Universidade Federal de Campina Grande
Centro de Ciências e Tecnologia
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Curso de Mestrado em Matemática

Iago Renan Valentim da Silva

A quantile regression model based on an Owen distribution

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por

Iago Renan Valentim da Silva †

sob orientação do

Prof. Dr. Manoel Ferreira dos Santos Neto

Dissertação apresentada ao Corpo Docente do Programa de Pós-Graduação em Matemática - CCT - UFCG, como requisito parcial para obtenção do título de Mestre em Matemática.

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Área de Concentração: Probabilidade e Estatística

Aprovada por:

Prof. Dr. Eliardo Guimarães da Costa

Prof. Dr. Carolina Ivonne Marchant Fuentes

Prof. Dr. Manoel Ferreira dos Santos Neto

Orientador

Universidade Federal de Campina Grande
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Resumo

Neste trabalho apresentamos uma distribuição generalizada de Birnbaum-Saunders que foi modificada para permitir a modelagem quantílica. Usamos a nova parametrização desta distribuição para gerar o modelo de regressão quantílica. Estimadores de máxima verossimilhança são submetidos a simulações de Monte Carlo. Os resultados numéricos mostram que o modelo proposto tem um desempenho admirável. Para demonstrar seu potencial, empregamos uma aplicação econômica de nossa metodologia.

Abstract

In this work we presented, a generalized Birnbaum-Saunders distribution that was modified to allow for quantile modeling. We use a new parameterization of this distribution to establish the quantile regression model. Maximum likelihood estimators were subjected to Monte Carlo simulations. The numerical results showed that the proposed model performs admirably. To demonstrate their potential, we employ an economical application of our methodology.

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A minha mãe (Neumann) e a minha esposa (Aline), pelo apoio incondicional em todos os momentos da minha trajetória acadêmica.

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Chapter 1

Introduction

The Birnbaum-Saunders (BS) distribution (Freudenthal & Shinozuka 1961, Birnbaum & Saunders 1969, Desmond 1985, 1986), which can be derived by a monotone transformation on the standard normal random variable (Balakrishnan & Kundu 2019), is undoubtedly one of the most important probabilistic models. It is possible to measure its importance through the numerous works derived from this distribution and applications in the most diverse areas (engineering, economics, social area, medicine, and environment, among others). Two excellent works listing several of these researches and carrying out a detailed review of the BS distribution are Leiva (2016) and Balakrishnan & Kundu (2019).

Some of the leading research on the BS distribution focuses on proposing generalizations of the original model for more flexible models. Some of these works are Díaz-García & Leiva-Sánchez (2005), Díaz-García & Domínguez-Molina (2006), Owen (2006), Sanhueza et al. (2008), Leiva et al. (2008), Gómez et al. (2009), Cordeiro & Lemonte (2014), Martínez-Flórez et al. (2014),Wang et al. (2018), Martínez-Flórez et al. (2019), Benkhelifa (2020), Thomas et al. (2021), Reyes et al. (2021), among others. Recently, some researchers have focused their research on developing approaches to model the quantiles of the BS distribution. For example, Sánchez et al. (2020), Sánchez et al. (2021), Gallardo et al. (2020), Gallardo & Santos-Neto (2021) and Mazucheli et al. (2021).

In the BS distribution, the assumption of independence of crack extensions from cycle to cycle can be pretty unrealistic. Therefore Owen (2006) derived a model relaxing this independence assumption. This new model considers that the sequence of crack extensions is modeled as a long memory process, and the characteristics of this development introduce a new third parameter (Owen 2006). There are still not many works based on this model. It will be called here Owen (OW) distribution.

This work proposes a quantile regression model based on the OW distribution. We will first use an exponentiated version of the OW distribution to build this model. Next, we will use a reparametrization that makes τth quantile the location parameter of the new distribution. This proposal is based on Gallardo et al. (2020) and Gallardo & Santos-Neto (2021), which use the same approach. Additionally, we analyze the performance of the maximum-likelihood estimator (MLEs) in small, moderate, and large samples via Monte Carlo simulations. The numerical results show that the estimators present good asymptotic properties. Finally, the application shows that the proposed model is very competitive.

The remainder of the work is organized as follows: Section 1.1 is presented a reason for the proposed methodology. Chapter 2 is devoted to the development of a quantile regression model based on the OW distribution. This Chapter presents a new distribution based on the OW distribution and its reparametrization. Furthermore, is formulated the quantile regression model based on the new distribution. In Section 2.6 is carried out in a simulation study to evaluate the performance of the proposed results. Chapter 3 is dedicated to the analysis of real-world dataset. Finally, the conclusions are presented in Section 4.

1.1 Motivation

Owen (2006) revisited the derivation of the BS distribution proposed in Birnbaum & Saunders (1969) because the assumption of independence of crack extensions from cycle to cycle can be quite unrealistic, and one new model was derived by relaxing this independence assumption. The sequence of crack extensions is modeled as a long memory process in Owen (2006), and the characteristics of this development introduced a new third parameter. In Martínez-Flórez et al. (2014) was obtained a more flexible extension of the BS distribution based on the asymmetric alpha-power family of distributions. Meanwhile, in this work, we generalize the proposal of Martínez-Flórez et al. (2014) considering the Owen distribution. In this way, we obtain a more general and realistic probabilistic model. Considering this class of distributions, we decided to create a regression model that models the τ-order quantiles.

Chapter 2

Quantile regression model

2.1 Exponencialized Owen distribution

The Owen (`OW`) distribution (Owen 2006) can be defined by its cumulative distribution function `CDF`, which is given by

$$F(t \mid \lambda, \beta, \kappa) = \Phi\left[\frac{1}{\lambda}\left(\frac{t^{1-\kappa}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{t^{\kappa}}\right)\right], \quad t \in \Re_{>0}, \tag{2.1}$$

where $\Phi(\cdot)$ is the `CDF` of the standard normal. For (2.1) to be well-defined, it is necessary $\lambda \in \Re_{>0}$, $\beta \in \Re_{>0}$ and $\kappa \in \Re \cap (0,1)$. Furthermore, we have that α and κ are shape parameters and β is the median of the distribution (Owen 2006). We denote $T \sim \texttt{OW}(\lambda, \beta, \kappa)$. It is easily shown that the `BS` distribution is a special case when $\kappa = 1/2$. For more details, see Owen (2006).

Now we present the Lehmann family of distributions proposed by Lehmann (1953), which is characterized by its `CDF` given by

$$G(z \mid \boldsymbol{\theta}, \alpha) = [F(z \mid \boldsymbol{\theta})]^{\alpha}, \quad z \in \Re,\ \alpha \in \Re_{>0},\ \boldsymbol{\theta} \in \Re^{p}, \tag{2.2}$$

where $p = \dim(\boldsymbol{\theta})$. This distribution is known as the standard α-exponentiated distribution, and it is denoted by the notation $Z \sim \alpha\text{-}\texttt{G}(\boldsymbol{\theta})$. If $\alpha = 1$, then the basal model is obtained. The probability density function (`PDF`) is defined as

$$g(z \mid \boldsymbol{\theta}, \alpha) = \alpha f(z|\boldsymbol{\theta}) F\left[(z|\boldsymbol{\theta})\right]^{\alpha-1}.$$

From (2.1) and (2.2) we obtain

$$G(z \mid \lambda, \beta, \kappa, \alpha) = \left[\Phi\left(a_z\right)\right]^{\alpha}, \quad z, \lambda, \beta, \alpha \in \Re_{>0}, \tag{2.3}$$

where $a_z = \lambda^{-1}\left(\frac{z^{1-\kappa}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{z^{\kappa}}\right)$. This distribution is a more general case and includes the BS ($\kappa = 1/2$ and $\alpha = 1$) and OW ($\alpha = 1$) distributions. Furthermore, we have that $Z \sim \alpha$-$\texttt{OW}(\lambda, \beta, \kappa, \alpha)$. The PDF of Z is given by

$$g(z \mid \lambda, \beta, \kappa, \alpha) = \alpha\, \phi(a_z)\, [\Phi(a_z)]^{\alpha-1} \frac{[\kappa\beta + (1-\kappa)z]}{\lambda\sqrt{\beta}z^{\kappa+1}}. \tag{2.4}$$

2.2 Exponencialized Owen distribution indexed by τth quantile

Here we will consider that in (2.3) $\alpha = \alpha_\tau = -{}^{\log(\tau)}/_{\log(2)}$ is fixed and consequently, the result below is obtained (Gallardo & Santos-Neto 2021)

$$G(\beta \mid \lambda, \beta, \kappa, \alpha_\tau) = [F(0)]^{\alpha_\tau} = (1/2)^{\alpha_\tau} = \tau,$$

i.e., fixing $\alpha = \alpha_\tau$, β denotes the τth quantile of the α_τ-$\texttt{OW}(\lambda, \beta, \kappa)$ distribution. As a result, we define a flexible distribution based on a generalized BS distribution to perform quantile regression for positive data (rather than just median regression). If $Z \sim \alpha_\tau$-$\texttt{OW}(\lambda, \beta, \kappa)$, then by (2.4) we have the following PDF

$$\begin{aligned} g(z \mid \lambda, \beta, \kappa, \alpha_\tau) = & \frac{\alpha_\tau}{\sqrt{2\pi}} \frac{1}{\lambda\sqrt{\beta}z^{\kappa}} \left(1 - \kappa + \frac{\beta\kappa}{z}\right) \exp\left[-\frac{1}{2\lambda^2}\frac{(z-\beta)^2}{\beta z^{2\kappa}}\right] \\ & \times \left\{\Phi\left[\frac{1}{\lambda}\left(\frac{z^{1-\kappa}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{z^{\kappa}}\right)\right]\right\}^{\alpha_\tau - 1} \end{aligned} \tag{2.5}$$

and the logarithm of (2.5) is given by

$$\begin{aligned} \log(g(z \mid \lambda, \beta, \kappa, \alpha_\tau)) = & \log(\alpha_\tau) - \log(\lambda) - \frac{1}{2}\log(\beta) - \frac{1}{2}\log(2\pi) + \log(z - \kappa z + \beta\kappa) \\ & - (\kappa+1)\log(z) - \frac{1}{2\beta\lambda^2}\left[\frac{(z-\beta)^2}{z^{2\kappa}}\right] + (\alpha_\tau - 1)\log(\Phi(a_z)). \end{aligned}$$

