A brief note on the limit $\omega \to \infty$ in Weyl geometrical scalar-tensor theory

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Abstract

We obtain vacuum solutions in the presence of a cosmological constant in the context of the Weyl geometrical scalar-tensor theory. We investigate the limit when ω goes to infinity and show by working out the solutions that in this limit there are some cases in which the scalar field tends to a constant (with the implicit consequence of the geometry becoming Riemannian), although the solutions do not reduce to the corresponding Einstein solutions. We have also extended a previous result, known in the literature, by showing that in the case of vacuum with cosmological constant the field equations of the Weyl geometrical scalar-tensor theory are formally identical to Brans-Dicke field equations, even though these theories are not physically equivalent.

Keywords: Weyl geometry; Scalar-tensor theory.

I. INTRODUCTION

Among the so-called modified gravity theories one that stands as the most simple and popular is that proposed by C. Brans and R. Dicke in 1961[1]. As is well known, the main motivation of the authors was to incorporate Mach's principle into a relativistic theory of gravity. With this purpose in mind they managed to formulate a scalar-tensor theory, in which the gravitational effects would be described both by a metric field $g_{\mu\nu}$ and a scalar field Φ, with the geometry of the underlying space-time manifold being assumed to be Riemannian. The scalar field, which is neither a geometrical nor a matter-related field, replaces the gravitational constant and is interpreted as the inverse of the gravitational coupling parameter. To date Brans-Dicke theory which is generally considered to be in agreement with observation, has been regarded with interest by many theoreticians[2].

A different approach to a scalar-tensor theory of gravity consists in considering the scalar field Φ to be of a geometrical nature, which would be more in accordance with the geometrization program of physics that Einstein started with general relativity. In this sense, the scalar field which appears in the so-called Weyl geometrical scalar-tensor theory is regarded as an essential element of the space-time geometry[3]. Indeed, in this theory space-time is assumed to be a very special case of Weyl (non-integral) space-time[4]. This has been referred to in the literature as a *Weyl integrable space-time* (WIST)[5]. Other gravity theories in which a scalar field plays a geometrical role have been proposed[6].

Due to some similarities between the Weyl geometrical scalar-tensor theory and the Brans-Dicke theory, it is interesting to see what happens when ω, the scalar field coupling constant, goes to infinity. Concerning this point, let us firstly recall that for quite a long time it was (erroneously) held that Brans-Dicke theory would reduce to general relativity when $\omega \to \infty$[7]. The question now seems to be settled after the publication of a series of articles on the subject[8]. In fact, it has been shown that under certain circumstances Brans-Dicke theory does not go over general relativity when $\omega \to \infty$. This result is not entirely inconsequential as it may affect the way we set limits on the values of ω from solar system experiments or even from cosmological observation. In this paper, we examine the same kind of asymptotic behaviour, however now in the context of another scalar-tensor theory which also contains an adimensional parameter, namely, Weyl geometrical scalar-tensor theory.

Inspired by previous work[9], our approach to this problem will consist in initially obtaining some solutions of the Weyl geometrical scalar-tensor theory, considering the vacuum field equations in the presence of a cosmological term Λ. With the help of these solutions, we shall study the limit $\omega \to \infty$, investigating the possibility of obtaining the general relativity equations.

Finally, a relevant question is concerned with the physical motivation of the present work. As a matter of fact, the problem we have examined in the present paper is of purely theoretical nature: We investigate a mathematical feature of a class of scalar-tensor theories which have a free parameter ω. The behaviour of different solutions when ω goes over infinity may have consequences in the interpretation of the theory. For instance, it was long believed that the original Brans-Dicke theory would approach general relativity for large ω, in which case the theory would completely lose its interest since on the basis of the Occam's rasor principle the simplest theory is always preferable. Now it is well known that the limit of Brans-Dicke solutions to general relativistic solutions corresponding to the same momentum-energy tensor is not unique, or may not yield a solution of Einstein's equations at all, in which case they must not simply be discarded as devoid of interest even for large ω. For additional motivation concerning this question, which apply both to the original Brans-Dicke theory and the geometrical scalar-tensor theory, we would like to refer the reader to Ref. 10.

