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On the theory of Gorenstein dimension with respect to a semidualizing module

por

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João Pessoa - PB

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sob orientação do

Prof. Dr. Cleto Brasileiro Miranda Neto

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Neste trabalho estudamos as noções de módulos k-*torsionless* e módulos G-perfeitos reduzidos em um contexto mais geral, relativo a um módulo semidualizante. Consideramos vários aspectos e obtemos novas caracterizações dessas propriedades, o que nos levou a generalizar resultados anteriores de diversos autores. Como uma aplicação especial, investigamos a Cohen-Macaulaycidade, sob certas condições, de um módulo celebrado em álgebra e geometria: o módulo de derivações. Finalmente, a intersecção das duas principais classes gerais de módulos trabalhadas nesta tese é considerada e exemplificada.

Palavras-chave: Dimensão de Gorenstein; módulo semidualizante; transposta de Auslander; módulo *torsionless*; módulo horizontalmente ligado; módulo de derivações.

Abstract

This work studies the notions of k-torsionless modules and reduced G-perfect modules in a more general setting, relative to a semidualizing module. We consider various aspects and obtain new characterizations of these properties, which led us to generalize previous results by several authors. As a special application, we investigate the Cohen-Macaulayness, under certain conditions, on a celebrated module in both algebra and geometry: the module of derivations. Finally, the intersection of the two main general classes of modules worked out in this thesis is taken into account and exemplified.

Keywords: Gorenstein dimension; semidualizing module; Auslander transpose; torsionless module; horizontally linked module; derivation module.

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"A Matemática, quando a compreendemos bem, possui não somente a verdade, mas também a suprema beleza."

Bertrand Russell

Dedicatória

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Contents

Introduction

The notion of “semidualizing module” is one of the main notions in the relative homological algebra. Since this notion was independently introduced by Foxby [14], Vasconcelos [49] and Golod [18], its various aspects have been investigated by many authors from different stand points; see for example [7], [11], [18], [21], [39], [41], [42], [46], [48], [51].

Among various research areas on semidualizing modules, one basically focuses on extending the “absolute” classical notions and results of homological algebra to the “relative” setting with respect to a semidualizing module. For instance, this has been done for the classical and Gorenstein homological dimensions mainly through the works of Golod [18], Holm and Jørgensen [21] and White [51], and (co)homological theories have been extended to the relative setting with respect to a semidualizing module mainly through the works of Takahashi and White [48], Sather-Wagstaff et al. [44], [45], Salimi et al. [37], [38]. Following this idea, the present thesis aims at studying several properties of the two notions, namely “k-torsionless modules” and “reduced G-perfect modules”, in the relative setting with respect to a semidualizing module.

In [3], Auslander and Bridger introduced the notion of “k-torsionless modules” under the name “k-torsionfree modules”. They proved many results in the most general setting, e.g. over possibly non-commutative, non-Noetherian rings. The techniques used are quite abstract and unfamiliar to many commutative algebraists. In [32], Maşek develop the theory in the context of commutative Noetherian rings, and showed that, in this important special case, the theory is fairly elementary and easy to build. In [39], Salimi et al. extended the classical k-torsionless theory to the relative setting with respect to a semidualizing module, but only for modules in Bass class. In this thesis, by using de notion of "ω-k-torsionfree modules" due to Huang [23], we introduce the notion of k-torsionless module with respect to a semidualizing module C, called "C-k-torsionless module". Such modules were studied under other names by, for example, Araya and Iima [2], and Dibaei and Sadeghi [10].

In the first main result of this thesis, over a commutative Noetherian ring with identity, it is proved a number of characterizations for "C-k-torsionless modules" with finite Gorenstein dimension with respect to a semidualizing module C (see Theorem 2.2.7). More precisely,

Theorem A *Let R be a commutative Noetherian ring with identity, M a finitely generated R-module and $k \geq 0$ an integer. Consider the following conditions:*

(i) *M is C-k-torsionless;*

(ii) *M is C-k-syzygy;*

(iii) *There exists an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ of finitely generated R-modules with $\mathrm{G}_C\text{-}\dim_R(X_i) = 0$ for every $i = 0, \ldots, k-1$;*

(iv) *$M_{\mathfrak{p}}$ is k-torsionfree over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Supp}_R(M)$;*

(v) *M satisfies $\widetilde{S}_k$;*

(vi) $\mathrm{grade}_R(\mathrm{Ext}^i_R(M, C)) \geq i + k$, *for all $i > 0$.*

Then the following implications hold true.

(a) *(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).*

(b) *If M has finite G_C-dimension on $X^{k-1}(R)$, then (v) $\Rightarrow$ (i).*

(c) *If M has finite G_C-dimension, then all the conditions above are equivalent.*

The theorem above generalizes Theorem 42 of Maşek [32] to the relative setting with respect to a semidualizing module. It also generalizes Theorem 4.11 of Salimi et al [39] and refines Proposition 2.4 of Dibaei and Sadeghi [10]. As consequence we obtain generalizations of the results of [30], [32], [39] [40], [41] and [50]. As one of the applications of this theory we obtained characterizations for maximal Cohen-Macaulay derivation modules, as described in Miranda-Neto–Souza [36].

In [9], Dibaei and Sadeghi introduced notion of "reduced G-perfect modules" over commutative semiperfect (e.g., local) Noetherian rings. In [10], they extended this notion to the relative setting with respect to a semidualizing module C, called "reduced G_C-perfect modules". In this thesis, we continue study these modules focusing on Auslander transpose with respect to a semidualizing module and obtain new results in this direction.

One of the main results of this part is, over a commutative semiperfect Noetherian ring with identity, to show how reduced G_C-perfect property is preserved under the Auslander transpose with respect to semidualizing module C (see Theorem 3.2.7). More precisely,

Theorem B *Let R be a commutative semiperfect Noetherian ring with identity, M a finitely generated R-module. Let n and t be two integers, then the following statements are equivalent:*

(i) *M is reduced G_C-perfect of G_C-dimension n and $\mathrm{Ext}^n_R(M,C)$ is G_C-perfect of G_C-dimension $n+t-1$.*

(ii) *$\mathrm{Tr}_C(M)$ is reduced G_C-perfect of G_C-dimension t and $\mathrm{Ext}^t_R(\mathrm{Tr}_C(M),C)$ is G_C-perfect of G_C-dimension $n+t-1$.*

The theorem above generalizes Corollary 3.6 of Dibaei and Sadeghi [9] to the relative setting with respect to a semidualizing module. As consequence of that theory we obtain generalizations of some results of [3], [9] and [31].

We now proceed to a more detailed description of the various parts of the thesis.

Chapter 1 is devoted to the preliminaries used throughout. In the first two sections, we recall the definition of "semidualizing module", "Gorenstein dimension with respect to a semidualizing module" and "Auslander transpose with respect to a semidualizing module", as well as some of its properties. *Sections* 1.3, 1.4, 1.5 and 1.6 are about Auslander and Bass classes, grade and G-perfect modules with respect to a semidualizing module, horizontally linked modules, and modules of derivations, respectively.

Chapter 2 is focused on extending the classical results of k-torsionless, k-syzygy and dual modules to the "relative" setting with respect to a semidualizing module. In *Section* 2.1, we introduce the relative notion of "C-k-syzygy and C-k-torsionless modules" where C is a semidualizing module; see Definitions 2.1.1 and 2.1.5. We study the connection between the relative and absolute notions. We also describe the behavior of "C-k-torsionless modules" along an exact sequence.

Section 2.2 is the core of the chapter, we prove Theorem 2.2.7 (Theorem A) that provides a number of characterizations for "C-k-torsionless module" with finite Gorenstein dimension with respect to a semidualizing module C, which generalizes [32, Theorem 42] and also can be viewed as a generalization of [39, Theorem 4.11] as well as a refinement of [10, Proposition 2.4]. Among many consequences, it is shown, for example, that every C-dual module with a certain property is C'-reflexive, where C' is another semidualizing module. In *Section* 2.3, we study the "C-q-Gorenstein rings" which are a "relative" notion of the q-Gorenstein rings from Maşek [32] with respect to a semidualizing module C. We prove Theorem 2.3.6 that improves Theorem 2.2.7 for some conditions, which generalizes [32, Theorem 43] and also can be viewed as a generalization of [39, Theorem 4.10]. As a consequence, it is shown that on "C-q-Gorenstein rings" the notions of "C-k-torsionless" and "C-k-syzygy" coincide up to a certain order (see Corollary 2.3.7).

In *Section* 2.4, we study the dual modules with respect to a semidualizing module C of finite Gorenstein dimension with respect to another semidualizing module C'. We prove Theorem 2.4.1 that provides an upper bound for the Gorenstein dimension with respect to C' for certain "C-dual modules" and, as a consequence, we generalize [40, Corollary 2.4] and [41, Proposition 3.9] (see Corollaries 2.4.2 and 2.4.3). We also study the module of derivations with respect to a semidualizing module, focusing on Cohen-Macaulay property. Finally, in *Section* 2.5, we return to the "absolute" setting when the semidualizing module is the base ring. We prove Theorem 2.5.3 that provides a sufficient condition for the module of derivations to be free on local domains. As a consequence, we have Corollary 2.5.6 and Corollary 2.5.7, both independently recover [30, Theorem 4]. Such results were published in Miranda-Neto–Souza [35].

Chapter 3 is dedicated to study the "reduced G_C-perfect modules" focusing on Auslander transpose with respect to a semidualizing module C. In *Section* 3.1, we prove Proposition 3.1.6 and Theorem 3.1.7 that provide information about the Ext module and the Gorenstein dimension with respect to a semidualizing module C of the relative Auslander transpose these modules. In *Section* 3.2, we prove Theorem 3.2.7 (Theorem B) that shows how reduced G_C-perfect property is preserved under the Auslander transpose with respect to a semidualizing module C. As a consequence, we generalize [9, Corollary 3.6].

In *Section* 3.3, we study the theory of linkage for G_C-perfect and reduced G_C-perfect modules. We prove Theorem 3.3.3 that provides a class of modules that is "horizontally linked", which generalizes [31, Theorem 1]. We also prove Theorem 3.3.10 that provides a characterization for the horizontal linkage of reduced G_C-perfect modules. Finally, in *Section* 3.4, we study the connection between "C-k-torsionless modules" and "reduced G_C-perfect modules". We prove Proposition 3.4.2 that provides a sufficient condition for a reduced G_C-perfect module to be C-k-torsionless. We also provide an example of module that is both C-k-torsionless and reduced G_C-perfect (see Example 3.4.6) what seems new in the literature.

Finally, to facilitate the reading of the thesis and make it more objective, as well as a little more self-sufficient, we add three appendices. *Appendix* A is dedicated to some introductory results of homological algebra. In *Appendix* B, we list some concepts and basic results of commutative algebra involving the three main classes of rings used in the thesis: regular, Gorenstein and Cohen-Macaulay. In *Appendix* C, we present the semiperfect rings where horizontal linkage can be defined for finitely generated modules.

Notation and terminology

We begin with a brief introduction to the notations used throughout this thesis.

- R is a commutative Noetherian ring with identity.
- M, N, P, K denote R-modules.
- k denotes both a ring and a non-negative integer, it depends on the context.
- C and C' denote semidualizing R-modules.
- $(-)^C = \mathrm{Hom}_R(-, C)$ is the relative duality functor with respect to C.
- $\mathrm{pd}_R\,(M)$ is the projective dimension of M.
- $\mathrm{G}_C\text{-}\mathrm{dim}_R\,(M)$ is the Gorenstein dimension of M with respect to C.
- $\mathrm{Tr}_C(M)$ is a transpose of M with respect to C.
- $\mathrm{Der}_k(R, M)$ is the module of k-derivations of R into M.
- $\lambda = \Omega\mathrm{Tr}$ is the operator horizontally linkage.
- $\mathcal{A}_C$ is the Auslander class.
- $\mathcal{B}_C$ is the Bass class.
- ω_R is the canonical module of a Cohen-Macaulay local ring R.

Also, unless otherwise stated, all rings will be assumed to be commutative Noetherian with identity.

Chapter 1

Preliminaries

This preliminary chapter records the blanket assumptions, some basic notation, and a few important results to be used throughout the thesis.

Throughout this thesis, unless otherwise stated, R is a commutative Noetherian ring with identity, and by "finite module" we mean "finitely generated module".

1.1 Gorenstein dimension with respect to a semidualizing module

We first recall one of the central notions of this work, the concept of semidualizing module.

Definition 1.1.1 A finite R-module C is called a *semidualizing* module if

(i) The homothety morphism $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism;

(ii) $\operatorname{Ext}_R^i(C, C) = 0$ for all $i > 0$.

If furthermore C has finite injective dimension then C is said to be a *dualizing* module.

It is obvious that R itself is a semidualizing R-module. Also if R is Cohen-Macaulay then its canonical module (if it exists) is a dualizing module

Remark 1.1.2 It is not obvious that a local ring admits semidualizing modules other than itself and, possibly, a canonical module. This was raised as a question in 1985 by Golod, see [19], and in 1987 Foxby gave examples of rings with three different semidualizing modules ([7], p. 1874). In 2001 Christensen [7] described a procedure for constructing Cohen-Macaulay local rings with any finite number of semidualizing modules.

Notation 1.1.3 In what follows, C always denotes a semidualizing R-module. Let $(-)^C = \mathrm{Hom}_R(-, C)$ be the relative duality functor with respect to C. As an exception, in the special case where $C = R$, we use $(-)^* = \mathrm{Hom}_R(-, R)$. For every R-module M we denote by $\sigma_M^C : M \to M^{CC}$, the natural biduality map with respect to C, defined by $\sigma_M^C(x)(f) = f(x)$ for any $x \in M$ and $f \in M^C$.

Now we can present our first relative definitions with respect to the semidualizing module C.

Definition 1.1.4 A finite R-module M is called *C-torsionless*, resp. *C-reflexive*, if the biduality map σ_M^C is injective, resp. an isomorphism.

Definition 1.1.5 A finite R-module M is said to be *totally C-reflexive* if

(i) M is C-reflexive;

(ii) $\mathrm{Ext}_R^i(M, C) = 0$, for all $i > 0$;

(iii) $\mathrm{Ext}_R^i(M^C, C) = 0$, for all $i > 0$.

In the special case where $C = R$, a totally R-reflexive module is simply called a *totally reflexive module.*

The next result describes the behavior of the totally C-reflexive modules along an exact sequence.

Proposition 1.1.6 [43, Proposition 5.1.1 and Proposition 5.1.3] *Consider an exact sequence of finite R-modules $0 \to M' \to M \to M'' \to 0$. If M', M'' or M, M'' are totally C-reflexive, then the third module is also totally C-reflexive. If M', M are totally C-reflexive and $\mathrm{Ext}_R^1(M'', C) = 0$ then M'' is totally C-reflexive.*

The Gorenstein dimension was extended to G_C-dimension by Foxby in [14] and by Golod in [18]. For a comprehensive treatment of the general theory of G_C-dimension, see Sather-Wagstaff [43].

Definition 1.1.7 Let M be a finite R-module. A G_C*-resolution* of M is an exact sequence

$$\cdots \to X_i \to X_{i-1} \to \cdots \to X_0 \to M \to 0,$$

such that X_i is totally C-reflexive for all $i \geq 0$. Note that every finite projective R-module is totally C-reflexive (see [43, Proposition 2.1.13]), and so every finite R-module admits a G_C-resolution. The smallest non-negative integer n for which there exists a G_C-resolution

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0,$$

is called the G_C*-dimension* of M. If n does not exist, we consider that the G_C-dimension of M is infinite. In the special case where $C = R$, the G_R-dimension of M, denoted by $\mathrm{G}\text{-}\dim_R(M)$, is simply the *Gorenstein dimension* of M.

Remark 1.1.8 If M has G_C-dimension n, we write $\mathrm{G}_C\text{-}\dim_R(M) = n$. Therefore M is totally C-reflexive if and only if $\mathrm{G}_C\text{-}\dim_R(M) = 0$. By [43, Proposition 2.1.4], $\mathrm{G}_C\text{-}\dim_R(M \oplus N) = 0$ if and only if $\mathrm{G}_C\text{-}\dim_R(M) = 0$ and $\mathrm{G}_C\text{-}\dim_R(N) = 0$.

In the following, we will present a list of results involving the G_C-dimension.

Proposition 1.1.9 [16, Corollary 3.4] *Let M be a finite R-module and n a non-negative integer. Then* $\mathrm{G}_C\text{-}\dim_R(M) \leq n$ *if and only if* $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq n$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$. *Therefore*

$$\mathrm{G}_C\text{-}\dim_R(M) = \sup\{\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

Proposition 1.1.10 [43, Proposition 6.1.7] *Let M be a non-zero finite R-module of finite* G_C*-dimension. Then*

$$\mathrm{G}_C\text{-}\dim_R(M) = \sup\{i \geq 0 \mid \mathrm{Ext}^i_R(M, C) \neq 0\}.$$

Proposition 1.1.11 [43, Proposition 6.1.8] *Consider an exact sequence of finite R-modules*

$$0 \to M' \to M \to M'' \to 0.$$

If two of them have finite G_C*-dimension, then so does the third.*

Proposition 1.1.12 [43, Proposition 6.1.10] *Consider an exact sequence of finite R-modules of finite* G_C*-dimension*

$$0 \to M' \to M \to M'' \to 0.$$

If $\mathrm{G}_C\text{-}\dim_R(M) = 0$ *and* $\mathrm{G}_C\text{-}\dim_R(M'') > 0$, *then* $\mathrm{G}_C\text{-}\dim_R(M') = \mathrm{G}_C\text{-}\dim_R(M'') - 1$.

The following theorem extends the Auslander-Bridger formula [3, Theorem 4.13(b)] to the G_C-dimension setting.

Theorem 1.1.13 [16, Theorem 4.4] *Let R be a local ring and M a non-zero finite R-module of finite* G_C*-dimension. Then*

$$\mathrm{G}_C\text{-}\dim_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M).$$

Proposition 1.1.14 [17, Proposition 1.3] *Let R be a local ring. Then C is a dualizing module if and only if* $\mathrm{G}_C\text{-}\dim_R(M) < \infty$ *for all finite R-module M.*

The following result shows that G_C-dimension is a refinement of projective dimension for finite modules.

Proposition 1.1.15 [43, Corollary 6.4.5] *Let M be a finite R-module. There is an inequality* $\mathrm{G}_C\text{-}\dim_R(M) \leq \operatorname{pd}_R(M)$ *with equality when* $\operatorname{pd}_R(M) < \infty$.

Proposition 1.1.16 [43, Remark 6.1.9] *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite R-modules. Then we have the following inequalities:*

(i) $\mathrm{G}_C\text{-}\dim_R(M') \leq \sup\{\mathrm{G}_C\text{-}\dim_R(M), \mathrm{G}_C\text{-}\dim_R(M'')\}$,

(ii) $\mathrm{G}_C\text{-}\dim_R(M) \leq \sup\{\mathrm{G}_C\text{-}\dim_R(M'), \mathrm{G}_C\text{-}\dim_R(M'')\}$,

(iii) $\mathrm{G}_C\text{-}\dim_R(M'') \leq \sup\{\mathrm{G}_C\text{-}\dim_R(M'), \mathrm{G}_C\text{-}\dim_R(M)\} + 1$.

The last results of this section involve regular sequences.

Proposition 1.1.17 [16, Corollary 3.2 and Remark 3.6] *Let $x_1, \ldots, x_n \in R$ be given elements. Then $\{x_1, \ldots, x_n\}$ is an R-regular sequence if and only if $\{x_1, \ldots, x_n\}$ is a C-regular sequence. Moreover, if* $\mathrm{G}_C\text{-}\dim_R(M) = 0$ *and $\{x_1, \ldots, x_n\}$ is an R-regular sequence, then $\{x_1, \ldots, x_n\}$ is an M-regular sequence.*

The following result seems to be well-known. However, we give a proof as we do not have a reference.

Lemma 1.1.18 *Let $(R, \mathfrak{m})$ be a local ring. Let M, N be R-modules such that the R-module* $\operatorname{Hom}_R(M, N)$ *is non-zero and finite. If $\{x, y\} \subseteq \mathfrak{m}$ is an N-regular sequence, then $\{x, y\}$ is also a* $\operatorname{Hom}_R(M, N)$*-regular sequence. In particular, if $\{x, y\} \subseteq \mathfrak{m}$ is an R-regular sequence, then $\{x, y\}$ is also a M^C-regular sequence.*

Proof. Set $H = \operatorname{Hom}_R(M, N)$. From the short exact sequence $0 \to N \xrightarrow{f_x} N \to N/xN \to 0$, where f_x is the multiplication by x on N, we get an exact sequence

$$0 \to H \xrightarrow{\mu_x} H \to \operatorname{Hom}_R(M, N/xN), \tag{1.1}$$

where $\mu_x = \mathrm{Hom}_R(M, f_x)$ is the multiplication by x on H. Therefore x is an H-regular element. Similarly, from the short exact sequence

$$0 \to N/xN \xrightarrow{f_y} N/xN \to N/(x,y)N \to 0,$$

we get that y is a $\mathrm{Hom}_R(M, N/xN)$-regular element. From (1.1), H/xH is isomorphic to a submodule of $\mathrm{Hom}_R(M, N/xN)$. Therefore, y is also H/xH-regular. Finally, by Nakayama's Lemma, $H/(x,y)H \neq 0$. In particular, if $\{x, y\} \subseteq \mathfrak{m}$ is an R-regular sequence then, by Proposition 1.1.17, $\{x, y\} \subseteq \mathfrak{m}$ is a C-regular sequence, and the result follows from what we proved above. □

1.2 Auslander transpose with respect to a semidualizing module

We now consider the following general concept of transpose of a finite module with respect to semidualizing module (cf. Geng [16]).

Definition 1.2.1 Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective presentation of a finite R-module M. Then, applying functor $(-)^C := \mathrm{Hom}_R(-, C)$ on the projective presentation of M, we get the exact sequence

$$0 \to M^C \to P_0^C \xrightarrow{f^C} P_1^C \to \mathrm{Coker}\,(f^C) \to 0.$$

We call $\mathrm{Coker}\,(f^C)$ a *transpose of* M *with respect to* C, and denote it by $\mathrm{Tr}_C(M)$. In the special case where $C = R$, this notion coincides with the usual Auslander transpose, denoted by $\mathrm{Tr}M$.

Definition 1.2.2 We write $\mathrm{add}_R(C)$ for the subclass consisting of all finite R-modules isomorphic to direct summands of finite direct sums of copies of C. Two finite R-modules M and N are said to be $\mathrm{add}_R(C)$-*equivalent*, denoted by $M \approx_C N$, if there exist X, Y in $\mathrm{add}_R(C)$ such that

$$M \oplus X \cong N \oplus Y.$$

In the special case where $C = R$, the notion of $\mathrm{add}_R(C)$-equivalence of modules coincides with projective equivalence of modules and we simply use $\approx$ instead of $\approx_C$.

Remark 1.2.3 Let M be a finite R-module. Then

(i) $\mathrm{Tr}_C(M)$ is unique up to $\mathrm{add}_R(C)$-equivalence, see [16, Proposition 2.2]. Thus, each $\mathrm{Ext}^i_R(\mathrm{Tr}_C(M), C)$ is unique up to isomorphisms for any $i > 0$.

(ii) It is easy to see that $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M))$ is well defined, that is, it does not depend on the choice of the representantive of $\mathrm{Tr}_C(M)$. Therefore, by Theorem 1.1.13, $\mathrm{depth}_R(\mathrm{Tr}_C(M))$ is well defined, if R is a local ring and $\mathrm{Tr}_C(M) \neq 0$ has finite G_C-dimension.

In the following, we will present a list of results involving the transpose with respect to C.

Proposition 1.2.4 [10, Lemma 2.2] *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite R-modules. Then, for a suitable choice of transposes, we have a long exact sequence:*

$$0 \to (M'')^C \to (M)^C \to (M')^C \to \mathrm{Tr}_C(M'') \to \mathrm{Tr}_C(M) \to \mathrm{Tr}_C(M') \to 0.$$

Proposition 1.2.5 [42, Lemma 2.12] *For a finite R-module M, there exists the following exact sequence*

$$0 \to M \to \mathrm{Tr}_C(\mathrm{Tr}_C(M)) \to X \to 0,$$

where $\mathrm{G}_C\text{-}\dim_R(X) = 0$.

Proposition 1.2.6 [16, Lemma 2.5 and Proposition 2.7] *Let M be a finite R-module.*

(i) *We have the following exact sequence*

$$0 \to \mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) \to M \xrightarrow{\sigma^C_M} M^{CC} \to \mathrm{Ext}^2_R(\mathrm{Tr}_C(M), C) \to 0.$$

(ii) *If M has G_C-dimension zero then M^C has G_C-dimension zero.*

(iii) *M has G_C-dimension zero if and only if $\mathrm{Tr}_C(M)$ has G_C-dimension zero.*

The following remark will be used in the proof of Theorem 2.2.7.