To generate pseudorandom numbers from a α_τ-$\texttt{OW}(\lambda, \beta, \kappa)$ distribution, we must find the positive root of the equation

$$\beta - z + 2Wz^{\kappa}\sqrt{\beta} = 0, \tag{2.6}$$

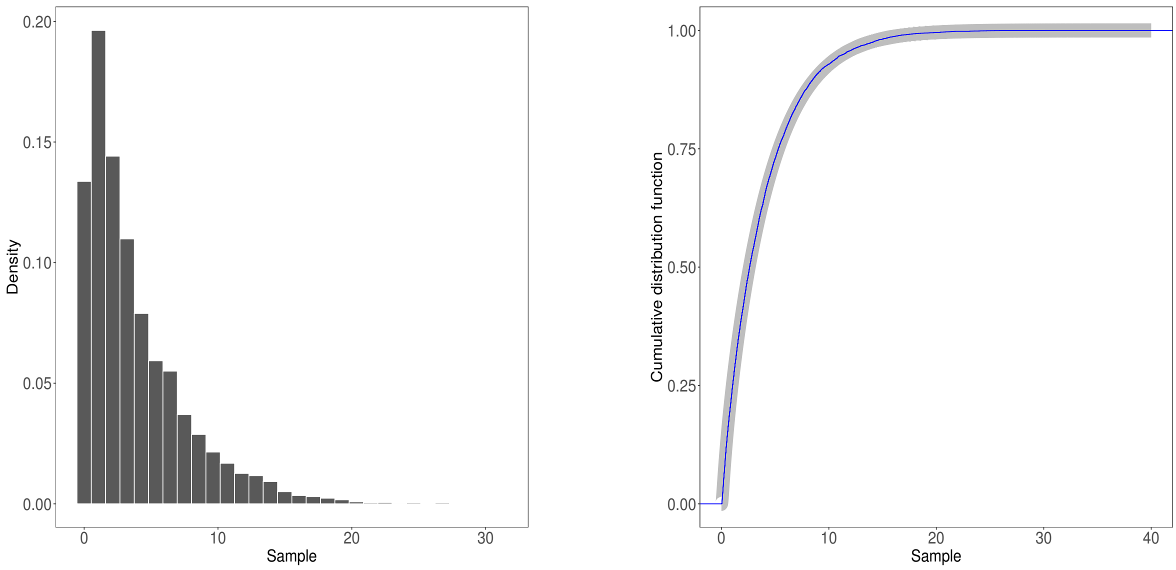
where $W \sim \alpha_\tau$-$\texttt{N}(0, {}^{\lambda^2}/_2)$. This can be easily done using the R programming language (R Core Team 2022) with the command `uniroot()`. In R codes below (Listing 2.1), we present a procedure for generating a sequence of pseudorandom numbers.

```r
r_alpha_owen <- function(n, family = "norm", lambda, nu, tau,
    kappa, ...){
  Q <- get(paste("q", family, sep = ""), mode = "function")
  a   <- -log(tau)/log(2)
  u <- NULL

  for(i in 1:n){
    unif_number <- runif(1)
    p.new <- unif_number^(1/a)
    z <- (lambda/2)*Q(p = p.new, ...)

    r <- function(x){
      2*sqrt(nu)*(x^kappa)*z - x + nu
    }

    u[i] <- uniroot(r, interval = c(0,1000), extendInt = "yes")
       $root
  }
  return(u)
}
```

Listing 2.1: R codes.

Figure 2.1(a) presents the histogram generated from $\alpha_{0.25}$-OW($\lambda = 2.0, \beta = 1.0, \kappa = 0.1$). Note that Figure 2.1(b) shows an excellent fit of the generated data (see empirical cumulative distribution function (ECDF)) to the theoretical cumulative distribution function. The R codes of the α_τ-OW CDF and of the plots are presented in Listing 2.2.

```r
cdf_alpha_owen<- function(q, family = "norm", lambda, nu, tau,
    kappa, log.p = FALSE, ...){
  CDF <- get(paste("p", family, sep = ""), mode = "function")
  a   <- -log(tau)/log(2)
  a.q <- (1.0/lambda) * (  ((q^(1-kappa))/sqrt(nu)) - (sqrt(nu)
      /(q^(kappa) ) ) )

  cdf <- CDF(a.q, log.p = log.p, ...)^a

  cdf
}

sample_alpha_owen <- data.frame(x = r_alpha_owen(10000,
                                                 lambda = 2,
                                                 nu = 1,
```



(a) Histogram of the generated values.

(b) ECDF (blue) and CDF (gray).

Figure 2.1: Histogram and cumulative distribution functions for simulated values.

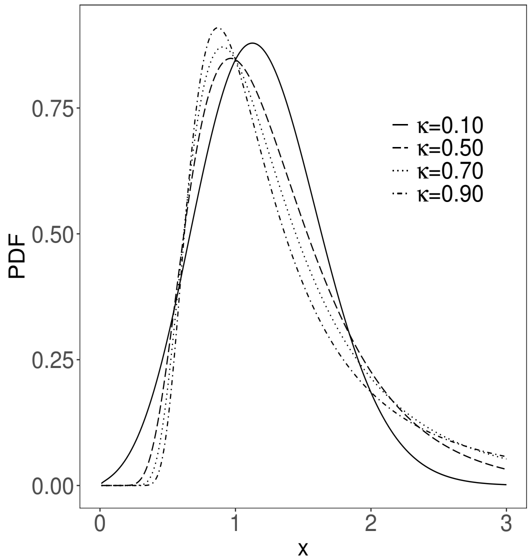
```
                                                   kappa = 0.4,
                                                   tau = 0.25),
                                q = seq(0 , 40, l = 10000),
                                y = cdf_alpha_owen(seq(0, 40, l
                                    = 10000),
                                lambda = 2,
                                                   nu = 1,
                                                   kappa = 0.4,
                                                   tau = 0.25
                                ))
ggplot(sample_alpha_owen, aes(x)) +
  geom_histogram(aes(y = ..density..), color = "white") +
  labs(x = "Sample", y = "Density") +
  theme_bw() +
  theme(axis.text  = element_text(size = 20),
        axis.title = element_text(size = 20),
        panel.grid.major = element_blank(),
        panel.grid.minor = element_blank()
  )

ggplot(sample_alpha_owen, aes(x)) +
  geom_line(aes(x = q, y = y), color = "gray",   size = 8) +
  stat_ecdf(color = "blue") +
  labs(x = "Sample", y = "Cumulative distribution function") +
  theme_bw() +
```

```r
  theme(axis.text  = element_text(size = 20),
        axis.title = element_text(size = 20),
        panel.grid.major = element_blank(),
        panel.grid.minor = element_blank())
```

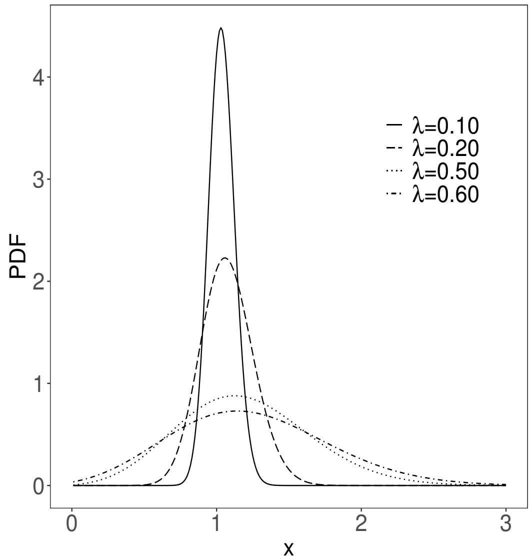
Listing 2.2: R codes.

Figure 2.2 contains α_τ-OW density plots for different values of λ, β, κ and α. It is noteworthy (Figure 2.2(a)) that the α_τ-OW density becomes slightly more asymmetrical as the values of κ increases. The increase λ impacts the variability of the distribution (Figure 2.2(b)). In Figure 2.2(c), note that the α_τ-OW density has its scale reduced with the increase of β. Notice that the asymmetry increases as τ increases (Figure 2.2(d)). The R codes of the (Figure 2.2(a)) are presented in Listing 2.3.

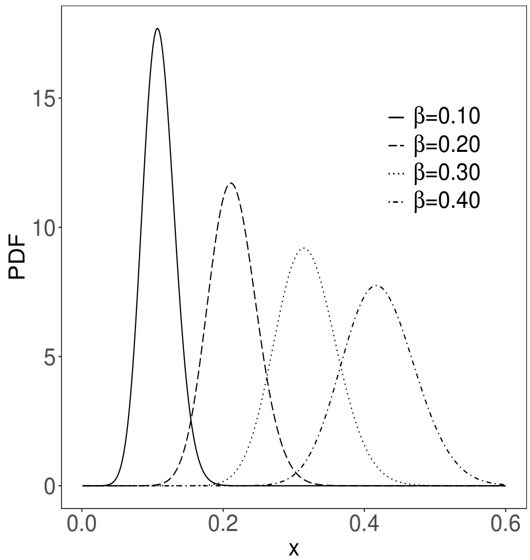
```r
pdf_alpha_owen <- function(x, family = "norm", tau, lambda, nu, kappa, log = FALSE, ...){
  g <- get(paste("d", family, sep = ""), mode = "function")
  G <- get(paste("p", family, sep = ""), mode = "function")

  a   <- -log(tau)/log(2)
  a.x <- (1.0/lambda) * (  ((x^(1-kappa))/sqrt(nu)) - (sqrt(nu)/(x^(kappa) ) ) )

    log.f <- log(a) + g(x = a.x, log = TRUE, ...) + (a - 1)*G(q = a.x, log.p = TRUE, ...) - (kappa + 1) * log(x) + log( ( (kappa*nu) + ((1-kappa)*x)) ) - log(lambda) - (1/2) * log(nu)

  if (!log) log.f <- exp(log.f)

  log.f
}

#Figure 2d
library(ggfortify)
p <- ggdistribution(pdf_alpha_owen, seq(0.01, 4, 0.01),
                    lambda = 0.75, nu= 1, tau = 0.08, kappa = 0.5,
                    linetype = "solid", xlab = "x", ylab = "PDF")

p <- ggdistribution(pdf_alpha_owen,  seq(0, 4, 0.01),
```



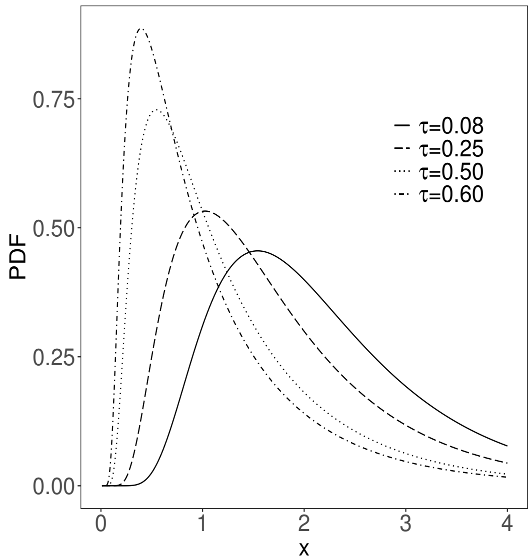
(a) Plots of the α_τ*-*`OW` `PDF` *with* $\lambda = 0.5, \beta = 1.0, \tau = 0.35$*, and varying* κ*.*



(b) Plots of the α_τ*-*`OW` `PDF` *with* $\kappa = 0.1, \beta = 1.0, \tau = 0.35$*, and varying* λ*.*



(c) Plots of the α_τ*-*`OW` `PDF` *with* $\lambda = 0.1, \kappa = 0.1, \tau = 0.35$*, and varying* β*.*



(d) Plots of the α_τ*-*`OW` `PDF` *with* $\lambda = 0.75, \kappa = 0.5, \beta = 1.0$*, and varying* τ*.*

Figure 2.2: Plots of the α_τ*-*`OW` `PDF` *for different combinations of the parameters.*

```
                     lambda = 0.75, nu = 1, tau = 0.25, kappa =
                        0.5,
                     linetype = "longdash", p = p)

p <- ggdistribution(pdf_alpha_owen , seq(0, 4, 0.01),
                     lambda = 0.75, nu = 1, tau = 0.5, kappa =
                        0.5,
                     linetype = "dotted", p = p)

```

```
ggdistribution(pdf_alpha_owen , seq(0, 4, 0.01),
               lambda = 0.75, nu = 1, tau = 0.60, kappa = 0.5,
               linetype = "dotdash", p = p) +
  theme_bw() +
  theme(axis.text  = element_text(size = 20),
        axis.title = element_text(size = 20),
        panel.grid.major = element_blank(),
        panel.grid.minor = element_blank())
```

Listing 2.3: R codes.

