Advanced Studies in Theoretical Physics
Vol. 10, 2016, no. 7, 311 - 318
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/astp.2016.639

Gravitomagnetism due to a Rotating Charged Body in Brans-Dicke Theory

A. Barros

Centro de Desenvolvimento Sustentável do Semiárido
Universidade Federal de Campina Grande, CEP 58540-000
Sumé, PB, Brazil

Abstract

We take into account a rotating charged body, obtaining the Kerr-Newman-type solution in Brans-Dicke theory, assuming the weak field approximation and considering a source with low rotating motion. The solution is exhibited by using the fact that one can establish a straightforward correspondence between weak fields solutions in General Relativity and Brans-Dicke theory. Then, we calculate the gravitomagnetic field and investigate some gravitomagnetic effects, showing the electric charge contribution in each case and comparing the results with those predicted by General Relativity.

PACS: 04.50.Kd, 04.25.Nx

Keywords: Gravitomagnetism, Brans-Dicke Theory, Weak field approximation

1 Introduction

In the weak field approximation of the General Relativity theory is possible to separate the gravitational effects in two parts: the first, connected with mass, and the second, linked with mass currents. Then, in a formal analogy with electrodynamics, we say that the rotation of a mass creates the gravitomagnetic field, while the rest mass only generates the gravitoelectric field. In this way, one defines the gravitomagnetic field in the context of the Kerr metric, which describes the curved spacetime geometry around a rotating mass [6]. The gravitomagnetism has been studied taking into account effects produced by the gravitational field of rotating astronomical sources [11, 4, 9]. In relation to the Earth, it produces a gravitomagnetic field that causes a precession in gyroscopes orbiting around the planet; this is the Lense-Thirring effect, which was verified by the GP-B experiment with an accuracy of 19% [7]. Also, gravitomagnetism have been investigated in the spacetime of a rotating, electrically charged body, which can represent a Kerr-Newman black hole [13, 3].

Additionally, it is known that there are alternative theories of gravity [15], such as the scalar-tensor theories, which are the simplest generalization of the General Relativity. In these theories, the gravitational effects are described by the spacetime metric $g_{\mu\nu}$ and also by a scalar field ϕ. They incorporate a coupling parameter ω of the scalar field with the geometry, which is a function $\omega = \omega(\phi)$. The Brans-Dicke theory corresponds to the case in that $\omega = constant$, being your value fixed from experimental observations [5]. The scalar-tensor theories are studied in several aspects [1, 16] and also because they admit key ingredients of string theories, such as a dilaton-like gravitational scalar field and its non-minimal coupling to the curvature [12].

In this work, we obtain the Kerr-Newman-type solution in Brans-Dicke theory, supposing the weak field approximation and considering a material source with low rotating motion. The metric is found by using the fact that one can establish a straightforward correspondence between weak field solutions in General Relativity and Brans-Dicke theory [2]. After, we calculate the gravitomagnetic field generated by the rotating charged body. Then, some gravitomagnetic effects are examined. In this way, the paper is organized as follows. We exhibit the Kerr-Newman solution in Section 2 and, in Section 3, the Kerr-Newman-type solution in Brans-Dicke theory is determined. The expression of the gravitomagnetic field is obtained in Section 4, being the effects of frame dragging and gravitomagnetic time delay studied in Section 5. Our conclusions are presented in Section 6.

2 The Kerr-Newman solution

In the framework of the General Relativity theory, the Kerr-Newman solution represents the gravitational field of a central mass M rotating with angular momentum $\overrightarrow{j}$ and electric charge Q. It can be written as [3]

$$\begin{aligned} \widetilde{ds}^2 &= -c^2\left(1-\frac{2GM}{c^2r}+\frac{GQ^2}{c^4r^2}\right)dt^2+\left(1+\frac{2GM}{c^2r}-\frac{GQ^2}{2c^4r^2}\right)\delta_{ij}dx^idx^j \\ &\quad -\frac{4}{c^2}\left(1-\frac{Q^2}{2c^2Mr}\right)(\overrightarrow{a}\cdot d\overrightarrow{r})\,(cdt)\,, \end{aligned} \tag{1}$$