Remark 1.2.7 Let M be a finite R-module. For a generating set $\{f_1, \ldots, f_n\}$ of M^C, denote $f : M \to C^n$ as the map $(f_1, \ldots, f_n)$. It follows from Proposition 1.2.6 (i) that f is injective if and only if $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) = 0$. In this case, there is an exact sequence

$$0 \to M \xrightarrow{f} C^n \to N \to 0,$$

which is dual exact with respect to $(-)^C$, hence $\mathrm{Ext}^1_R(N, C) = 0$. Such an exact sequence is called a *universal pushforward of M with respect to C* (see [10, p. 4462]).

Lemma 1.2.8 [28, Lemma 2.7] *Let $0 \to M \to X_1 \to X_0 \to N \to 0$ be an exact sequence of finite R-modules where X_0 and X_1 have G_C-dimension zero. Then there exists the following exact sequence*

$$0 \to M \to Y \to X \to N \to 0,$$

where Y is in $\mathrm{add}_R(C)$ and X has G_C-dimension zero.

Proposition 1.2.9 [16, Proposition 3.1 and Proposition 3.3] *Let M be a finite R-module. Then the following assertions hold:*

(i) *For any $\mathfrak{p} \in \operatorname{Spec}(R)$, $C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-module.*

(ii) $(\operatorname{Tr}_C(M))_{\mathfrak{p}} \approx_{C_{\mathfrak{p}}} \operatorname{Tr}_{C_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *for any $\mathfrak{p} \in \operatorname{Spec}(R)$.*

The following result is well-known when R is semiperfect [10, Remark 2.1 (i)]. However, we give a proof as we do not have a reference in the general setting.

Proposition 1.2.10 *Let M be a finite R-module. Then,* $\operatorname{Tr}_C(M) \approx_C \operatorname{Tr}M \otimes_R C$. *Moreover, if* $\operatorname{Tr}M$ *and* $\operatorname{Tr}_C(M)$ *come from the same projective presentation of M, then* $\operatorname{Tr}_C(M) \cong \operatorname{Tr}M \otimes_R C$.

Proof. Assume that $P_1 \xrightarrow{f} P_0 \to M \to 0$ is a projective presentation of M. For each $i = 0, 1$, let $\omega_i : (P_i)^* \otimes_R C \to \operatorname{Hom}_R(P_i, R \otimes_R C)$ be the homomorphism defined as $\omega_i(\psi \otimes m)(p) = \psi(p) \otimes m$, for each $\psi \in (P_i)^*$, $m \in C$ and $p \in P_i$. Let $\mu : R \otimes_R C \to C$ be the isomorphism, given by $\mu(a \otimes m) = am$, for each $a \in R$ and $m \in C$. Being P_i a finite projective module, we have that ω_i is an isomorphism for each $i = 0, 1$. Thus, $\varphi_i := \operatorname{Hom}_R(P_i, \mu) \circ \omega_i : (P_i)^* \otimes_R C \to (P_i)^C$ is an isomorphism for each $i = 0, 1$. By applying the functors $(-)^* := \operatorname{Hom}_R(-, R)$, $(-) \otimes_R C$, and $(-)^C := \operatorname{Hom}_R(-, C)$ on the projective presentation of M, we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & (P_0)^* \otimes_R C \arrow[r, "f^* \otimes \mathrm{id}_C"] \arrow[d, "\varphi_0"] & (P_1)^* \otimes_R C \arrow[r] \arrow[d, "\varphi_1"] & \operatorname{Coker}(f^*) \otimes_R C \arrow[r] & 0 \\
0 \arrow[r] & M^C \arrow[r] & (P_0)^C \arrow[r, "f^C"] & (P_1)^C \arrow[r] & \operatorname{Coker}(f^C) \arrow[r] & 0
\end{tikzcd}$$

Therefore, $\operatorname{Coker}(f^*) \otimes_R C \cong \operatorname{Coker}(f^* \otimes \mathrm{id}_C)$ and $\operatorname{Coker}(f^* \otimes \mathrm{id}_C) \cong \operatorname{Coker}(f^C)$. As $\operatorname{Coker}(f^*) \approx \operatorname{Tr}M$, we have $\operatorname{Coker}(f^*) \otimes_R C \approx_C \operatorname{Tr}M \otimes_R C$. Now, as $\operatorname{Tr}_C(M) \approx_C \operatorname{Coker}(f^C)$, the assertion follows. $\square$

1.3 Auslander and Bass classes

A semidualizing module defines two important classes of modules, namely the Auslander and Bass classes, with a certain nice duality property. These two classes, defined next, were introduced by Foxby [14].

Definition 1.3.1 The *Auslander class with respect to* C, denoted by $\mathcal{A}_C$, consists of all R-modules M satisfying the following conditions:

(i) The natural map $\gamma_M^C : M \to \operatorname{Hom}_R(C, M \otimes_R C)$ is an isomorphism;

(ii) $\operatorname{Tor}_i^R(C, M) = 0 = \operatorname{Ext}_R^i(C, M \otimes_R C)$ for all $i > 0$.

Dually, the *Bass class with respect to* C, denoted by $\mathcal{B}_C$, consists of all R-modules M satisfying the following conditions:

(i) The natural evaluation map $\xi_M^C : C \otimes_R \operatorname{Hom}_R(C, M) \to M$ is an isomorphism;

(ii) $\operatorname{Ext}_R^i(C, M) = 0 = \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M))$ for all $i > 0$.

In the following, we collect some basic properties and examples of modules in $\mathcal{A}_C$, respectively in $\mathcal{B}_C$.

Remark 1.3.2 The following hold:

(i) If any two R-modules in a short exact sequence are in $\mathcal{A}_C$, respectively $\mathcal{B}_C$, then so is the third one, see [48, 1.9].

(ii) Every module of finite flat dimension is in $\mathcal{A}_C$. Also, the class $\mathcal{B}_C$ contains all modules of finite injective dimension, see [48, 1.9].

(iii) Over a Cohen-Macaulay local ring R with canonical module ω_R, $M \in \mathcal{A}_{\omega_R}$ if and only if $\operatorname{G-dim}_R(M) < \infty$, see [15, Theorem 1].

(iv) Let M, N be finite R-modules such that $M \approx N$. Then $M \in \mathcal{A}_C$ if and only if $N \in \mathcal{A}_C$.

Now, we present some basic results involving Auslander and Bass classes.

Proposition 1.3.3 [10, Lemma 2.11] *Let R be a local ring and M a finite R-module. If $M \in \mathcal{A}_C$, then* $\operatorname{depth}_R(M) = \operatorname{depth}_R(M \otimes_R C)$.

Theorem 1.3.4 [48, Theorem 4.1 and Corollary 4.2] *Let M and N be R-modules. If M and N are in $\mathcal{A}_C$, then*

$$\operatorname{Ext}_R^i(M \otimes_R C, N \otimes_R C) \cong \operatorname{Ext}_R^i(M, N),$$

for all $i \geq 0$.

Remark 1.3.5 Let M be a finite R-module such that $\operatorname{Tr}M \in \mathcal{A}_C$. Then, by Proposition 1.2.10 and Theorem 1.3.4, we have the following isomorphisms

$$\operatorname{Ext}_R^i(\operatorname{Tr}_C(M), C) \cong \operatorname{Ext}_R^i(\operatorname{Tr}M \otimes_R C, C) \cong \operatorname{Ext}_R^i(\operatorname{Tr}M, R),$$

for all $i > 0$.

Theorem 1.3.6 [48, Theorem 2.8] *Let M be an R-module. Then the following hold:*

(i) $M \in \mathcal{A}_C$ *if and only if* $M \otimes_R C \in \mathcal{B}_C$.

(ii) $M \in \mathcal{B}_C$ *if and only if* $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C$.

We can now describe, in another way, the modules in $\mathrm{add}_R\,(C)$.

Lemma 1.3.7 $\mathrm{add}_R\,(C) = \{P \otimes_R C \mid P$ *is a finite projective R-module*$\}$.

Proof. Let P be a finite projective R-module. Then, there exists Q an R-module such that $P \oplus Q \cong R^n$ for some integer $n \geq 0$. Applying $(-) \otimes_R C$ to $P \oplus Q \cong R^n$ implies

$$(P \otimes_R C) \oplus (Q \otimes_R C) \cong \oplus^n C.$$

Thus $P \otimes_R C \in \mathrm{add}_R\,(C)$. Conversely, assume that $M \in \mathrm{add}_R\,(C)$. So there exists N an R-module such that $M \oplus N \cong \oplus^n C$ for some integer $n \geq 0$. Applying $\mathrm{Hom}_R(C, -)$ to $M \oplus N \cong \oplus^n C$ implies

$$\mathrm{Hom}_R(C, M) \oplus \mathrm{Hom}_R(C, N) \cong \mathrm{Hom}_R(C, M \oplus N) \cong \mathrm{Hom}_R(C, \oplus^n C) \cong R^n.$$

Thus, $\mathrm{Hom}_R(C, M)$ is a finite projective R-module. By Remark 1.3.2 (ii), we have $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C$. Also we have, by Theorem 1.3.6, $M \in \mathcal{B}_C$. Therefore $M \cong \mathrm{Hom}_R(C, M) \otimes_R C$, where $\mathrm{Hom}_R(C, M)$ is a finite projective R-module. $\square$

A finite R-module M is said to be of *finite* G_C*-dimension on* $X \subseteq \mathrm{Spec}\,(R)$, if $\mathrm{G}_{C_\mathfrak{p}}\text{-}\mathrm{dim}_{R_\mathfrak{p}}\,(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in X$. We denote

$$X^n(R) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid \mathrm{depth}\,(R_\mathfrak{p}) \leq n\}.$$

Theorem 1.3.8 [10, Theorem 2.12] *Let M be a finite R-module. Assume that $M \in \mathcal{A}_C$ and that k is a positive integer. Consider the following statements:*

(i) $\mathrm{Ext}^i_R(\mathrm{Tr}M, R) = 0$ *for all* $1 \leq i \leq k$.

(ii) $\mathrm{Ext}^i_R(\mathrm{Tr}M, C) = 0$ *for all* $1 \leq i \leq k$.

Then we have (i) $\Rightarrow$ (ii). If M has finite G_C*-dimension on $X^{k-1}(R)$ the statements are equivalent.*

In [39] Salimi et al generalized the notion of Auslander transpose (in a different way from Geng's) for finite R-modules admitting a "C-projective presentation" in the sense defined below.

Definition 1.3.9 Let M be a finite R-module.

(i) A *C-projective presentation* of M is an exact sequence

$$X_1 \to X_0 \to M \to 0,$$

where each X_i is in $\mathrm{add}_R(C)$.

(ii) If M has a C-projective presentation

$$X_1 \xrightarrow{g} X_0 \to M \to 0,$$

the *Auslander dual of M with respect to C*, denoted by $\mathrm{D}_C(M)$, is defined as $\mathrm{D}_C(M) := \mathrm{Coker}\,(g^C)$.

Remark 1.3.10 Let M be a finite R-module such that $M \in \mathcal{B}_C$. Then

(i) $\mathrm{D}_C(M)$ is unique up to projective equivalence, see [39, Theorem 3.5].

(ii) $\mathrm{D}_C(M) \approx \mathrm{Tr}(\mathrm{Hom}_R(C, M))$, see [39, Proposition 3.11].

1.4 Grade and G_C-perfect modules

The grade is an important invariant of modules and it is closely connected with the G_C-dimension.

We begin with the definition of grade and present some characterizations.

Definition 1.4.1 The *grade* of a module M is defined as

$$\mathrm{grade}_R\,(M) = \inf\{i \geq 0 \mid \mathrm{Ext}^i_R(M, R) \neq 0\}.$$

Proposition 1.4.2 [3, Corollary 4.6] *Let M be a non-zero finite R-module. Then*

$$\mathrm{grade}_R\,(M) = \min\{\mathrm{depth}\,(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Supp}_R\,(M)\}.$$

Proposition 1.4.3 *Let M be a finite R-module. Then,*

$$\mathrm{grade}_R\,(M) = \inf\{i \geq 0 \mid \mathrm{Ext}^i_R(M, C) \neq 0\}.$$

In particular, $\mathrm{grade}_R\,(M) \leq \mathrm{G}_C\text{-}\mathrm{dim}_R\,(M)$.

Proof. We may assume that $M \neq 0$. By [43, Proposition 2.1.16(c)], $\mathrm{Ann}_R(M)C \neq C$. By Proposition 1.1.17, a sequence $\{x_1, \ldots, x_n\}$ is an R-regular sequence if and only if $\{x_1, \ldots, x_n\}$ is a C-regular sequence. Thus, by [4, Proposition 1.2.10(e)], we have

$$\begin{aligned} \mathrm{grade}_R(M) &= \mathrm{grade}_R(\mathrm{Ann}_R(M), R) \\ &= \mathrm{grade}_R(\mathrm{Ann}_R(M), C) \\ &= \inf\{i \geq 0 \mid \mathrm{Ext}^i_R(M, C) \neq 0\}. \end{aligned}$$

Therefore the assertion is clear. □

We recall the following definition from [18].

Definition 1.4.4 Let M be a finite R-module, we say that M is a G_C-*perfect* module if

$$\mathrm{grade}_R(M) = \mathrm{G}_C\text{-}\mathrm{dim}_R(M).$$

Such modules were introduced under the name *quasi-perfect* by Foxby [15], and they were studied by Golod [18]. In the special case where $C = R$, the G_R-perfect module is simply called G-*perfect*.

Proposition 1.4.5 [18, 10] *Let R be a local ring and M a G_C-perfect R-module of grade n. Then $\mathrm{Ext}^n_R(M, C)$ is a G_C-perfect R-module of grade n.*

1.5 Horizontally linked modules

Throughout this section, R is a semiperfect ring (e.g. R local; see Appendix C) and all modules are finite R-modules.

Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective presentation of M. By Remark 1.2.3, the transpose of M with respect to R, denoted by $\mathrm{Tr}M$, is unique up to projective equivalence. Thus, by Lemma C.5, minimal projective presentations of M represent isomorphic transposes of M.

Let $P \xrightarrow{\alpha} M$ be an epimorphism such that P is a projective module. The *syzygy module of* M, denoted by ΩM, is the kernel of α which is unique up to projective equivalence, by Schanuel's lemma (see e.g. [47, Proposition 3.12]). Thus ΩM is uniquely determined, up to isomorphism, by a projective cover of M.

Martsinkovsky and Strooker [31] generalized the notion of linkage for modules over non-commutative semiperfect Noetherian rings (i.e. finite modules over such rings have projective covers). They introduced the operator $\lambda = \Omega\mathrm{Tr}$, where $\Omega\mathrm{Tr}M$ is defined assuming that $P_1 \xrightarrow{f} P_0 \to M \to 0$ is a minimal projective presentation of M, and assuming that $P \to \mathrm{Tr}M$ is a projective cover of $\mathrm{Tr}M$. Based on this generalization, several works have been done on studying the linkage theory in the context of modules; see for example [8], [9], [10].

Definition 1.5.1 [31, Definition 3] Two R-modules M and N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$. Equivalently, M is called horizontally linked (to λM) if $M \cong \lambda^2 M$.

A *stable module* is a module with no non-zero projective direct summands. In the following, we collect some basic properties, examples and a characterization of horizontally linked modules.

Theorem 1.5.2 [31, Theorem 2] *An R-module M is horizontally linked if and only if it is stable and* $\mathrm{Ext}^1_R(\mathrm{Tr}M, R) = 0$.

Theorem 1.5.3 [31, Theorem 1] *Let M be a stable R-module of* G*-dimension zero. Then M is horizontally linked and its link is a stable R-module of* G*-dimension zero.*

Proposition 1.5.4 [31, Proposition 3] *A horizontally linked R-module is stable.*

Proposition 1.5.5 [31, Proposition 4] *Suppose M is horizontally linked. Then λM is also horizontally linked and, in particular, λM is stable.*

Theorem 1.5.6 [1, Theorem 32.13] *Let M be a stable R-module. Then* $\mathrm{Tr}M$ *is stable. In particular, λM is non-zero.*

1.6 The module of derivations

If R is a ring and M is an R-module, a *derivation* of R into M is an additive map $\Delta\colon R \to M$ such that

$$\Delta(ab) = a\Delta(b) + b\Delta(a), \quad \forall\, a, b \in R.$$

We denote by $\mathrm{Der}(R, M)$ the set of all derivations of R into M, which is an R-module in a natural way. If R is a k-algebra via a ring homomorphism $\psi\colon k \to R$, an element of $\mathrm{Der}(R, M)$ is a k-*derivation* if it vanishes on the image of ψ (a typical situation is when ψ is an inclusion). The set formed by all k-derivations of R into M is denoted by

$\mathrm{Der}_k(R, M)$, which is seen to be an R-submodule of $\mathrm{Der}(R, M)$. If $\Omega_{R/k}$ is the module of Kähler k-differentials of R, it is well-known that

$$\mathrm{Hom}_R(\Omega_{R/k}, M) \cong \mathrm{Der}_k(R, M).$$

We refer, e.g., to Matsumura [33, Chapter 9]. In the case $M = R$ we simplify the notation to $\mathrm{Der}_k(R)$ (which is then the R-dual of $\Omega_{R/k}$). If for instance R is a polynomial ring $k[X_1, \ldots, X_n]$ – or a localization thereof – then $\mathrm{Der}_k(R)$ is a free R-module on the partial derivations $\partial_1, \ldots, \partial_n$.

We invoke a few specific concepts and facts that will play a fundamental role in the sequel.

Definition 1.6.1 Given a k-algebra R, we say that an ideal $I \subseteq R$ is $\mathrm{Der}_k(R)$-*differential* (*differential*, for short) if

$$\Delta(I) \subseteq I, \quad \forall \Delta \in \mathrm{Der}_k(R).$$

Remark 1.6.2 By Zorn's lemma, the family

$$\mathfrak{F}_{R/k} = \{J \mid J \text{ is a proper differential ideal of } R\}$$

contains maximal elements. If R is local then $\mathfrak{F}_{R/k}$ has a unique maximal element (cf. Maloo [29, p. 82]), the so-called *maximally differential ideal* of the k-algebra R, denoted herein by $\mathfrak{P}_{R/k}$.

Theorem 1.6.3 [29, Theorem 5] *If R is a local k-algebra such that $\mathrm{Der}_k(R)$ is a finitely generated R-module, then $\mathrm{Der}_k(R)/\mathfrak{P}_{R/k}\mathrm{Der}_k(R)$ is free as an $R/\mathfrak{P}_{R/k}$-module. In particular, if $\mathfrak{P}_{R/k} = (0)$ then $\mathrm{Der}_k(R)$ is free as an R-module.*

Theorem 1.6.4 [26, Theorem 1.4] *Let $(R, \mathfrak{m})$ be a local k-algebra.*

(i) *If $R/\mathfrak{P}_{R/k}$ has positive characteristic, then $\mathrm{rad}(\mathfrak{P}_{R/k}) = \mathfrak{m}$;*

(ii) *If $R/\mathfrak{P}_{R/k}$ has characteristic zero, then $\mathfrak{P}_{R/k}$ is prime.*

Lemma 1.6.5 [29, Lemma 9] *Let R be a integral domain containing a field k such that $\mathrm{Der}_k(K)$ is a finite K-vector space, where K denote the quotient field of R. Then $\mathrm{Der}_k(R)$ is a finite R-module.*

Theorem 1.6.6 [33, Theorem 30.7] *Let $(R, \mathfrak{m})$ be a local integral domain containing both $\mathbb{Q}$ and a field k such that $\mathrm{tr.deg}_k(R/\mathfrak{m}) < \infty$. Then $\mathrm{Der}_k(R)$ is a finite R-module.*

Chapter 2

C-k-torsionless modules with finite G_C-dimension

Throughout this chapter C and C' are semidualizing R-modules. Our main goal in this chapter is to generalize Theorem 42 of Maşek [32] into the context of G_C-dimension. As consequences we obtain generalizations of the results of [30], [32], [39] [40], [41] and [50]. Such results will appear in Miranda-Neto–Souza [36] (the results given in Section 2.5 are established in [35]).

2.1 C-k-syzygy and C-k-torsionless modules

First, we need the definitions of k-syzygy and k-torsionless, presented in [32], in a more general context. The following definition, which generalizes the definition of k-syzygy, is due to Huang [22, Definition 1.2].

Definition 2.1.1 Let M be a finite R-module and $k \geq 0$ an integer. Then M is called *C-k-syzygy* ($k \geq 1$) if there exists an exact sequence of R-modules

$$0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1},$$

with X_i in $\mathrm{add}_R\,(C)$ for all $i = 0, \ldots, k-1$. By convention, if $k = 0$ then every module is regarded as a C-0-syzygy.

Remark 2.1.2 (i) In the special case where $C = R$, the notion of "C-k-syzygy module" coincides with the well-known notion of "k-syzygy module" defined in [32, Definition 9].

(ii) By Lemma 1.3.7, our definition of C-k-syzygy coincides with the definition of "k-*th* C-syzygy" by Salimi et al [39, Definition 4.2].

(iii) A finite R-module M is called an "nth C-syzygy" in [10, p. 4462] if there is an exact sequence $0 \to M \to C_1 \to \cdots \to C_n$, where $C_i \cong \oplus^{m_i} C$ for some m_i. In particular, each C_i is in $\operatorname{add}_R(C)$, thus every nth C-syzygy is C-n-syzygy.

Example 2.1.3 Let M be a finite R-module. Then M^C is a C-2-syzygy. In fact, let $R^m \to R^n \to M \to 0$ be a free resolution of M. Applying the functor $(-)^C = \operatorname{Hom}_R(-, C)$ on the free resolution of M, gives the exact sequence $0 \to M^C \to \oplus^n C \to \oplus^m C$.

The following result, the first item of which is Theorem 4.3 of [39], provides a relation between the classical and the generalized definitions of k-syzygy.

Proposition 2.1.4 *Let M be a finite R-module and $k \geq 0$ an integer. Then the following statements hold true:*

(i) *If $M \in \mathcal{A}_C$, then M is a k-syzygy if and only if $C \otimes_R M$ is a C-k-syzygy.*

(ii) *If $M \in \mathcal{B}_C$, then M is a C-k-syzygy if and only if $\operatorname{Hom}_R(C, M)$ is a k-syzygy.*

Proof. We may assume that $k > 0$.

(i) It follows from [39, Theorem 4.3] and Remark 2.1.2(ii): We only prove the assertion for $k = 1$, as the general statement easily follows by induction on k. Let M be 1-syzygy, that is, there exists a short exact sequence

$$0 \to M \to P \to N \to 0 \tag{2.1}$$

of R-modules with P finite and projective. By Remark 1.3.2(i), $N \in \mathcal{A}_C$. Therefore, applying the functor $C \otimes_R (-)$ to (2.1) yields a short exact sequence

$$0 \to C \otimes_R M \to C \otimes_R P \to C \otimes_R N \to 0$$

of R-modules, indicating that $C \otimes_R M$ is C-1-syzygy.

Conversely, assume that $C \otimes_R M$ is C-1-syzygy. Then there exists a short exact sequence

$$0 \to C \otimes_R M \to C \otimes_R Q \to K \to 0 \tag{2.2}$$

of R-modules with Q finite and projective. Applying the functor $\mathrm{Hom}_R(C,-)$ to (2.2), and using that $M \in \mathcal{A}_C$, we obtain the following short exact sequence

$$0 \to \mathrm{Hom}_R(C, C \otimes_R M) \to \mathrm{Hom}_R(C, C \otimes_R Q) \to \mathrm{Hom}_R(C, K) \to 0$$

Since $M, Q \in \mathcal{A}_C$, we have $M \cong \mathrm{Hom}_R(C, C \otimes_R M)$ and $Q \cong \mathrm{Hom}_R(C, C \otimes_R Q)$. Hence, M is 1-syzygy.

(ii) Set $N = \mathrm{Hom}_R(C, M)$. As $M \in \mathcal{B}_C$, by Theorem 1.3.6(ii), $N \in \mathcal{A}_C$. Then, by definition of Bass class, $M \cong N \otimes_R C$. Now, by item (i), the assertion follows. $\square$

The following definition, which generalizes the notion of k-torsionless, is due to Huang [23, Definition 2].

Definition 2.1.5 Let M be a finite R-module and $k \geq 0$ an integer. Then M is called *C-k-torsionless* if $\mathrm{Ext}^i_R(\mathrm{Tr}_C(M), C) = 0$ for all $1 \leq i \leq k$. By definition, every module is C-0-torsionless.

