Gravitomagnetic Monopole in a Scalar-Tensor Theory of Gravity

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Abstract

We consider the Brans-Dicke scalar-tensor theory, supposing a weak gravitational field generated by a material source with low rotating motion and nonzero magnetic monopole. In this context, we determine a Kerr-Taub-NUT-type solution and the gravitomagnetic field. After, the effects of frame dragging and gravitomagnetic time delay are explored and the results obtained are compared with those predicted by General Relativity.

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1   Introduction

When we take into account the formalism of the gravitomagnetism [8], the weak field approximation of the General Relativity is admitted and, in a formal analogy with electrodynamics, one shows that the rotation of a mass creates the gravitomagnetic field, while the rest mass only generates the gravitoelectric field. However, other factors, as for example a magnetic monopole charge, also produce gravitational effects. In fact, Newman, Tamburino and Unti obtained

a solution of the coupled Einstein-Maxwell equations that considers the magnetic monopole charge of the source [14, 11]. In a more general context, the Kerr-Taub-NUT metric describes the solution of the Einstein-Maxwell equations for a rotating body with nonzero magnetic monopole associated with the NUT parameter ℓ [9, 12]. On the other hand, the Kerr-Taub-NUT solution can be interpreted as possessing a source with mass, gravitomagnetic dipole moment (angular momentum) and gravitomagnetic monopole moment [4]. Currently, there is no evidence for the existence of magnetic monopole as well as of gravitomagnetic monopole, but the Kerr-Taub-NUT spacetime has been investigated in various aspects [7, 17].

The scalar-tensor theories of gravity are the simplest generalization of the General Relativity [18]. These theories are popular, among other reasons [1, 19, 15], because incorporate some ingredients of string theories, such as a dilaton-like gravitational scalar field and its non-minimal coupling to the curvature [13]. In the scalar-tensor theories context, the gravitational effects are described by two fields: the spacetime metric $g_{\mu\nu}$ and the scalar field ϕ. A coupling parameter $\omega = \omega(\phi)$ of the scalar field with the geometry is introduced, being your value fixed from experimental observations; the case in that $\omega = constant$ corresponds to the Brans-Dicke theory [6].

In this paper, we will consider the Brans-Dicke scalar-tensor theory, supposing a weak gravitational field generated by a material source with low rotating motion and nonzero magnetic monopole. Using the fact that one can establish a straightforward correspondence between weak field solutions in General Relativity and Brans-Dicke theory [2], we will determine a Kerr-Taub-NUT-type solution and the gravitomagnetic field. After, some gravitomagnetic effects will be explored. Therefore, in Section 2, we present some results about the Brans-Dicke weak field and, in Section 3, the Kerr-Taub-NUT-type solution is determined. The expression of the gravitomagnetic field is obtained in Section 4, being the effects of frame dragging and gravitomagnetic time delay studied in Section 5. Finally, Section 6 is devoted to our conclusions.

2 The weak field approximation in Brans-Dicke Theory

The field equations of the Brans-Dicke theory are [6]:

$$G_{\mu\nu} = \frac{8\pi}{\phi c^4} T_{\mu\nu} + \frac{\omega}{\phi^2} \left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha} \right) + \frac{1}{\phi} \left(\phi_{;\mu;\nu} - g_{\mu\nu}\Box\phi \right), \quad (1)$$

$$\Box\phi = \frac{8\pi T}{(2\omega + 3)c^4}, \quad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor and $\Box\phi = \phi^{;\sigma}{}_{;\sigma} = g^{\sigma\gamma}\phi_{;\gamma;\sigma}$. The energy-momentum tensor associated with the material content is $T_{\mu\nu}$ and $T = T^{\mu}{}_{\mu}$.