where $\overrightarrow{a} = \frac{G}{cr^3}(\overrightarrow{j} \times \overrightarrow{r})$ and $\overrightarrow{j} = j\widehat{z}$, being G the Newton's gravitational constant and c the speed of light in vacuum. We use a Cartesian-like coordinate system $x^\alpha = (ct, \overrightarrow{r})$ with $\overrightarrow{r} = (x, y, z)$ and $\alpha = 0, 1, 2, 3$. The expression (1) is obtained considering the weak field approximation conditions $\frac{GM}{c^2 r} \ll 1$ and $\frac{GQ^2}{c^4 r^2} \ll 1$, besides the fact that the localized and slowly rotating source satisfies the relation $\frac{j}{cMr} \ll 1$.

It is useful to introduce the definitions

$$\Phi_1 = \frac{GM}{r} - \frac{GQ^2}{2c^2 r^2}, \tag{2}$$

$$\Phi_2 = \frac{GM}{r} - \frac{GQ^2}{4c^2 r^2}, \tag{3}$$

$$\overrightarrow{A} = \left(1 - \frac{Q^2}{2c^2 Mr}\right) \overrightarrow{a}. \tag{4}$$

Then, the metric (1) stays

$$\widetilde{ds}^2 = -c^2 \left(1 - \frac{2\Phi_1}{c^2}\right) dt^2 + \left(1 + \frac{2\Phi_2}{c^2}\right) \delta_{ij} dx^i dx^j - \frac{4}{c} \left(\overrightarrow{A} \cdot d\overrightarrow{r}\right) dt. \tag{5}$$

3 The Kerr-Newman-type solution in Brans-Dicke theory

The field equations of the Brans-Dicke theory are [5]:

$$G_{\mu\nu} = \frac{8\pi}{\phi c^4} T_{\mu\nu} + \frac{\omega}{\phi^2} \left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right) + \frac{1}{\phi} \left(\phi_{;\mu;\nu} - g_{\mu\nu}\Box\phi\right), \tag{6}$$

$$\Box\phi = \frac{8\pi T}{(2\omega + 3)\, c^4}, \tag{7}$$

where $G_{\mu\nu}$ is the Einstein tensor and $\Box\phi = \phi^{;\sigma}{}_{;\sigma} = g^{\sigma\gamma}\phi_{;\gamma;\sigma}$. The energy-momentum tensor associated with the material content is $T_{\mu\nu}$ and $T = T^\mu{}_\mu$.

Now, considering the weak field regime of the Brans-Dicke theory, we admit that the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{8}$$

being $\eta_{\mu\nu}$ the Minkowski metric and $h_{\mu\nu}$ a small perturbation, such that only first-order terms in $h_{\mu\nu}$ are conserved. Furthermore, the scalar field is

$$\phi = \phi_0 + \varepsilon = \phi_0 \left(1 + \frac{\varepsilon}{\phi_0}\right), \tag{9}$$

where ϕ_0 is a constant and ε a first-order term in the density of matter, so that $|\varepsilon/\phi_0| \ll 1$. Thus, we keep only the terms of first order in ε/ϕ_0. Then, equation (7) yields

$$\Box\varepsilon = \frac{8\pi T}{(2\omega+3)\,c^4}. \tag{10}$$

In the weak field approximation context, the solutions of the Brans-Dicke equations are related to the solutions of the General Relativity equations with the same $T_{\mu\nu}$ [2]. Indeed, if the metric $\widetilde{g}_{\mu\nu}(G, x^\alpha)$ is a known solution of Einstein's equations for a given $T_{\mu\nu}$, then the Brans-Dicke solution, corresponding to the same $T_{\mu\nu}$, is given by

$$g_{\mu\nu}(x^\alpha) = [1 - G_0\varepsilon(x^\alpha)]\,\widetilde{g}_{\mu\nu}(G_0, x^\alpha), \tag{11}$$

where $G_0 = 1/\phi_0 = \left(\frac{2\omega+3}{2\omega+4}\right) G$.