Remark 2.1.6 (i) In the special case where $C = R$, the notion of "C-k-torsionless module" coincides with the well-known notion of "k-torsionless module" defined in [32, Definition 7].

(ii) Huang uses the term "ω-k-torsionfree" (see [23, Definition 2]) for what we call "C-k-torsionless". We chose this terminology because it is the one used by Maşek [32] when $C = R$.

(iii) By the exact sequence in Proposition 1.2.6(i), we have the following isomorphisms

$$\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) \cong \mathrm{Ker}\,(\sigma^C_M) \text{ and } \mathrm{Ext}^2_R(\mathrm{Tr}_C(M), C) \cong \mathrm{Coker}\,(\sigma^C_M).$$

Thus, a finite R-module M is C-2-torsionless (resp. C-1-torsionless) if and only if M is C-reflexive (resp. C-torsionless).

(iv) By the exact sequence in Definition 1.2.1, we have the following isomorphisms

$$\mathrm{Ext}^{i+2}_R(\mathrm{Tr}_C(M), C) \cong \mathrm{Ext}^i_R(M^C, C),$$

for all $i > 0$. Thus, for each $k \geq 3$, M is C-k-torsionless when it is C-reflexive and $\mathrm{Ext}^i_R(M^C, C) = 0$ for all $1 \leq i \leq k-2$. Therefore, our definition of C-k-torsionless coincides with the definition of "$C_{(k)}$-torsionless" in [39, Definition 4.5].

(v) Let M be a finite R-module such that $M \in \mathcal{B}_C$. Then M is C-k-torsionless if and only if $\mathrm{Ext}^i_R(\mathrm{D}_C(M), C) = 0$ for all $1 \leq i \leq k$. See [39, Remark 4.6 (ii)].

Example 2.1.7 Let M be a finite R-module of G_C-dimension zero. Then M is a C-k-torsionless for all $k \geq 0$. In fact, by Proposition 1.2.6, $\mathrm{Tr}_C(M)$ has G_C-dimension zero which implies $\mathrm{Ext}^i_R(\mathrm{Tr}_C(M), C) = 0$ for all $i > 0$.

When an Auslander transpose is in $\mathcal{A}_C$ the classical and generalized properties of being k-torsionless coincide.

Proposition 2.1.8 *Let M be a finite R-module and $k \geq 0$ an integer. Suppose that $\mathrm{Tr}M \in \mathcal{A}_C$. Then M is C-k-torsionless if and only if M is k-torsionless.*

Proof. By Proposition 1.2.10, $\mathrm{Tr}_C(M) \approx_C \mathrm{Tr}M \otimes_R C$. Since $\mathrm{Ext}^i_R(C, C) = 0$, for all $i > 0$, we get $\mathrm{Ext}^i_R(\mathrm{Tr}_C(M), C) \cong \mathrm{Ext}^i_R(\mathrm{Tr}M \otimes_R C, C)$, for all $i > 0$. By Theorem 1.3.4, $\mathrm{Ext}^i_R(\mathrm{Tr}M \otimes_R C, C) \cong \mathrm{Ext}^i_R(\mathrm{Tr}M, R)$, for all $i > 0$. Thus, we have the following isomorphisms

$$\mathrm{Ext}^i_R(\mathrm{Tr}_C(M), C) \cong \mathrm{Ext}^i_R(\mathrm{Tr}M, R),$$

for all $i > 0$. Now, by the isomorphisms above, the assertion follows. $\square$

The following result is a partial version of Proposition 2.1.4 for C-k-torsionless modules.

Proposition 2.1.9 *Let M be a finite R-module and $k \geq 0$ an integer. Then the following statements hold true:*

(i) *Assume that $M \in \mathcal{B}_C$. If $\mathrm{Hom}_R(C, M)$ is k-torsionless, then M is C-k-torsionless. The converse holds when $\mathrm{Hom}_R(C, M)$ has finite G_C-dimension on $X^{k-1}(R)$.*

(ii) *Assume that $M \in \mathcal{A}_C$. If M is k-torsionless, then $M \otimes_R C$ is C-k-torsionless. The converse holds when M has finite G_C-dimension on $X^{k-1}(R)$.*

Proof. (i) By Remark 1.3.10(ii), $\mathrm{D}_C(M) \approx \mathrm{Tr}(\mathrm{Hom}_R(C, M))$, and so we have

$$\mathrm{Ext}^i_R(\mathrm{D}_C(M), C) \cong \mathrm{Ext}^i_R(\mathrm{Tr}(\mathrm{Hom}_R(C, M)), C),$$

for all $i > 0$, where $\mathrm{D}_C(M)$ is the Auslander dual, relative to C, of M. By Theorem 1.3.6, $\mathrm{Hom}_R(C, M) \in \mathcal{A}_C$. By Theorem 1.3.8, $\mathrm{Ext}^i_R(\mathrm{Tr}(\mathrm{Hom}_R(C, M)), C) = 0$ for all $1 \leq i \leq k$, so $\mathrm{Ext}^i_R(\mathrm{D}_C(M), C) = 0$ for all $1 \leq i \leq k$. By Remark 2.1.6(v), it follows that M is C-k-torsionless. Conversely, assume that M is C-k-torsionless and that $\mathrm{Hom}_R(C, M)$ has finite G_C-dimension on $X^{k-1}(R)$. By isomorphisms above, $\mathrm{Ext}^i_R(\mathrm{Tr}(\mathrm{Hom}_R(C, M)), C) = 0$, for all $1 \leq i \leq k$. Thus, by Theorem 1.3.8, $\mathrm{Hom}_R(C, M)$ is k-torsionless.

(ii) By Theorem 1.3.6, $M \otimes_R C \in \mathcal{B}_C$. Note that $\mathrm{Hom}_R(C, M \otimes_R C) \cong M$, since $M \in \mathcal{A}_C$. Applying the item (i) on $M \otimes_R C$, the assertion follows. $\square$

Now, we generalize the results [32, Proposition 8] and [39, Proposition 4.7].

Proposition 2.1.10 *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules, $k \geq 0$ integer, and $L = \mathrm{Coker}\,((M)^C \to (M')^C)$.*

(i) *If M' and M'' are C-k-torsionless and* $\mathrm{grade}_R\,(L) \geq k$, *then M is C-k-torsionless.*

(ii) *If M is C-k-torsionless, M'' is C-$(k-1)$-torsionless and* $\mathrm{grade}_R\,(L) \geq k-1$, *then M' is C-k-torsionless.*

(iii) *If M' is C-$(k+1)$-torsionless, M is C-k-torsionless and* $\mathrm{grade}_R\,(L) \geq k+1$, *then M'' is C-k-torsionless.*

Proof. By Proposition 1.2.4, there exists a natural long exact sequence

$$0 \to (M'')^C \to (M)^C \to (M')^C \to \mathrm{Tr}_C(M'') \to \mathrm{Tr}_C(M) \to \mathrm{Tr}_C(M') \to 0.$$

Then, we get an exact sequence

$$0 \to L \to \mathrm{Tr}_C(M'') \to \mathrm{Tr}_C(M) \to \mathrm{Tr}_C(M') \to 0,$$

which we can split into short exact sequences

$$0 \to L \to \mathrm{Tr}_C(M'') \to L' \to 0,$$

and

$$0 \to L' \to \mathrm{Tr}_C(M) \to \mathrm{Tr}_C(M') \to 0,$$

where $L' = \mathrm{Coker}\,(L \to \mathrm{Tr}_C(M''))$. We will only prove item (i) since the other items follow in a completely analogous way. The two short exact sequences above induce the following long exact sequences

$$\cdots \to \mathrm{Ext}_R^{i-1}(L, C) \to \mathrm{Ext}_R^i(L', C) \to \mathrm{Ext}_R^i(\mathrm{Tr}_C(M''), C) \to \cdots \tag{2.3}$$

and

$$\cdots \to \mathrm{Ext}_R^i(\mathrm{Tr}_C(M'), C) \to \mathrm{Ext}_R^i(\mathrm{Tr}_C(M), C) \to \mathrm{Ext}_R^i(L', C) \to \cdots \tag{2.4}$$

By hypothesis, $\mathrm{Ext}_R^i(\mathrm{Tr}_C(M'), C) = 0 = \mathrm{Ext}_R^i(\mathrm{Tr}_C(M''), C)$ for all $1 \leq i \leq k$. As $\mathrm{grade}_R\,(L) \geq k$, by Proposition 1.4.3, it follows that $\mathrm{Ext}_R^i(L, C) = 0$ for all $0 \leq i \leq k-1$. So, by exact sequence (2.3), $\mathrm{Ext}_R^i(L', C) = 0$ para $1 \leq i \leq k$. It follows from the exact sequence (2.4) that $\mathrm{Ext}_R^i(\mathrm{Tr}_C(M), C) = 0$ for all $1 \leq i \leq k$. Therefore, M is C-k-torsionless. $\square$

2.2 A generalization of Maşek's theorem on C-k-torsionless modules

In this section we study the relationship between C-k-torsionless, C-k-syzygy and several other conditions for finite modules. We will show a generalization of Theorem 42 of Maşek [32] to the relative setting with respect to a semidualizing module C.

Now we present more two definitions, due to Maşek [32], which will be necessary for the statement of the main result of this section.

Definition 2.2.1 Let R be a local ring, and let M be a finite R-module. Fix an integer $k \geq 0$. M is *k-torsionfree* if every R-regular sequence of length at most k is also M-regular.

Definition 2.2.2 An R-module M is said to satisfy the property $\widetilde{S}_k$ if $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{k, \operatorname{depth}(R_\mathfrak{p})\}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proposition 2.2.3 *Let $(R, \mathfrak{m})$ be a local ring and let M be a k-torsionfree R-module. Then,*

$$\operatorname{depth}_R(M) \geq \min\{k, \operatorname{depth}(R)\}.$$

Proof. Set $r = \operatorname{depth}(R)$. Let $\{x_1, \ldots, x_r\}$ be a maximal R-regular sequence in $\mathfrak{m}$. If $r \leq k$, by definition, $\{x_1, \ldots, x_r\}$ is a M-regular sequence, so $\operatorname{depth}_R(M) \geq r$. If $r > k$, then $\{x_1, \ldots, x_k\}$ is an R-regular sequence. By definition, $\{x_1, \ldots, x_k\}$ is a M-regular sequence. Thus, $\operatorname{depth}_R(M) \geq k$. $\square$

To prove our main result of this section we will need three more lemmas.

Lemma 2.2.4 [32, Lemma 39] *Let R be a local ring. If $0 \to M \to X \to N \to 0$ is a short exact sequence of finite R-modules with X k-torsionfree and N $(k-1)$-torsionfree, then M is k-torsionfree.*

Proof. If $k = 1$ and x_1 is R-regular, then x_1 is X-regular and therefore M-regular as well. Thus M is at least 1-torsionfree in this case.

If $k \geq 2$ and $x_1, \ldots, x_s$ is an R-regular sequence with $2 \leq s \leq k$, then x_1 is N-regular, X-regular and M-regular, and hence, by Proposition B.3, we have an exact sequence of $\bar{R}$-modules,

$$0 \to \bar{M} \to \bar{X} \to \bar{N} \to 0,$$

where $\bar{R} = R/x_1R$, $\bar{M} = M/x_1M$, and $\bar{N} = N/x_1N$.

As X is k-torsionfree over R and x_1 is R-regular, we see that $\bar{X}$ is $(k-1)$-torsionfree over $\bar{R}$. Similarly, $\bar{N}$ is $(k-2)$-torsionfree over $\bar{R}$. By induction, $\bar{M}$ is $(k-1)$-torsionfree over $\bar{R}$, and in particular $x_2, \ldots, x_s$ is $\bar{M}$-regular. But then $x_1, x_2, \ldots, x_s$ is M-regular, as required. □

Lemma 2.2.5 *Let M be a finite R-module and $k \geq 0$ an integer. Assume that M has finite G_C-dimension and that M satisfies $\widetilde{S}_k$. Then,*

$$\operatorname{grade}_R(\operatorname{Ext}^i_R(M, C)) \geq i + k,$$

for all $i > 0$.

Proof. By Proposition 1.4.2, if N is a non-zero R-module, then

$$\operatorname{grade}_R(N) = \min\{\operatorname{depth}(R_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(N)\}. \tag{2.5}$$

If $i > 0$ is an integer, we may assume that $\operatorname{Ext}^i_R(M, C) \neq 0$. Let $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Ext}^i_R(M, C))$, we will show that $\operatorname{depth}(R_\mathfrak{p}) \geq i+k$. Thus, by (2.5), we will get $\operatorname{grade}_R(\operatorname{Ext}^i_R(M, C)) \geq i + k$, as asserted. Since

$$\operatorname{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) \cong (\operatorname{Ext}^i_R(M, C))_\mathfrak{p} \neq 0,$$

we have $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq i$. By Theorem 1.1.13 and by hypothesis, we get

$$\operatorname{depth}(R_\mathfrak{p}) = \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq i + \min\{k, \operatorname{depth}(R_\mathfrak{p})\}.$$

As $i > 0$, it follows that $\operatorname{depth}(R_\mathfrak{p}) > k$. Therefore, $\operatorname{depth}(R_\mathfrak{p}) \geq i + k$. □

Lemma 2.2.6 *Let M be a finite R-module and $k \geq 0$ an integer. Assume that M has finite G_C-dimension n, and that $\operatorname{grade}_R(\operatorname{Ext}^i_R(M, C)) \geq i + k$, for all $1 \leq i \leq n$. Then M is C-k-torsionless.*

Proof. We argue by induction on n. If $n = 0$ then, by Proposition 1.2.6, for each $i > 0$ we have $\operatorname{Ext}^i_R(\operatorname{Tr}_C M, C) = 0$, and so M is C-l-torsionless for all $l \geq 0$. Now, assume that $n > 0$. Consider a short exact sequence

$$0 \to K \to X \to M \to 0 \tag{2.6}$$

of finite R-modules such that $\mathrm{G}_C\text{-}\dim_R(X) = 0$ and $\mathrm{G}_C\text{-}\dim_R(K) = n - 1$. In particular, $\operatorname{Ext}^i_R(X, C) = 0$ for all $i > 0$, and by the exact sequence (2.6), we get the isomorphisms $\operatorname{Ext}_R^i(K, C) \cong \operatorname{Ext}_R^{i+1}(M, C)$ for all $i > 0$. For each $1 \leq i \leq n-1$, we have

$$\operatorname{grade}_R(\operatorname{Ext}_R^i(K, C)) = \operatorname{grade}_R(\operatorname{Ext}_R^{i+1}(M, C)) \geq i + (k+1).$$

By the induction hypothesis, K is C-$(k+1)$-torsionless. Since X has G_C-dimension zero, it follows that X is C-l-torsionless for all $l \geq 0$, and, in particular, X is C-k-torsionless. Note that $L = \operatorname{Coker}(X^C \to K^C) \cong \operatorname{Ext}_R^1(M, C)$. Therefore, by hypothesis,

$$\operatorname{grade}_R(L) = \operatorname{grade}_R(\operatorname{Ext}_R^1(M, C)) \geq k+1.$$

By Proposition 2.1.10, M is C-k-torsionless. □

We are now in a position to state the main theorem of this section, which generalizes Theorem 42 of Maşek [32] for the context of G_C-dimension. It also generalizes Theorem 4.11 of Salimi et al [39] and refines Proposition 2.4 of Dibaei and Sadeghi [10].

Theorem 2.2.7 *Let M be a finite R-module and $k \geq 0$ an integer. Consider the following conditions:*

(i) *M is C-k-torsionless;*

(ii) *M is C-k-syzygy;*

(iii) *There exists an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ of finite R-modules with $\mathrm{G}_C\text{-}\dim_R(X_i) = 0$ for every $i = 0, \ldots, k-1$;*

(iv) *$M_{\mathfrak{p}}$ is k-torsionfree over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$;*

(v) *M satisfies $\widetilde{S}_k$;*

(vi) $\operatorname{grade}_R(\operatorname{Ext}_R^i(M, C)) \geq i + k$, *for all $i > 0$.*

Then the following implications hold true.

(a) *(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).*

(b) *If M has finite G_C-dimension on $X^{k-1}(R)$, then (v) $\Rightarrow$ (i).*

(c) *If M has finite G_C-dimension, then all the conditions above are equivalent.*

Proof. (a). (i) $\Rightarrow$ (ii) It follows from [10, Proposition 2.4]: Applying $(-)^C := \mathrm{Hom}_R(-, C)$ to a projective resolution $\cdots \to P_{k-1} \to \cdots \to P_0 \to M^C \to 0$ of M^C implies the complex $0 \to M^{CC} \to (P_0)^C \to (P_1)^C \to \cdots \to (P_{k-1})^C$, where each $(P_i)^C$ is in $\mathrm{add}_R\,(C)$. By Proposition 1.2.6, M is embedded in M^{CC} if $k = 1$ and $M \cong M^{CC}$ if $k > 1$. Therefore M is always C-1-syzygy and, for $k = 2$, M is C-2-syzygy. Assume that $k > 2$. By Remark 2.1.6, $\mathrm{Ext}^i_R(M^C, C) \cong \mathrm{Ext}^{i+2}_R(\mathrm{Tr}_C(M), C) = 0$ for all $0 < i < k-1$. Therefore the complex

$$0 \to M^{CC} \to (P_0)^C \to (P_1)^C \to \cdots \to (P_{k-1})^C$$

is exact, i.e. M is a C-k-syzygy.

(ii) $\Rightarrow$ (iii) By definition, there exists the following exact sequence

$$0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1},$$

where each X_i is in $\mathrm{add}_R\,(C)$. By Remark 1.1.8, every module in $\mathrm{add}_R\,(C)$ has zero G_C-dimension.

(iii) $\Rightarrow$ (ii) It follows from [28, Lemma 2.8]: We proceed by induction on k. When $k = 1$, we have an exact sequence $0 \to M \to X_0$ of finite R-modules with $\mathrm{G}_C\text{-}\dim_R(X_0) = 0$. Since X_0 is a C-reflexive module, it follows that M is a C-1-syzygy by Example 2.1.3. Now assume that $k \geq 2$ and we have an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ of finite R-modules with $\mathrm{G}_C\text{-}\dim_R(X_i) = 0$ for every $i = 0, \ldots, k-1$. Put $K = \mathrm{Coker}\,(X_0 \to X_1)$. By Lemma 1.2.8, we get an exact sequence

$$0 \to M \to Y \to X \to K \to 0,$$

where Y is in $\mathrm{add}_R\,(C)$ and X has G_C-dimension zero. Put $M' = \mathrm{Im}\,(Y \to X)$. Then we get an exact sequence $0 \to M' \to X \to X_2 \to \cdots \to X_{k-1}$. So, by the induction hypothesis, we get the assertion.

(iii) $\Rightarrow$ (iv) By localizing at $\mathfrak{p} \in \mathrm{Supp}_R\,(M)$, it suffices to show that (iii) $\Rightarrow$ M is k-torsionfree over a *local* ring R. We proceed by induction on k. The case $k = 0$ being trivial. Assume that $k > 0$, set $N = \mathrm{Coker}\,(M \to X_0)$. Then we have a short exact sequence

$$0 \to N \to X_1 \to X_2 \to \cdots \to X_{k-1}.$$

By the induction hypothesis, N is $(k-1)$-torsionfree. As X_0 has G_C-dimension zero, by Proposition 1.1.17, X_0 is l-torsionfree for all $l \geq 0$. In particular, X_0 is k-torsionfree. Considering the exact sequence

$$0 \to M \to X_0 \to N \to 0,$$

the conclusion follows from Lemma 2.2.4.

(iv) $\Rightarrow$ (v) Let $\mathfrak{p} \in \operatorname{Spec}(R)$, we will show that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{k, \operatorname{depth}(R_\mathfrak{p})\}$. If $\mathfrak{p} \notin \operatorname{Supp}_R(M)$, then $M_\mathfrak{p} = 0$, and the result is immediate. So, we may assume that $\mathfrak{p} \in \operatorname{Supp}_R(M)$. By hypothesis, $M_\mathfrak{p}$ is k-torsionfree $R_\mathfrak{p}$-module. By Proposition 2.2.3,

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{k, \operatorname{depth}(R_\mathfrak{p})\}.$$

As $\mathfrak{p}$ was taken arbitrarily in $\operatorname{Spec}(R)$, it follows that M satisfies $\widetilde{S}_k$.

(b). It follows from [10, Proposition 2.4]: We argue by induction on k. If $k = 1$ then, by Theorem 1.1.13, M has G_C-dimension zero on $X^0(R)$ and so, by Proposition 1.2.6, $\operatorname{Tr}_C(M)$ has G_C-dimension zero on $X^0(R)$. Hence $\operatorname{Ext}^1_R(\operatorname{Tr}_C(M), C)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in X^0(R)$. Thus $X^0(R) \cap \operatorname{Supp}_R(\operatorname{Ext}^1_R(\operatorname{Tr}_C(M), C)) = \emptyset$. It is enough to show that $\operatorname{Ass}_R(\operatorname{Ext}^1_R(\operatorname{Tr}_C(M), C)) = \emptyset$. Assume contrarily that $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Ext}^1_R(\operatorname{Tr}_C(M), C)) \subseteq \operatorname{Supp}_R(\operatorname{Ext}^1_R(\operatorname{Tr}_C(M), C))$. By Proposition 1.2.6, $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(\operatorname{Ext}^1_R(\operatorname{Tr}_C(M), C)_\mathfrak{p}) \subseteq \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$, and so $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$. As M satisfies $\widetilde{S}_1$, we get $\mathfrak{p} \in X^0(R)$, which is a contradiction.

Now, let $k > 1$. By Theorem 1.1.13, $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) - \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})$ for all $\mathfrak{p} \in X^{k-1}(R)$. As M satisfies $\widetilde{S}_{k-1}$, we obtain that M has G_C-dimension zero on $X^{k-1}(R)$ and in particular M is C-1-torsionless, i.e., $\operatorname{Ext}^1_R(\operatorname{Tr}_C(M), C) = 0$. Consider a universal pushforward

$$0 \to M \to C^m \to N \to 0 \tag{2.7}$$

of M with respect to C. As $\operatorname{Ext}^1_R(N, C) = 0$, by the exact sequence (2.7) and Proposition 1.1.6, it follows that N has G_C-dimension zero on $X^{k-1}(R)$. Since M satisfies $\widetilde{S}_k$, it is easy to see that N satisfies $\widetilde{S}_{k-1}$. By induction hypothesis, N is C-$(k-1)$-torsionless. Finally, by Proposition 2.1.10(ii), the exact sequence (2.7) implies that M is C-k-torsionless.

(c). Finally, the implications (v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (i) follow from Lemma 2.2.5 and Lemma 2.2.6, respectively. $\square$

Remark 2.2.8 Our item (iv) is stated differently in [32, Theorem 42(d)], which takes $\mathfrak{p}$ in $\mathrm{Spec}\,(R)$. This is however equivalent to assuming $\mathfrak{p} \in \mathrm{Supp}_R\,(M)$, as we did, since by convention $\mathrm{depth}_R\,(0) = \infty$, i.e., the zero module is k-torsionfree for all k.

On local Cohen-Macaulay rings with canonical module we can conclude the following:

Corollary 2.2.9 *Let R be a Cohen-Macaulay local ring with canonical module ω_R. For an R-module M of finite G_C-dimension on $X^{k-1}(R)$, the following are equivalent:*

(i) *M is C-k-torsionless;*

(ii) *M is ω_R-k-torsionless;*

(iii) *M satisfies $\widetilde{S}_k$;*

(iv) *M is C-k-syzygy;*

(v) *M is ω_R-k-syzygy.*

Proof. Since M has finite G_C-dimension on $X^{k-1}(R)$, by Theorem 2.2.7(b), we have the equivalences (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iii). By Proposition 1.1.14, M has finite G_{ω_R}-dimension, and so, by Theorem 2.2.7(c), we have the equivalences (ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (iii). $\square$

Recall that a ring R is said to be *quasi-normal* if it satisfies Serre's condition (S_2) and for every prime ideal $\mathfrak{p}$ of height at most one, $R_\mathfrak{p}$ is Gorenstein. In [50, Corollary 1.5], it is shown that if R is a quasi-normal ring and N is a reflexive R-module, then for every finite R-module M, $\mathrm{Hom}_R(M, N)$ is also reflexive. In [41, Proposition 3.8], this result is generalized to the context of D-reflexive modules, where D is a dualizing R-module. Now, as a consequence of Theorem 2.2.7, we obtain a generalization of these results to C-reflexive modules which have finite $\mathrm{G}_{C'}$-dimension on $X^1(R)$, where C and C' are semidualizing R-modules.