Consider a random sample $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^\top$, where each $Z_i \sim \alpha_\tau\text{-}\texttt{OW}(\lambda, \beta, \kappa)$. Then, the corresponding log-likelihood function for $\boldsymbol{\theta} = (\lambda, \beta, \kappa)^\top$, is given by

$$\ell(\boldsymbol{\theta} \mid \mathbf{z}) = \sum_{i=1}^{n} \log(g(z_i \mid \lambda, \beta, \kappa, \alpha_\tau)). \tag{2.7}$$

The components of the score vector, $\mathbf{U}(\boldsymbol{\theta})$, obtained by differentiating the log-likelihood function, given in (2.7), with respect to the parameters, are

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta} \mid \mathbf{z})}{\partial \lambda} &= -\frac{n}{\lambda} + \frac{1}{\lambda}\sum_{i=1}^{n} a_{z_i}^2 - \frac{(\alpha_\tau - 1)}{\lambda}\sum_{i=1}^{n} a_{z_i} w_i; \\
\frac{\partial \ell(\boldsymbol{\theta} \mid \mathbf{z})}{\partial \beta} &= -\frac{n}{2\beta} + \kappa \sum_{i=1}^{n} \frac{1}{z_i - \kappa z_i + \beta\kappa} + \frac{1}{2\beta^2\lambda^2}\sum_{i=1}^{n}\left[\frac{{z_i}^2 - \beta^2}{{z_i}^{2\kappa}}\right] \\
&\quad - \frac{(\alpha_\tau - 1)}{2\lambda\beta^{3/2}}\sum_{i=1}^{n}\frac{(z_i + \beta)}{z_i^{\kappa}} w_i; \\
\frac{\partial \ell(\boldsymbol{\theta} \mid \mathbf{z})}{\partial \kappa} &= \sum_{i=1}^{n} \frac{(\beta - z_i)}{z_i - \kappa z_i + \beta\kappa} - \sum_{i=1}^{n}\log(z_i) + \frac{1}{\beta\lambda^2}\sum_{i=1}^{n}\frac{(z_i - \beta)^2}{z_i^{2\kappa}}\log(z_i) \\
&\quad - (\alpha_\tau - 1)\sum_{i=1}^{n} a_{z_i} w_i \log(z_i),
\end{aligned}$$

where $w_i = {}^{\phi(a_{z_i})}/_{\Phi(a_{z_i})},\ i = 1, \ldots, n,$

The MLEs $\widehat{\boldsymbol{\theta}} = (\hat{\lambda}, \hat{\beta}, \hat{\kappa})^\top$ of the parameters that index the model are obtained by solving the equations $\frac{\partial \ell(\boldsymbol{\theta}|\mathbf{z})}{\partial \lambda} = 0$, $\frac{\partial \ell(\boldsymbol{\theta}|\mathbf{z})}{\partial \beta} = 0$ and $\frac{\partial \ell(\boldsymbol{\theta}|\mathbf{z})}{\partial \kappa} = 0$. They do not have a closed form. Estimates can be obtained by numerically maximizing of the log-likelihood given in (2.7) using a non-linear optimization algorithm.

The asymptotic inference for the parameter vector $\boldsymbol{\theta}$ can be based on the normal approximation of the MLE of $\boldsymbol{\theta}$, $\widehat{\boldsymbol{\theta}} = (\hat{\lambda}, \hat{\beta}, \hat{\kappa})^\top$. Under the usual regularity conditions (Cox & Hinkley 1974, Chapter 9), we have $\widehat{\boldsymbol{\theta}} \overset{a}{\sim} \texttt{N}_3(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, for large n, where $\overset{a}{\sim}$ means approximately distributed, and $\boldsymbol{\Sigma}$ is approximated by $\mathbf{J}(\widehat{\boldsymbol{\theta}})^{-1}$, where $\mathbf{J}(\widehat{\boldsymbol{\theta}})$ is the 3×3

observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, obtained from

$$\mathbf{J}(\boldsymbol{\theta}) = -\mathbf{H}(\boldsymbol{\theta}) = -\begin{bmatrix} h_{\lambda\lambda} & h_{\lambda\beta} & h_{\lambda\kappa} \\ & h_{\beta\beta} & h_{\beta\kappa} \\ & & h_{\kappa\kappa} \end{bmatrix},$$

whose elements are given by

$$h_{\lambda\lambda} = \frac{n}{\lambda^2} - \frac{3}{\lambda^2}\sum_{i=1}^{n} a_{z_i}^2 - \frac{(\alpha-1)}{\lambda^2}\sum_{i=1}^{n} a_{z_i} w_i \left[-2 + a_{z_i}^2 + a_{z_i} w_i\right];$$

$$h_{\lambda\beta} = -\frac{1}{\lambda^3\beta^2}\sum_{i=1}^{n}\frac{{z_i}^2-\beta^2}{z_i^{2\kappa}} - \frac{(\alpha-1)}{\lambda^2\beta^{3/2}}\sum_{i=1}^{n}\frac{z_i+\beta}{{z_i}^{\kappa}} w_i\left(-1 + a_{z_i}^2 + a_{z_i} w_i\right);$$

$$h_{\lambda\kappa} = -\frac{2}{\beta\lambda^3}\sum_{i=1}^{n}\frac{(z_i-\beta)^2}{z_i^{2\kappa}}\log(z_i) - \frac{(\alpha-1)}{\lambda}\sum_{i=1}^{n} a_{z_i} w_i\left(-1 + a_{z_i}^2 + a_{z_i} w_i\right)\log(z_i);$$

$$h_{\beta\beta} = \frac{n}{2\beta^2} - \frac{\kappa^2}{(z_i - \kappa z_i + \beta\kappa)^2} - \frac{1}{\lambda^2\beta^3}\sum_{i=1}^{n} {z_i}^{2(1-k)} + \frac{(\alpha-1)}{4\lambda\beta^{5/2}}\sum_{i=1}^{n}\frac{(3z_i+\beta)}{{z_i}^{\kappa}} w_i$$
$$- \frac{(\alpha-1)}{4\lambda^2\beta^3}\sum_{i=1}^{n}\frac{(z_i+\beta)^2}{z_i^{2\kappa}} w_i\left(a_{z_i} + w_i\right);$$

$$h_{\beta\kappa} = \sum_{i=1}^{n}\frac{z_i}{(z_i - \kappa z_i + \beta\kappa)^2} - \frac{1}{\beta^2\lambda^2}\sum_{i=1}^{n}\frac{(z^2-\beta^2)}{{z_i}^{2\kappa}}\log(z_i)$$
$$+ \frac{(\alpha-1)}{2\lambda\beta^{3/2}}\sum_{i=1}^{n}\frac{(z_i+\beta)}{{z_i}^{k}}\log(z_i) w_i - \frac{(\alpha-1)}{2\lambda\beta^{3/2}}\sum_{i=1}^{n}\frac{(z_i+\beta)}{{z_i}^{\kappa}} a_{z_i} w_i \log(z_i)\left(2a_{z_i} + w_i\right);$$

$$h_{\kappa\kappa} = -\sum_{i=1}^{n}\frac{(\beta-z)^2}{(z_i - \kappa z_i + \beta\kappa)^2} - \frac{2}{\beta\lambda^2}\sum_{i=1}^{n}\frac{(z_i-\beta)^2}{z_i^{2\kappa}}\log^2(z_i)$$
$$-(\alpha-1)\sum_{i=1}^{n}\log^2(z_i)\, a_{z_i} w_i\left(-1 + 2a_{z_i}^2 + a_{z_i} w_i\right).$$

The multivariate normal distribution, $\mathtt{N}_3(\boldsymbol{\theta}, \mathbf{J}(\widehat{\boldsymbol{\theta}})^{-1})$, can be used to construct approximate confidence intervals for the parameters λ, β and κ, which are given, respectively, by $\hat{\lambda} \pm z_{\gamma/2}\cdot\sqrt{\widehat{\mathrm{var}}(\hat{\lambda})}$, $\hat{\beta} \pm z_{\gamma/2}\cdot\sqrt{\widehat{\mathrm{var}}(\hat{\beta})}$ and $\hat{\kappa} \pm z_{\gamma/2}\cdot\sqrt{\widehat{\mathrm{var}}(\hat{\kappa})}$, where $\widehat{\mathrm{var}}(\cdot)$ is the diagonal element of $\mathbf{J}(\widehat{\boldsymbol{\theta}})^{-1}$ corresponding to each parameter, and $z_{\gamma/2}$ is the quantile $100(1-\gamma/2)\%$ of the standard normal distribution.

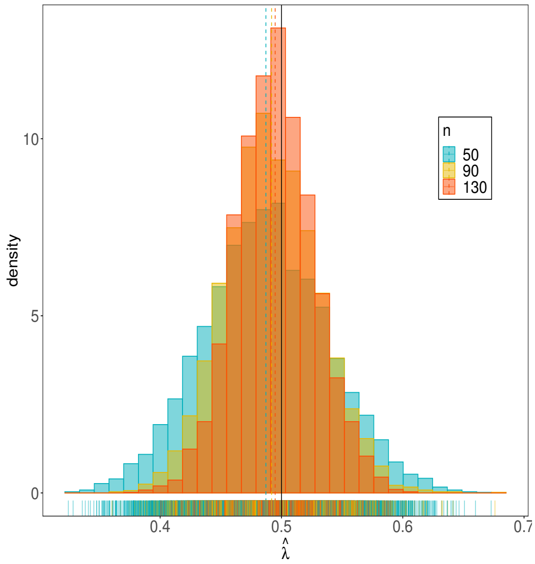
We can easily check if the fit using the α_τ-OW model is statistically "superior" to a fit using the BS model by testing the null hypothesis $\mathcal{H}_0 : \kappa = 0.5$ against $\mathcal{H}_1 : \kappa \neq 0.5$. For testing $\mathcal{H}_0$, consider the likelihood ratio (LR) statistic that is given by $S_{LR} = 2\cdot[\ell(\hat{\boldsymbol{\theta}} \mid \mathbf{z}) - \ell(\tilde{\boldsymbol{\theta}} \mid \mathbf{z})]$, where $\hat{\boldsymbol{\theta}}$ are the unrestricted MLEs obtained from the maximization of $\ell(\boldsymbol{\theta} \mid \mathbf{z})$ under $\mathcal{H}_1$ and $\tilde{\boldsymbol{\theta}}$ are the restricted MLEs obtained from the maximization of $\ell(\boldsymbol{\theta} \mid \mathbf{z})$ under $\mathcal{H}_0$. Under the null hypothesis, the limiting distribution of S_{LR} is χ_1^2. The null hypothesis must be rejected if S_{LR} exceeds the upper $100(1-\gamma)\%$ quantile of the χ_1^2 distribution.