The paper is organized as follows. In Section 2, we show that if one considers the vacuum field equations with cosmological constant, the field equations of the Weyl geometrical scalar-tensor theory and Brans-Dicke theory are formally identical. Based on this result, we exhibit solutions of the Weyl geometrical scalar-tensor theory in Section 3. We proceed in Section 4 to examine the limit $\omega \to \infty$ for some cases. Finally, Section 5 is devoted to our conclusions.

II. THE WEYL GEOMETRICAL SCALAR-TENSOR THEORY

The field equations of the Weyl geometrical scalar-tensor theory in its most general form are given by[11]

$$\bar{G}_{\mu\nu} = \omega(\phi)\left(\frac{\phi_{,\alpha}\phi^{,\alpha}}{2}g_{\mu\nu} - \phi_{,\mu}\phi_{,\nu}\right) - \frac{1}{2}e^{\phi}g_{\mu\nu}V(\phi) - 8\pi T_{\mu\nu} \tag{1}$$

$$\bar{\Box}\phi = -\left(1 + \frac{1}{2\omega}\frac{d\omega}{d\phi}\right)\phi_{,\mu}\phi^{,\mu} - \frac{e^{\phi}}{\omega}\left(\frac{1}{2}\frac{dV}{d\phi} + V\right), \tag{2}$$

where ϕ is a scalar field, ω is a function of ϕ, $V(\phi)$ corresponds to the scalar field potential and $T_{\mu\nu}$ represents the Weyl invariant energy-momentum tensor of the matter fields.[3] Moreover, we are denoting by $\bar{G}_{\mu\nu}$ and $\bar{\Box}$ the Einstein tensor and the d'Alembertian operator, respectively, as defined with respect to the Weyl connection, whose coefficients in a local coordinate basis read

$$\Gamma^{\alpha}_{\mu\nu} = \{^{\alpha}_{\mu\nu}\} - \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu}\phi_{,\nu} + g_{\beta\nu}\phi_{,\mu} - g_{\mu\nu}\phi_{,\beta}), \tag{3}$$

with $\{^{\alpha}_{\mu\nu}\}$ representing the usual Christoffel symbols. At this point, we should mention that the scalar field ϕ is regarded as a purely geometrical field. Indeed, ϕ is a basic ingredient essential of the Weyl nonmetricity condition

$$\nabla_{\alpha}g_{\mu\nu} = g_{\mu\nu}\phi_{,\alpha} \tag{4}$$

which in this form characterizes the space-time manifold as a Weyl integrable space-time[5].

Let us now restrict ourselves to the particular case when $\omega(\phi) = \omega = const.$ The field equations written above then becomes

$$\begin{aligned} G_{\mu\nu} = & -\frac{(\omega - \frac{3}{2})}{\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{g_{\mu\nu}}{2}\Phi_{,\alpha}\Phi^{,\alpha}\right) \\ & - \frac{1}{\Phi}(\Phi_{,\mu;\nu} - g_{\mu\nu}\Box\Phi\ ) - \frac{g_{\mu\nu}}{2\Phi}V(\Phi) - 8\pi T_{\mu\nu}, \end{aligned} \tag{5}$$

$$\Box\Phi\ = \frac{1}{\omega}\left(-\frac{1}{2}\frac{dV}{d\Phi}\Phi + V(\Phi)\right), \tag{6}$$

where we are using the field variable $\Phi = e^{-\phi}$. Note also that we are expressing the Weylian geometric quantities $\bar{G}_{\mu\nu}$ and $\bar{\Box}\phi$ in terms of their Riemannian counterparts, the latter being denoted by $G_{\mu\nu}$ and $\Box\phi$ calculated from the metric $g_{\mu\nu}$ and the Christoffel symbols $\{^{\alpha}_{\mu\nu}\}$. If we take $V(\Phi) = 2\Lambda\Phi$, which is equivalent to introduce the cosmological constant Λ, then the vacuum field equations can be written as