Corollary 2.2.10 *Let M and N be finite R-modules such that N is C-reflexive and $\mathrm{Hom}_R(M, N)$ has finite $\mathrm{G}_{C'}$-dimension on $X^1(R)$. Then, $\mathrm{Hom}_R(M, N)$ is C'-reflexive.*

Proof. Let $\mathfrak{p} \in \mathrm{Supp}_R\,(\mathrm{Hom}_R(M, N))$, so $(\mathrm{Hom}_R(M, N))_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ is a non-zero finite $R_\mathfrak{p}$-module. Since $N_\mathfrak{p}$ is $C_\mathfrak{p}$-reflexive, we have $N_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}^{C_\mathfrak{p}}, C_\mathfrak{p})$. Let $\{x, y\} \subseteq \mathfrak{p}R_\mathfrak{p}$ be an $R_\mathfrak{p}$-regular sequence. By Lemma 1.1.18, $\{x, y\}$ is an $N_\mathfrak{p}$-regular sequence. Again by Lemma 1.1.18, $\{x, y\}$ is a $(\mathrm{Hom}_R(M, N))_\mathfrak{p}$-regular sequence.

Thus, $(\mathrm{Hom}_R(M,N))_{\mathfrak{p}}$ is 2-torsionfree over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{Hom}_R(M,N))$. By Theorem 2.2.7(b), $\mathrm{Hom}_R(M,N)$ is C'-2-torsionless. Therefore, by Remark 2.1.6(iii), $\mathrm{Hom}_R(M,N)$ is C'-reflexive. □

As an immediate consequence of the corollary above we derive that every C-dual of a finite module, with finite $\mathrm{G}_{C'}$-dimension on $X^1(R)$, is C'-reflexive. We point out that, on local rings with low depth, the total C'-reflexivity can be guaranteed without requiring M to be finite, as we shall see later in Proposition 2.4.3.

Corollary 2.2.11 *Let M be a finite R-module such that M^C has finite $\mathrm{G}_{C'}$-dimension on $X^1(R)$. Then, M^C is C'-reflexive.*

Proof. Apply Corollary 2.2.10 with $N = C$. □

The following result is a generalization of [40, Theorem 4.5].

Theorem 2.2.12 *Let M be a finite R-module of finite G_C-dimension on $X^{k-1}(R)$. Then M is C-k-torsionless if and only if*

(i) *$M_{\mathfrak{p}}$ is $C_{\mathfrak{p}}$-k-torsionless for $\mathfrak{p} \in \mathrm{Spec}\,(R)$ with $\mathrm{depth}\,(R_{\mathfrak{p}}) \leq k-1$, and*

(ii) *$\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq k$ for $\mathfrak{p} \in \mathrm{Spec}\,(R)$ with $\mathrm{depth}\,(R_{\mathfrak{p}}) \geq k$.*

Furthermore, if M has finite G_C-dimension, then M is C-k-torsionless if and only if $M_{\mathfrak{p}}$ is $C_{\mathfrak{p}}$-k-torsionless for $\mathfrak{p} \in \mathrm{Spec}\,(R)$ with $\mathrm{depth}\,(R_{\mathfrak{p}}) \leq k-1$, and $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\mathrm{dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathrm{depth}\,(R_{\mathfrak{p}}) - k$ for $\mathfrak{p} \in \mathrm{Spec}\,(R)$ with $\mathrm{depth}\,(R_{\mathfrak{p}}) \geq k$.

Proof. Assume that M is C-k-torsionless. Since

$$(\mathrm{Ext}^i_R(\mathrm{Tr}_C(M),C))_{\mathfrak{p}} \cong \mathrm{Ext}^i_{R_{\mathfrak{p}}}(\mathrm{Tr}_{C_{\mathfrak{p}}}(M_{\mathfrak{p}}),C_{\mathfrak{p}}),$$

for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ and $i > 0$, it follows that $M_{\mathfrak{p}}$ is a $C_{\mathfrak{p}}$-k-torsionless $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$. In particular, (i) holds. By Theorem 2.2.7, M satisfies $\widetilde{S}_k$. If $\mathfrak{p} \in \mathrm{Spec}\,(R)$ satisfies $\mathrm{depth}\,(R_{\mathfrak{p}}) \geq k$, then $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq k$, and so (ii) holds.

Conversely, by Theorem 2.2.7, it suffices to show that M satisfies $\widetilde{S}_k$. Let $\mathfrak{p} \in \mathrm{Spec}\,(R)$. If $\mathrm{depth}\,(R_{\mathfrak{p}}) \geq k$, by (ii), $\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq k = \min\{k, \mathrm{depth}\,(R_{\mathfrak{p}})\}$. If $\mathrm{depth}\,(R_{\mathfrak{p}}) < k$, by (i), $M_{\mathfrak{p}}$ is $C_{\mathfrak{p}}$-k-torsionless. By Theorem 2.2.7, $M_{\mathfrak{p}}$ satisfies $\widetilde{S}_k$, so

$$\mathrm{depth}_{(R_{\mathfrak{p}})_Q}(M_{\mathfrak{p}})_Q \geq \min\{k, \mathrm{depth}\,(R_{\mathfrak{p}})_Q\},$$

for all $Q \in \operatorname{Spec}(R_{\mathfrak{p}})$. In particular, taking $Q = \mathfrak{p}R_{\mathfrak{p}}$ the maximal ideal of $R_{\mathfrak{p}}$, we get $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{k, \operatorname{depth}(R_{\mathfrak{p}})\}$. As $\mathfrak{p}$ was taken arbitrarily in $\operatorname{Spec}(R)$, it follows that M satisfies $\widetilde{S}_k$. $\square$

An immediate consequence is that Theorem 2.2.12 recovers [41, Theorem 3.3].

Corollary 2.2.13 [41, Theorem 3.3] *Let M be a finite R-module of finite G_C-dimension. Then the following statements hold:*

(a) *M is C-torsionless if and only if*

  (i) *$M_{\mathfrak{p}}$ is $C_{\mathfrak{p}}$-torsionless for $\mathfrak{p} \in \operatorname{Ass}(R)$, and*

  (ii) $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \operatorname{depth}(R_{\mathfrak{p}}) - 1$ *for* $\mathfrak{p} \in \operatorname{Spec}(R)$ *with* $\operatorname{depth}(R_{\mathfrak{p}}) \geq 1$.

(b) *M is C-reflexive if and only if*

  (i) *$M_{\mathfrak{p}}$ is $C_{\mathfrak{p}}$-reflexive for* $\mathfrak{p} \in \operatorname{Spec}(R)$ *with* $\operatorname{depth}(R_{\mathfrak{p}}) \leq 1$, *and*

  (ii) $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \operatorname{depth}(R_{\mathfrak{p}}) - 2$ *for* $\mathfrak{p} \in \operatorname{Spec}(R)$ *with* $\operatorname{depth}(R_{\mathfrak{p}}) \geq 2$.

Proof. It is an immediate consequence of Theorem 2.2.12 with $k = 1, 2$ and Remark 2.1.6(iii). $\square$

As another consequence of Theorem 2.2.12, we can generalize [40, Corollary 4.6] as follows.

Corollary 2.2.14 *Let R be a Cohen-Macaulay local ring of dimension k with canonical module ω_R and let M be a non-zero finite R-module. Then the following statements are equivalent:*

(i) *M is ω_R-k-torsionless.*

(ii) $\mathrm{G}_{\omega_R}\text{-}\dim_R(M) = 0$.

(iii) *M is maximal Cohen-Macaulay.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathfrak{m}$ be the maximal ideal of R. Since $\operatorname{depth}(R_{\mathfrak{m}}) = k$, by Theorem 2.2.12, $\mathrm{G}_{\omega_{R_{\mathfrak{m}}}}\text{-}\dim_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = 0$, and so $\mathrm{G}_{\omega_R}\text{-}\dim_R(M) = 0$.

(ii) $\Rightarrow$ (iii) It follows from Theorem 1.1.13.

(iii) $\Rightarrow$ (i) Since M is maximal Cohen-Macaulay, it follows that $M_{\mathfrak{p}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$. In particular, $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \operatorname{depth}(R_{\mathfrak{p}}) \geq \min\{k, \operatorname{depth}(R_{\mathfrak{p}})\}$, for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Therefore, M satisfies $\widetilde{S}_k$. By Theorem 2.2.7, the assertion follows. $\square$

2.3 C-q-Gorenstein rings

In this section we present the concept of C-q-Gorenstein ring, which is a generalization of the notion of q-Gorenstein ring introduced in Maşek [32]. We prove an extension of [32, Theorem 43], and explore some consequences.

Definition 2.3.1 A ring R is *C-q-Gorenstein* ($q \geq 0$ an integer) if there exists a semidualizing R-module C such that $\mathrm{id}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) < \infty$ for every prime ideal $\mathfrak{p}$ of R with $\operatorname{depth}(R_{\mathfrak{p}}) \leq q-1$.

Remark 2.3.2 (i) Note that in the special case where $C = R$, the definition of "C-q-Gorenstein" coincides with the definition of "q-Gorenstein" in [32, Definition 44], and this definition, in turn, includes the reduced rings and the Gorenstein rings. Moreover, when $C = R$ and $q = 2$, this definition coincides with that of "quasi-normal" ring given in [50, Definition 1.2].

(ii) Every Cohen-Macaulay local ring R with canonical module ω_R is ω_R-q-Gorenstein for all $q \geq 0$.

(iii) A ring R is C-1-Gorenstein if and only if $C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Ass}(R)$.

(iv) The class of C-q-Gorenstein rings is studied by Araya-Iima [2] in terms of a certain condition denoted by (G^{C}_{q-1}).

As an application of Theorem 2.2.7 we have several characterizations for C-k-torsionless modules over C-q-Gorenstein rings.

Proposition 2.3.3 *Fix $q \geq 0$ integer. Let R be a C-q-Gorenstein ring and let M be a finite R-module. For $0 \leq k \leq q$, the following statements are equivalent:*

(i) *M is C-k-torsionless;*

(ii) *M is C-k-syzygy;*

(iii) *There exists an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ of finite R-modules with $\mathrm{G}_C\text{-}\dim_R(X_i) = 0$ for every $i = 0, \ldots, k-1$;*

(iv) *$M_{\mathfrak{p}}$ is k-torsionfree over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$;*

(v) *M satisfies $\widetilde{S}_k$.*

Proof. Let $0 \leq k \leq q$ be a fixed integer. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $\operatorname{depth}(R_{\mathfrak{p}}) \leq k-1 \leq q-1$. Since R is a C-q-Gorenstein ring, by definition, $C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-module with $\operatorname{id}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) < \infty$, so that $C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$-module. By Proposition 1.1.14, $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$. Therefore, M has finite G_C-dimension on $X^{k-1}(R)$. Now the assertion is obvious by Theorem 2.2.7 items (a) and (b). $\square$

As an immediate consequence we obtain a result similar to [50, Theorem 1.4].

Corollary 2.3.4 *Let R be a C-2-Gorenstein ring and M a finite R-module. A necessary and sufficient condition for M to be C-reflexive is that $M_{\mathfrak{p}}$ be 2-torsionfree over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$.*

Another immediate byproduct is a recovering of [2, Corollary 3.2].

Corollary 2.3.5 *Fix an integer $q \geq 0$. Let R be a ring that is both C-q-Gorenstein and C'-q-Gorenstein. Then the following conditions are equivalent for any finite R-module M and any integer k with $0 \leq k \leq q$:*

(i) *M is C-k-torsionless;*

(ii) *M is C'-k-torsionless;*

(iii) *M satisfies $\widetilde{S}_k$;*

(iv) *M is C-k-syzygy;*

(v) *M is C'-k-syzygy.*

As it turns out, the equivalence of the conditions (i) and (ii) in Proposition 2.3.3 remain valid for $k = q+1$. We will prove this as a consequence of the following result, which is a generalization and refinement of both [32, Theorem 43] and [39, Theorem 4.10].

Theorem 2.3.6 *Let M be a finite R-module and $k \geq 0$ an integer. Assume that M has finite G_C-dimension on $X^{k-2}(R)$. Then the following are equivalent:*

(i) *M is C-k-torsionless;*

(ii) *M is C-k-syzygy;*

(iii) *There exists an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ of finite R-modules with $\mathrm{G}_C\text{-}\dim_R(X_i) = 0$ for every $i = 0, \ldots, k-1$.*

In particular, for any given finite R-module M, we get that M is C-1-torsionless if and only if M is C-1-syzygy.

Proof. By Theorem 2.2.7 (a), it suffices to show that (ii) $\Rightarrow$ (i). Assume that M is C-k-syzygy. We will show that M is C-k-torsionless by induction on k. For $k = 0$ there is nothing to prove. If $k = 1$, M is C-1-syzygy, so there is an exact sequence

$$0 \to M \to X \to N \to 0,$$

where $N = \operatorname{Coker}(M \to X)$ and X is in $\operatorname{add}_R(C)$. In particular, $\mathrm{G}_C\text{-}\dim_R(X) = 0$, and so X is C-1-torsionless. By Proposition 2.1.10, M is C-1-torsionless. Assume that $k \geq 2$. Since M is C-k-syzygy, we have an exact sequence

$$0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}, \tag{2.8}$$

where each X_i is in $\operatorname{add}_R(C)$. Set $N = \operatorname{Coker}(M \to X_0)$. From the exact sequence (2.8), N is C-$(k-1)$-syzygy. By the induction hypothesis, N is C-$(k-1)$-torsionless. Now to finish the proof, by Proposition 2.1.10, it is enough to show that $\operatorname{grade}_R(\operatorname{Ext}^1_R(N, C)) \geq k - 1$. If $\mathfrak{p} \in \operatorname{Spec}(R)$ is such that $\operatorname{depth}(R_\mathfrak{p}) \leq k - 2$, then by hypothesis, $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$, which implies that $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(N_\mathfrak{p}) < \infty$. Since $N_\mathfrak{p}$ is $C_\mathfrak{p}$-$(k-1)$-torsionless, by Theorem 2.2.7(c), $\operatorname{grade}_{R_\mathfrak{p}}(\operatorname{Ext}^1_{R_\mathfrak{p}}(N_\mathfrak{p}, C_\mathfrak{p})) \geq k$. Now we claim that $\mathfrak{p} \notin \operatorname{Supp}_R(\operatorname{Ext}^1_R(N, C))$. Assume contrarily that $\operatorname{Ext}^1_{R_\mathfrak{p}}(N_\mathfrak{p}, C_\mathfrak{p}) \neq 0$. Then

$$k \leq \operatorname{grade}_{R_\mathfrak{p}}(\operatorname{Ext}^1_{R_\mathfrak{p}}(N_\mathfrak{p}, C_\mathfrak{p})) \leq \operatorname{depth}(R_\mathfrak{p}) \leq k - 2,$$

which is an absurd. Thus, we have shown that if $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Ext}^1_R(N, C))$ then $\operatorname{depth}(R_\mathfrak{p}) \geq k - 1$. By Proposition 1.4.2, $\operatorname{grade}_R(\operatorname{Ext}^1_R(N, C)) \geq k - 1$. $\square$

As a generalization of [32, Corollary 45], we conclude that for C-q-Gorenstein rings, the definitions of k-torsionless and k-syzygy, with respect to C, coincide up to order $k = q + 1$.

Corollary 2.3.7 *Fix an integer $q \geq 0$. Let R be a C-q-Gorenstein ring and M a finite R-module. For all $0 \leq k \leq q + 1$, M is C-k-torsionless if and only if M is C-k-syzygy.*

Proof. Let $0 \leq k \leq q + 1$ be a fixed integer. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $\operatorname{depth}(R_\mathfrak{p}) \leq k - 2 \leq q - 1$. Since $C_\mathfrak{p}$ is a semidualizing $R_\mathfrak{p}$-module with $\operatorname{id}_{R_\mathfrak{p}}(C_\mathfrak{p}) < \infty$, it follows that $C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$-module. By Proposition 1.1.14, $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$. Now the assertion is clear by Theorem 2.3.6. $\square$

We know that every C-reflexive module is a C-dual. The result below shows that for certain C-q-Gorenstein rings the converse holds.

Corollary 2.3.8 *Fix $q \geq 1$ integer. Let R be a C-q-Gorenstein ring and let M be a finite R-module. Then M^C is C-reflexive.*

Proof. By Example 2.1.3 and Corollary 2.3.7, we have that M^C is C-2-torsionless. Therefore, by Remark 2.1.6(iii), M^C is C-reflexive. $\square$

Now we can prove [42, Remark 2.15] without using spectral sequences.

Corollary 2.3.9 *Let M be 1-syzygy. Then $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) = 0$. In particular, if R is a semiperfect ring and M is horizontally linked, then $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) = 0$.*

Proof. Since M is 1-syzygy, we have a exact sequence $0 \to M \to P$ where P is a finite and projective R-module. Applying the functor $(-)^C := \mathrm{Hom}_R(-, C)$ on the exact sequence $R^n \to P^C \to 0$, we get an exact sequence $0 \to P \to \oplus^n C$. Therefore, the sequence $0 \to M \to \oplus^n C$ is exact, and so M is C-1-syzygy. Now the assertion is clear by Theorem 2.3.6. $\square$

2.4 C-dual and k-derivation modules

In this section we explore more results with C-dual modules of finite $\mathrm{G}_{C'}$-dimension, and we will give some applications involving certain relative derivation modules, without requiring any hypothesis about Kähler differential modules.

We begin by presenting the key result of the section, which links the theory studied so far to the important topic of derivation modules.

Theorem 2.4.1 *Let $(R, \mathfrak{m})$ be a local ring with $\mathrm{depth}(R) \leq n$, where $n \geq 2$ is an integer. Assume that M^C is a finite R-module of finite $\mathrm{G}_{C'}$-dimension, for some R-module M. Then $\mathrm{G}_{C'}\text{-}\dim_R(M^C) \leq n - 2$.*

Proof. If $M^C = 0$, there is nothing to prove, so we may assume that M^C is non-zero. Let $\{x, y\}$ be an R-regular sequence. In particular, it is contained in the maximal ideal

$\mathfrak{m}$. By Lemma 1.1.18, $\{x, y\}$ is a M^C-regular sequence. Thus, M^C is a 2-torsionfree R-module, and so, by Proposition 2.2.3, we have the following bound

$$\operatorname{depth}_R(M^C) \geq \min\{2, \operatorname{depth}(R)\}.$$

If $n = 2$, then $\operatorname{depth}_R(M^C) \geq \operatorname{depth}(R)$. Since M^C has finite $\mathrm{G}_{C'}$-dimension, by Theorem 1.1.13, it follows that

$$\mathrm{G}_{C'}\text{-}\dim_R(M^C) = \operatorname{depth}(R) - \operatorname{depth}_R(M^C) = 0.$$

If $n > 2$, then we may assume that $\operatorname{depth}(R) > 2$. By using Theorem 1.1.13 again, we get

$$\mathrm{G}_{C'}\text{-}\dim_R(M^C) = \operatorname{depth}(R) - \operatorname{depth}_R(M^C) \leq n - 2,$$

as desired. $\square$

As a consequence we have the following generalization of [41, Proposition 3.9].

Corollary 2.4.2 *Let R be a ring with $\dim(R) \leq n$, where $n \geq 2$ is an integer. Assume that M^C has finite $\mathrm{G}_{C'}$-dimension, for some finite R-module M. Then $\mathrm{G}_{C'}\text{-}\dim_R(M^C) \leq n - 2$.*

Proof. By Corollary 2.2.11, M^C is C'-reflexive. In particular, $(M^C)_{\mathfrak{p}} \cong (M_{\mathfrak{p}})^{C_{\mathfrak{p}}}$ is $C'_{\mathfrak{p}}$-reflexive for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $\mathfrak{p}$ be a prime ideal of R in the support of M. We have a local ring $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ with $\operatorname{depth}(R_{\mathfrak{p}}) \leq n$. Since $(M_{\mathfrak{p}})^{C_{\mathfrak{p}}}$ has finite $\mathrm{G}_{C'_{\mathfrak{p}}}$-dimension, by Theorem 2.4.1, we have $\mathrm{G}_{C'_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}}(M^C)_{\mathfrak{p}} \leq n - 2$. Now the assertion follows from Proposition 1.1.9. $\square$

As we will see below, another corollary of Theorem 2.4.1 is that, in low dimensions, we can guarantee the total reflexivity of C-dual modules with finite $\mathrm{G}_{C'}$-dimension. In particular, we generalize [40, Corollary 2.4].

Corollary 2.4.3 *Let $(R, \mathfrak{m})$ be a local ring with $\operatorname{depth}(R) \leq 2$. Assume that M^C is a finite R-module of finite $\mathrm{G}_{C'}$-dimension, for some R-module M. Then M^C is totally C'-reflexive. In particular, every C-reflexive R-module of finite $\mathrm{G}_{C'}$-dimension is totally C'-reflexive.*

Proof. By Theorem 2.4.1 with $n = 2$, we have $\mathrm{G}_{C'}\text{-}\dim_R(M^C) = 0$. Thus, by Remark 1.1.8, M^C is totally C'-reflexive. $\square$

As a consequence of Corollary 2.4.3 we have the following result, which generalizes [30, Lemma 3] to the context of G_C-dimension.

Corollary 2.4.4 *Let k be a ring and let R be a local k-algebra with* $\operatorname{depth}(R) \leq 2$*. If* $\operatorname{Der}_k(R, C)$ *is a finite R-module of finite* $\mathrm{G}_{C'}$*-dimension. Then,* $\operatorname{Der}_k(R, C)$ *is totally C'-reflexive as an R-module.*

Proof. We know that $\operatorname{Der}_k(R, C) \cong \operatorname{Hom}_R(\Omega_k(R), C) = (\Omega_k(R))^C$, where $\Omega_k(R)$ is the module of Kähler k-differentials of R (see Section 1.6). Now the assertion is clear by Corollary 2.4.3. $\square$

Remark 2.4.5 Assume that R is a regular local ring with $\dim(R) \leq 2$ such that $\operatorname{Der}_k(R)$ is a finite R-module. In particular, $\operatorname{Der}_k(R)$ has finite projective dimension. By Proposition 1.1.15, we have $\operatorname{pd}_R(\operatorname{Der}_k(R)) = \operatorname{G-dim}_R(\operatorname{Der}_k(R))$. By Corollary 2.4.4 with $C = C' = R$, we get that $\operatorname{Der}_k(R)$ is a free R-module, which is the conclusion of [30, Lemma 3].

For Cohen-Macaulay k-algebras with canonical module, we have:

Corollary 2.4.6 *Let k be a ring and let R be a Cohen-Macaulay local k-algebra with* $\dim(R) \leq 2$ *and with canonical module ω_R. Assume that* $\operatorname{Der}_k(R, C)$ *is a finite R-module, for some semidualizing R-module C. Then* $\operatorname{Der}_k(R, C)$ *is either maximal Cohen-Macaulay or zero.*

Proof. Since $\operatorname{Der}_k(R, C)$ is a finite R-module, it follows from Proposition 1.1.14 that $\operatorname{Der}_k(R, C)$ has finite G_{ω_R}-dimension. By Corollary 2.4.4 with $C' = \omega_R$, $\operatorname{Der}_k(R, C)$ has zero G_{ω_R}-dimension. Assume that $\operatorname{Der}_k(R, C)$ is non-zero. By Theorem 1.1.13, we get

$$\operatorname{depth}_R(\operatorname{Der}_k(R, C)) = \operatorname{depth}(R) = \dim(R).$$

Therefore, $\operatorname{Der}_k(R, C)$ is a maximal Cohen-Macaulay R-module. $\square$

Corollary 2.4.7 *Let k be a ring and let R be a Gorenstein local k-algebra with* $\dim(R) \leq 2$*. Assume that* $\operatorname{Der}_k(R)$ *is a non-zero finite R-module. Then* $\operatorname{Der}_k(R)$ *is maximal Cohen-Macaulay.*

Corollary 2.4.8 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local integral domain with canonical module ω_R and* $\dim(R) \leq 2$*. Consider the following statements:*

(i) *R contains $\mathbb{Q}$ and a field k such that* $\operatorname{tr.deg}_k(R/\mathfrak{m}) < \infty$*.*

(ii) *R contains a field k such that* $\operatorname{Der}_k(K)$ *is a finite K-vector space, where K denote the quotient field of R.*

If R satisfies (i) or (ii), then $\operatorname{Der}_k(R)$ *is either maximal Cohen-Macaulay or zero.*

Proof. If R satisfies (i), by Theorem 1.6.6, $\operatorname{Der}_k(R)$ is a finite R-module. If R satisfies (ii), by Lemma 1.6.5, $\operatorname{Der}_k(R)$ is a finite R-module. Now the assertion is clear by Corollary 2.4.6. □

For local integral domains with low depth, we obtain a characterization for the Cohen-Macaulay property by means of existence of a suitable C-dual.