2.3 Numerical evaluation of the maximum likelihood estimators

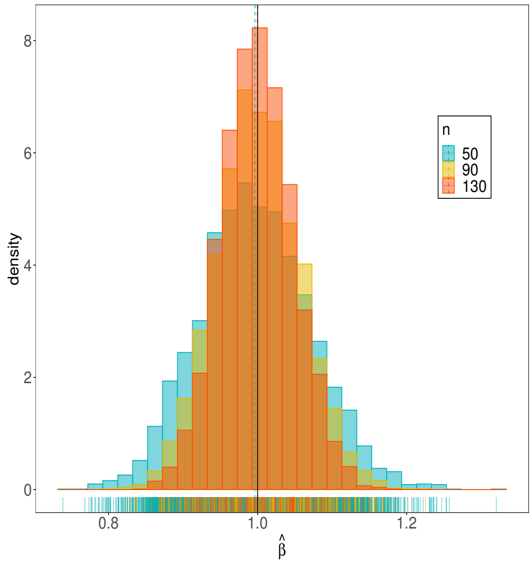
Now, a small Monte Carlo simulation is conducted to evaluate the behavior of the maximum likelihood estimators of the parameters of the α_τ-`OW` distribution. The simulation was performed using the `R` language (R Core Team 2022). The number of Monte Carlo replications was 5000. For maximizing the log-likelihood function, we use the command `mle` from package `stats4` (R Core Team 2022) with numeric derivatives. Also, for the Monte Carlo simulations, we use the `MonteCarlo` package (Leschinski 2019). The `MonteCarlo` package for `R` provides tools for quickly and easily creating simulation studies, as well as summarizing the results in LaTeX tables. The evaluation of point estimation was performed based on the following quantities for each sample size: the bias and the root mean squared error, $\sqrt{\texttt{MSE}}$, where `MSE` is the mean squared error estimated from 5000 Monte Carlo replications. These measures were calculated using the `simhelpers` package (Joshi & Pustejovsky 2022). We set the sample size at $n = 50, 90$ and 130, the parameter λ at $\lambda = 0.5, 1.0$ and 1.5, the parameter κ at $\kappa = 0.3, 0.5$ and 0.7 and τ was fixed at 0.5. Without loss of generality, the scale parameter β was fixed at 1.0. It can be seen from Table 2.1 that the estimates are quite stable and, more importantly, are close to the true values for the sample sizes considered. Figure 2.3 presents the empirical distribution of the `MLEs`.

Table 2.1: Bias and root mean squared error in parentheses

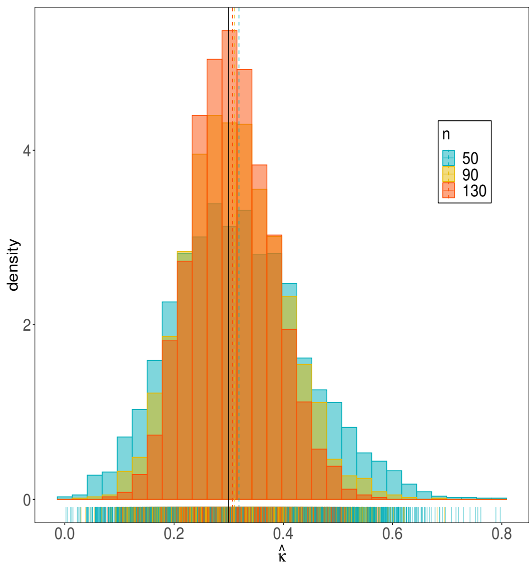
λ	κ	$n=50$			$n=90$			$n=130$		
		$\hat{\lambda}$	$\hat{\beta}$	$\hat{\kappa}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\kappa}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\kappa}$
0.5	0.3	-0.0129 (0.0526)	-0.0036 (0.0749)	0.0189 (0.1210)	-0.0081 (0.0395)	0.0001 (0.0566)	0.0112 (0.0891)	-0.0052 (0.0325)	-0.0003 (0.0478)	0.0070 (0.0745)
	0.5	-0.0135 (0.0530)	0.0033 (0.0760)	0.0002 (0.1260)	-0.0078 (0.0387)	0.0029 (0.0576)	0.0015 (0.0954)	-0.0049 (0.0317)	0.0007 (0.0486)	0.0011 (0.0785)
	0.7	-0.0131 (0.0540)	0.0088 (0.0772)	-0.0177 (0.1220)	-0.0077 (0.0388)	0.0033 (0.0573)	-0.0080 (0.0892)	-0.0056 (0.0321)	0.0022 (0.0474)	-0.0077 (0.0735)
1.0	0.3	-0.0228 (0.1110)	-0.0028 (0.1510)	0.0121 (0.0703)	-0.0122 (0.0833)	-0.0022 (0.1130)	0.0071 (0.0516)	-0.0083 (0.0690)	-0.0026 (0.0957)	0.0047 (0.0425)
	0.5	-0.0289 (0.1030)	0.0140 (0.1560)	-0.0024 (0.0791)	-0.0152 (0.0759)	0.0051 (0.1150)	0.0016 (0.0578)	-0.0115 (0.0629)	0.0040 (0.0956)	0.0003 (0.0478)
	0.7	-0.0242 (0.1120)	0.0269 (0.1620)	-0.0122 (0.0698)	-0.0110 (0.0835)	0.0114 (0.1150)	-0.0052 (0.0496)	-0.0075 (0.0689)	0.0097 (0.0962)	-0.0048 (0.0414)
1.5	0.3	-0.0311 (0.1760)	0.0106 (0.2280)	0.0072 (0.0569)	-0.0132 (0.1300)	0.0038 (0.1710)	0.0050 (0.0422)	-0.0112 (0.1090)	0.0007 (0.1420)	0.0037 (0.0345)
	0.5	-0.0433 (0.1590)	0.0242 (0.2330)	0.0007 (0.0662)	-0.0223 (0.1150)	0.0163 (0.1770)	-0.0005 (0.0492)	-0.0149 (0.0936)	0.0071 (0.1410)	-0.0001 (0.0404)
	0.7	-0.0315 (0.1790)	0.0470 (0.2610)	-0.0124 (0.0585)	-0.0106 (0.1330)	0.0311 (0.1890)	-0.0076 (0.0428)	-0.0104 (0.1080)	0.0190 (0.1510)	-0.0045 (0.0350)



(a) Empirical distribution of $\hat{\lambda}$ for different values of n and $\lambda = 0.5, \beta = 1.0, \kappa = 0.3, \tau = 0.5$.



(b) Empirical distribution of $\hat{\beta}$ for different values of n and $\lambda = 0.5, \beta = 1.0, \kappa = 0.3, \tau = 0.5$.



(c) Empirical distribution of $\hat{\kappa}$ for different values of n and $\lambda = 0.5, \beta = 1.0, \kappa = 0.3, \tau = 0.5$.

Figure 2.3: Empirical distribution of the MLEs.

2.4 The regression model

If α_τ is fixed, then we can model the τth quantile through a regression framework. Consider the following quantile α_τ-exponentiated Owen regression model

$$l(\beta_i) = \eta_i = \mathbf{x}_i^\top \boldsymbol{\vartheta}, \quad (i = 1, \dots, n),$$

where $\boldsymbol{\vartheta} = (\vartheta_0, \vartheta_1, \ldots, \vartheta_j, \ldots, \vartheta_{p-1})^\top$ is a parameter vector, $\boldsymbol{\vartheta} \in \mathbb{R}^p$, with $j = 0, \ldots, (p-1); p < n$, $\mathbf{x} = (1, \ldots, x_{ij}, \ldots, x_{ip})^\top$ contains values of explanatory variables, and $l(\cdot)$ is a link function, such as in generalized linear models (Nelder & Wedderburn 1972). For the parameters vector $\boldsymbol{\theta} = (\lambda, \boldsymbol{\vartheta}, \kappa)^\top$, the logarithm of the likelihood function considering a random sample is

$$\ell(\boldsymbol{\theta} \mid \mathbf{z}) = n\left[\log(\alpha_\tau) - \log(\lambda) - \frac{1}{2}\log(\beta_i) - \frac{1}{2}\log(2\pi)\right] + \sum_{i=1}^{n}\log\left(z_i - \kappa z_i + \beta_i\kappa\right)$$
$$-(\kappa+1)\sum_{i=1}^{n}\log(z_i) - \frac{1}{2\beta_i\lambda^2}\sum_{i=1}^{n}\left[\frac{(z_i-\beta_i)^2}{z_i^{2\kappa}}\right]$$
$$+(\alpha_\tau - 1)\sum_{i=1}^{n}\log\left(\Phi(a_{z_i})\right), \quad (2.8)$$

where $\beta_i = l^{-1}(\mathbf{x}_i^\top \boldsymbol{\vartheta})$. The score function, obtained by differentiating the log-likelihood function $\ell(\boldsymbol{\theta} \mid \mathbf{z})$ concerning the model's parameters is, $\mathbf{U}(\boldsymbol{\theta}) = (\mathbf{U}(\lambda), \mathbf{U}(\boldsymbol{\vartheta}), \mathbf{U}(\kappa))^\top$ where

$$\mathbf{U}(\lambda) = \mathbf{X}^\top\mathbf{D}_c(\mathbf{z}^\circ - \boldsymbol{\mu}^\circ), \quad \mathbf{U}(\boldsymbol{\vartheta}) = \mathrm{Tr}\Big(\mathbf{D}_{z_i^\bullet}\Big), \quad \text{and,} \quad \mathbf{U}(\kappa) = \mathrm{Tr}\Big(\mathbf{D}_{z_i^*}\Big),$$

where all of the above elements are defined in Appendix A. The `MLE` of the parameters that index the model are obtained by solving $\mathbf{U}(\boldsymbol{\theta}) = \mathbf{0}$. They cannot be expressed in closed form. Estimates can be obtained by numerically maximizing (2.8) using a nonlinear optimization algorithm.

After some algebra, it is possible to demonstrate that the Hessian matrix is given by

$$\mathbf{H}(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\boldsymbol{\vartheta}\partial\boldsymbol{\vartheta}^\top} & \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\boldsymbol{\vartheta}\partial\lambda^\top} & \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\boldsymbol{\vartheta}\partial\kappa^\top} \\ \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\boldsymbol{\vartheta}\partial\lambda^\top} & \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\lambda\partial\lambda^\top} & \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\lambda\partial\kappa^\top} \\ \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\boldsymbol{\vartheta}\partial\kappa^\top} & \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\lambda\partial\kappa^\top} & \frac{\partial^2\ell(\boldsymbol{\theta}|\mathbf{z})}{\partial\kappa\partial\kappa^\top} \end{pmatrix} = \begin{pmatrix} \mathbf{X}^\top\mathbf{D}_a\mathbf{X} & \mathbf{X}^\top\mathbf{D}_c\mathbf{v} & \mathbf{X}^\top\mathbf{D}_c\mathbf{b} \\ \mathbf{v}^\top\mathbf{D}_c\mathbf{X} & \mathrm{Tr}\left(\mathbf{D}_m\right) & \mathrm{Tr}\left(\mathbf{D}_r\right) \\ \mathbf{b}^\top\mathbf{D}_c\mathbf{X} & \mathrm{Tr}\left(\mathbf{D}_r\right) & \mathrm{Tr}\left(\mathbf{D}_d\right) \end{pmatrix}. \quad (2.9)$$

Note that all elements of the matrix block $\mathbf{H}(\boldsymbol{\theta})$ are obtained by partial derivatives, where $\mathbf{D}_a = \mathrm{diag}(a_i, \ldots, a_n)$, $\mathbf{D}_c = \mathrm{diag}(c_i, \ldots, c_n)$, $\mathbf{b} = (b_1, \ldots, b_n)^\top$, $\mathbf{v} = (v_1, \ldots, v_n)^\top$, $\mathbf{D}_m = \mathrm{diag}(m_i, \ldots, m_n)$ $\mathbf{D}_r = \mathrm{diag}(r_i, \ldots, r_n)$ and $\mathbf{D}_d = \mathrm{diag}(d_i, \ldots, d_n)$, the elements are defined in Appendix A

It can be shown that under some regularity conditions, $\hat{\boldsymbol{\theta}}$ is asymptotically distributed as $\mathtt{N}_{p+2}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ can be approximated by $\mathbf{J}(\widehat{\boldsymbol{\theta}})^{-1}$, where $\mathbf{J}(\widehat{\boldsymbol{\theta}}) = -\mathbf{H}(\widehat{\boldsymbol{\theta}})$ is the $(p+2) \times (p+2)$ observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, with $\mathbf{H}(\boldsymbol{\theta})$ defined in (2.9). As a result, based on the asymptotic normality of $\hat{\boldsymbol{\theta}}$ we have that it is possible to obtain an approximated $100 \times (1-\gamma/2)\%$ confidence region for $\boldsymbol{\theta}$. This region is defined as being the set of values of $\boldsymbol{\theta}$ such that

$$(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top \mathbf{J}(\widehat{\boldsymbol{\theta}})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \chi^2_{p+2}(\gamma),$$

where $\chi^2_{p+2}(\gamma)$ denotes the $1-\gamma$ quantile of the chi-square distribution with $p+2$ degrees of freedom.