Therefore, according to (11), the Kerr-Newman-type solution in Brans-Dicke theory must be

$$ds^2 = [1 - G_0\varepsilon(x^\alpha)]\,[\widetilde{ds}(G_0, x^\alpha)]^2. \tag{12}$$

The term $[\widetilde{ds}(G_0, x^\alpha)]^2$ represents the equation (5), but with the replacement of G by G_0. Hence, we have that

$$\begin{aligned} ds^2 = [1 - G_0\varepsilon(x^\alpha)] &\left[-c^2\left(1 - \frac{2\Phi_1(G_0)}{c^2}\right) dt^2 - \frac{4}{c}\left(\overrightarrow{A}(G_0)\cdot d\overrightarrow{r}\right) dt \right. \\ &\left. + \left(1 + \frac{2\Phi_2(G_0)}{c^2}\right)\delta_{ij}dx^i dx^j\right], \end{aligned} \tag{13}$$

or yet,

$$\begin{aligned} ds^2 &= -c^2\left[1 - \frac{2\Phi_1(G_0)}{c^2} - G_0\varepsilon\right] dt^2 - \frac{4}{c}\left(\vec{A}(G_0)\cdot d\overrightarrow{r}\right) dt \\ &\quad + \left[1 + \frac{2\Phi_2(G_0)}{c^2} - G_0\varepsilon\right]\delta_{ij}dx^i dx^j. \end{aligned} \tag{14}$$

Finally, with the definitions

$$2\frac{\Lambda}{c^2} = \frac{2\Phi_1(G_0)}{c^2} + G_0\varepsilon, \tag{15}$$

$$2\frac{\Psi}{c^2} = \frac{2\Phi_2(G_0)}{c^2} - G_0\varepsilon, \tag{16}$$

the Kerr-Newman-type metric can be expressed as

$$ds^2 = -c^2\left(1 - 2\frac{\Lambda}{c^2}\right) dt^2 - \frac{4}{c}\left(\vec{A}(G_0)\cdot d\overrightarrow{r}\right) dt + \left(1 + 2\frac{\Psi}{c^2}\right)\delta_{ij}dx^i dx^j. \tag{17}$$

4 Gravitomagnetic field

Let us consider the motion of a test particle of mass m in the spacetime given by (17). The Lagrangian $L = -mc\frac{ds}{dt}$ can be calculated, keeping first-order terms in Λ, Ψ and $\vec{A}(G_0)$. Then, we obtain

$$L = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\gamma\Lambda + m\gamma\frac{v^2}{c^2}\Psi - \frac{2m}{c}\gamma\vec{A}(G_0)\cdot\vec{v}, \tag{18}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and v is the particle velocity. Also, in the weak gravitational field context, we assume that the material particle has a small velocity [8] and only terms until second-order in $\frac{v}{c}$ are maintained. Thereby, taking into account all approximations, one gets

$$L = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\Lambda - \frac{2m}{c}\vec{A}(G_0)\cdot\vec{v}, \tag{19}$$

which is analogous to the electromagnetic Lagrangian [10]. Hence, being the equation of motion $\frac{d\vec{p}}{dt} = \vec{F}$, with the linear momentum $\vec{p} = \gamma m\vec{v}$, we will have

$$\vec{F} = -m\left(-\nabla\Lambda\right) - \frac{2m}{c}\vec{v}\times(\nabla\times\vec{A}(G_0)). \tag{20}$$

Using equation (15), it follows that

$$\vec{F} = -m\left[-\nabla\left(\Phi_1(G_0) + \frac{c^2}{2}G_0\varepsilon\right)\right] - \frac{2m}{c}\vec{v}\times(\nabla\times\vec{A}(G_0)). \tag{21}$$

Now, if we define [14]: $\overrightarrow{E} = -\nabla\Phi_1(G_0)$ and $\overrightarrow{B} = \nabla\times\vec{A}(G_0)$, equation (21) becomes

$$\vec{F} = -m\vec{E} - \frac{2m}{c}\vec{v}\times\vec{B} + \frac{mc^2}{2}G_0\nabla\varepsilon. \tag{22}$$

Therefore, we see that the equation of motion does not take a Lorentz force law form because of the scalar field term.