Proposition 2.4.9 *Let R be a local integral domain with* $\operatorname{depth}(R) \leq 2$. *Assume that M^C is a non-zero finite R-module with finite $\mathrm{G}_{C'}$-dimension, for some R-module M. Then the following are equivalent:*

(i) *R is a Cohen-Macaulay ring;*

(ii) *M^C is a maximal Cohen-Macaulay R-module;*

(iii) *M^C is a Cohen-Macaulay R-module.*

Proof. By Corollary 2.4.3 and Theorem 1.1.13, we have

$$\operatorname{depth}_R(M^C) = \operatorname{depth}(R). \tag{2.9}$$

(i)⇒(ii) Assume that R is a Cohen-Macaulay ring, so $\operatorname{depth}(R) = \dim(R)$. By (2.9), M^C is maximal Cohen-Macaulay R-module.

(ii)⇒(iii) We have nothing to prove.

(iii)⇒(i) Assume that M^C is a Cohen-Macaulay R-module, so

$$\operatorname{depth}_R(M^C) = \dim_R(M^C). \tag{2.10}$$

Since R is an integral domain and M^C is non-zero finite R-module, it follows that M^C has finite and positive rank, say $r > 0$. By Proposition A.12, M^C has a free submodule with rank r, then we obtain the following exact sequence

$$0 \to R^r \to M^C \to X \to 0,$$

where $X = M^C/R^r$. Then $\dim_R(M^C) = \max\{\dim_R(R^r), \dim_R(X)\}$. As $\dim_R(R^r) = \dim(R)$, we have

$$\dim_R(M^C) = \dim(R). \tag{2.11}$$

Now the assertion is clear by (2.9), (2.10) and (2.11). $\square$

As an immediate consequence we have the following Cohen-Macaulayness characterization:

Corollary 2.4.10 *Let k be a ring and let R be a local k-domain with* $\operatorname{depth}(R) \leq 2$. *Assume that* $\operatorname{Der}_k(R)$ *is a non-zero finite R-module with finite* G*-dimension. Then R is a Cohen-Macaulay ring if and only if* $\operatorname{Der}_k(R)$ *is (maximal) Cohen-Macaulay.*

Proof. It follows from Proposition 2.4.9 with $C = C' = R$ and $M = \Omega_k(R)$. $\square$

If M is an R-module that is not finite (finitely generated), then it is well-known that we do not necessarily have $(M^C)_\mathfrak{p} \cong (M_\mathfrak{p})^{C_\mathfrak{p}}$, for a given $\mathfrak{p} \in \operatorname{Spec}(R)$, see for example [47, Example 7.38]. Thus, $(M^C)_\mathfrak{p}$ may not be a dual module. The result below gives sufficient conditions for a localization of a C-dual module as well as a C-dual of a localization (at primes with low height) to be $C'_\mathfrak{p}$-reflexive.

Proposition 2.4.11 *Let R be a local ring and M an R-module. For given a prime ideal $\mathfrak{p} \subseteq R$, the following statements hold:*

(i) *If* $\operatorname{depth}(R_\mathfrak{p}) \leq 2$ *and* $(M_\mathfrak{p})^{C_\mathfrak{p}}$ *is a finite $R_\mathfrak{p}$-module of finite* $\mathrm{G}_{C'_\mathfrak{p}}$*-dimension, then* $(M_\mathfrak{p})^{C_\mathfrak{p}}$ *is totally $C'_\mathfrak{p}$-reflexive.*

(ii) *If* $\operatorname{grade}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}) \leq 2$ *and M^C is a finite R-module such that $(M^C)_\mathfrak{p}$ has finite* $\mathrm{G}_{C'_\mathfrak{p}}$*-dimension, then $(M^C)_\mathfrak{p}$ is totally $C'_\mathfrak{p}$-reflexive.*

Proof. (i) The assertion follows immediately from Corollary 2.4.3

(ii) We may assume that $M^C \neq 0$. Set $r = \operatorname{grade}(\mathfrak{p}, R) = \operatorname{grade}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p})$. By definition, there exists $\mathbf{x}$ a maximal R-regular sequence of length $r \leq 2$ contained in $\mathfrak{p}$. By Lemma 1.1.18, $\mathbf{x}$ is a M^C-regular sequence in $\mathfrak{p}$. If $\mathfrak{p} \notin \operatorname{Supp}_R(M^C)$, we have nothing to prove. Assume that $\mathfrak{p} \in \operatorname{Supp}_R(M^C)$. By Corollary B.2, $\mathbf{x}$ is a $(M^C)_\mathfrak{p}$-regular sequence, and so

$$\operatorname{depth}_{R_\mathfrak{p}}((M^C)_\mathfrak{p}) \geq r = \operatorname{ht}(\mathfrak{p}) = \dim(R_\mathfrak{p}) \geq \operatorname{depth}(R_\mathfrak{p}).$$

By Theorem 1.1.13, we have

$$\mathrm{G}_{C'_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}((M^C)_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) - \operatorname{depth}_{R_\mathfrak{p}}((M^C)_\mathfrak{p}) = 0.$$

Therefore, $(M^C)_\mathfrak{p}$ is totally $C'_\mathfrak{p}$-reflexive. $\square$

Corollary 2.4.12 *Let k be a ring and let R be a Cohen-Macaulay local k-algebra with canonical module ω_R and let $\mathfrak{p}$ be a prime ideal of R. Assume that $\mathfrak{p}$ has height at most 2. Then the following statements hold:*

(i) *If* $\operatorname{Der}_k(R_{\mathfrak{p}})$ *is a non-zero finite $R_{\mathfrak{p}}$-module, then* $\operatorname{Der}_k(R_{\mathfrak{p}})$ *is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module.*

(ii) *If* $\operatorname{Der}_k(R)$ *is a finite R-module and* $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Der}_k(R))$, *then* $\operatorname{Der}_k(R)_{\mathfrak{p}}$ *is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module.*

Proof. It follows from Proposition 2.4.11 with $C = R$, $C' = \omega_R$, $M = \Omega_k(R)$, and from Theorem 1.1.13. $\square$

2.5 A generalization of Maloo's theorem on freeness of derivation modules

In this section we will work with k-derivation modules, without requiring any hypothesis about the module of Kähler k-differentials. We will show a generalization of Theorem 4 of Maloo [30], stated originally for regular rings, for more general integral domains and derive some important corollaries. For details about such generalization see Miranda-Neto–Souza [35] on which this section is based.

As a first step, we employ Corollary 2.4.4 to get a generalization of [30, Lemma 3] to the non-regular context.

Lemma 2.5.1 *Let k be a ring and let R be a local k-algebra with* $\operatorname{depth}(R) \leq 2$. *If* $\operatorname{Der}_k(R)$ *is a finite R-module of finite projective dimension, then* $\operatorname{Der}_k(R)$ *is free as an R-module.*

Proof. By Proposition 1.1.15, we have $\operatorname{pd}_R(\operatorname{Der}_k(R)) = \text{G-dim}_R(\operatorname{Der}_k(R))$. The conclusion follows from Corollary 2.4.4 with $C = C' = R$. $\square$

Lemma 2.5.2 *Let k be a ring and let R be a local k-algebra. Let $\mathfrak{p}$ be a prime ideal of R with height at most 2. If* $\operatorname{grade}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p})$ *and* $\operatorname{Der}_k(R)$ *is a finite R-module such that* $\operatorname{Der}_k(R)_{\mathfrak{p}}$ *has finite projective dimension, then* $\operatorname{Der}_k(R)_{\mathfrak{p}}$ *is free as an $R_{\mathfrak{p}}$-module.*

Proof. It follows from Proposition 2.4.11 with $C = C' = R$, $M = \Omega_k(R)$, and Proposition 1.1.15. $\square$

Our central result is the following:

Theorem 2.5.3 *Let k be a ring and let R be a local k-domain such that* $\mathrm{Der}_k(R)$ *is a finite R-module and* $\mathrm{grade}(\mathfrak{P}_{R/k}) = \mathrm{ht}(\mathfrak{P}_{R/k}) \leq 2$. *If* $\mathrm{Der}_k(R)$ *has finite projective dimension, then* $\mathrm{Der}_k(R)$ *is free as an R-module.*

Proof. Set $D = \mathrm{Der}_k(R)$ and $\mathfrak{P} = \mathfrak{P}_{R/k}$. Let p denote the characteristic of $R/\mathfrak{P}$. We may assume that $D \neq 0$.

If $p > 0$, by Theorem 1.6.4(i), $\mathrm{rad}(\mathfrak{P})$ is the maximal ideal of R. Then

$$\dim(R) = \mathrm{ht}(\mathrm{rad}(\mathfrak{P})) = \mathrm{ht}(\mathfrak{P}) \leq 2.$$

By Lemma 2.5.1, it follows that D is free.

If $p = 0$, by Theorem 1.6.4(ii), $\mathfrak{P}$ is a prime ideal of R. By Lemma 2.5.2, $D_{\mathfrak{P}}$ is a free $R_{\mathfrak{P}}$-module. By Theorem 1.6.3, we have

$$D/\mathfrak{P}D \;\cong\; (R/\mathfrak{P})^{\oplus r}$$

for some integer $r \geq 1$. Thus

$$\nu_R(D) \;=\; \nu_R(D/\mathfrak{P}D) = \nu_{R/\mathfrak{P}}(D/\mathfrak{P}D) \;=\; r,$$

where $\nu_R(D)$ denotes the minimal number of generators of R-module D. On the other hand, if we denote $\kappa(\mathfrak{P}) \;=\; R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ (the residue field of $R_{\mathfrak{P}}$), we can write

$$\nu_{R_{\mathfrak{P}}}(D_{\mathfrak{P}}) \;=\; \nu_{R_{\mathfrak{P}}}(D_{\mathfrak{P}}/\mathfrak{P}D_{\mathfrak{P}}) \;=\; \nu_{R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}}(D_{\mathfrak{P}}/\mathfrak{P}D_{\mathfrak{P}}) \;=\; \nu_{\kappa(\mathfrak{P})}(\kappa(\mathfrak{P})^{\oplus r}) \;=\; r$$

so that $\nu_{R_{\mathfrak{P}}}(D_{\mathfrak{P}}) = \nu_R(D)$. As $D_{\mathfrak{P}}$ is a free $R_{\mathfrak{P}}$-module, it follows, by Proposition A.13, that $\nu_{R_{\mathfrak{P}}}(D_{\mathfrak{P}}) \leq \mathrm{rk}_R(D)$. Therefore, we have inequalities

$$\nu_{R_{\mathfrak{P}}}(D_{\mathfrak{P}}) \leq \mathrm{rk}_R(D) \leq \nu_R(D) = \nu_{R_{\mathfrak{P}}}(D_{\mathfrak{P}}),$$

and hence $\mathrm{rk}_R(D) = \nu_R(D)$. The conclusion follows from Lemma A.14. $\square$

Remark 2.5.4 If we assume that $k \subset R$ and

$$\mathfrak{P}_{R/k} \cap k \;=\; (0)$$

(e.g., if the subring k is a field) then we have a natural embedding

$$k \hookrightarrow R/\mathfrak{P}_{R/k}$$

and hence $\mathrm{char}(R/\mathfrak{P}_{R/k}) = \mathrm{char}(k)$. Now notice that, in the case $p > 0$ of the proof of Theorem 2.5.3, the hypothesis $\mathrm{grade}(\mathfrak{P}_{R/k}) \;=\; \mathrm{height}(\mathfrak{P}_{R/k})$ is not needed. An immediate byproduct of these observations is Corollary 2.5.5 below.

Corollary 2.5.5 *Let R be a local domain containing a field k with $\mathrm{char}(k) > 0$, such that $\mathrm{Der}_k(R)$ is a finite R-module and $\mathrm{ht}(\mathfrak{P}_{R/k}) \leq 2$. If $\mathrm{pd}_R(\mathrm{Der}_k(R)) < \infty$ then $\mathrm{Der}_k(R)$ is free as an R-module.*

It is a standard fact that in a Cohen-Macaulay local ring R we have $\mathrm{grade}(I) = \mathrm{ht}(I)$ for every ideal $I \subset R$. Thus, we also readily derive from Theorem 2.5.3 the following consequence:

Corollary 2.5.6 *Let k be a ring and let R be a Cohen-Macaulay local k-domain such that $\mathrm{Der}_k(R)$ is a finite R-module and $\mathrm{ht}(\mathfrak{P}_{R/k}) \leq 2$. If $\mathrm{Der}_k(R)$ has finite projective dimension, then $\mathrm{Der}_k(R)$ is free as an R-module.*

Corollary 2.5.7 *Let k be a ring and let R be a factorial local k-domain such that $\mathrm{Der}_k(R)$ is a finite R-module and $\mathrm{ht}(\mathfrak{P}_{R/k}) \leq 2$. If $\mathrm{Der}_k(R)$ has finite projective dimension, then $\mathrm{Der}_k(R)$ is free as an R-module.*

Proof. Write $\mathfrak{P} = \mathfrak{P}_{R/k}$. If $\mathrm{char}(R/\mathfrak{P}) > 0$, then, by Theorem 1.6.4(i), $\mathrm{rad}(\mathfrak{P}_{R/k}) = \mathfrak{m}$ and so, by Lemma 2.5.1, $\mathrm{Der}_k(R)$ is free as an R-module. Now, we may assume that $\mathrm{char}(R/\mathfrak{P}) = 0$, so that $\mathfrak{P}$ is prime by Theorem 1.6.4(ii). If $\mathrm{ht}(\mathfrak{P}) \leq 1$, then (since R is a domain) we must have $\mathrm{grade}(\mathfrak{P}) = \mathrm{ht}(\mathfrak{P})$, and the assertion follows from Theorem 2.5.3. So we may assume that $\mathrm{ht}(\mathfrak{P}) = 2$. Therefore, we can guarantee that $\mathfrak{q} \subset \mathfrak{P}$ for some prime ideal $\mathfrak{q} \subset R$ with

$$\mathrm{height}(\mathfrak{q}) \ = \ 1.$$

Since R is factorial, we have $\mathfrak{q} = (a)$ for some (prime) element $a \in R$ (cf. Matsumura [33, Theorem 20.1]). It follows that any given

$$b \in \mathfrak{P} \setminus (a)$$

is a non-zero-divisor of $R/(a)$, i.e., $\{a, b\} \subset \mathfrak{P}$ is an R-sequence and hence $\mathrm{grade}(\mathfrak{P}) \geq 2$, which forces $\mathrm{grade}(\mathfrak{P}) = 2 = \mathrm{height}(\mathfrak{P})$, so that we can once again apply Theorem 2.5.3. $\square$

Remark 2.5.8 In virtue of the cases treated in Corollary 2.5.5 and Corollary 2.5.6, it is natural to ask about the existence of factorial non-Cohen-Macaulay domains of characteristic zero (and dimension necessarily greater than or equal to 3, since a factorial – hence normal – domain of dimension 2 is Cohen-Macaulay). This is a non-trivial problem but fortunately such rings do exist, as shown by Freitag-Kiehl [13, Theorem 5.8], which thus justifies our Corollary 2.5.7. We also refer the reader to the survey given in Lipman [27].

Finally, recall that every regular local ring R is a factorial Cohen-Macaulay domain, and that every finite R-module has finite projective dimension. Thus, both Corollary 2.5.6 and Corollary 2.5.7 independently recover Maloo [30, Theorem 4], which we state below.

Corollary 2.5.9 *Let k be a ring and let R be a regular local k-algebra such that $\mathrm{Der}_k(R)$ is a finite R-module and $\mathrm{ht}(\mathfrak{P}_{R/k}) \leq 2$. Then $\mathrm{Der}_k(R)$ is free as an R-module.*

Remark 2.5.10 As expected, the converse of Corollary 2.5.9 does not hold. A simple example is the non-regular 2-dimensional local domain

$$R = k[X,Y,Z]_{(X,Y,Z)}/(XY - Z^p)$$

where k is a field with $\mathrm{char}(k) = p > 0$. Letting x, y, z denote the residue classes of the variables, the R-module $\mathrm{Der}_k(R)$ is seen to be free, a basis being $\{\Delta_1, \Delta_2\}$, where

$$\Delta_1 = (p-1)x\partial_x + y\partial_y, \qquad \Delta_2 = \partial_z,$$

see [34, Proposition 2.3]. The case of characteristic zero is much harder, but Maloo [29, p. 84] presents a 1-dimensional non-regular Noetherian local ring containing a field k with $\mathrm{char}(k) = 0$, such that $\mathrm{Der}_k(R)$ is a finite free R-module.

Chapter 3

Reduced G_C-perfect modules and transpose with respect to C

Throughout this chapter, R is a commutative semiperfect Noetherian ring with identity, C is a semidualizing R-module and all modules are finite R-modules. Our main goal in this chapter is to study reduced G_C-perfect modules focusing on their Auslander transpose with respect to C. As a consequence we obtain generalizations of some results given in [3], [9] and [31].

3.1 A formula for reduced G_C-perfect modules

In this section, we will study the concept of reduced G_C-perfect module and present a formula that relates the G_C-dimension of a reduced G_C-perfect module M and the G_C-dimensions of $\mathrm{Tr}_C(M)$ and of $\mathrm{Ext}_R^{\mathrm{G}_C\text{-}\dim_R(M)}(M,C)$. We conclude the section with some consequences involving the operator $\lambda = \Omega\mathrm{Tr}$.

Definition 3.1.1 The *reduced grade* of a module M with respect to a semidualizing C is defined as follows

$$\mathrm{r.grade}_R\,(M,C) = \inf\{i > 0 \mid \mathrm{Ext}_R^i(M,C) \neq 0\}.$$

Remark 3.1.2 Let M, N be R-modules. Then

(i) $\mathrm{grade}_R\,(M) = \mathrm{r.grade}_R\,(M,C)$, if $\mathrm{grade}_R\,(M) > 0$.

(ii) $\mathrm{Ext}_R^i(M,C) = 0$ for all $i > 0$ if and only if $\mathrm{r.grade}_R\,(M,C) = \infty$.

(iii) If M has positive G_C-dimension, then

$$\operatorname{grade}_R(M) \leq \operatorname{r.grade}_R(M, C) \leq \mathrm{G}_C\text{-}\dim_R(M).$$

(iv) If $M \approx_C N$, then $\operatorname{r.grade}_R(M, C) = \operatorname{r.grade}_R(N, C)$. In particular, the reduced grade of $\operatorname{Tr}_C(M)$ with respect to C is well defined.

Definition 3.1.3 Let M be an R-module of finite G_C-dimension, we say that M is *reduced* G_C*-perfect* if its G_C-dimension is equal to its reduced grade with respect to C, i.e.

$$\operatorname{r.grade}_R(M, C) = \mathrm{G}_C\text{-}\dim_R(M).$$

Such modules were introduced by Dibaei and Sadeghi [10]. In the special case where $C = R$, the reduced G_R-perfect module is simply called *reduced* G-*perfect*.

Note that every reduced G_C-perfect module has a finite and positive G_C-dimension.

Example 3.1.4 Every G_C-perfect module of positive grade is reduced G_C-perfect. In particular, every perfect module of positive grade is reduced G_C-perfect.

We will present another example of reduced G_C-perfect module that is not necessarily G_C-perfect (see Example 3.2.6 in the next section).

For an integer $n > 0$, we consider the composition $\mathcal{T}_n^C := \operatorname{Tr}_C \Omega^{n-1}$, where Ω^{n-1} is obtained from a minimal projective resolution.

Lemma 3.1.5 *Let M be an R-module and n a positive integer. Then, there exist R-modules L and P such that P is projective and the following sequences, for a suitable choice of transposes, are exact*

$$0 \to \operatorname{Ext}_R^n(M, C) \to \mathcal{T}_n^C(M) \to L \to 0,$$

and

$$0 \to L \to \operatorname{Tr}_C(P) \to \mathcal{T}_{n+1}^C(M) \to 0.$$

Moreover, if $\operatorname{r.grade}_R(M, C) \geq n$*, then there exists an exact sequence*

$$0 \to \operatorname{Tr}_C(M) \to (P_2)^C \to \cdots \to (P_n)^C \to \mathcal{T}_n^C(M) \to 0,$$

where P_i is a projective R-module for all $i = 2, \ldots, n$.

Proof. Consider the short exact sequence

$$0 \to \Omega^n M \xrightarrow{f} P \to \Omega^{n-1} M \to 0, \tag{3.1}$$

where P is a projective R-module. By Proposition 1.2.4, there is an exact sequence

$$0 \to (\Omega^{n-1}M)^C \to P^C \xrightarrow{f^C} (\Omega^n M)^C \to \mathrm{Tr}_C(\Omega^{n-1}M) \xrightarrow{\psi} \mathrm{Tr}_C(P) \to \mathrm{Tr}_C(\Omega^n M) \to 0. \tag{3.2}$$

Note that $\mathrm{Tr}_C(\Omega^{n-1}M) = \mathcal{T}_n^C(M)$ and $\mathrm{Tr}_C(\Omega^n M) = \mathcal{T}_{n+1}^C(M)$. On the other hand, from the short exact sequence (3.1) we get an exact sequence

$$0 \to (\Omega^{n-1}M)^C \to P^C \xrightarrow{f^C} (\Omega^n M)^C \to \mathrm{Ext}_R^n(M, C) \to 0. \tag{3.3}$$

Therefore, by (3.2) and (3.3), we get $\mathrm{Ext}_R^n(M, C) \cong \mathrm{Ker}\,(\psi)$. Thus, we have an exact sequence

$$0 \to \mathrm{Ext}_R^n(M, C) \to \mathcal{T}_n^C(M) \xrightarrow{\psi} \mathrm{Tr}_C(P) \to \mathcal{T}_{n+1}^C(M) \to 0.$$

Taking $L := \mathrm{Im}\,(\psi)$ we get the two desired exact sequences.

Now assume that $\mathrm{r.grade}_R\,(M, C) \geq n$, i.e. $\mathrm{Ext}_R^i(M, C) = 0$ for all $0 < i < n$. Let

$$\cdots \to P_n \xrightarrow{\varphi_n} P_{n-1} \xrightarrow{\varphi_{n-1}} P_{n-2} \xrightarrow{\varphi_{n-2}} \cdots \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to M \to 0 \tag{3.4}$$

be the minimal projective resolution of M. In particular, we have exact sequences

$$P_1 \xrightarrow{\varphi_1} P_0 \to M \to 0 \quad \text{and} \quad P_n \xrightarrow{\varphi_n} P_{n-1} \to \Omega^{n-1}M \to 0$$

which are projective presentations of M and $\Omega^{n-1}M$, respectively. Then we get exact sequences

$$0 \to M^C \to (P_0)^C \xrightarrow{\varphi_1^C} (P_1)^C \to \mathrm{Tr}_C(M) \to 0 \tag{3.5}$$

and

$$0 \to (\Omega^{n-1}M)^C \to (P_{n-1})^C \xrightarrow{\varphi_n^C} (P_n)^C \to \mathcal{T}_n^C(M) \to 0. \tag{3.6}$$

Applying the functor $(-)^C := \mathrm{Hom}_R(-, C)$ on the exact sequence (3.4), we get a complex

$$0 \to M^C \to (P_0)^C \xrightarrow{\varphi_1^C} (P_1)^C \xrightarrow{\varphi_2^C} (P_2)^C \xrightarrow{\varphi_3^C} \cdots \xrightarrow{\varphi_{n-1}^C} (P_{n-1})^C \xrightarrow{\varphi_n^C} (P_n)^C \to \cdots. \tag{3.7}$$

As $\mathrm{Ext}_R^i(M, C) = 0$ for all $0 < i < n$, the complex (3.7) is exact until the module $(P_{n-1})^C$. From the exact sequences (3.5) and (3.6), we get $\mathrm{Tr}_C(M) \cong \mathrm{Ker}\,(\varphi_3^C)$ and $\mathcal{T}_n^C(M) \cong \mathrm{Coker}\,(\varphi_n^C)$. Now the assertion is obvious by (3.7). $\square$

The following result provides an important isomorphism.