2.5 Diagnostic Measures

Following the model fit, diagnostic analyses should be performed to ensure the estimated model's goodness of fit. We will present a global measure of explained variation and graphical tools for detecting deviations from the proposed model and influential observations. A global goodness-of-fit measure can be defined. We define based on the work of Nagelkerke et al. (1991) (see also Cribari-Neto & Fonseca (2019)) the following pseudo-R^2:

$$R^2_N = \frac{1 - \exp\left\{\frac{2}{n}[\ell(\hat{\vartheta}_0 \mid \mathbf{z}) - \ell(\hat{\boldsymbol{\vartheta}} \mid \mathbf{z})]\right\}}{1 - \exp\left\{\frac{2}{n}\ell(\hat{\vartheta}_0 \mid \mathbf{z})\right\}},$$

where $\ell(\hat{\vartheta}_0 \mid \mathbf{z})$ and $\ell(\hat{\boldsymbol{\vartheta}} \mid \mathbf{z})$ are the log-likelihoods for the model obtained only using the intercept in the linear predictor and the full model, respectively. Note that R^2_N assumes values between $[0, 1]$, and the closer to one, the better the model fit.

The quantile residuals (Dunn & Smyth 1996) can also be defined as follows:

$$r^q_i = \Phi^{-1}\left\{G(z_i \mid \hat{\lambda}, \hat{\beta}_i, \hat{\kappa}, \alpha_\tau)\right\}, \quad i = 1, \ldots, n,$$

where $G(z \mid \lambda, \beta, \kappa, \alpha)$ is defined in (2.3). The residuals of the fitted model are normally distributed with zero mean and unitary variance. This can be verified using different normality tests. Furthermore, when performing a residual analysis, applying the concept outlined by Atkinson (1982) is standard practice. He proposed creating confidence bands for residual quantile-quantile plots. The confidence bands for the residual r^q_i can then be computed and used to determine whether the fitted model accurately represents the data.

The generalized Cook's distance (Cook 1977, 1986) measures the impact of each observation on the regression parameter estimates. In the case of the proposed model, we shall use the usual approximation to generalized Cook's distance given by

$$\mathrm{GCD}_i(\boldsymbol{\theta}) \approx \frac{1}{p}\left(\mathbf{U}_{(i)}(\boldsymbol{\theta})^\top \, \mathbf{H}(\boldsymbol{\theta})^{-1}_{(i)} \, \mathbf{J}(\boldsymbol{\theta}) \, \mathbf{H}(\boldsymbol{\theta})^{-1}_{(i)} \, \mathbf{U}(\boldsymbol{\theta})_{(i)}\right), \quad i = 1, \ldots, n,$$

where $\mathbf{U}(\boldsymbol{\theta})_{(i)}$ and $\mathbf{H}(\boldsymbol{\theta})_{(i)}$ are the score vector and the Hessian matrix, respectively, without considering the case i, evaluated at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$.

2.6 Simulations

We shall use Monte Carlo simulations to investigate the performance of the maximum likelihood estimates. To realize these studies, we must first estimate the model's parameters. We do this by computing the estimates for each set of values of $n_{obs} = 90, 600, 3000$, generated from an $\alpha_{0.5}$-`OW`$(2.0, \beta_i, 0.2)$, considering.

$$\beta_i = 0.5 + 1.5x_{i1} - 0.5x_{i2}, \quad i = 1, \ldots, n_{obs},$$

where x_{i1} and x_{i2} are distributed through a uniform distribution over the interval $(0, 1)$. Maximum likelihood estimates were processed by Newton-type maximization methods through the function `maxLik` (Henningsen & Toomet 2011) in the `R` software (R Core Team 2022). Finally, we observed some properties of the estimators by calculating the relative bias (RB), and the root mean squared error (RMSE).

Table 2.2: RB and RMSE estimates considering $\lambda = 2.0$, $\kappa = 0.5$ and $\tau = 0.2$.

n	RB					n	RMSE				
	$\hat{\lambda}$	$\hat{\kappa}$	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$		$\hat{\lambda}$	$\hat{\kappa}$	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$
90	-0.068	0.191	0.065	-0.038	0.026	90	0.309	0.063	0.224	0.363	0.220
600	-0.018	0.006	0.009	0.013	-0.022	600	0.095	0.016	0.081	0.154	0.092
3000	0.000	0.002	0.008	-0.010	0.000	3000	0.041	0.008	0.032	0.075	0.040

In the Table 2.2 the `RB` of the maximum likelihood estimators of $\hat{\lambda}$, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ were negative, however for $\hat{\gamma}_2$ it was negative only with $n_{obs} = 600$. Note that the `RB` were relatively low. This effect can be noticed when we observe that as the sample size increases, the `RB` decreases for all scenarios. The reduction of the `RB` of the estimators were: 100%($\hat{\lambda}$), 98%($\hat{\kappa}$), 87%($\hat{\gamma_0}$) , 73%($\hat{\gamma_1}$) and 100%($\hat{\gamma_2}$). As we can see, the biggest reduction happens in $\hat{\lambda}$. In the case of `RMSE`, there is also a significant drop in bias when the sample increases, with emphasis on $\hat{\kappa}$ showing a reduction of 87%. This result is something expected of good estimators.

Chapter 3

Application

We present one empirical application demonstrating that the quantile α_τ- exponentiated Owen regression model is flexible and applicable. We compare the results of this model with the quantile `BS` regression model proposed by Sánchez et al. (2020). The `R` programming language was used for all computations (see Appendix C). This application presents a data set related to Chilean household income, as illustrated in Sánchez et al. (2020). This data set corresponds to the Chilean household income survey conducted by the National Institute of Statistics in 2016 (the most recent study available at the time), which is available at https://www.ine.cl/estadisticas/sociales/ingresos-y-gastos/encuesta-de-presupuestos-familiares. Sánchez et al. (2020) considers a subsample of $n = 100$ cases drawn at random from the entire data set. The variables are as follows: household income (Z), the total income due to salaries (X_1), the total income due to independent work (X_2) and the total income due to retirements (X_3).

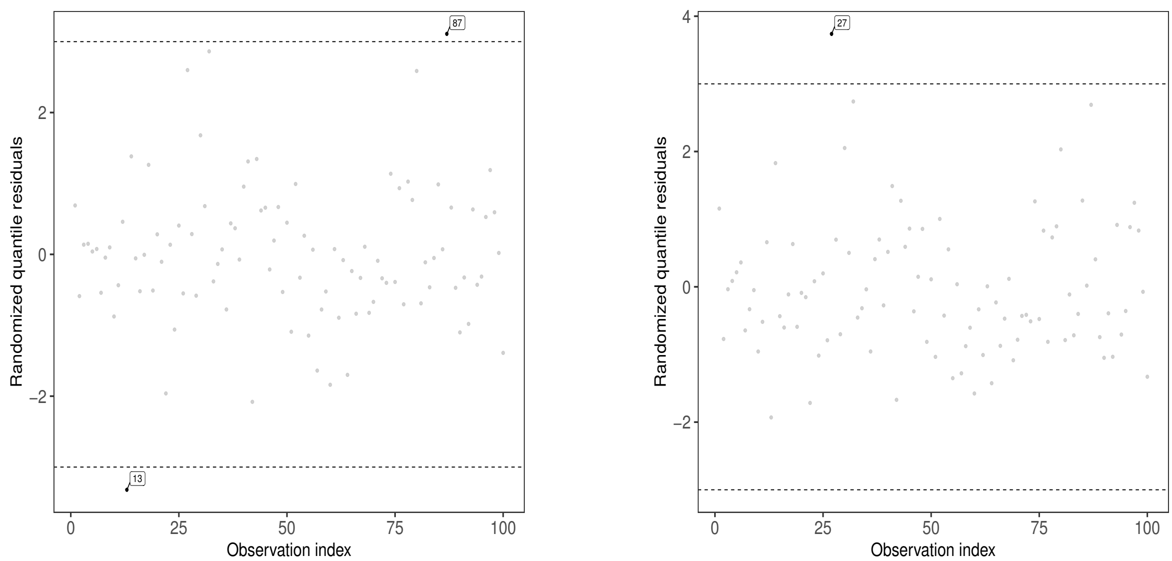
It is suggested that $Z_i \sim \alpha_\tau\text{-}\mathtt{OW}(\lambda, \beta_i, \kappa)$ where

$$\beta_i = \vartheta_0 + \vartheta_1 x_{i1} + \vartheta_2 x_{i2} + \vartheta_3 x_{i3}, \quad i = 1, \ldots, 100.$$

To compare this proposal to the model in Sánchez et al. (2020), we use $\tau = 0.5$ (the median). The `MLEs` and standard errors are presented in Table 3.1. Here we consider the `identity` link function because it is used in Sánchez et al. (2020). Note that the exploratory variable are relevant to the analysis in both models. Figure 3.2 presents the envelopes for the quantile residuals. Analyzing Figures 3.2(a)-3.2(b), apparently,

Table 3.1: Estimates and SE for parameters of the α_τ-OW model.