On the other hand, in the weak field approximation, we consider that the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3}$$

being $\eta_{\mu\nu}$ the Minkowski metric and $h_{\mu\nu}$ a small perturbation, such that only first-order terms in $h_{\mu\nu}$ are conserved. Moreover, the scalar field is

$$\phi = \phi_0 + \varepsilon = \phi_0\left(1 + \frac{\varepsilon}{\phi_0}\right), \tag{4}$$

where ϕ_0 is a constant and ε a first-order term in the density of matter, so that $|\varepsilon/\phi_0| \ll 1$. Thus, we keep only the terms of first order in ε/ϕ_0. Then, equation (2) stays

$$\Box\varepsilon = \frac{8\pi T}{(2\omega + 3)\, c^4}. \tag{5}$$

In this approximation, the solutions of the Brans-Dicke equations are related to the solutions of the General Relativity equations with the same $T_{\mu\nu}$. Really, if the metric $\widetilde{g}_{\mu\nu}(G, x^\alpha)$ is a known solution of Einstein's equations for a given $T_{\mu\nu}$, then the Brans-Dicke solution, corresponding to the same $T_{\mu\nu}$, is given by [2]

$$g_{\mu\nu}(x^\alpha) = [1 - G_0\varepsilon(x^\alpha)]\,\widetilde{g}_{\mu\nu}(G_0, x^\alpha), \tag{6}$$

where $G_0 = 1/\phi_0 = \left(\frac{2\omega+3}{2\omega+4}\right)G$. Therefore, the line element takes the form

$$ds^2 = [1 - G_0\varepsilon(x^\alpha)]\,\widetilde{ds}^2(G_0, x^\alpha). \tag{7}$$

3   The Kerr-Taub-NUT-type solution

The Kerr-Taub-NUT metric represents the solution of Einstein's field equations for a mass M rotating with angular momentum $\overrightarrow{j}$ and possessing non zero magnetic monopole, corresponding to the gravitomagnetic monopole moment $\mu = -\ell$, where ℓ is the NUT parameter. It can be written as [5]

$$\begin{aligned}
\widetilde{ds}^2 &= -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)\delta_{ij}dx^i dx^j \\
&\quad - \frac{4}{c^2}\left[\frac{G}{cr^3}(\overrightarrow{j}\times\overrightarrow{r}) - \frac{\mu c^2 z}{r\,(x^2+y^2)}\,(\widehat{z}\times\overrightarrow{r})\right]\cdot d\overrightarrow{r}\,(cdt),
\end{aligned} \tag{8}$$

where $\overrightarrow{j} = j\widehat{z}$, being G the Newtonian gravitational constant and c the speed of light in vacuum. We use a Cartesian-like coordinate system $x^\alpha = (ct, \overrightarrow{r})$

with $\overrightarrow{r} = (x, y, z)$ and $\alpha = 0, 1, 2, 3$. The expression (8) is obtained considering the weak field approximation conditions $\frac{GM}{c^2 r} \ll 1$ and $\frac{\mu}{r} \ll 1$, and also the fact that the localized and slowly rotating source satisfies the relation $\frac{j}{cMr} \ll 1$.

If we define

$$\Phi = \frac{GM}{r}, \tag{9}$$

$$\overrightarrow{A} = \frac{G}{cr^3}(\overrightarrow{j} \times \overrightarrow{r}) - \frac{\mu c^2 z}{r(x^2 + y^2)}(\widehat{z} \times \overrightarrow{r}), \tag{10}$$

the metric (8) reduces to

$$\widetilde{ds}^2 = -c^2\left(1 - \frac{2\Phi}{c^2}\right)dt^2 + \left(1 + \frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j - \frac{4}{c^2}\left(\overrightarrow{A} \cdot d\overrightarrow{r}\right)cdt. \tag{11}$$