Proposition 3.1.6 *Let M be a reduced* G_C*-perfect* R*-module of* G_C*-dimension* n*. Then,*

$$\mathrm{Ext}_R^i(\mathrm{Tr}_C(M), C) \cong \mathrm{Ext}_R^{i+n-1}(\mathrm{Ext}_R^n(M, C), C),$$

for all $i > 0$*.*

Proof. By Lemma 3.1.5, we get exact sequences:

$$0 \to \mathrm{Tr}_C(M) \to (P_2)^C \to \cdots \to (P_n)^C \to \mathcal{T}_n^C(M) \to 0, \tag{3.8}$$

$$0 \to \mathrm{Ext}_R^n(M, C) \to \mathcal{T}_n^C(M) \to L \to 0, \tag{3.9}$$

$$0 \to L \to \mathrm{Tr}_C(P) \to \mathcal{T}_{n+1}^C(M) \to 0, \tag{3.10}$$

where P and P_i are projective R-modules for all $i = 2, \ldots, n$. As $\mathrm{G}_C\text{-}\dim_R(M) = n$, it follows that $\mathrm{G}_C\text{-}\dim_R(\Omega^n M) = 0$. Then, by Proposition 1.2.6, we have

$$\mathrm{G}_C\text{-}\dim_R(\mathcal{T}_{n+1}^C(M)) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(\Omega^n M)) = 0.$$

Since $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(P)) = 0$, we obtain $\mathrm{G}_C\text{-}\dim_R(L) = 0$ by exact sequence (3.10) and Proposition 1.1.6. In particular, $\mathrm{Ext}_R^i(L, C) = 0$ for all $i > 0$. From exact sequence (3.9), we get a long exact sequence

$$\cdots \to \mathrm{Ext}_R^i(L, C) \to \mathrm{Ext}_R^i(\mathcal{T}_n^C(M), C) \to \mathrm{Ext}_R^i(\mathrm{Ext}_R^n(M, C), C) \to \mathrm{Ext}_R^{i+1}(L, C) \to \cdots.$$

Then

$$\mathrm{Ext}_R^i(\mathcal{T}_n^C(M), C) \cong \mathrm{Ext}_R^i(\mathrm{Ext}_R^n(M, C), C), \text{ for all } i > 0. \tag{3.11}$$

We can rewrite the exact sequence (3.8) of the following form

$$0 \to \mathrm{Tr}_C(M) \to G_{n-2} \to \cdots \to G_0 \to \mathcal{T}_n^C(M) \to 0, \tag{3.12}$$

where $G_j = (P_{n-j})^C$ with $j = 0, \ldots, n-2$. As each P_{n-j} has G_C-dimension zero, we have

$$\mathrm{Ext}_R^i(G_j, C) = 0, \text{ for all } i > 0, j = 0, \ldots, n-2.$$

Define $K_0 = \mathcal{T}_n^C(M)$, $K_1 = \mathrm{Ker}\,(G_0 \to \mathcal{T}_n^C(M))$, and $K_{j+1} = \mathrm{Ker}\,(G_j \to G_{j-1})$ for all $1 \leq j \leq n-2$. Note that $K_{n-1} \cong \mathrm{Tr}_C(M)$. Let $j \in \{0, \ldots, n-2\}$ be a fixed integer. Thus, we have a short exact sequence

$$0 \to K_{j+1} \to G_j \to K_j \to 0,$$

which induces a long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(G_j, C) \to \operatorname{Ext}^i_R(K_{j+1}, C) \to \operatorname{Ext}^{i+1}_R(K_j, C) \to \operatorname{Ext}^{i+1}_R(G_j, C) \to \cdots.$$

As $\operatorname{Ext}^i_R(G_j, C) = 0$ for all $i > 0$, we get

$$\operatorname{Ext}^i_R(K_{j+1}, C) \cong \operatorname{Ext}^{i+1}_R(K_j, C), \text{ for all } i > 0.$$

Therefore, for each $i > 0$, we have

$$\operatorname{Ext}^i_R(K_{n-1}, C) \cong \operatorname{Ext}^{i+1}_R(K_{n-2}, C) \cong \operatorname{Ext}^{i+2}_R(K_{n-3}, C) \cong \cdots \cong \operatorname{Ext}^{i+n-1}_R(K_0, C).$$

Thus,

$$\operatorname{Ext}^i_R(\operatorname{Tr}_C(M), C) \cong \operatorname{Ext}^{i+n-1}_R(\mathcal{T}^C_n(M), C), \text{ for all } i > 0. \tag{3.13}$$

Now the assertion is obvious by (3.11) and (3.13). $\square$

The next result establishes a connection between the G_C-dimension of a reduced G_C-perfect module M and the G_C-dimensions of $\operatorname{Tr}_C(M)$ and $\operatorname{Ext}^{\mathrm{G}_C\text{-}\dim_R(M)}_R(M, C)$.

Theorem 3.1.7 *Let M be a reduced G_C-perfect R-module of G_C-dimension n. Then,*

$$\mathrm{G}_C\text{-}\dim_R(M) + \mathrm{G}_C\text{-}\dim_R(\operatorname{Tr}_C(M)) = \mathrm{G}_C\text{-}\dim_R(\operatorname{Ext}^n_R(M, C)) + 1.$$

Proof. By Lemma 3.1.5, we get exact sequences

$$0 \to \operatorname{Tr}_C(M) \to G_{n-2} \to \cdots \to G_0 \to \mathcal{T}^C_n(M) \to 0, \tag{3.14}$$

and

$$0 \to \operatorname{Ext}^n_R(M, C) \to \mathcal{T}^C_n(M) \to L \to 0, \tag{3.15}$$

where $\mathrm{G}_C\text{-}\dim_R(L) = 0 = \mathrm{G}_C\text{-}\dim_R(G_j)$ for all $j = 0, \ldots, n-2$. Applying Proposition 1.1.11 on exact sequence (3.14), we get

$$\mathrm{G}_C\text{-}\dim_R(\operatorname{Tr}_C(M)) < \infty \Leftrightarrow \mathrm{G}_C\text{-}\dim_R(\mathcal{T}^C_n(M)) < \infty. \tag{3.16}$$

Again applying Proposition 1.1.11, now on exact sequence (3.15), we have

$$\mathrm{G}_C\text{-}\dim_R(\mathcal{T}^C_n(M)) < \infty \Leftrightarrow \mathrm{G}_C\text{-}\dim_R(\operatorname{Ext}^n_R(M, C)) < \infty. \tag{3.17}$$

By (3.16) and (3.17)

$$\mathrm{G}_C\text{-}\dim_R(\operatorname{Tr}_C(M)) < \infty \Leftrightarrow \mathrm{G}_C\text{-}\dim_R(\operatorname{Ext}^n_R(M, C)) < \infty.$$

Therefore, we can assume that $\mathrm{Tr}_C(M)$ and $\mathrm{Ext}^n_R(M,C)$ have finite G_C-dimensions. Thus, by Proposition 1.1.10, we can write

$$\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^n_R(M,C)) = \sup\{j \geq 0 \mid \mathrm{Ext}^j_R(\mathrm{Ext}^n_R(M,C),C) \neq 0\}, \tag{3.18}$$

and

$$\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) = \sup\{j > 0 \mid \mathrm{Ext}^j_R(\mathrm{Tr}_C(M),C) \neq 0\}. \tag{3.19}$$

Observe that the "sup" in (3.19) above is stricly positive since M is reduced G_C-perfect (this is why we wrote $j > 0$). More precisely, by Proposition 1.2.6 we would have $\mathrm{G}_C\text{-}\dim_R(M) = 0$ which is a contradiction. Taking $j = i+n-1$ on Proposition 3.1.6, we have

$$\mathrm{Ext}^j_R(\mathrm{Ext}^n_R(M,C),C) \cong \mathrm{Ext}^{j-n+1}_R(\mathrm{Tr}_C(M),C), \text{ for all } j > n-1. \tag{3.20}$$

Define $m = \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M))$ and $t = \mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^n_R(M,C))$. Taking $j = m+n-1 > n-1$ in (3.20), we have

$$\mathrm{Ext}^{m+n-1}_R(\mathrm{Ext}^n_R(M,C),C) \cong \mathrm{Ext}^m_R(\mathrm{Tr}_C(M),C) \neq 0,$$

by (3.19). For $j > m+n-1$, we have $j-n+1 > m$, so

$$\mathrm{Ext}^j_R(\mathrm{Ext}^n_R(M,C),C) \cong \mathrm{Ext}^{j-n+1}_R(\mathrm{Tr}_C(M),C) = 0,$$

by (3.20) and (3.19). Now the assertion is clear by (3.18). $\square$

Corollary 3.1.8 *Let R be a local ring and M a reduced G_C-perfect R-module of G_C-dimension n. Assume that $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) < \infty$. Then*

$$\mathrm{depth}_R(M) + \mathrm{depth}_R(\mathrm{Tr}_C(M)) = \mathrm{depth}(R) + \mathrm{depth}_R(\mathrm{Ext}^n_R(M,C)) - 1.$$

Proof. As $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) < \infty$, it follows that $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^n_R(M,C)) < \infty$, by Theorem 3.1.7. Set $d = \mathrm{depth}(R)$. By Theorem 1.1.13 and Theorem 3.1.7, we get

$$\begin{aligned}
\mathrm{depth}_R(M) + \mathrm{depth}_R(\mathrm{Tr}_C(M)) &= -(d - \mathrm{depth}_R(M) + d - \mathrm{depth}_R(\mathrm{Tr}_C(M))) + 2d \\
&= -(\mathrm{G}_C\text{-}\dim_R(M) + \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M))) + 2d \\
&= -\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^n_R(M,C)) - 1 + 2d \\
&= -d + \mathrm{depth}_R(\mathrm{Ext}^n_R(M,C)) - 1 + 2d \\
&= d + \mathrm{depth}_R(\mathrm{Ext}^n_R(M,C)) - 1,
\end{aligned}$$

as desired. $\square$

In order to employ the operator λ in the next results, we begin with the following lemma.

Lemma 3.1.9 *Let M be an R-module such that $\operatorname{Tor}_1^R(\operatorname{Tr}M, C) = 0$. Then, there exist a projective R-module P and a suitable choice of $\operatorname{Tr}_C(M)$ such that we have an exact sequence*

$$0 \to \lambda M \otimes_R C \to P \otimes_R C \to \operatorname{Tr}_C(M) \to 0. \tag{3.21}$$

In particular, we get the isomorphism

$$\operatorname{Ext}_R^i(\lambda M \otimes_R C, C) \cong \operatorname{Ext}_R^{i+1}(\operatorname{Tr}_C(M), C), \tag{3.22}$$

for all $i > 0$.

Proof. Consider the exact sequence

$$0 \to \lambda M \to P \to \operatorname{Tr}M \to 0,$$

where P is a projective R-module. Then we get an exact sequence

$$0 \to \operatorname{Tor}_1^R(\operatorname{Tr}M, C) \to \lambda M \otimes_R C \to P \otimes_R C \to \operatorname{Tr}M \otimes_R C \to 0.$$

By Proposition 1.2.10, $\operatorname{Tr}M \otimes_R C \cong \operatorname{Tr}_C(M)$. As $\operatorname{Tor}_1^R(\operatorname{Tr}M, C) = 0$, we have an exact sequence

$$0 \to \lambda M \otimes_R C \to P \otimes_R C \to \operatorname{Tr}_C(M) \to 0.$$

Since $\operatorname{Ext}_R^i(P \otimes_R C, C) = 0$ for all $i > 0$, we get

$$\operatorname{Ext}_R^i(\lambda M \otimes_R C, C) \cong \operatorname{Ext}_R^{i+1}(\operatorname{Tr}_C(M), C),$$

for all $i > 0$. $\square$

The following result is a generalization of [9, Proposition 3.5(i)].

Corollary 3.1.10 *Let M be a reduced G_C-perfect R-module of G_C-dimension n. Assume that $\operatorname{Tor}_1^R(\operatorname{Tr}M, C) = 0$. Then*

$$\operatorname{Ext}_R^i(\lambda M \otimes_R C, C) \cong \operatorname{Ext}_R^{i+n}(\operatorname{Ext}_R^n(M, C), C),$$

for all $i > 0$.

Proof. It is an immediate consequence of Proposition 3.1.6 and Lemma 3.1.9. $\square$

Corollary 3.1.11 *Let M be a reduced G_C-perfect R-module of G_C-dimension n. Assume that $\mathrm{Tor}_1^R(\mathrm{Tr}M, C) = 0$. Then*

$$\mathrm{G}_C\text{-}\dim_R(M) + \mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}_R^n(M, C)).$$

Proof. By Theorem 3.1.7, we have

$$\mathrm{G}_C\text{-}\dim_R(M) + \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) - 1 = \mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}_R^n(M, C)). \tag{3.23}$$

As $\mathrm{G}_C\text{-}\dim_R(P \otimes_R C) = 0$, Lemma 3.1.9 and Proposition 1.1.11 give

$$\mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) < \infty \Leftrightarrow \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) < \infty.$$

Therefore, if $\mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) = \infty$, then $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) = \infty$, which implies that $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}_R^n(M, C)) = \infty$ and the claim follows.
Now, assume that $\mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) < \infty$. As $\mathrm{G}_C\text{-}\dim_R(M) > 0$ we have $0 < \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) < \infty$. By Proposition 1.1.12, we get

$$\mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) - 1. \tag{3.24}$$

Now the assertion is obvious by (3.23) and (3.24). $\square$

Corollary 3.1.12 *Let R be a local ring and M a reduced G_C-perfect R-module of G_C-dimension n. Assume that $\mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) < \infty$ and $\mathrm{Tor}_1^R(\mathrm{Tr}M, C) = 0$. Then*

$$\mathrm{depth}_R(M) + \mathrm{depth}_R(\lambda M \otimes_R C) = \mathrm{depth}(R) + \mathrm{depth}_R(\mathrm{Ext}_R^n(M, C)).$$

Proof. Since $\mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) < \infty$, it follows that $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) < \infty$ and $\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}_R^n(M, C)) < \infty$. Set $d = \mathrm{depth}(R)$. By Theorem 1.1.13 and Corollary 3.1.11, we get

$$\begin{aligned}
\mathrm{depth}_R(M) + \mathrm{depth}_R(\lambda M \otimes_R C) &= -2d + \mathrm{depth}_R(M) + \mathrm{depth}_R(\lambda M \otimes_R C) + 2d \\
&= -(\mathrm{G}_C\text{-}\dim_R(M) + \mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C)) + 2d \\
&= -\mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}_R^n(M, C)) + 2d \\
&= -d + \mathrm{depth}_R(\mathrm{Ext}_R^n(M, C)) + 2d \\
&= d + \mathrm{depth}_R(\mathrm{Ext}_R^n(M, C)),
\end{aligned}$$

as desired. $\square$

3.2 Reduced G_C-perfectness and the Auslander transpose

We will show some results of [3] in the context of G_C-dimension and present formulas that relate the grade and the reduced grade with respect to C. We will show how the reduced G_C-perfect property is preserved under the Auslander transpose with respect to C. As a main consequence, we will generalize [9, Corollary 3.6].

The following proposition is a generalization of [3, Proposition 4.16].

Proposition 3.2.1 *Let M be an R-module such that $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in \bigcup_{i\in\Lambda} \mathrm{Supp}_R(\mathrm{Ext}^i_R(M,C))$, where $\Lambda = \{i > 0 \mid \mathrm{Ext}^i_R(M,C) \neq 0\}$. Then*

$$\mathrm{grade}_R(\mathrm{Ext}^i_R(M,C)) \geq i,$$

for all $i > 0$.

Proof. Let N be a non-zero finite R-module, by Proposition 1.4.2, we have

$$\mathrm{grade}_R(N) = \min\{\mathrm{depth}(R_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}_R(N)\}. \tag{3.25}$$

If $\mathrm{Ext}^i_R(M,C) = 0$ for all i, there is nothing to prove, since by convention $\mathrm{grade}_R(0) = \infty$. Otherwise, let $i > 0$ be such that $\mathrm{Ext}^i_R(M,C) \neq 0$. Let $\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{Ext}^i_R(M,C))$. We will show that $\mathrm{depth}(R_\mathfrak{p}) \geq i$, and thus by (3.25) it will follow that $\mathrm{Ext}^i_R(M,C)$ has grade at least i. Since

$$\mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) \cong (\mathrm{Ext}^i_R(M,C))_\mathfrak{p} \neq 0,$$

and $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$, by Proposition 1.1.10, we have $\mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq i$. Therefore, by Theorem 1.1.13 we get

$$\mathrm{depth}(R_\mathfrak{p}) = \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) + \mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \mathrm{G}_{C_\mathfrak{p}}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq i,$$

as desired. □

As a consequence we can generalize [3, Corollary 4.17] as follows.

Corollary 3.2.2 *Let M be an R-module of finite G_C-dimension, then*

$$\mathrm{grade}_R(\mathrm{Ext}^i_R(M,C)) \geq i,$$

for all $i > 0$.

Proof. By Proposition 1.1.9, $\mathrm{G}_{C_{\mathfrak{p}}}\text{-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Now the claim follows from Proposition 3.2.1. Another way to prove this fact is by using directly Theorem 2.2.7(c) with $k = 0$. □

The next result establishes a connection between the r.grade with respect to C of a reduced G_C-perfect module M, the grade of $\operatorname{Ext}_R^{\operatorname{r.grade}_R(M,C)}(M, C)$ and the r.grade of $\operatorname{Tr}_C(M)$ with respect to C.

Proposition 3.2.3 *Let M be a reduced G_C-perfect R-module of G_C-dimension n. Then,*

$$\operatorname{r.grade}_R(M, C) + \operatorname{r.grade}_R(\operatorname{Tr}_C(M), C) = \operatorname{grade}_R(\operatorname{Ext}_R^n(M, C)) + 1.$$

Proof. Taking $j = i + n - 1$ in Proposition 3.1.6, we have

$$\operatorname{Ext}_R^j(\operatorname{Ext}_R^n(M, C), C) \cong \operatorname{Ext}_R^{j-n+1}(\operatorname{Tr}_C(M), C), \text{ for all } j > n - 1. \tag{3.26}$$

Assume that $\operatorname{r.grade}_R(\operatorname{Tr}_C(M), C) < \infty$. Set $m = \operatorname{r.grade}_R(\operatorname{Tr}_C(M), C)$. Taking $j = m + n - 1 > n - 1$ in (3.26), we get

$$\operatorname{Ext}_R^{m+n-1}(\operatorname{Ext}_R^n(M, C), C) \cong \operatorname{Ext}_R^m(\operatorname{Tr}_C(M), C) \neq 0.$$

Let $0 \leq j < m + n - 1$. We consider two distinct cases:

If $n \leq j < m + n - 1$, then $1 \leq j - n + 1 < m$. Thus, by (3.26), we have

$$\operatorname{Ext}_R^j(\operatorname{Ext}_R^n(M, C), C) \cong \operatorname{Ext}_R^{j-n+1}(\operatorname{Tr}_C(M), C) = 0.$$

If $0 \leq j < n$, since M has finite G_C-dimension, by Corollary 3.2.2 we get $\operatorname{grade}_R(\operatorname{Ext}_R^n(M, C)) \geq n$. So, by Proposition 1.4.3, $\operatorname{Ext}_R^j(\operatorname{Ext}_R^n(M, C), C) = 0$. Therefore, $m + n - 1 = \operatorname{grade}_R(\operatorname{Ext}_R^n(M, C))$.

Finally, if $\operatorname{r.grade}_R(\operatorname{Tr}_C(M), C) = \infty$, then $\operatorname{Ext}_R^i(\operatorname{Tr}_C(M), C) = 0$ for all $i > 0$. Thus,

$$\operatorname{Ext}_R^j(\operatorname{Ext}_R^n(M, C), C) \cong \operatorname{Ext}_R^{j-n+1}(\operatorname{Tr}_C(M), C) = 0, \text{ for all } j \geq n.$$

As $\operatorname{grade}_R(\operatorname{Ext}_R^n(M, C)) \geq n$, it follows that $\operatorname{Ext}_R^j(\operatorname{Ext}_R^n(M, C), C) = 0$ for all $0 \leq j < n$. Therefore, $\operatorname{grade}_R(\operatorname{Ext}_R^n(M, C)) = \infty$ and the result follows. □

Adding some hypotheses we can include the operator λ in the above formula. To this end, the following lemma will be necessary.

Lemma 3.2.4 *Let* $0 \to N \to X_k \to \cdots \to X_1 \to M \to 0$ *be an exact sequence of* R*-modules such that* $\mathrm{Ext}^i_R(X_j, C) = 0$ *for all* $i > 0$, $j = 1, \ldots, k$. *If* $\mathrm{r.grade}_R\,(M, C) > k$, *then*

$$\mathrm{r.grade}_R\,(N, C) = \mathrm{r.grade}_R\,(M, C) - k.$$

Proof. We prove the claim by induction on k. If $k = 1$, we obtain the isomorphisms

$$\mathrm{Ext}^i_R(N, C) \cong \mathrm{Ext}^{i+1}_R(M, C), \text{ for all } i > 0. \tag{3.27}$$

Let us separate the proof in two cases:

If $\mathrm{r.grade}_R\,(M, C) = \infty$, then $\mathrm{Ext}^i_R(M, C) = 0$ for all $i > 0$. For $i > 0$, (3.27) gives $\mathrm{Ext}^i_R(N, C) \cong \mathrm{Ext}^{i+1}_R(M, C) = 0$. Thus, $\mathrm{r.grade}_R\,(N, C) = \infty$.

If $\mathrm{r.grade}_R\,(M, C) < \infty$, by (3.27), we get

$$\mathrm{Ext}^{t-1}_R(N, C) \cong \mathrm{Ext}^t_R(M, C) \neq 0, \tag{3.28}$$

where $t := \mathrm{r.grade}_R\,(M, C)$. Therefore, $\mathrm{r.grade}_R\,(N, C) \leq t - 1$. Let $0 < i < t - 1$ and note that $1 < i + 1 < t$. By (3.27), we have

$$\mathrm{Ext}^i_R(N, C) \cong \mathrm{Ext}^{i+1}_R(M, C) = 0. \tag{3.29}$$

Thus, by (3.28) and (3.29), $\mathrm{r.grade}_R\,(N, C) = t - 1$.

Now assume that $k > 1$. Set $K = \mathrm{Ker}\,(X_{k-1} \to X_{k-2})$, then we get an exact sequence $0 \to K \to X_{k-1} \to \cdots \to X_1 \to M \to 0$. Applying the induction hypothesis on K implies that

$$\mathrm{r.grade}_R\,(K, C) = \mathrm{r.grade}_R\,(M, C) - k + 1. \tag{3.30}$$

Applying the case $k = 1$ on the exact sequence $0 \to N \to X_k \to K \to 0$, we get

$$\mathrm{r.grade}_R\,(N, C) = \mathrm{r.grade}_R\,(K, C) - 1. \tag{3.31}$$

Now, using (3.30) and (3.31), we obtain the result. □

The next consequence connects to the concept of horizontal linkage:

Corollary 3.2.5 *Let* M *be a horizontally linked and reduced* G_C*-perfect* R*-module of* G_C*-dimension* n. *Assume that* $\mathrm{Tor}^R_1(\mathrm{Tr}M, C) = 0$. *Then,*

$$\mathrm{r.grade}_R\,(M, C) + \mathrm{r.grade}_R\,(\lambda M \otimes_R C, C) = \mathrm{grade}_R\,(\mathrm{Ext}^n_R(M, C)).$$

Proof. By Lemma 3.1.9, we have a short exact sequence

$$0 \to \lambda M \otimes_R C \to P \otimes_R C \to \mathrm{Tr}_C(M) \to 0,$$

where P is a projective R-module. Since M is a horizontally linked R-module, by Corollary 2.3.9, we have $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) = 0$ which implies that $\mathrm{r.grade}_R\,(\mathrm{Tr}_C(M), C) > 1$. By Lemma 3.2.4, we have the following equality

$$\mathrm{r.grade}_R\,(\lambda M \otimes_R C, C) = \mathrm{r.grade}_R\,(\mathrm{Tr}_C(M), C) - 1.$$

Hence the assertion is obvious by Proposition 3.2.3. $\square$

In Example 3.2.6(i) below we generalize [9, Example 3.2].

Example 3.2.6 Let n be a positive integer and M a non-zero R-module such that $\mathrm{grade}_R\,(M) \geq n$. Then,

(i) $\mathcal{T}_n^C(M)$ is a reduced G_C-perfect R-module of G_C-dimension n.

(ii) If R is local and $\mathrm{grade}_R\,(M) > n$, then $\mathcal{T}_n^C(M)$ is not a G_C-perfect R-module.

(iii) If $\mathrm{grade}_R\,(M) > n$ and $\mathrm{G}_C\text{-}\mathrm{dim}_R\,(M) = \infty$, then $\mathrm{Tr}_C(\mathcal{T}_n^C(M))$ is not a reduced G_C-perfect.

Proof. (i) We prove the claim by induction on n. If $n = 1$, then $\mathrm{grade}_R\,(M) \geq 1$, which implies that $M^C = 0$. Let $P_1 \to P_0 \to M \to 0$ be the minimal projective presentation of M. Applying the functor $(-)^C = \mathrm{Hom}_R\,(-, C)$ to such resolution gives the following exact sequence

$$0 \to P_0^C \to P_1^C \to \mathrm{Tr}_C(M) \to 0.$$

Thus, $\mathrm{G}_C\text{-}\mathrm{dim}_R\,(\mathrm{Tr}_C(M)) \leq 1$. As $\mathrm{G}_C\text{-}\mathrm{dim}_R\,(M)$ is positive, the same happens to $\mathrm{G}_C\text{-}\mathrm{dim}_R\,(\mathrm{Tr}_C(M))$. Therefore, $\mathcal{T}_1^C(M) = \mathrm{Tr}_C(M)$ is reduced G_C-perfect of G_C-dimension 1.