$$\begin{aligned} G_{\mu\nu} = & -\frac{W}{\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right) \\ & - \frac{1}{\Phi}(\Phi_{,\mu;\nu} - g_{\mu\nu}\Box\Phi\ ) - \Lambda g_{\mu\nu}, \end{aligned} \tag{7}$$

$$\Box\Phi\ = \frac{2\Lambda\Phi}{2W+3}, \tag{8}$$

where $W = \omega - \frac{3}{2}$. On the other hand, the Brans-Dicke vacuum field equations with a cosmological term Λ are given by[12]

$$G_{\mu\nu} = -\frac{\omega}{\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right) - \frac{1}{\Phi}(\Phi_{,\mu;\nu} - g_{\mu\nu}\Box\Phi\ ) - \Lambda g_{\mu\nu}, \tag{9}$$

$$\Box\Phi\ = \frac{2\Lambda\Phi}{2\omega+3}, \tag{10}$$

which are formally identical to (7) and (8) if we just set $W = \omega - \frac{3}{2}$. It should be noted, however, that the two theories are not physically equivalent, since in the Weyl geometrical scalar-tensor theory test particles follow affine Weyl geodesics (auto-parallels) and not metric geodesics[3].

III. SOME SOLUTIONS OF THE WEYL GEOMETRICAL SCALAR-TENSOR THEORY

As a consequence of the formal equality of field equations in the vacuum regime plus cosmological constant, it is clear that a solution of the Weyl geometrical scalar-tensor theory can be obtained if we make the change $\omega \to \omega - 3/2$ in a known solution of Brans-Dicke theory. Thus, let us consider some solutions of Brans-Dicke theory and obtain the corresponding solutions in Weyl geometrical scalar-tensor theory.

Initially, let us consider a Friedmann-Robertson-Walker metric with flat spatial section

$$ds^2 = dt^2 - R^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{11}$$

where $R(t)$ denotes the scale factor. A known class of solutions of the Brans-Dicke vacuum field equations with a cosmological constant is given by[13]

$$R(t) = R_0 \exp[(1+\omega)\Psi_0 t], \tag{12}$$

$$\Phi(t) = \Phi_0 \exp[\Psi_0 t], \tag{13}$$

where R_0, Φ_0 are constants, and

$$\Psi_0 = \pm\sqrt{\frac{2\Lambda}{(2\omega+3)(3\omega+4)}}, \tag{14}$$

with $\Lambda > 0$, $\omega > -\frac{4}{3}$ or $\omega < -\frac{3}{2}$. In this case, it is not difficult to see that the corresponding solutions of the Weyl geometrical scalar-tensor theory are simply

$$R(t) = R_0 \exp\left[\left(\omega - \frac{1}{2}\right)\Psi_1 t\right], \tag{15}$$

$$\Phi(t) = \Phi_0 \exp[\Psi_1 t], \tag{16}$$

where, for $\omega > \frac{1}{6}$ or $\omega < 0$,

$$\Psi_1 = \pm\sqrt{\frac{\Lambda}{\omega(3\omega - \frac{1}{2})}}. \tag{17}$$

As a second example, let us consider a Friedmann-Robertson-Walker geometry in the form

$$ds^2 = dt^2 - R^2(t)[d\chi^2/(1 - k\chi^2) + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{18}$$

For $k = 1$ and $\omega > -1$, a class of static solutions of the Brans-Dicke vacuum field equations with $\Lambda > 0$ is given by[14],

$$R(t) = \sqrt{\frac{2\omega + 3}{\Lambda(\omega + 1)}} = const, \tag{19}$$

$$\Phi(t) = \Phi_0 \exp\left[\pm\sqrt{\frac{2\Lambda}{2\omega + 3}}\, t\right], \tag{20}$$

where Φ_0 is a constant. Again, in the Weyl geometrical scalar-tensor theory we can obtain the following solutions, which are valid for $\omega > \frac{1}{2}$:

$$R(t) = const = \sqrt{\frac{2\omega}{\Lambda(\omega - \frac{1}{2})}}, \tag{21}$$

$$\Phi(t) = \Phi_0 \exp\left[\pm\sqrt{\frac{\Lambda}{\omega}}\, t\right]. \tag{22}$$

