B}. \tag{26}$$

For the stationary case, we have $\dfrac{\partial\vec{A}}{\partial t} = 0$. Then, equation (26) reduces to

$$\vec{F} = -m\vec{E} - \frac{2m}{c}\vec{v}\times\vec{B} + \frac{mc^2}{2}\left(\frac{2\omega+3}{2\omega+4}\right)G\nabla\varepsilon, \tag{27}$$

where we utilize (9). It is interesting to mention that, in the approach to gravitoelectromagnetism in General Relativity, the equation of motion takes a Lorentz force law form, when one considers the stationary situation [3]. However, in Brans-Dicke theory context, this is not possible because of the scalar field term in (27).

4 Conclusion

It was found that, in the context of Brans-Dicke theory, is possible to write the equations of the gravitational field in a similar way to Maxwell's equations, by considering weak field approximation and low rotating velocity of the source. On the other hand, the equation of motion of a particle under the action of gravitoelectric and gravitomagnetic fields does not have a Lorentz force law form, even in the stationary case, because of a scalar field dependent term. In the limit $\omega \to \infty$, the field equations (11)-(14) as well as the gravitoelectromagnetic force (27) are reduced to the corresponding expressions in the General Relativity scenery.

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