Now assume that $n > 1$. Applying the induction hypothesis to $\mathcal{T}_{n-1}^C(M)$ implies that $\mathcal{T}_{n-1}^C(M)$ is a reduced G_C-perfect R-module of G_C-dimension $n - 1$. As $\mathrm{Ext}^{n-1}_R(M, C) = 0$, Lemma 3.1.5 gives a short exact sequence

$$0 \to \mathcal{T}_{n-1}^C(M) \to \mathrm{Tr}_C(P) \to \mathcal{T}_n^C(M) \to 0. \tag{3.32}$$

Since $\mathrm{G}_C\text{-}\dim_R(\mathcal{T}_{n-1}^C(M)) = n-1$ and $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(P)) = 0$, the exact sequence (3.32) yields $\mathrm{G}_C\text{-}\dim_R(\mathcal{T}_n^C(M)) \leq n$. To conclude, it suffices to show the following inequalities: $\mathrm{r.grade}_R(\mathcal{T}_n^C(M), C) \geq n$ and $\mathrm{G}_C\text{-}\dim_R(\mathcal{T}_n^C(M)) > 0$. Since $\Omega^{n-1}M$ is a 1-syzygy module, it follows, by Corollary 2.3.9, that $\mathrm{Ext}_R^1(\mathcal{T}_n^C(M), C) = 0$, thus $\mathrm{r.grade}_R(\mathcal{T}_n^C(M), C) > 1$. Applying Lemma 3.2.4 on the exact sequence (3.32), we get $\mathrm{r.grade}_R(\mathcal{T}_n^C(M), C) = n$. Finally, note that if we assume $\mathrm{G}_C\text{-}\dim_R(\mathcal{T}_n^C(M)) = 0$, then $\mathrm{G}_C\text{-}\dim_R(\Omega^{n-1}M) = 0$, which implies that M has G_C-dimension less than n, contradicting the hypothesis that $\mathrm{grade}_R(M) \geq n$. By Remark 3.1.2, we have inequalities

$$n = \mathrm{r.grade}_R(\mathcal{T}_n^C(M), C) \leq \mathrm{G}_C\text{-}\dim_R(\mathcal{T}_n^C(M)) \leq n.$$

Therefore, $\mathcal{T}_n^C(M)$ is a reduced G_C-perfect R-module of G_C-dimension n.

(ii) Set $N = \mathcal{T}_n^C(M)$. By Proposition 3.2.3, we have

$$\mathrm{r.grade}_R(N, C) + \mathrm{r.grade}_R(\mathrm{Tr}_C(N), C) = \mathrm{grade}_R(\mathrm{Ext}_R^n(N, C)) + 1. \tag{3.33}$$

By Proposition 1.2.5, there exists the following exact sequence

$$0 \to \Omega^{n-1}M \to \mathrm{Tr}_C(N) \to X \to 0, \tag{3.34}$$

where $\mathrm{G}_C\text{-}\dim_R(X) = 0$. In particular, $\mathrm{Ext}_R^i(X, C) = 0$ for all $i > 0$, which implies that

$$\mathrm{Ext}_R^i(\Omega^{n-1}M, C) \cong \mathrm{Ext}_R^i(\mathrm{Tr}_C(N), C), \text{ for all } i > 0. \tag{3.35}$$

Thus,

$$\mathrm{r.grade}_R(\Omega^{n-1}M, C) = \mathrm{r.grade}_R(\mathrm{Tr}_C(N), C). \tag{3.36}$$

As $\mathrm{r.grade}_R(M, C) > n$, by Lemma 3.2.4 we get

$$\mathrm{r.grade}_R(\Omega^{n-1}M, C) = \mathrm{r.grade}_R(M, C) - n + 1. \tag{3.37}$$

By (3.33), (3.36) and (3.37), we have

$$\mathrm{r.grade}_R(M, C) = \mathrm{grade}_R(\mathrm{Ext}_R^n(N, C)). \tag{3.38}$$

Assume by absurd that N is a G_C-perfect R-module. So, it has grade n. By Proposition 1.4.5, $\mathrm{Ext}_R^n(N, C)$ is a G_C-perfect R-module of grade n. Thus, by (3.38), $\mathrm{r.grade}_R(M, C) = n$ which forces $\mathrm{grade}_R(M) = n$, a contradiction.

(iii) Let us keep the notation used in the previous item. Applying the inequalities from Proposition 1.1.16 on the exact sequence (3.34), we get $\mathrm{G}_C\text{-}\dim_R(\Omega^{n-1}M) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(N))$. Therefore, $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(N)) = \infty$. In particular, $\mathrm{Tr}_C(N)$ is not a reduced G_C-perfect. $\square$

The following theorem shows how the property of being reduced G_C-perfect is preserved under transpose with respect to C.

Theorem 3.2.7 *Let M be an R-module. Let n and t be two integers. Then the following statements are equivalent:*

(i) *M is reduced G_C-perfect of G_C-dimension n and $\mathrm{Ext}^n_R(M,C)$ is G_C-perfect of G_C-dimension $n+t-1$.*

(ii) *$\mathrm{Tr}_C(M)$ is reduced G_C-perfect of G_C-dimension t and $\mathrm{Ext}^t_R(\mathrm{Tr}_C(M),C)$ is G_C-perfect of G_C-dimension $n+t-1$.*

Proof. Set $N = \mathrm{Tr}_C(M)$. By Proposition 1.2.5, we have an exact sequence

$$0 \to M \to \mathrm{Tr}_C(N) \to X \to 0 \tag{3.39}$$

where $\mathrm{G}_C\text{-}\dim_R(X) = 0$. As $\mathrm{Ext}^i_R(X,C) = 0$ for all $i > 0$, by the exact sequence (3.39) we have the isomorphisms $\mathrm{Ext}^i_R(M,C) \cong \mathrm{Ext}^i_R(\mathrm{Tr}_C(N),C)$, for all $i > 0$. Thus,

$$\mathrm{r.grade}_R(\mathrm{Tr}_C(N),C) = \mathrm{r.grade}_R(M,C). \tag{3.40}$$

Also, applying the inequalities from Proposition 1.1.16 on the exact sequence (3.39), we get a following equality

$$\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(N)) = \mathrm{G}_C\text{-}\dim_R(M). \tag{3.41}$$

(i)⇒(ii) By Theorem 3.1.7 and Proposition 3.2.3, we have

$$\begin{aligned}
\mathrm{G}_C\text{-}\dim_R(N) &= \mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^n_R(M,C)) + 1 - \mathrm{G}_C\text{-}\dim_R(M) \\
&= \mathrm{grade}_R(\mathrm{Ext}^n_R(M,C)) + 1 - \mathrm{r.grade}_R(M,C) \\
&= \mathrm{r.grade}_R(N,C).
\end{aligned}$$

Therefore, N is reduced G_C-perfect of G_C-dimension t. Again applying Theorem 3.1.7 and Proposition 3.2.3, we get

$$\mathrm{G}_C\text{-}\dim_R(N) + \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(N)) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Ext}^t_R(N,C)) + 1 \tag{3.42}$$

and

$$\text{r.grade}_R\,(N,C) + \text{r.grade}_R\,(\text{Tr}_C(N),C) = \text{grade}_R\,(\text{Ext}^t_R(N,C)) + 1. \tag{3.43}$$

Thus, by (3.40), (3.41), (3.42) and (3.43), we have

$$\begin{aligned}
\text{G}_C\text{-dim}_R\,(\text{Ext}^t_R(N,C)) &= \text{G}_C\text{-dim}_R\,(N) + n - 1 \\
&= \text{r.grade}_R\,(N,C) + n - 1 \\
&= \text{grade}_R\,(\text{Ext}^t_R(N,C)).
\end{aligned}$$

Therefore, $\text{Ext}^t_R(N,C)$ is G_C-perfect of G_C-dimension $n+t-1$.

(ii)⇒(i) By Theorem 3.1.7 and Proposition 3.2.3, we get

$$\text{G}_C\text{-dim}_R\,(\text{Tr}_C(N)) = \text{r.grade}_R\,(\text{Tr}_C(N),C). \tag{3.44}$$

Thus, by (3.40), (3.41) and (3.44), M is reduced G_C-perfect of G_C-dimension n. Once again, applying Theorem 3.1.7 and Proposition 3.2.3 gives

$$\begin{aligned}
\text{G}_C\text{-dim}_R\,(\text{Ext}^n_R(M,C)) &= \text{G}_C\text{-dim}_R\,(M) + t - 1 \\
&= \text{r.grade}_R\,(M,C) + t - 1 \\
&= \text{grade}_R\,(\text{Ext}^n_R(M,C)).
\end{aligned}$$

Therefore, $\text{Ext}^n_R(M,C)$ is G_C-perfect of G_C-dimension $n+t-1$. □

Corollary 3.2.8 *Let R be a local ring and M a G_C-perfect R-module of grade $n>0$. Then $\text{Tr}_C(M)$ is a reduced G_C-perfect R-module of G_C-dimension 1, and the R-module $\text{Ext}^1_R(\text{Tr}_C(M),C)$ is G_C-perfect of G_C-dimension n.*

Proof. By Example 3.1.4 and Proposition 1.4.5, M is reduced G_C-perfect of G_C-dimension n and $\text{Ext}^n_R(M,C)$ is a G_C-perfect R-module of G_C-dimension $n=n+1-1$. Now the assertion is clear by Theorem 3.2.7 with $t=1$. □

Corollary 3.2.9 *Let $n>0$ and $t>2$ be two integers. Let M be a reduced G_C-perfect R-module of G_C-dimension n be such that $\text{Ext}^n_R(M,C)$ is a G_C-perfect R-module of G_C-dimension $n+t-1$. Then M^C is reduced G_C-perfect of G_C-dimension $t-2$ and $\text{Ext}^{t-2}_R(M^C,C)$ is G_C-perfect of G_C-dimension $n+t-1$.*

Proof. By Theorem 3.2.7, $\mathrm{Tr}_C(M)$ is reduced G_C-perfect of G_C-dimension t and $\mathrm{Ext}^t_R(\mathrm{Tr}_C(M), C)$ is G_C-perfect of G_C-dimension $n+t-1$. Consider the exact sequence

$$0 \to M^C \to (P_0)^C \to (P_1)^C \to \mathrm{Tr}_C(M) \to 0, \tag{3.45}$$

where P_0 and P_1 are projective R-modules. Applying Proposition 1.1.12 on the exact sequence (3.45), we get an equality

$$\mathrm{G}_C\text{-}\dim_R(M^C) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) - 2 = t - 2. \tag{3.46}$$

Now, applying Lemma 3.2.4 on the exact sequence (3.45), we have

$$\mathrm{r.grade}_R(M^C, C) = \mathrm{r.grade}_R(\mathrm{Tr}_C(M), C) - 2 = t - 2. \tag{3.47}$$

By (3.46) and (3.47), M^C is reduced G_C-perfect of G_C-dimension $t-2$. Finally, by the isomorphism

$$\mathrm{Ext}^{t-2}_R(M^C, C) \cong \mathrm{Ext}^t_R(\mathrm{Tr}_C(M), C),$$

we conclude that $\mathrm{Ext}^{t-2}_R(M^C, C)$ is G_C-perfect of G_C-dimension $n+t-1$. $\square$

As another consequence of Theorem 3.2.7, we generalize [9, Corollary 3.6] as follows:

Corollary 3.2.10 *Let M be a horizontally linked R-module. Suppose $\mathrm{Tor}^R_1(\mathrm{Tr}M, C)$ is zero. Let n and t be two positive integers. Then the following statements are equivalent:*

(i) *M is reduced G_C-perfect of G_C-dimension n and $\mathrm{Ext}^n_R(M, C)$ is G_C-perfect of G_C-dimension $n+t$.*

(ii) *$\lambda M \otimes_R C$ is reduced G_C-perfect of G_C-dimension t and $\mathrm{Ext}^t_R(\lambda M \otimes_R C, C)$ is G_C-perfect of G_C-dimension $n+t$.*

Proof. By Theorem 3.2.7, it is sufficient to prove that the following statements are equivalent:

(ii)' $\mathrm{Tr}_C(M)$ is reduced G_C-perfect of G_C-dimension $t+1$ and $\mathrm{Ext}^{t+1}_R(\mathrm{Tr}_C(M), C)$ is G_C-perfect of G_C-dimension $n+t$.

(ii) $\lambda M \otimes_R C$ is reduced G_C-perfect of G_C-dimension t and $\mathrm{Ext}^t_R(\lambda M \otimes_R C, C)$ is G_C-perfect of G_C-dimension $n+t$.

By Lemma 3.1.9, we have an exact sequence

$$0 \to \lambda M \otimes_R C \to P \otimes_R C \to \mathrm{Tr}_C(M) \to 0, \tag{3.48}$$

where P is a projective R-module. Since M is horizontally linked R-module, it follows that $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) = 0$, by Corollary 2.3.9. As $\mathrm{G}_C\text{-}\dim_R(P \otimes_R C) = 0$, by the exact sequence (3.48), Proposition 1.1.11 and Proposition 1.1.6, we have

$$0 < \mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) < \infty \Leftrightarrow 0 < \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) < \infty.$$

In this case, by Proposition 1.1.12, we have

$$\mathrm{G}_C\text{-}\dim_R(\lambda M \otimes_R C) = \mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) - 1, \tag{3.49}$$

and, by Lemma 3.2.4,

$$\mathrm{r.grade}_R(\lambda M \otimes_R C, C) = \mathrm{r.grade}_R(\mathrm{Tr}_C(M), C) - 1. \tag{3.50}$$

Thus, by (3.49) and (3.50), $\mathrm{Tr}_C(M)$ is reduced G_C-perfect of G_C-dimension $t+1$ if and only if $\lambda M \otimes_R C$ is reduced G_C-perfect of G_C-dimension t. Finally, since $t > 0$, by the exact sequence (3.48), we obtain the isomorphism

$$\mathrm{Ext}^t_R(\lambda M \otimes_R C, C) \cong \mathrm{Ext}^{t+1}_R(\mathrm{Tr}_C(M), C).$$

Therefore the assertion is obvious. $\square$

3.3 Horizontal linkage for G_C-perfect and reduced G_C-perfect modules

In this section we will prove some results involving horizontal linkage, Auslander class, G_C-perfect and reduced G_C-perfect modules. We present generalizations of some results of Martsinkovsky and Strooker [31], and we conclude the section with a generalization of a result due to Dibaei and Sadeghi [9] that characterizes the horizontal linkage of reduced G_C-perfect modules.

We first bring the following lemma.

Lemma 3.3.1 *Let M be a non-zero horizontally linked R-module. Then, the dual M^* is non-zero, i.e.* $\mathrm{grade}_R(M) = 0$.

Proof. Assume contrarily that M^* is zero. Then, the bidual module M^{**} is zero. By Proposition 1.2.6, there is an exact sequence

$$0 \to \mathrm{Ext}^1_R(\mathrm{Tr}M, R) \to M \xrightarrow{\sigma_M} M^{**} \to \mathrm{Ext}^2_R(\mathrm{Tr}M, R) \to 0$$

which implies that $\mathrm{Ext}^1_R(\mathrm{Tr}M, R) \cong M \neq 0$. Therefore, by Theorem 1.5.2, M can not be a horizontally linked module, which is a contradiction. Thus, M^* is non-zero. Finally, as

$$\mathrm{grade}_R\,(M) = \inf\{i \geq 0 \mid \mathrm{Ext}^i_R(M, R) \neq 0\},$$

it follows that $M^* \neq 0$ if and only if $\mathrm{grade}_R\,(M) = 0$. $\square$

The following result is an immediate consequence of Lemma 3.3.1.

Corollary 3.3.2 *Let M be a non-zero horizontally linked and G_C-perfect R-module. Then, M has G_C-dimension zero.*

Proof. $\mathrm{G}_C\text{-}\dim_R\,(M) = \mathrm{grade}_R\,(M) = 0$ by Lemma 3.3.1. $\square$

To get a reciprocal of Corollary 3.3.2 we will use the notion of Auslander class.

The following result is a generalization of Theorem 1.5.3 ([31, Theorem 1]). In particular, we get a converse for Corollary 3.3.2.

Theorem 3.3.3 *Let M be a stable R-module of G_C-dimension zero. Assume that $\lambda M \in \mathcal{A}_C$. Then M is horizontally linked, λM is a stable R-module and $\lambda M \otimes_R C$ has G_C-dimension zero.*

Proof. As $\mathrm{G}_C\text{-}\dim_R\,(M) = 0$, it follows that $\mathrm{G}_C\text{-}\dim_R\,(\mathrm{Tr}_C(M)) = 0$, by Proposition 1.2.6. In particular, $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) = 0$. Consider the exact sequence

$$0 \to \lambda M \to P \to \mathrm{Tr}M \to 0,$$

where P is a projective R-momule. Since $\lambda M \in \mathcal{A}_C$, it follows, by Remark 1.3.2(i), that $\mathrm{Tr}M \in \mathcal{A}_C$. Thus, by Remark 1.3.5, $\mathrm{Ext}^1_R(\mathrm{Tr}M, R) = 0$. By Theorem 1.5.2, M is horizontally linked. By Proposition 1.5.5, λM is a stable R-module. As $\mathrm{Tr}M \in \mathcal{A}_C$, by definition, $\mathrm{Tor}^R_1(\mathrm{Tr}M, C) = 0$. By Lemma 3.1.9, there exists an exact sequence

$$0 \to \lambda M \otimes_R C \to Q \otimes_R C \to \mathrm{Tr}_C(M) \to 0,$$

where Q is a projective R-module. Therefore, by Proposition 1.1.6, we conclude that $\lambda M \otimes_R C$ has G_C-dimension zero. $\square$

As a consequence we obtain a generalization of [31, Corollary 2].

Corollary 3.3.4 *Let R be a Cohen-Macaulay local ring with canonical module ω_R, and M a stable maximal Cohen-Macaulay R-module. Assume that* $\text{G-dim}_R(\lambda M) < \infty$. *Then M is horizontally linked and λM is again a stable maximal Cohen-Macaulay R-module.*

Proof. By Remark 1.3.2(iii), $\lambda M \in \mathcal{A}_{\omega_R}$. By Proposition 1.3.3, we get $\operatorname{depth}_R(\lambda M) = \operatorname{depth}_R(\lambda M \otimes_R \omega_R)$. Since M is stable, it follows that $\lambda M \neq 0$, by Theorem 1.5.6. Thus, by Proposition 1.1.14 and Theorem 1.1.13, we have

$$\mathrm{G}_{\omega_R}\text{-}\mathrm{dim}_R(\lambda M) = \mathrm{G}_{\omega_R}\text{-}\mathrm{dim}_R(\lambda M \otimes_R \omega_R).$$

Now the assertion is obvious by Theorem 3.3.3. $\square$

Another immediate consequence tells us that, under a certain condition, the only G_C-perfect modules that are horizontally linked are the stable totally C-reflexive modules.

Corollary 3.3.5 *Let M be a* G_C*-perfect R-module. Assume that $\lambda M \in \mathcal{A}_C$. Then M is horizontally linked if and only if M is stable of* G_C*-dimension zero.*

Proof. Assume that M is horizontally linked, then $\mathrm{G}_C\text{-}\mathrm{dim}_R(M) = 0$ and M is stable, by Corollary 3.3.2 and Proposition 1.5.4. Conversely, if M is stable with G_C-dimension zero, then the result follows by Theorem 3.3.3. $\square$

Over a Cohen-Macaulay local ring with canonical module, the horizontal linkage of certain Cohen-Macaulay modules is characterized via the maximal Cohen-Macaulay property.

Corollary 3.3.6 *Let R a Cohen-Macaulay local ring with canonical module ω_R, and M a non-zero Cohen-Macaulay R-module. Assume that* $\text{G-dim}_R(\lambda M) < \infty$. *Then M is horizontally linked if and only if M is stable maximal Cohen-Macaulay.*

Proof. By Proposition 1.1.14, M has finite G_{ω_R}-dimension. As depth $(R) = \dim(R)$ and $\operatorname{depth}_R(M) = \dim_R(M)$, it follows, by Theorem 1.1.13 and Corollary B.9, that

$$\mathrm{G}_{\omega_R}\text{-}\dim_R(M) = \dim(R) - \dim_R(M) = \operatorname{ht}(\operatorname{Ann}_R(M)) = \operatorname{grade}_R(M).$$

Therefore, M is G_{ω_R}-perfect. Note that $\mathrm{G}_{\omega_R}\text{-}\dim_R(M) = 0$ if and only if M is maximal Cohen-Macaulay. Since $\mathrm{G}\text{-}\dim_R(\lambda M) < \infty$, we have $\lambda M \in \mathcal{A}_{\omega_R}$, by Remark 1.3.2(iii). Now by Corollary 3.3.5 we obtain the result. $\square$

As we will see below, over a Gorenstein local ring the only Cohen-Macaulay modules that are horizontally linked are the stable maximal Cohen-Macaulay modules.

Corollary 3.3.7 *Let R be a Gorenstein local ring and M a non-zero Cohen-Macaulay R-module. Then M is horizontally linked if and only if M is stable maximal Cohen-Macaulay.*

We will now relate the reduced G_C-perfect modules with the Auslander class. For this purpose we will need the following results.

Lemma 3.3.8 *Let M be a reduced G_C-perfect R-module of G_C-dimension n. Assume that* $\operatorname{grade}_R(\operatorname{Ext}_R^n(M,C)) \geq n+1$. *Then* $\operatorname{Ext}_R^1(\operatorname{Tr}_C(M), C) = 0$.

Proof. As $\operatorname{grade}_R(\operatorname{Ext}_R^n(M,C)) \geq n+1$, it follows that

$$\operatorname{Ext}_R^j(\operatorname{Ext}_R^n(M,C), C) = 0, \tag{3.51}$$

for all $0 \leq j \leq n$. Taking $j = i+n-1$ in Proposition 3.1.6, we have

$$\operatorname{Ext}_R^j(\operatorname{Ext}_R^n(M,C), C) \cong \operatorname{Ext}_R^{j-n+1}(\operatorname{Tr}_C(M), C), \tag{3.52}$$

for all $j \geq n$. Taking $j = n$ in (3.52) and (3.51), we obtain the result. Another way to prove this fact is by using directly Lemma 2.2.6 with $k = 1$. $\square$

Lemma 3.3.9 *Let M be a horizontally linked R-module of finite G_C-dimension. Then*

$$\operatorname{grade}_R(\operatorname{Ext}_R^i(M,C)) \geq i+1,$$

for all $i > 0$.

Proof. By Corollary 2.3.9, M is C-1-torsionless. Now the assertion is clear by Theorem 2.2.7(c) with $k = 1$. $\square$

Finally, the following result provides a characterization for the horizontal linkage of reduced G_C-perfect modules.

Theorem 3.3.10 *Let M be a stable reduced G_C-perfect R-module of G_C-dimension n. Assume that $\lambda M \in \mathcal{A}_C$. The following conditions are equivalent:*

(i) *M is horizontally linked.*

(ii) $\mathrm{r.grade}_R\,(M, C) + \mathrm{r.grade}_R\,(\lambda M) = \mathrm{grade}_R\,(\mathrm{Ext}^n_R\,(M, C))$.

(iii) $\mathrm{grade}_R\,(\mathrm{Ext}^i_R(M, C)) \geq i + 1$, *for all $i > 0$.*

(iv) $\mathrm{grade}_R\,(\mathrm{Ext}^n_R(M, C)) \geq n + 1$.