For our purposes, it is also interesting to consider some vacuum solutions with $\Lambda = 0$. Let us now consider the following well known spherically symmetric solution in Brans-Dicke theory[1]

$$ds^2 = e^{2\alpha}dt^2 - e^{2\beta}[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{23}$$

$$e^{2\alpha} = \left(\frac{1 - B/\chi}{1 + B/\chi}\right)^{2/\sigma}, \tag{24}$$

$$e^{2\beta} = (1 + B/\chi)^4 \left(\frac{1 - B/\chi}{1 + B/\chi}\right)^{(2/\sigma)(\sigma - C - 1)}, \tag{25}$$

$$\Phi = \Phi_0 \left(\frac{1 - B/\chi}{1 + B/\chi} \right)^{-C/\sigma}, \tag{26}$$

with,

$$\sigma = \left[(C+1)^2 - C \left(1 - \frac{1}{2}\omega C \right) \right]^{1/2},$$
$$B = \frac{M}{2\Phi_0} \left(\frac{2\omega + 4}{2\omega + 3} \right)^{1/2}, C = -\frac{1}{\omega + 2}, \tag{27}$$

where Φ_0 is constant and M is the mass of a spherical matter distribution. Thus, the analogous solution in the Weyl geometrical scalar-tensor theory will given by Eqs. (23)-(26), with the new redefined constants

$$\sigma = \left[(C+1)^2 - C \left(1 - \frac{1}{2}\omega C + \frac{3}{4}C \right) \right]^{1/2},$$
$$B = \frac{M}{2\Phi_0} \left(\frac{2\omega + 1}{2\omega} \right)^{1/2}, C = -\frac{1}{\omega + \frac{1}{2}}. \tag{28}$$

Another vacuum solution with $\Lambda = 0$ is provided by the O'Hanlon-Tupper solutions[15], with line element given by (11) with

$$R(t) = R_0 t^q, \tag{29}$$

$$\Phi(t) = \Phi_0 t^r, \tag{30}$$

where R_0, Φ_0, q, r are constants, $q = \frac{1}{3}(1-r)$ and $\frac{1}{r} = -\frac{1}{2}\left[1 \pm \sqrt{3(2\omega+3)}\right]$, $\omega > -\frac{3}{2}$. The corresponding solutions in the Weyl geometrical scalar-tensor theory are also expressed by Eqs. (29) and (30), with the redefinition

$$\frac{1}{r} = -\frac{1}{2}\left[1 \pm \sqrt{6\omega}\right], \tag{31}$$

where $\omega > 0$.

IV. THE LIMIT $\omega \to \infty$

In Weyl geometrical scalar-tensor theory, let us consider a solution of the scalar field Φ that takes the form[1]

[1] Here we would like to point out that to our knowledge as far as the literature is concerned the precise meaning of the limit $\omega \to \infty$ has not been explicitly given. In fact, the focus has been always on the

$$\Phi = \Phi_0 + O\left(\frac{1}{\omega}\right) \tag{32}$$

for large ω. Now, in the limit $\omega \to \infty$, we have $\Phi \to \Phi_0 = const$ and the equation (7) may be written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{33}$$

Furthermore, as $\Phi = e^{-\phi}$, we get for large ω:

$$\phi_{,\alpha} = -\frac{\Phi_{,\alpha}}{\Phi} \sim O\left(\frac{1}{\omega}\right). \tag{34}$$

Then, for $\omega \to \infty$, $\phi_{,\alpha} \to 0$ and the Eqs. (3) and (4) will be given by

$$\Gamma^{\alpha}_{\mu\nu} = \{^{\alpha}_{\mu\nu}\}, \qquad \nabla_{\alpha} g_{\mu\nu} = 0. \tag{35}$$

Let us remark that, in the limit $\omega \to \infty$, we have a Riemannian space-time manifold and we recover the Einstein vacuum field equations with cosmological constant. Moreover, the space-time geometry becomes Riemannian as can be seen from the behaviour of both the compatibility conditions and the coefficients of the affine connection.