τ	Parameter	Base distribution					
		Birnbaum-Saunders			Owen		
		Estimate	SE	p-value	Estimate	SE	p-value
0.5	ϑ_0	198.148	11.727	0.000	238.640	20.083	0.000
	ϑ_1	1.044	0.119	0.000	0.985	0.083	0.000
	ϑ_2	1.109	0.218	0.000	1.071	0.153	0.000
	ϑ_3	1.086	0.158	0.000	1.217	0.171	0.000
	λ	0.365	0.019	×	3.296	2.342	×
	κ	×	×	×	0.145	0.105	0.000
	AIC		1395.675			1379.763	
	R_N^2		0.799			0.832	

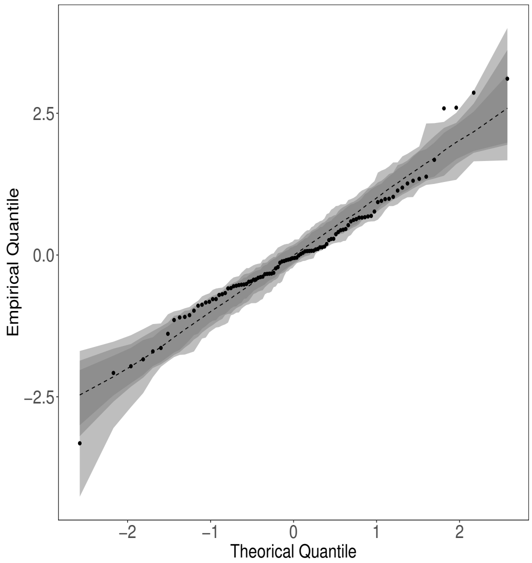


(a) BS quantile regression model. *(b) α_τ-OW quantile regression model.*

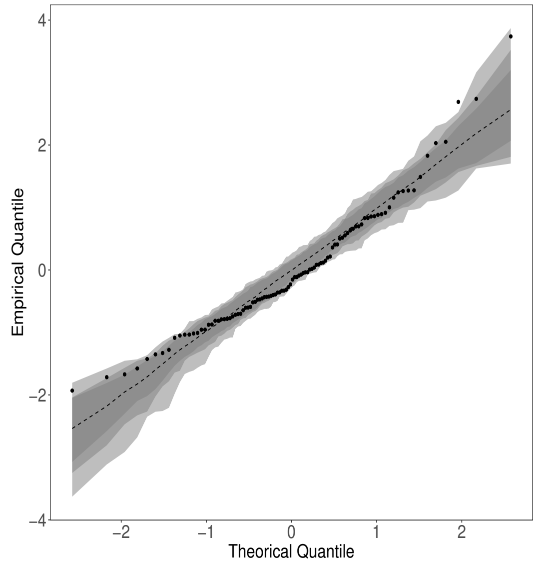
Figure 3.1: Plots of the quantile residuals versus observations index.

both models are suitable. However, when we perform the likelihood ratio test statistic for testing $\mathcal{H}_0 : \kappa = 1/2$ against $\mathcal{H}_1 : \kappa \neq 1/2$, it equals to 17.911($p$-value < 0.001). Therefore, there is strong evidence in the data, at the 5% significance level, that the quantile α_τ-exponentiated Owen regression model is superior to the model proposed by Sánchez et al. (2020). For the quantile α_τ-exponentiated Owen regression model, the value of the pseudo-R^2 is 0.832.

The generalized Cook's distance was calculated using the BS quantile regression

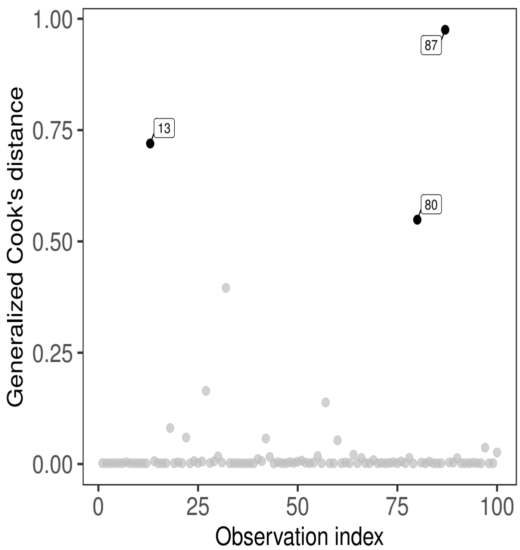


(a) BS quantile regression model.

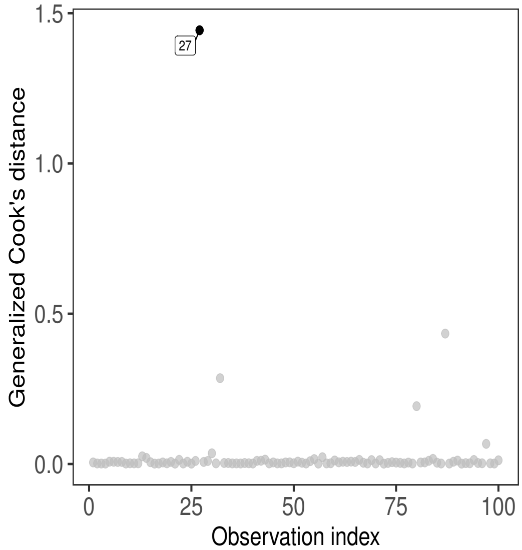


(b) α_τ-OW quantile regression model.

Figure 3.2: Simulated envelopes with bands of confidence for the quantile residual.



(a) BS quantile regression model.



(b) α_τ-OW quantile regression model.

Figure 3.3: Plots of generalized Cook's distance.

model and the α_τ-OW quantile regression model. We noticed that the α_τ-OW quantile regression model has only one potentially influential observation (#27), whereas the other fitted model has three (#13, #80 e #87) (see Figure 3.3).

We removed each atypical observation from the data in sequence and fitted the model after each data point removal. In each case, we calculated the absolute relative change (RC) in the estimates, i.e., $\left|\hat{\boldsymbol{\theta}}_{k(j)}-\hat{\boldsymbol{\theta}}_k/\hat{\boldsymbol{\theta}}_k\right|$, where $\hat{\boldsymbol{\theta}}_k$ is the kth MLE estimate considering the complete data and $\hat{\boldsymbol{\theta}}_{k(j)}$ is the estimate obtained removing the jth observation. Considering the idea of Cribari-Neto & Fonseca (2019), let's test the hypotheses $\mathcal{H}_0 : \kappa = 1/2$ against $\mathcal{H}_1 : \kappa \neq 1/2$ when we remove a particular observation. Table 3.2 displays the relative changes in parameter estimates and test p-values. Note that there were no inferential changes.

Table 3.2: RCs (in %) in MLEs and their corresponding SE's for the indicated parameter and dropped cases and respective p-values for the α_τ-OW model.

Dropped Cases	Component	Parameter					
		ϑ_0	ϑ_1	ϑ_2	ϑ_3	λ	κ
27	$\mathrm{RC}_{\widehat{\theta}_{t(i)}}$	3.476	0.228	1.139	11.311	78.593	71.741
	p-value	<0.001	<0.001	<0.001	<0.001	×	<0.001

Chapter 4

Concluding remarks

Quantile regression models have been frequently used. In this work, we propose a quantile regression model based on an α_τ-`OW` distribution. The model proposed by Sánchez et al. (2020) is a particular case of our model. Maximum likelihood estimation is also implemented, and the observed information matrix is derived. We present a pseudo-R^2 measure and obtain a generalized Cook's distance. In addition, we investigated the performances of the maximum likelihood estimators via Monte Carlo simulation. In an economical application, the proposed quantile regression model was shown to be more adequate than the quantile regression model proposed by Sánchez et al. (2020). In future research, we shall extend it to deal with censored observations. Furthermore, we can consider an α_τ-`OW` distribution with different kernels to construct a new quantile regression model.

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Appendices

Appendix A

In this Appendix, we obtain the score function and the observed information matrix for $(\lambda, \boldsymbol{\vartheta}, \kappa)$. From (2.8) we obtain, for $j = 1, \ldots, p$,

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta} \mid \mathbf{z})}{\partial \vartheta_j} &= \sum_{i=1}^{n} \frac{\partial \log(g(z_i|\lambda, \beta_i, \kappa, \alpha_\tau))}{\partial \beta_i} \frac{d\beta_i}{d\eta_i} \frac{\partial \eta_i}{\partial \vartheta_j} = \sum_{i=1}^{n} \frac{\partial \log(g(z_i|\lambda, \beta_i, \kappa, \alpha_\tau))}{\partial \beta_i} \frac{1}{h'(\beta_i)} x_{ij}, \\
&= \sum_{i=1}^{n} \left[-\underbrace{\frac{1}{2\beta_i}}_{\mu_i^\circ} + \underbrace{\frac{\kappa}{z_i - \kappa z_i + \beta_i \kappa} + \frac{(z_i + \beta_i)\left[a_{z_i} - (\alpha_\tau - 1) w_i\right]}{2\lambda \beta_i^{3/2} z_i^{\kappa}}}_{z_i^\circ} \right] \underbrace{\frac{1}{h'(\beta_i)}}_{c_i} x_{ij}, \\
&= \sum_{i=1}^{n} (z_i^\circ - \mu_i^\circ) c_i x_{ij},
\end{aligned}$$

and the matrix expression for the score function for $\boldsymbol{\vartheta}$ is $\mathbf{U}(\boldsymbol{\vartheta}) = \mathbf{X}^\top \mathbf{D}_c (\mathbf{z}^\circ - \boldsymbol{\mu}^\circ)$ where $\mathbf{X}^\top$ is an $n \times p$ matrix whose ith row is $\mathbf{x}_i^\top$, $\mathbf{z}^\circ = (z_1^\circ, \ldots, z_n^\circ)^\top$, $\boldsymbol{\mu}^\circ = (\mu_1^\circ, \ldots, \mu_n^\circ)^\top$ and $\mathbf{D}_c = \mathrm{diag}(c_1, \ldots, c_n)$. Similarly, it can be shown that the score function for λ can be written

$$\frac{\partial \ell(\boldsymbol{\theta} \mid \mathbf{z})}{\partial \lambda} = \frac{1}{\lambda} \sum_{i=1}^{n} \left[\underbrace{a_{z_i}^2 - (\alpha_\tau - 1) a_{z_i} w_i - 1}_{z_i^\bullet} \right] = \frac{1}{\lambda} \sum_{i=1}^{n} z_i^\bullet,$$

and in matrix form, we have that $\mathbf{U}(\lambda) = \mathrm{Tr}\Big(\mathbf{D}_{z_i^\bullet}\Big)$ where $\mathbf{D}_{z_i^\bullet} = \mathrm{diag}(z_1^\bullet, \ldots, z_n^\bullet)$. Finally, the score function for κ is given by

$$\begin{aligned}
\frac{\partial \ell(\boldsymbol{\theta} \mid \mathbf{z})}{\partial \kappa} &= \sum_{i=1}^{n} \left\{ \underbrace{\frac{(\beta_i - z_i)}{z_i - \kappa z_i + \beta_i \kappa} + \log(z_i) \left[a_{z_i}^2 - (\alpha_\tau - 1) a_{z_i} w_i - 1 \right]}_{z_i^*} \right\} \\
&= \sum_{i=1}^{n} z_i^*,
\end{aligned}$$

and in matrix form, we have that $\mathbf{U}(\kappa) = \mathrm{Tr}\Big(\mathbf{D}_{z_i^*}\Big)$ where $\mathbf{D}_{z_i^*} = \mathrm{diag}(z_1^*, \ldots, z_n^*)$.