Now, using equations (7) and (11), we find the Kerr-Taub-NUT-type solution in Brans-Dicke theory:

$$ds^2 = [1 - G_0 \varepsilon(x^\alpha)]\left[-c^2\left(1 - \frac{2\Phi(G_0)}{c^2}\right)dt^2 - \frac{4}{c^2}\left(\overrightarrow{A}(G_0) \cdot d\overrightarrow{r}\right)cdt \right.$$
$$\left. + \left(1 + \frac{2\Phi(G_0)}{c^2}\right)\delta_{ij}dx^i dx^j\right], \tag{12}$$

or yet,

$$ds^2 = -c^2\left[1 - \frac{2\Phi(G_0)}{c^2} - G_0 \varepsilon\right]dt^2 - \frac{4}{c^2}\left(\overrightarrow{A}(G_0) \cdot d\overrightarrow{r}\right)cdt$$
$$+ \left[1 + \frac{2\Phi(G_0)}{c^2} - G_0 \varepsilon\right]\delta_{ij}dx^i dx^j. \tag{13}$$

For the sake of simplicity, one has

$$2\frac{\Lambda}{c^2} = \frac{2\Phi(G_0)}{c^2} + G_0 \varepsilon, \tag{14}$$

$$2\frac{\Psi}{c^2} = \frac{2\Phi(G_0)}{c^2} - G_0 \varepsilon, \tag{15}$$

and the metric can be expressed as

$$ds^2 = -c^2\left(1 - 2\frac{\Lambda}{c^2}\right)dt^2 - \frac{4}{c^2}\left(\overrightarrow{A}(G_0) \cdot d\overrightarrow{r}\right)cdt + \left(1 + 2\frac{\Psi}{c^2}\right)\delta_{ij}dx^i dx^j. \tag{16}$$

4   Gravitomagnetic field

Let us consider the motion of a test particle of mass m in the spacetime given by (16). As is well known, the Lagrangian is given by $L = -mc\dfrac{ds}{dt}$. Then, we conclude that

$$L = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\gamma\Lambda + m\gamma\frac{v^2}{c^2}\Psi - \frac{2m}{c}\gamma\overrightarrow{A}(G_0)\cdot\vec{v}, \qquad (17)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, v is the particle velocity and only first-order terms in Λ, Ψ and $\overrightarrow{A}(G_0)$ were kept. Besides, in the weak gravitational field context, we assume that the material particle has a small velocity and only terms until second-order in $\dfrac{v}{c}$ are preserved. Thus, taking into account all approximations, we obtain

$$L = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\Lambda - \frac{2m}{c}\overrightarrow{A}(G_0)\cdot\vec{v}, \qquad (18)$$

which is analogous to the electromagnetic Lagrangian [10]. Hence, noting that the equation of motion is $\dfrac{d\vec{p}}{dt} = \vec{F}$, with the linear momentum $\vec{p} = \gamma m\vec{v}$, one has

$$\vec{F} = -m\left(-\nabla\Lambda\right) - \frac{2m}{c}\vec{v}\times\left[\nabla\times\overrightarrow{A}(G_0)\right]. \qquad (19)$$

Using equation (14), it follows that

$$\vec{F} = -m\left[-\nabla\left(\Phi(G_0) + \frac{c^2}{2}G_0\varepsilon\right)\right] - \frac{2m}{c}\vec{v}\times\left[\nabla\times\overrightarrow{A}(G_0)\right]. \qquad (20)$$

The gravitoelectric field $\overrightarrow{E} = -\nabla\Phi(G_0)$ and the gravitomagnetic field $\overrightarrow{B} = \nabla\times\vec{A}(G_0)$ can be introduced [16]. In this way, equation (20) becomes

$$\vec{F} = -m\vec{E} - \frac{2m}{c}\vec{v}\times\vec{B} + \frac{mc^2}{2}G_0\nabla\varepsilon. \qquad (21)$$

Thus, the equation of motion does not take a Lorentz force law form due to the scalar field term.