As a first example, we note that the solution (16) presents the behavior indicated in Eq. (32) for large ω. Thus, in the limit $\omega \to \infty$, we have from (15) that

$$R(t) = R_0 \exp\left[\pm\sqrt{\frac{\Lambda}{3}}t\right], \tag{36}$$

becoming identical to de Sitter's solution of general relativity . A second example, for the particular case $\Lambda = 0$, is furnished by the solution (26), since from Eqs. (28) when ω is large we obtain

$$\frac{C}{\sigma} \sim O\left(\frac{1}{\omega}\right). \tag{37}$$

behaviour of the solutions with regard solely to the parameter ω. Although not explicitly mentioned, the procedure seems to take for granted that in the limit $\omega \to \infty$ the scalar field Φ becomes $\Phi = \Phi_0 + O(1/\omega)$ or $\Phi = \Phi_0 + O(1/\sqrt{\omega})$, where both $O(1/\omega)$ and $O(1/\sqrt{\omega})$ goes to zero for fixed coordinates. More formally, the meaning of this limit is the following: given an arbitrary $\varepsilon > 0$, there exists a number N such that if $\omega > N$, then $|\psi(\omega, x)| < \varepsilon$ for any fixed values of the coordinates x, where $\psi(\omega, x) = O(1/\omega)$ or $O(1/\sqrt{\omega})$. In other words, what is required in this procedure is *pointwise convergence* instead *of uniform convergence.*

It is not difficult to see that when $\omega \to \infty$ the solution given by Eqs. (23)-(26) and (28) goes over to the Schwarzschild metric of general relativity, with the identification $1/\Phi_0 = G$, G denoting the gravitational constant.

Let us now examine the case in which the solution of the scalar field Φ for large ω behaves like

$$\Phi = \Phi_0 + O\left(\frac{1}{\sqrt{\omega}}\right). \tag{38}$$

In this case, the Eq. (7) in the limit $\omega \to \infty$ may be written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \lim_{\omega\to\infty}\left[-\frac{\omega}{\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right)\right] \neq 0. \tag{39}$$

Also, we verify that

$$\phi_{,\alpha} = -\frac{\Phi_{,\alpha}}{\Phi} \sim O\left(\frac{1}{\sqrt{\omega}}\right) \tag{40}$$

when ω is large. It is clear that in this case the Eqs. (3) and (4) becomes identical to (35) when $\omega \to \infty$, which then implies that the space-time manifold becomes Riemannian. However, according to (39), the Einstein vacuum field equations with cosmological constant (or in the particular case $\Lambda = 0$) are not recovered.

Let us now consider examples of solutions which behave in accordance to (38). It is immediately seen that the solutions (22) and (30)-(31) satisfy (38) and that, in these cases, the space-time manifold again becomes Riemannian in the limit $\omega \to \infty$. Nevertheless, in the context of the solution (22), the static solution (21) does not coincide with the general relativity vacuum solution with $\Lambda \neq 0$ when the limit mentioned is taken. The same conclusion is true in the context of the solution (30)-(31), i.e., the solution (29) does not represent the Einstein vacuum solution if $\omega \to \infty$.

Finally, let us briefly comment on the case in which $T_{\mu\nu} \neq 0$ and $V(\Phi) = 0$. We note that if a given solution Φ satisfies the Eq. (32), then in the limit $\omega \to \infty$ the field equations (5) become

$$G_{\mu\nu} = -8\pi T_{\mu\nu}. \tag{41}$$

In addition, the Eqs. (35) would also be valid in this limit.