The first element of the block matrix $\mathbf{H}(\boldsymbol{\theta})$ presented in (2.9), it is obtained by the second derivative of $\ell(\boldsymbol{\theta})$ with respect to ϑ_j and ϑ_l one has that

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\vartheta_j\partial\vartheta_l} &= \sum_{i=1}^{n}\left[\frac{\partial^2\log(g(z_i|\lambda,\beta_i,\kappa,\alpha_\tau))}{\beta_i^2}\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\right.\\ &\quad\left.+\frac{\partial\log(g(z_i|\lambda,\beta_i,\kappa,\alpha_\tau))}{\beta_i}\frac{\partial}{\partial\beta_i}\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\right]\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}x_{ij}x_{il}\\ &= \sum_{i=1}^{n}\left[\frac{\partial^2\log(g(z_i|\lambda,\beta_i,\kappa,\alpha_\tau))}{\beta_i^2}\left(\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\right)^2\right.\\ &\quad\left.+\frac{\partial\log(g(z_i|\lambda,\beta_i,\kappa,\alpha_\tau))}{\beta_i}\left(\frac{\partial}{\partial\beta_i}\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\right)\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\right]\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}x_{ij}x_{il},\end{aligned}$$

being

$$\frac{\partial^2\log(g(z_i|\lambda,\beta_i,\kappa,\alpha_\tau))}{\beta_i^2} = \frac{1}{2\beta_i^2}-\frac{\kappa^2}{(z_i-\kappa z_i+\beta_i\kappa)^2}-\frac{1}{\lambda^2\beta_i^3}{z_i}^{2(1-k)}+\frac{(\alpha-1)}{4\lambda\beta_i^{5/2}}\frac{(3z_i+\beta_i)}{{z_i}^{\kappa}}w_i.$$

be $\mathbf{D}_a=\mathrm{diag}(a_1,\ldots,a_n)$ with

$$a_i = \frac{\partial^2\log(g(z_i|\lambda,\beta_i,\kappa,\alpha_\tau))}{\beta_i^2}\left(\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\right)^2+\frac{\partial\log(g(z_i|\lambda,\beta_i,\kappa,\alpha_\tau))}{\beta_i}\left(\frac{\partial}{\partial\beta_i}\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\right)\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i},$$

then in matrix form we have that the Hessian matrix for vector $\boldsymbol{\vartheta}$ will be $\mathbf{X}^\top\mathbf{D}_a\mathbf{X}$. The second and fourth element of the block matrix $\mathbf{H}(\boldsymbol{\theta})$ it is obtained by the second derivative of $\ell(\boldsymbol{\theta})$ with respect to ϑ_j and λ is given by

$$\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\vartheta_j\partial\lambda} = \sum_{i=1}^{n}\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\beta_i\partial\lambda}\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\frac{\partial\eta_i}{\partial\vartheta_j},$$

but being

$$\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\beta_i\partial\lambda} = -\frac{1}{\lambda^3\beta_i^2}\frac{{t_i}^2-\beta_i^2}{{t_i}^{2\kappa}}-\frac{(\alpha-1)}{\lambda^2\beta_i^{3/2}}\frac{t_i+\beta_i}{{t_i}^{\kappa}}w_i\left(-1+a_{t_i}^2+a_{t_i}w_i\right),$$

with this result, we have

$$\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\vartheta_j\partial\lambda} = \sum_{i=1}^{n}\left[-\frac{1}{\lambda^3\beta_i^2}\frac{{t_i}^2-\beta_i^2}{{t_i}^{2\kappa}}-\frac{(\alpha-1)}{\lambda^2\beta_i^{3/2}}\frac{t_i+\beta_i}{{t_i}^{\kappa}}w_i\left(-1+a_{t_i}^2+a_{t_i}w_i\right)\right]\underbrace{\frac{1}{h'(\beta_i)}}_{c_i}x_{ij}.$$

be $v=\mathrm{diag}(v_1,\ldots,v_n)^\top$, with

$$v_i = -\frac{1}{\lambda^3\beta_i^2}\frac{{t_i}^2-\beta_i^2}{{t_i}^{2\kappa}}-\frac{(\alpha-1)}{\lambda^2\beta_i^{3/2}}\frac{t_i+\beta_i}{{t_i}^{\kappa}}w_i\left(-1+a_{t_i}^2+a_{t_i}w_i\right),$$

then in matrix form we have that the Hessian matrix for vector will be $\mathbf{X}^\top\mathbf{D}_c\mathbf{v}$

The third and seventh element of the block matrix $\mathbf{H}(\boldsymbol{\theta})$ it is obtained by the second derivative of $\ell(\boldsymbol{\theta})$ with respect to ϑ_j and κ is given by

$$\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\vartheta_i\partial\kappa} = \sum_{i=1}^{n}\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\beta_i\partial\kappa}\frac{\mathrm{d}\beta_i}{\mathrm{d}\eta_i}\frac{\partial\eta_i}{\partial\vartheta_j},$$

but being

$$\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\beta_i\partial\kappa}=\frac{t_i}{(t_i-\kappa t_i+\beta_i\kappa)^2}-\frac{1}{\beta_i^2\lambda^2}\frac{(t^2-\beta_i^2)}{t_i{}^{2\kappa}}\log(t_i)+\frac{(\alpha-1)}{2\lambda\beta_i^{3/2}}\frac{(t_i+\beta_i)}{t_i{}^k}\log(t_i)w_i,$$

with this result, we have

$$\begin{aligned}\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\vartheta_j\partial\kappa}=&\sum_{i=1}^{n}\Bigg[\frac{t_i}{(t_i-\kappa t_i+\beta_i\kappa)^2}-\frac{1}{\beta_i^2\lambda^2}\frac{(t^2-\beta_i^2)}{t_i{}^{2\kappa}}\log(t_i)\\&+\frac{(\alpha-1)}{2\lambda\beta_i^{3/2}}\frac{(t_i+\beta_i)}{t_i{}^k}\log(t_i)w_i\Bigg]\underbrace{\frac{1}{h'(\beta_i)}}_{c_i}x_{ij}.\end{aligned}$$

be $b=\mathrm{diag}(b_1,\ldots,b_n)^\top$, with

$$b_i=\frac{t_i}{(t_i-\kappa t_i+\beta_i\kappa)^2}-\frac{1}{\beta_i^2\lambda^2}\frac{(t^2-\beta_i^2)}{t_i{}^{2\kappa}}\log(t_i)+\frac{(\alpha-1)}{2\lambda\beta_i^{3/2}}\frac{(t_i+\beta_i)}{t_i{}^k}\log(t_i)w_i,$$

then in matrix form we have that the Hessian matrix for vector will be $\mathbf{X}^\intercal\mathbf{D}_c\mathbf{b}$. The fifth element of the block matrix $\mathbf{H}(\boldsymbol{\theta})$ it is obtained by the second derivative of $\ell(\boldsymbol{\theta})$ with respect to λ is given by

$$\sum_{i=1}^{n}\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\lambda^2}=\sum_{i=1}^{n}\left[\frac{1}{\lambda^2}-\frac{3}{\lambda^2}a_{t_i}^2-\frac{(\alpha-1)}{\lambda^2}a_{t_i}w_i\left[-2+a_{t_i}^2+a_{t_i}w_i\right]\right].$$

be $\mathbf{D}_m=\mathrm{diag}(m_1,\ldots,m_n)^\top$, with

$$m_i=\frac{1}{\lambda^2}-\frac{3}{\lambda^2}a_{t_i}^2-\frac{(\alpha-1)}{\lambda^2}a_{t_i}w_i\left[-2+a_{t_i}^2+a_{t_i}w_i\right],$$

then in matrix form we have that the Hessian matrix for vector will be $\mathrm{Tr}(\mathbf{D}_m)$. The sixth and eighth element of the block matrix $\mathbf{H}(\boldsymbol{\theta})$ it is obtained by the second derivative of $\ell(\boldsymbol{\theta})$ with respect to κ is given by

$$\begin{aligned}\sum_{i=1}^{n}\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\kappa^2}=&\sum_{i=1}^{n}\Bigg[-\frac{(\beta_i-t)^2}{(t_i-\kappa t_i+\beta_i\kappa)^2}-\frac{2\,(t_i-\beta_i)^2}{\beta_i\lambda^2t_i{}^{2\kappa}}\log^2(t_i)\\&-(\alpha-1)\log^2(t_i)\,a_{t_i}w_i\left(-1+2a_{t_i}^2+a_{t_i}w_i\right)\Bigg].\end{aligned}$$

be $\mathbf{D}_d=\mathrm{diag}(d_1,\ldots,d_n)^\intercal$, with

$$d_i=-\frac{(\beta_i-t)^2}{(t_i-\kappa t_i+\beta_i\kappa)^2}-\frac{2\,(t_i-\beta_i)^2}{\beta_i\lambda^2t_i^{2\kappa}}\log^2(t_i)-(\alpha-1)\log^2(t_i)\,a_{t_i}w_i\left(-1+2a_{t_i}^2+a_{t_i}w_i\right),$$

then in matrix form we have that the Hessian matrix for vector will be $\mathrm{Tr}\,(\mathbf{D}_d)$. The last element of the block matrix $\mathbf{H}(\boldsymbol{\theta})$ it is obtained by the second derivative of $\ell(\boldsymbol{\theta})$ with respect to λ and κ is given by

$$\sum_{i=1}^{n}\frac{\partial^2\ell(\boldsymbol{\theta}\mid\mathbf{z})}{\partial\lambda\partial\kappa}=\sum_{i=1}^{n}\left[-\frac{2}{\beta_i\lambda^3}\frac{(t_i-\beta_i)^2}{t_i^{2\kappa}}\log(t_i)-\frac{(\alpha-1)}{\lambda}a_{t_i}w_i\left(-1+a_{t_i}^2+a_{t_i}w_i\right)\log(t_i)\right].$$

be $\mathbf{D}_r = \text{diag}(r_1, \ldots, r_n)^\top$, with

$$r_i = -\frac{2}{\beta_i \lambda^3} \frac{(t_i - \beta_i)^2}{t_i^{2\kappa}} \log(t_i) - \frac{(\alpha - 1)}{\lambda} a_{t_i} w_i \left(-1 + a_{t_i}^2 + a_{t_i} w_i\right) \log(t_i),$$

then in matrix form we have that the Hessian matrix for vector will be $\text{Tr}\left(\mathbf{D}_r\right)$.

Appendix B

The CDF of the α-OW distribution is:

$$G(z \mid \lambda, \beta, \kappa, \alpha) = \left[\Phi\left(a_z\right)\right]^{\alpha}, \quad z, \lambda, \beta, \alpha \ \in \ \Re_{>0},$$

where $a_z = \lambda^{-1}\left(\frac{z^{1-\kappa}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{z^{\kappa}}\right)$. Because the CDF is strictly monotonically, it is possible to obtain its inverse. The inversion method is based on the principle that continuous CDF are distributed uniformly over the open interval $(0, 1)$. If u is a uniform random number on $(0, 1)$, then $z = G^{-1}(u \mid \lambda, \beta, \kappa, \alpha)$ generates a pseudorandom number z from α-OW distribution. In this way

$$\begin{aligned}
G(z \mid \lambda, \beta, \kappa, \alpha) &= u \\
\left[\Phi\left(a_z\right)\right]^{\alpha} &= u \\
\Phi(a_z) &= u^{1/\alpha} \\
({}^1\!/\!{}_2) \times \lambda^{-1}\left(\frac{z^{1-\kappa}}{\sqrt{\beta}} - \frac{\sqrt{\beta}}{z^{\kappa}}\right) &= ({}^1\!/\!{}_2) \times \Phi^{-1}\left(u^{1/\alpha}\right) \\
\frac{z-\beta}{2\sqrt{\beta}z^{\kappa}} &= \underbrace{\frac{\lambda}{2}\Phi^{-1}\left(u^{1/\alpha}\right)}_{W} \\
\frac{z-\beta}{2\sqrt{\beta}z^{\kappa}} &= W,
\end{aligned}$$

where $W \sim \alpha\text{-N}(0, {}^{\lambda^2}\!/\!{}_2)$ and consequently, we must find the positive root of the equation

$$z = \beta + 2\sqrt{\beta}Wz^{\kappa} \Rightarrow \beta + 2\sqrt{\beta}Wz^{\kappa} - z = 0.$$