The gravitoelectric field $\overrightarrow{E}$ and the gravitomagnetic field $\overrightarrow{B}$ can be calculated from (2) and (4), with the exchange G by G_0. The results are

$$\overrightarrow{E} = \left(\frac{2\omega+3}{2\omega+4}\right)\left(\frac{GM}{r^2} - \frac{GQ^2}{c^2r^3}\right)\widehat{r} \tag{23}$$

and

$$\overrightarrow{B} = \overrightarrow{b} - \frac{Q^2}{2c^2Mr}\left[\left(\frac{2\omega+3}{2\omega+4}\right)\frac{G}{c}\left(\frac{4\widehat{r}(\widehat{r}\cdot\overrightarrow{j}) - 2\overrightarrow{j}}{r^3}\right)\right], \tag{24}$$

where

$$\overrightarrow{b} = \nabla \times \overrightarrow{a}(G_0) = \left(\frac{2\omega+3}{2\omega+4}\right)\frac{G}{cr^3}\left[3\widehat{r}(\widehat{r}\cdot\overrightarrow{j}) - \overrightarrow{j}\right] \tag{25}$$

is the gravitomagnetic field of a rotating non-charged mass. If $\omega \to \infty$, the fields given by equations (23) and (24) are reduced to the corresponding expressions in General Relativity [3]. The factor $\frac{2\omega+3}{2\omega+4}$ represents the contribution of the Brans-Dicke scalar field since if ω is finite in (10) then $\varepsilon \neq 0$.

5 Gravitomagnetic Effects

A first effect caused by the gravitomagnetic field is the dragging of inertial frames around a rotating material source. In this case, the angular velocity of precession of gyroscopes relative to distant stars will be given by [6]

$$\overrightarrow{\Omega} = \frac{\overrightarrow{B}}{c} = \overrightarrow{\beta} - \frac{Q^2}{2c^2Mr}\left[\left(\frac{2\omega+3}{2\omega+4}\right)\frac{G}{c^2}\left(\frac{4\widehat{r}(\widehat{r}\cdot\overrightarrow{j}) - 2\overrightarrow{j}}{r^3}\right)\right], \tag{26}$$

with $\overrightarrow{\beta} = \dfrac{\overrightarrow{b}}{c}$. The expression obtained shows the electric charge contribution to the gravitational effect of frame dragging and also how $\overrightarrow{\Omega}$ depends on the Brans-Dicke factor $\frac{2\omega+3}{2\omega+4}$.

On the other hand, the gravitomagnetic time delay, when a ray of light propagates from a point $P_1 : (ct_1, \vec{r}_1)$ to a point $P_2 : (ct_2, \vec{r}_2)$ in the spacetime given by (8) and (17), is defined by [3]

$$\Delta_B = \frac{1}{c}\int_{P_1}^{P_2} h_{0i}k^0k^i dl, \tag{27}$$

where $k^\mu = (1, \hat{k})$, $\hat{k}$ is the constant unit propagation vector of the signal and $dl =\mid d\vec{r} \mid= (\delta_{ij}dx^i dx^j)^{1/2}$ designates the Euclidean length element along the straight line that joins P_1 to P_2. Thus, with $\widehat{k}dl = d\overrightarrow{r}$, we have that

$$\Delta_B = \frac{1}{c}\int_{P_1}^{P_2}\left[-\frac{2}{c^2}\left(1 - \frac{Q^2}{2c^2Mr}\right)\overrightarrow{a}(G_0)\right]\cdot d\overrightarrow{r}, \tag{28}$$

which can also be written as

$$\Delta_B = -\frac{2}{c^3}\left(\frac{2\omega+3}{2\omega+4}\right)\int_{P_1}^{P_2}\overrightarrow{a}\cdot d\overrightarrow{r} + \frac{Q^2}{c^5M}\left(\frac{2\omega+3}{2\omega+4}\right)\int_{P_1}^{P_2}\frac{\overrightarrow{a}}{r}\cdot d\overrightarrow{r}. \tag{29}$$

The result shows how the gravitomagnetic time delay depends on the electric charge. In turn, when (26) and (29) are considered, we recovered in the limit $\omega \to \infty$ the corresponding equations in the General Relativity theory [3].