The fields $\overrightarrow{E}$ and $\overrightarrow{B}$ can be calculated from (9) and (10), with the exchange G by G_0. They are given by

$$\overrightarrow{E} = \left(\frac{2\omega + 3}{2\omega + 4}\right)\left(\frac{GM}{r^2}\right)\hat{r} \qquad (22)$$

and

$$\overrightarrow{B} = \overrightarrow{b} + \frac{\mu c^2 \overrightarrow{r}}{r^3}, \qquad (23)$$

where

$$\overrightarrow{b} = \left(\frac{2\omega + 3}{2\omega + 4} \right) \frac{G}{cr^3} \left[3\widehat{r}(\widehat{r} \cdot \overrightarrow{j}) - \overrightarrow{j} \right]. \tag{24}$$

It is interesting to note that when $\omega \to \infty$ the fields in the equations (22) and (23) are reduced to the corresponding expressions in General Relativity [5]. The factor $\frac{2\omega+3}{2\omega+4}$ represents the contribution of the Brans-Dicke scalar field since if ω is finite in (5) then $\varepsilon = \varepsilon(x^\alpha)$. The vector $\overrightarrow{b}$ in the expression (23) is the gravitomagnetic field in an ordinary approach [3], while the term with μ is the gravitomagnetic monopole contribution, which is identical to the General Relativity case.

5   Frame dragging and gravitomagnetic time delay

An effect caused by the gravitomagnetic field is the dragging of inertial frames around a rotating material source. As a consequence, the angular velocity of precession of gyroscopes relative to distant stars will be given by [8]

$$\overrightarrow{\Omega} = \frac{\overrightarrow{B}}{c} = \frac{\overrightarrow{b}}{c} + \frac{\mu c \overrightarrow{r}}{r^3}. \tag{25}$$

The expression above exhibits the gravitomagnetic monopole contribution to the gravitational effect of frame dragging. This contribution is similar to the General Relativity case.

When a ray of light propagates from a point $P_1 : (ct_1, \vec{r}_1)$ to a point $P_2 : (ct_2, \vec{r}_2)$ in the spacetime given by (3) and (16), the gravitomagnetic time delay is defined by [3]

$$\Delta_B = -\frac{2}{c^3} \int_{P_1}^{P_2} \overrightarrow{A}(G_0) \cdot d\overrightarrow{r}, \tag{26}$$

where $| d\vec{r} | = (\delta_{ij}dx^i dx^j)^{1/2}$ designates the Euclidean length element along the straight line that joins P_1 to P_2. Then, using (10) with $G \to G_0$, we will have

$$\begin{aligned}
\Delta_B &= -\frac{2}{c^3} \left(\frac{2\omega+3}{2\omega+4} \right) \int_{P_1}^{P_2} \left[\frac{G}{cr^3} (\overrightarrow{j} \times \overrightarrow{r}) \right] \cdot d\overrightarrow{r} \\
&\quad + \frac{2\mu}{c} \int_{P_1}^{P_2} \left[\frac{z}{r(x^2 + y^2)} (\widehat{z} \times \overrightarrow{r}) \right] \cdot d\overrightarrow{r}.
\end{aligned} \tag{27}$$

The result indicates how the gravitomagnetic time delay depends on the gravitomagnetic monopole. On the other hand, if $\mu = 0$ and we take the limit $\omega \to \infty$ the corresponding expression in the General Relativity theory is obtained [3].

6 Conclusion

We obtain the Kerr-Taub-NUT-type solution in the scalar-tensor theory of Brans-Dicke, considering the weak field approximation. In sequence, the gravitoelectric and the gravitomagnetic fields were introduced and we show that the equation of motion does not take a Lorentz force law form due to the presence of a term containing the scalar field ε. In the expression of the gravitomagnetic field, we note that the term related with the gravitomagnetic monopole is identical to the General Relativity case.

We approach gravitomagnetic effects as frame dragging and gravitomagnetic time delay, exhibiting the dependence of these effects in relation to the gravitomagnetic monopole and the Brans-Dicke factor $\frac{2\omega+3}{2\omega+4}$. Finally, the corresponding expressions in General Relativity were recovered in the limit $\omega \to \infty$.

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