Appendix C (R code)

In this Appendix is represented R codes used in the application of this work.

```r
library(maxLik)
quant_reg_owen <- function(formula, link = 'identity', tau = 0.5, data = NULL, out.list = FALSE, ...){

  linkstr   <- link
  linkobj   <- make.link(linkstr)
  linkinv   <- linkobj$linkinv
  mu.eta    <- linkobj$mu.eta
  model_aux <- glm(formula, family = Gamma(link = link), data = data)
  Y <- model_aux$y
  X <- model.matrix(model_aux)

  if (length(Y) < 1)
    stop("empty model")
  if ((min(Y) < 0))
    stop("invalid dependent variable, all observations must be positive")

  n <- length(Y)
  p <- ncol(X)
  xb <- mean(Y)
  xh <- 1/mean(1/Y)
  shape_start <- sqrt(2*(sqrt(xb/xh) - 1 ))
  kappa_start <- 0.5
  beta_start <- model_aux$coefficients[1:p]

    log_lik <- function(par, y, x, tau, link){
      linkstr   <- link
      linkobj   <- make.link(linkstr)
      linkinv   <- linkobj$linkinv
      lambda    <- par[1]
      kappa     <- par[2]
      coef_beta <- par[-c(1,2)]
      eta       <- as.vector(x %*% coef_beta)
      nu    <- linkinv(eta)

      pdf_alpha_owen(x = y, lambda = lambda, nu = nu, tau = tau, kappa = kappa, log = TRUE)
```

```
    }

    start_par <- c(lambda= shape_start, kappa = kappa_start, beta_start)


    options(warn = -1)   # Disable warning messages globally
    fit <- maxLik(logLik = log_lik, start = start_par, method = "BHHH", y = Y, x = X, tau = tau, link = link, control = list(iterlim = 600))


  if (out.list == TRUE) {
    coef <- fit$estimate
    npar <- length(coef)
    par.df <- npar - p
    eta <- drop(X %*% coef[(par.df + 1):npar])
    mu <- linkinv(eta)
    residuals <- (Y - mu)/mu.eta(eta)
    conv <- fit$code
    aic <- AIC(fit)


    list(coefficients = coef, residuals = residuals, fitted.values = mu, family = family,
         linear.predictors = eta, y = Y, converged = conv, hess = fit$hessian, vcov = solve(-fit$hessian),
         AIC = aic, tau = tau, link = link, formula = formula)
  } else return(summary(fit))

}
```

Listing 1: R codes - quantile regression model.

```
envelope_owen <- function(model, k = 100, color = "grey50", xlabel = "Theorical Quantile", ylabel = "Empirical Quantile", font = "serif", cex.axis=1, cex.lab=1)
{
  par(mar = c(4,4.5,0.1,0.1))

  n   <- length(model$y)
  td  <- res_quant_owen(model)
  re  <- matrix(0,n,k)
```

```

for (i in 1:k)
{
  y1 <- rnorm(n)
  re[, i] <- sort(y1)
}

e10 <- numeric(n)
e20 <- numeric(n)
e11 <- numeric(n)
e21 <- numeric(n)
e12 <- numeric(n)
e22 <- numeric(n)

for (l in 1:n)
{
  eo <- sort(re[l,])
  e10[l] <- eo[ceiling(k*0.01)]
  e20[l] <- eo[ceiling(k*(1 - 0.01))]
  e11[l] <- eo[ceiling(k*0.05)]
  e21[l] <- eo[ceiling(k*(1 - 0.05))]
  e12[l] <- eo[ceiling(k*0.1)]
  e22[l] <- eo[ceiling(k*(1 - 0.1))]
}

a   <- qqnorm(e10, plot.it = FALSE)$x
r   <- qqnorm(td, plot.it = FALSE)$x
xb  <- apply(re, 1, mean)
rxb <- qqnorm(xb, plot.it = FALSE)$x

df  <- data.frame(r = r,
                  xab = a,
                  emin = cbind(e10,e11,e12),
                  emax = cbind(e20,e21,e22),
                  xb = xb,
                  td = td,
                  rxb = rxb)
ggplot(df, aes(r,td)) +
  geom_ribbon(aes(x = xab,
                  ymin = emin.e10,
                  ymax = emax.e20),
              fill = color,
```

```
                alpha = 0.5)  +
    geom_ribbon(aes(x = xab,
                    ymin = emin.e11,
                    ymax = emax.e21),
                fill = color,alpha = 0.5) +
    geom_ribbon(aes(x = xab,
                    ymin = emin.e12,
                    ymax = emax.e22),
                fill = color,alpha = 0.5) +
    scale_fill_gradient(low = "grey25",
                        high = "grey75") +
    geom_point() + geom_line(aes(rxb,xb),lty = 2) +
    xlab(xlabel) + ylab(ylabel) + theme_bw() +
    theme(panel.grid.major = element_blank(),
          panel.grid.minor = element_blank())  +
    theme(text = element_text(size =20, family = font))
}
```

Listing 2: R codes - Envelope.

```
library(gghighlight)
cooks_dist_owen <- function(model,  plot = FALSE, data, cutting
    = 0.5 )
{

  y <- model$y
  theta <- model$coefficients
  inv.lpp <- solve(model$vcov)
  n <- length(y)
  values <- vector()
  npar <- length(theta)

  for (i in 1:n){
  datai <- as.data.frame(data[-i,])
  fit <- quant_reg_owen(model$formula, link = model$link, tau =
      model$tau, out.list = TRUE, data = datai)
  thetai <- fit$coefficients
  values[i] <- (1/(npar))*t(theta - thetai) %*% inv.lpp %*% (
     theta - thetai)
    }

  if (plot == TRUE){
    df <- data.frame(index = 1:NROW(data), cook = values)
```

```
    ggplot(df) + geom_point(aes(x = index, y = cook), size = 3)
        +
      xlab("Observation index") + ylab("Generalized Cook's
         distance") +
      gghighlight(cook > cutting, label_key = index) +
      theme_test(base_size = 18)
  } else return(values)
}
```

Listing 3: R codes - Generalizaed Cook's distance.

```
res_quant_owen <- function(model){
y <- model$y
tau <- model$tau
estimates <- model$coefficients
scale_est <- model$fitted.values
shape_est <- estimates[1]
kappa_est <- estimates[2]
prob <- cdf_alpha_owen(q = y, lambda = shape_est, nu = scale_
   est, tau = tau, kappa =kappa_est)
res <- qnorm(prob)

res

}
```

Listing 4: R codes - Quantile residuals.

```
my.mle <- function(...) tryCatch(expr = mle(...), error =
   function(e) NA)

est <- function(n, lambda, k){

repeat{
y <- r_alpha_owen(n, lambda = lambda, nu = 1.0, tau = 0.5,
   kappa = k)
xb <- mean(y)
xh <- 1/mean(1/y)
shape_start <- sqrt(2*(sqrt(xb/xh) - 1 ))
nLL0 <- function(scale, shape) -sum(dbisa( y, scale, shape, log
    = TRUE))
fit0 <- mle(nLL0, start = list(scale= 1, shape = shape_start),
   nobs = NROW(y))

```

```
nLL <- function(lambda, nu, k) -sum(pdf_alpha_owen(x = y, tau =
    0.5, lambda = lambda, nu = nu, kappa = k, log = TRUE))
fit <- my.mle(nLL, start = list(lambda = coef(fit0)[2], nu =
   coef(fit0)[1], k = 0.5),
           nobs = NROW(y),
           method = "L-BFGS-B", lower = rep(0, 3), upper  = c(
              Inf, Inf, 1))

if (is.na(fit) == FALSE){
  coef_mle <- as.vector(coef(fit))
  if (coef_mle[3] < 1 & coef_mle[3] > 0) break

}

}

coef_mle <- as.vector(coef(fit))

return(list("shape" = coef_mle[1], "scale" = coef_mle[2], "
   kappa" = coef_mle[3] ))
}

library(MonteCarlo)
library(tidyverse)
library(ggpubr)
library(simhelpers)

n_grid <- c(50, 90, 130)
lambda_grid <- c(0.5, 1.0, 1.5)
kappa_grid <- c(0.3, 0.5, 0.7)
param_list <- list("n" = n_grid, "lambda" = lambda_grid, "k" =
   kappa_grid)
MC_result <- MonteCarlo(func = est, nrep = 5000, param_list =
   param_list)

df <- MakeFrame(MC_result)
df1 <- df %>% filter(lambda == 0.5, k == 0.3)  %>% mutate(n =
   factor(n, ordered = TRUE ) )
gghistogram(df1, x = "shape", add = "mean",
            color = "n", rug = TRUE,
            y = "..density..",
            palette =c("#00AFBB", "#E7B800", "#FC4E07"),
```

```
            fill = "n",
            xlab = expression(hat(lambda)),
            ggtheme = theme_bw() +  theme(legend.position = c(0.87, 0.7),
              axis.text  = element_text(size = 20),
                                        axis.title = element_text(size = 20),
                                        legend.text = element_text(size = 20),
                                        legend.title = element_text(size = 20),
              legend.background = element_rect(fill = "white", color = "black"),
              panel.grid = element_blank() )) +
            geom_vline(xintercept = 0.5)


gghistogram(df1, x = "scale", add = "mean",
            color = "n", rug = TRUE,
            y = "..density..",
            palette =c("#00AFBB", "#E7B800", "#FC4E07"),
            fill = "n",
            xlab = expression(hat(beta)),
            ggtheme = theme_bw() +  theme(legend.position = c(0.87, 0.7),
                                        axis.text  = element_text(size = 20),
                                        axis.title = element_text(size = 20),
                                        legend.text = element_text(size = 20),
                                        legend.title = element_text(size = 20),
                                        legend.background = element_rect(fill = "white", color = "black"),
                                        panel.grid = element_blank() )) +
  geom_vline(xintercept = 1.0)
```

```
gghistogram(df1, x = "kappa", add = "mean",
            color = "n", rug = TRUE,
            y = "..density..",
            palette =c("#00AFBB", "#E7B800", "#FC4E07"),
            fill = "n",
            bins = 30,
            xlab = expression(hat(kappa)),
            ggtheme = theme_bw() +  theme(legend.position = c(0.87, 0.7),
                                          axis.text  = element_text(size = 20),
                                          axis.title = element_text(size = 20),
                                          legend.text = element_text(size = 20),
                                          legend.title = element_text(size = 20),
                                          legend.background = element_rect(fill = "white", color = "black"),
                                          panel.grid = element_blank() )) +
  geom_vline(xintercept = 0.3)


resul_lambda <- df %>% group_by(n, lambda, k) %>% do(calc_absolute(., estimates = shape, true_param = lambda,  perfm_criteria = c("bias", "rmse"))) %>% round(4); print(resul_lambda, n = 27)
resul_beta <- df %>% mutate(b = 1.0) %>% group_by(n, lambda, k, b) %>% do(calc_absolute(., estimates = scale, true_param = b,  perfm_criteria = c("bias", "rmse"))) %>% round(4); print(resul_beta, n = 27)
resul_kappa <- df %>% group_by(n, lambda, k) %>% do(calc_absolute(., estimates = kappa, true_param = k,  perfm_criteria = c("bias", "rmse"))) %>% round(4); print(resul_kappa, n = 27)
```

Listing 5: R codes - Monte Carlo simulation.