V. CONCLUSION

In this note we have shown that the Weyl geometrical scalar-tensor theory possesses some similarities with Brans-Dicke theory. Indeed, the two theories are formally identical if we restrict ourselves to the vacuum field equations[3]. This formal identity also holds if we include the cosmological constant in the field equations. Due to this identification, we were able to investigate the limit $\omega \to \infty$ in the Weyl geometrical scalar-tensor theory simply by getting new solutions from known solutions of the Brans-Dicke theory. Surely the two theories are not entirely equivalent, but this simple method allowed us to find solutions in a straightforward way. We thereby verify that, depending on the behavior of Φ for large ω, two possibilities appear when we take the limit $\omega \to \infty$. Depending on the asymptotic behaviour of Φ with respect to large values of ω, i.e., in accordance to Eq. (32) or to Eq. (38), the field equations reduce or do not reduce to the general relativistic equations. The space-time geometry, however, becomes Riemannian in any case.

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[1] C. Brans and R. H. Dicke, *Phys. Rev.* **124** (1961) 925.

[2] C. M. Will, *Living Rev. Relativ.* **17** (2014) 4.

[3] T. S. Almeida, M. L. Pucheu, C. Romero and J. B. Formiga, *Phys. Rev. D* **89** (2014) 064047.

[4] H. Weyl, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.* **1918** (1918) 465; *Space, Time, Matter*, Dover Books on Advanced Mathematics (Dover, New York, 1952).

[5] M. Novello and H. Heintzmann, *Phys. Lett. A* **98** (1983) 10; K. A. Bronnikov, M. Y. Konstantinov and V. Melnikov, *Gravitation Cosmol.* **1** (1995) 60; F. Dahia, G. A. T. Gomez and C. Romero, *J. Math. Phys.* (N.Y.) **49** (2008) 102501; T. Moon, J. Lee and P. Oh, *Mod. Phys. Lett. A* **25** (2010) 3129. For an excellent review of Weyl geometry, see E. Scholz, The Unexpected Resurgence of Weyl Geometry in late 20th-Century Physics in *Beyond Einstein*, eds. D. Rowe, T. Sauer and S. Walter, Einstein Studies, vol 14 (Birkhäuser, New York, 2018).

[6] J. B. Fonseca-Neto, C. Romero and S. P. G. Martinez, *Gen Relativ Gravit* **45** (2013) 1579; M. Novello, E. Bittencourt, U. Moschella, E. Goulart, J. M. Salim and J. D. Toniato, *JCAP* **6** (2013) 14. See also, D. K. Sen and K. A. Dunn, *J. Math. Phys.* **14** (1971) 58.

[7] See, for instance, S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[8] See, for instance, V. Faraoni, *Phys. Rev. D* **59** (1999) 084021 and the references therein; A. Bhadra and K. K. Nandi, *Phys. Rev. D* **64** (2001) 087501. For an excellent and almost complete discussion of the subject, see V. Faraoni, *Cosmology in Scalar-Tensor Gravity* (Kluwer Academic Publishers, Dordrecht, 2004).

[9] C. Romero and A. Barros, *Phys. Lett. A* **173** (1993) 243.

[10] V. Faraoni and J. Côté, *Phys. Rev. D* **99** (2019) 064013. B. Chauvineau, *Class. Quantum Grav.* **20** (2003) 2617.

[11] M. L. Pucheu, F. A. P. Alves Junior, A. B. Barreto and C. Romero, *Phys. Rev. D* **94** (2016) 064010.

[12] K. Uehara and C W. Kim, *Phys. Rev. D* **26** (1982) 2575.

[13] C. Romero and A. Barros, *Astrophys. Space Sci.* **192** (1992) 263.

[14] C. Romero and A. Barros, *Gen. Relativ. Gravit.* **25** (1993) 491.

[15] J. O'Hanlon and B. O. J. Tupper, *Nuovo Cimento* **7** (1972) 305.