6 Conclusion

We obtain the Kerr-Newman-type solution in Brans-Dicke theory. Then, adopting the framework of the gravitomagnetism, the gravitoelectric and the gravitomagnetic fields were calculated. Also, we show that the equation of motion does not take a Lorentz force law form due to the presence of a term containing the scalar field ε.

In sequence, gravitomagnetic effects as frame dragging and gravitomagnetic time delay were approached. The dependence of these effects in relation to the electric charge and the Brans-Dicke factor $\frac{2\omega+3}{2\omega+4}$ was exposed; in the limit $\omega \to \infty$, the corresponding results in General Relativity are retrieved.

References

[1] M. Artymowski, Z. Lalak and M. Lewicki, Inflation and Dark Energy from the Brans-Dicke Theory, *Journal of Cosmology and Astroparticle Physics*, **2015** (2015), 031. http://dx.doi.org/10.1088/1475-7516/2015/06/031

[2] A. Barros and C. Romero, On the Weak Field Approximation of Brans-Dicke Theory of Gravity, *Phys. Lett. A*, **245** (1998), 31-34. http://dx.doi.org/10.1016/S0375-9601(98)00382-X

[3] A. Barros, Gravitomagnetic Effects in the Kerr-Newman Spacetime, *Advan. Stud. Theor. Phys.*, **10** (2016), 81-87. http://dx.doi.org/10.12988/astp.2016.512114

[4] D. Bini, L. Iorio and D. Giordano, Orbital effects due to gravitational induction, *Gen. Rel. Grav.*, **47** (2015), 130. http://dx.doi.org/10.1007/s10714-015-1977-2

[5] C. Brans and R. H. Dicke, Mach's Principle and a Relativistic Theory of Gravitation, *Phys. Rev.*, **124** (1961), 925-935. http://dx.doi.org/10.1103/PhysRev.124.925

[6] I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia*, Princeton University Press, Princeton, 1995.

[7] C. W. F. Everitt, D. B. DeBra, B. W. Parkinson et al., Gravity Probe B: final results of a space experiment to test General Relativity, *Phys. Rev. Lett.*, **106** (2011), 221101. http://dx.doi.org/10.1103/PhysRevLett.106.221101

[8] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Elsevier Butterworth-Heinemann, Oxford, 2005.

[9] P. Landry and E. Poisson, Gravitomagnetic response of an irrotational body to an applied tidal field, *Phys. Rev. D*, **91** (2015), 104026. http://dx.doi.org/10.1103/PhysRevD.91.104026

[10] J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1999.

[11] B. Mashhoon, Time-varying gravitomagnetism, *Class. Quantum Grav.*, **25** (2008), 085014. http://dx.doi.org/10.1088/0264-9381/25/8/085014

[12] J. R. Morris, Dilatonic Effects on a Falling Test Mass in Scalar-Tensor Theory, *Gen. Rel. Grav.*, **43** (2011), 2821-2832. http://dx.doi.org/10.1007/s10714-011-1204-8

[13] P. Pani, E. Berti and L. Gualtieri, Gravitoelectromagnetic Perturbations of Kerr-Newman Black Holes: Stability and Isospectrality in the Slow-Rotation Limit, *Phys. Rev. Lett.*, **110** (2013), 241103. http://dx.doi.org/10.1103/physrevlett.110.241103

[14] J. R. Silva and A. Barros, The Gravitoelectromagnetic Analogy in the Context of Brans-Dicke Theory, *Advanced Studies in Theoretical Physics*, **9** (2015), 787-793. http://dx.doi.org/10.12988/astp.2015.5983

[15] C. Will, The Confrontation between General Relativity and Experiment, *Living Rev. Rel.*, **17** (2014), 4. http://dx.doi.org/10.12942/lrr-2014-4

[16] X. Zhang and Y. Ma, Nonperturbative Loop Quantization of Scalar-Tensor Theories of Gravity, *Phys. Rev. D*, **84** (2011), 104045. http://dx.doi.org/10.1103/PhysRevD.84.104045

Received: March 21, 2016; Published: July 3, 2016