Proof. (i) $\Rightarrow$ (ii) As $\lambda M \in \mathcal{A}_C$, it follows, by Theorem 1.3.4, that

$$\mathrm{Ext}^i_R\,(\lambda M \otimes_R C, C) \cong \mathrm{Ext}^i_R\,(\lambda M, R),$$

for all $i > 0$. Then $\mathrm{r.grade}_R\,(\lambda M) = \mathrm{r.grade}_R\,(\lambda M \otimes_R C, C)$. Consider the exact sequence

$$0 \to \lambda M \to P \to \mathrm{Tr} M \to 0,$$

where P is a projective R-module. By Remark 1.3.2(i), $\mathrm{Tr} M \in \mathcal{A}_C$. In particular, $\mathrm{Tor}^R_1(\mathrm{Tr} M, C) = 0$. Therefore, by Corollary 3.2.5, we have

$$\mathrm{r.grade}_R\,(M, C) + \mathrm{r.grade}_R\,(\lambda M \otimes_R C, C) = \mathrm{grade}_R\,(\mathrm{Ext}^n_R\,(M, C)).$$

(ii) $\Rightarrow$ (iv) As $\mathrm{r.grade}_R\,(M, C) = n$ and $\mathrm{r.grade}_R\,(\lambda M) \geq 1$ we get

$$\mathrm{grade}_R\,(\mathrm{Ext}^n_R\,(M, C)) = \mathrm{r.grade}_R\,(M, C) + \mathrm{r.grade}_R\,(\lambda M) \geq n + 1.$$

(iv) $\Rightarrow$ (i) By Lemma 3.3.8, we have $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) = 0$. As $\lambda M \in \mathcal{A}_C$, it follows that $\mathrm{Tr} M \in \mathcal{A}_C$. Thus, by Remark 1.3.5, $\mathrm{Ext}^1_R(\mathrm{Tr} M, R) = 0$. Now the assertion is obvious by Theorem 1.5.2.

(iii) $\Rightarrow$ (iv) It is obvious. Thus, to conclude, it is enough to prove (i) $\Rightarrow$ (iii), but this implication follows from Lemma 3.3.9. $\square$

The following corollary is a generalization of [9, Proposition 3.5].

Corollary 3.3.11 *Let M be a reduced G_C-perfect R-module of G_C-dimension n. Assume that $\lambda M \in \mathcal{A}_C$. Then the following statements hold true:*

(i) $\mathrm{Ext}^i_R(\lambda M, R) \cong \mathrm{Ext}^{i+n}_R(\mathrm{Ext}^n_R(M, C), C)$, *for all $i > 0$.*

(ii) *Assume that M is a stable R-module. Then M is horizontally linked if and only if* $\mathrm{r.grade}_R(M, C) + \mathrm{r.grade}_R(\lambda M) = \mathrm{grade}_R(\mathrm{Ext}^n_R(M, C))$.

Proof. (i) Consider the exact sequence

$$0 \to \lambda M \to P \to \mathrm{Tr}M \to 0,$$

where P is a projective R-module. By Remark 1.3.2(i), $\mathrm{Tr}M \in \mathcal{A}_C$. In particular, $\mathrm{Tor}^R_1(\mathrm{Tr}M, C) = 0$, and, by Theorem 1.3.4,

$$\mathrm{Ext}^i_R(\lambda M \otimes_R C, C) \cong \mathrm{Ext}^i_R(\lambda M, R),$$

for all $i > 0$. Therefore, the result follows by Corollary 3.1.10.

(ii) It is an immediate consequence of Theorem 3.3.10. □

3.4 Reduced G_C-perfect modules versus C-k-torsionless modules

In this section, we will prove some results involving reduced G_C-perfect and C-k-torsionless modules. We will give examples of such modules.

The class of C-k-torsionless modules and the class of reduced G_C-perfect modules are distinct, as the example below illustrates.

Example 3.4.1 Let n, k be positive integers and M a non-zero R-module.

(i) If M has G_C-dimension zero, then M is C-k-torsionless, but it is not reduced G_C-perfect.

(ii) If M is G_C-perfect of grade n, then M is reduced G_C-perfect, but, by Example 3.2.6(i), $\mathrm{G}_C\text{-}\dim_R(\mathrm{Tr}_C(M)) = 1$ which implies $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M), C) \neq 0$, then M is not C-k-torsionless.

The next result provides a necessary and sufficient condition for a reduced G_C-perfect module to be C-k-torsionless.

Proposition 3.4.2 *Let M be a reduced G_C-perfect R-module of G_C-dimension n, and let $k \geq 0$ be an integer. Then M is C-k-torsionless if and only if*

$$\mathrm{grade}_R\left(\mathrm{Ext}^n_R(M, C)\right) \geq n + k.$$

Proof. Assume that M is C-k-torsionless. By definition, $\mathrm{Ext}^i_R(\mathrm{Tr}_C(M), C) = 0$ for all $1 \leq i \leq k$, and so $\mathrm{r.grade}_R\left(\mathrm{Tr}_C(M), C\right) \geq k + 1$. By Proposition 3.2.3, we have

$$\mathrm{grade}_R\left(\mathrm{Ext}^n_R(M, C)\right) = n - 1 + \mathrm{r.grade}_R\left(\mathrm{Tr}_C(M), C\right) \geq n + k.$$

Conversely, assume that $\mathrm{grade}_R\left(\mathrm{Ext}^n_R(M, C)\right) \geq n + k$. By using Proposition 3.2.3 again, we get

$$\mathrm{r.grade}_R\left(\mathrm{Tr}_C(M), C\right) = \mathrm{grade}_R\left(\mathrm{Ext}^n_R(M, C)\right) - n + 1 \geq k + 1.$$

Therefore $\mathrm{Ext}^i_R(\mathrm{Tr}_C(M), C) = 0$ for all $1 \leq i \leq k$, i.e., M is C-k-torsionless. □

Question 3.4.3 Let M be a C-k-torsionless R-module. Under what conditions is M reduced G_C-perfect?

Corollary 3.4.4 *Let M be a stable reduced G_C-perfect R-module of G_C-dimension n. Assume that $\lambda M \in \mathcal{A}_C$. The following conditions are equivalent:*

(i) *M is horizontally linked.*

(ii) *M is C-1-torsionless.*

(iii) *M is C-1-syzygy.*

Proof. The assertion is clear by Theorem 3.3.10, Proposition 3.4.2 and Theorem 2.2.7(c) with $k = 1$. □

Putting together all results about C-k-torsionless and reduced G_C-perfect modules, we get:

Proposition 3.4.5 *Let M be a C-k-torsionless and reduced G_C-perfect R-module of G_C-dimension n. Then M satisfies the following conditions:*

(i) *M is C-k-syzygy;*

(ii) *There exists an exact sequence $0 \to M \to X_0 \to X_1 \to \cdots \to X_{k-1}$ of finite R-modules with G_C-$\dim_R(X_i) = 0$ for every $i = 0, \ldots, k-1$;*

(iii) $M_{\mathfrak{p}}$ *is* k*-torsionfree over* $R_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{Supp}_R(M)$*;*

(iv) M *satisfies* $\widetilde{S}_k$*;*

(v) $\operatorname{grade}_R(\operatorname{Ext}^i_R(M,C)) \geq i+k$*, for all* $i>0$*;*

(vi) $\operatorname{Ext}^i_R(\operatorname{Tr}_C(M),C) \cong \operatorname{Ext}^{i+n-1}_R(\operatorname{Ext}^n_R(M,C),C)$*, for all* $i>0$*;*

(vii) $\mathrm{G}_C\text{-}\dim_R(M) + \mathrm{G}_C\text{-}\dim_R(\operatorname{Tr}_C(M)) = \mathrm{G}_C\text{-}\dim_R(\operatorname{Ext}^n_R(M,C)) + 1$*;*

(viii) $\operatorname{r.grade}_R(M,C) + \operatorname{r.grade}_R(\operatorname{Tr}_C(M),C) = \operatorname{grade}_R(\operatorname{Ext}^n_R(M,C)) + 1$.

Proof. It follows directly from Theorem 2.2.7, Proposition 3.1.6, Theorem 3.1.7, and Proposition 3.2.3. $\square$

In the example below, we present a module that is both C-k-torsionless and reduced G_C-perfect.

Example 3.4.6 Let n be a positive integer, k a non-negative integer and M a non-zero R-module such that $\operatorname{grade}_R(M) \geq n+k$. Then $\mathcal{T}^C_n(M)$ is both C-k-torsionless and reduced G_C-perfect of G_C-dimension n. In particular, $\mathcal{T}^C_n(M)$ satisfies all the conditions of Proposition 3.4.5.

Proof. Set $N = \mathcal{T}^C_n(M)$. By Example 3.2.6, N is a reduced G_C-perfect R-module of G_C-dimension n. We will show that $\operatorname{grade}_R(\operatorname{Ext}^n_R(N,C)) \geq n+k$, which, by Proposition 3.4.2, will give the result. By Proposition 3.2.3, we have

$$\operatorname{grade}_R(\operatorname{Ext}^n_R(N,C)) = n + \operatorname{r.grade}_R(\operatorname{Tr}_C(N),C) - 1. \tag{3.53}$$

By Proposition 1.2.5, there exists the following exact sequence

$$0 \to \Omega^{n-1}M \to \operatorname{Tr}_C(N) \to X \to 0,$$

where $\mathrm{G}_C\text{-}\dim_R(X) = 0$. Thus,

$$\operatorname{r.grade}_R(\Omega^{n-1}M,C) = \operatorname{r.grade}_R(\operatorname{Tr}_C(N),C). \tag{3.54}$$

Consider the exact sequence

$$0 \to \Omega^{n-1}M \to P_{n-2} \to \cdots \to P_0 \to M \to 0,$$

where P_j is a projective R-module for all $j = 0, \ldots, n-2$. Since $\operatorname{grade}_R(M) \geq n+k \geq n > n-1$ and $\operatorname{Ext}^i_R(P_j, C) = 0$ for all $i > 0$ and $j = 0, \ldots, n-2$, by Lemma 3.2.4, we have

$$\operatorname{r.grade}_R(\Omega^{n-1}M, C) = \operatorname{r.grade}_R(M, C) - n + 1. \tag{3.55}$$

Therefore, by (3.53), (3.54) and (3.55),

$$\operatorname{grade}_R(\operatorname{Ext}^n_R(N, C)) = \operatorname{r.grade}_R(M, C) = \operatorname{grade}_R(M) \geq n + k,$$

as required. $\square$

Let M be an R-module with $\operatorname{pd}_R(M) \geq n > 0$, so $\Omega^{n-1}M$ is not a projective R-module. Then, with the objective of verifying the statement given in the example below, we may assume that its transpose is stable, i.e. $\mathcal{T}_n(M) = \operatorname{Tr}\Omega^{n-1}M$ is stable. Note that, for this purpose, we use Corollary 3.4.4 and Example 3.4.6.

Example 3.4.7 Let n, k be positive integers and M a non-zero R-module such that $\operatorname{grade}_R(M) \geq n + k$. Then $\mathcal{T}_n(M)$ is horizontally linked, k-torsionless and reduced G-perfect with Gorenstein dimension n.

Appendix

Appendix A

Some homological algebra

We will state some of the basic concepts and results used in this work.

Definition A.1 A *complex* $\mathbb{G}$ is a sequence of modules and homomorphisms, called *differentials*

$$\mathbb{G}: \quad \cdots \to G_{n+1} \xrightarrow{d_{n+1}} G_n \xrightarrow{d_n} G_{n-1} \to \cdots,$$

such that the composition $d_n \circ d_{n+1}$ is zero for every n integer.

If we wish to specify the differentials without writing out the complex, we write $(\mathbb{G}, d)$. The condition $d_n \circ d_{n+1} = 0$ is equivalent to $\mathrm{Im}\,(d_{n+1}) \subseteq \mathrm{Ker}(d_n)$, and therefore one can consider the quotient module $\mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$.

Definition A.2 Let n be an integer. For a complex $(\mathbb{G}, d)$ its *nth homology* is

$$\mathrm{H}_n(\mathbb{G}) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1}).$$

If a complex has no homology, that is, $\mathrm{H}_n(\mathbb{G}) = 0$ for all n then it is called an *exact* or *acyclic* complex. This is equivalent to saying that $\mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$ for all n. Therefore, the homology of a complex measures how far the complex is from being exact. If the complex

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

is exact, we say it is a *short exact sequence*. Note that this implies that i is an injective map and p is a projection. A short exact sequence is called *split* if there exists a map $h: M'' \to M$ such that $p \circ h = 1_{M''}$. If this holds, then $M \cong M' \oplus M''$.

We also consider complexes with increasing indexes, also known as cocomplexes,

$$\mathbb{G}: \qquad \cdots \to G^{n-1} \xrightarrow{d_{n-1}} G^n \xrightarrow{d_n} G^{n+1} \to \cdots,$$

and for these complexes we define *nth cohomology* as $\mathrm{H}^n(\mathbb{G}) = \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1})$.

Definition A.3 We say that an R-module P is *projective* if P is a direct summand of a free module.

From this definition it follows that the dual of a projective module is also projective. It is clear that every free module is projective, and the reciprocal is valid, for example, when the ring is local (see [33, Theorem 2.5]).

Definition A.4 We say that an R-module I is *injective* if I is a direct summand of any R-module that contains it.

Definition A.5 A *projective resolution* of an R-module M is an exact sequence

$$\mathbb{P}: \qquad \cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

with P_i projective, for all $i \geq 0$. If we omit the module M in the resolution, then it is called a *deleted projective resolution* and no information is lost by doing this since $M \cong \mathrm{Coker}(P_1 \to P_0)$. If this complex is finite of length n, say

$$\mathbb{P}: \qquad 0 \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

then we say that M has *finite projective dimension* denoted $\mathrm{pd}_R(M)$. Define $\mathrm{pd}_R(M) = n$, if n is the smallest number such that M has a projective resolution of length n. If no such n exists, then $\mathrm{pd}_R(M) = \infty$.

Recall that an R-module F is called *free* if it is isomorphic to a direct sum of copies of the ring R. This can actually be an infinite sum, however we will only consider finite ones and thus we can define the *rank* of F, denoted by $\mathrm{rk}_R(F)$, to be the number of copies of the ring comprising F. In addition, every module M is the quotient of a free module (see [47, Theorem 2.35]). That is, for every module M, there is a free module F and a projection $\pi : F \to M$.

A *free resolution* is defined in the same way as a projective resolution, except projective modules are replaced by free modules. Every module has a free resolution, and since free modules are also projective, every module has a projective resolution. A free resolution can simply be built, as we now describe.

Since every module is the quotient of a free module, we start with the natural projection $\pi : F_0 \to M$, where F_0 is free, so we have a short exact sequence

$$0 \to K_0 \xrightarrow{i_0} F_0 \xrightarrow{\pi} M \to 0,$$

where where i_0 the natural injection, and K_0 is the kernel of π. We repeat this process with K_0 instead of M now and take d_1 to be the composition $i_0 \circ \pi_1$. This process is continued possibly indefinitely, or until one has $K_n = 0$ for any n.

Definition A.6 An *injective resolution* of an R-module M is an exact sequence

$$I : \quad 0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

with I^i injective, for all $i \geq 0$. The *injective dimension* of M, denoted $\mathrm{id}_R(M)$, is the smallest number n such that there exists an injective resolution I of M with $I^m = 0$ for all $m > n$. If no such n exists, then $\mathrm{id}_R(M) = \infty$.

Every module can be immersed in an injective module (see [4, Theorem 3.1.8]). Thus, for each module M, there is an injective module I and an injection $\iota : M \to I$. By using a procedure similar to that which provides the existence of a free resolution, we can obtain an injective resolution of M.

Definition A.7 Let M and N be R-modules and suppose that $(\mathbb{F}, d)$ is a deleted projective resolution of M. We obtain a complex,

$$\mathbb{F} \otimes_R N : \quad \cdots \to F_i \otimes_R N \xrightarrow{d_i \otimes N} F_{i-1} \otimes_R N \xrightarrow{d_{i-1} \otimes N} \cdots \to F_1 \otimes_R N \xrightarrow{d_1 \otimes N} F_0 \otimes_R N \to 0,$$

where for $a \otimes b \in F_i \otimes_R N$ we have $a \otimes b \mapsto d_i(a) \otimes b$. The *ith Tor of a pair* M and N is the R-module

$$\mathrm{Tor}_i^R(M, N) := \mathrm{H}_i(\mathbb{F} \otimes_R N).$$

This definition is independent of the choice of the projective resolution (see [47, Corollary 6.21]). We also have, by [47, Theorem 6.29 and Theorem 7.1], the following isomorphisms $\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N$ and $\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^R(N, M)$ for all $i > 0$.

Corollary A.8 [47, Corollary 6.30] *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then there is a long exact sequence for every R-module N*

$$\cdots \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M'', N) \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0.$$

In a similar way, we can define the modules Ext.

Definition A.9 Let M and N be R-modules and suppose that $(\mathbb{F}, d)$ is a deleted projective resolution of M. We obtain a complex,

$$\operatorname{Hom}_R(\mathbb{F}, N): \quad 0 \to \operatorname{Hom}_R(F_0, N) \xrightarrow{d_1^N} \operatorname{Hom}_R(F_1, N) \xrightarrow{d_2^N} \operatorname{Hom}_R(F_2, N) \to \cdots,$$

where $d_i^N = \operatorname{Hom}_R(d_i, N)$ and for $f \in \operatorname{Hom}_R(F_i, N)$ we have $f \mapsto f \circ d_{i+1}$. The *ith Ext of a pair* M and N is the R-module

$$\operatorname{Ext}^i_R(M, N) := \mathrm{H}^i(\operatorname{Hom}_R(\mathbb{F}, N)).$$

This definition is independent of the choice of the projective resolution (see [47, Corollary 6.57]). We also have the following isomorphism $\operatorname{Hom}_R(M, N) \cong \operatorname{Ext}^0_R(M, N)$. It follows from [47, Proposition 7.21 and Proposition 7.22] that Ext commutes with finite direct sums in either variable.

Corollary A.10 [47, Corollary 6.46 and Corollary 6.62]

(i) *If* $0 \to N' \to N \to N'' \to 0$ *is a short exact sequence of* R*-modules, then there is a long exact sequence for every* R*-module* M

$$\begin{gathered} 0 \to \operatorname{Hom}_R(M, N') \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N'') \to \operatorname{Ext}^1_R(M, N') \to \cdots \\ \cdots \to \operatorname{Ext}^i_R(M, N') \to \operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^i_R(M, N'') \to \cdots \end{gathered}$$

(ii) *If* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of* R*-modules, then there is a long exact sequence for every* R*-module* N

$$\begin{gathered} 0 \to \operatorname{Hom}_R(M'', N) \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M', N) \to \operatorname{Ext}^1_R(M'', N) \to \cdots \\ \cdots \to \operatorname{Ext}^i_R(M'', N) \to \operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^i_R(M', N) \to \cdots \end{gathered}$$

Proposition A.11 [47, Proposition 7.39] *Let* $S \subseteq R$ *be a multiplicative subset. If* M *is a finite* R*-module, then there are isomorphisms*

$$S^{-1}\operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

for all $i \geq 0$ *and all* R*-modules* N.

The notion of rank of a free module may be extended to any module as follows: Let M be an R-module and Q the total ring of fractions of R. Then M has *rank* r, denoted by $\operatorname{rk}_R(M) = r$, if $M \otimes_R Q$ is a free Q-module of rank r. By [33, Remark p. 84], the rank of a module M over an integral domain R with field of fractions K is equal to the dimension of the K-vector space $M \otimes_R K$.

Proposition A.12 [4, Proposition 1.4.3] *Let M be an R-module. Then the following are equivalent:*

(i) *M has rank r;*

(ii) *M has a free submodule N of rank r such that M/N is a torsion module.*

When $(R, \mathfrak{m}, k)$ is a local ring, we denote by $\nu_R(M)$ the minimal number of generators of a finite R-module M that, by definition, is the dimension of the k-vector space $M \otimes_R k$.

Proposition A.13 [5, Proposition 16.3 and Proposition 16.10] *Let M be a finite R-module of rank r. Assume that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for some prime ideal $\mathfrak{p}$. Then $\nu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq r$.*

It is a standard fact that $\nu_R(M) \geq \mathrm{rk}_R(M)$, a condition for equality is given bellow.

Lemma A.14 [20, Lemma 8.9, p. 174] *Let R be a local integral domain. If M is a finite R-module such that $\nu_R(M) = \mathrm{rk}_R(M)$, then M is free as R-module.*

Appendix B

Regular, Gorenstein and Cohen-Macaulay rings

For a ring R, the supremum of the length of all strictly decreasing chains

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n,$$

of prime ideals of R is called the *Krull dimension* of R, denoted $\dim(R)$. For an R-module M, define $\dim_R(M) := \dim(R/\mathrm{Ann}_R(M))$. If not specified, dimension will always mean Krull dimension.

Let M be a module over a ring R. We say that $x \in R$ is an *M-regular element* if $xz = 0$ for $z \in M$ implies $z = 0$, in other words, if x is not a zero-divisor on M. Regular sequences are composed of successively regular elements:

Definition B.1 Let M be an R-module. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called an *M-regular sequence* or simply an *M-sequence* if the following conditions are satisfied:

(i) x_1 is an M-regular element and x_i is an $M/(x_1, \ldots, x_{i-1})M$-regular element for $i = 2, \ldots, n$;

(ii) $M/(\mathbf{x})M \neq 0$.

A *weak M-sequence* is only required to satisfy condition (i).

If R is a local ring with maximal ideal $\mathfrak{m}$, M is a non-zero finite R-module, and $\mathbf{x} \subseteq \mathfrak{m}$, then condition (ii) is satisfied automatically because of *Nakayama's lemma*, which in the local case says the following: If M is a finite R-module and $I \subseteq \mathfrak{m}$ is an ideal such that $IM = M$, then $M = 0$.

Corollary B.2 [4, Corollary 1.1.3] *Let M be a finite R-module, and $\boldsymbol{x}$ an M-sequence. Suppose that a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(M)$ contains $\boldsymbol{x}$. Then $\boldsymbol{x}$ (as a sequence in $R_{\mathfrak{p}}$) is an $M_{\mathfrak{p}}$-sequence.*

Proposition B.3 [4, Proposition 1.1.5] *Let*

$$\mathbb{F} : \cdots \to N_m \to N_{m-1} \to \cdots \to N_0 \to N_{-1} \to 0$$

be an exact complex of R-modules. If $\boldsymbol{x}$ is weakly N_i-regular for all i then $\mathbb{F} \otimes_R R/(\boldsymbol{x})$ is exact again.

An M-sequence $\mathbf{x} = x_1, \ldots, x_n$ contained in an ideal I is *maximal* in I if $x_1, \ldots, x_n, y$ is not an M-sequence for any $y \in I$.

Theorem B.4 (Rees) [4, Theorem 1.2.5] *Let M a finite R-module, and I an ideal such that $IM \neq M$. Then all maximal M-sequences in I have the same length n given by*

$$n = \min\{i \geq 0 \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Definition B.5 Let M be a finite R-module, and I an ideal such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the *grade of I on M*, denoted by $\operatorname{grade}_R(I, M)$. We complement this definition by setting $\operatorname{grade}_R(I, M) = \infty$ if $IM = M$. When $(R, \mathfrak{m}, k)$ is a local ring, the grade of $\mathfrak{m}$ on M is called the *the depth of M*, denoted by $\operatorname{depth}_R(M)$.

It is customary to set $\operatorname{grade}(I) = \operatorname{grade}_R(I, R) = \operatorname{grade}_R(R/I)$.

Proposition B.6 [4, Proposition 1.2.10(a)] *Let $(R, \mathfrak{m})$ be a local ring and M a finite R-module. Then* $\operatorname{depth}_R(M) = \operatorname{depth}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$.

The depth of a non-zero module is always limited by its dimension, as the following result ensures:

Proposition B.7 [4, Proposition 1.2.12 and Proposition 1.2.13] *Let $(R, \mathfrak{m})$ be a local ring and $M \neq 0$ a finite R-module. Then* $\operatorname{depth}_R(M) \leq \dim_R(M)$. *Moreover* $\operatorname{depth}_R(M) \leq \dim(R/\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Ass}_R(M)$. *In particular, if* $\mathfrak{m} \in \operatorname{Ass}_R(M)$*, then* $\operatorname{depth}_R(M) = 0$.

Now, we will introduce the class of Cohen-Macaulay rings and two subclasses: regular and Gorenstein rings.

Definition B.8 Let R be a local ring. A finite R-module $M \neq 0$ is a *Cohen-Macaulay module* if $\operatorname{depth}_R(M) = \dim_R(M)$. If R itself is a Cohen-Macaulay module, then it is called a *Cohen-Macaulay ring.* A *maximal Cohen-Macaulay module* is a module M such that $\operatorname{depth}_R(M) = \dim(R)$.

Corollary B.9 [4, Corollary 2.1.4] *Let R be a Cohen-Macaulay local ring and $I \subset R$ an ideal. Then* $\operatorname{grade}(I) = \operatorname{ht}(I)$ *and*

$$\operatorname{ht}(I) + \dim(R/I) = \dim(R).$$

The most distinguished of all local rings are those whose maximal ideal can be generated by a system of parameters:

Definition B.10 A local ring $(R, \mathfrak{m})$ is *regular* if it has a system of parameters generating $\mathfrak{m}$; such a system of parameters is called a *regular system of parameters.*

If k is a field, then k, $k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ and $k[[X_1, \ldots, X_n]]$ are examples of regular local rings (see [4, Section 2.2]).

Theorem B.11 (Auslander-Buchsbaum-Serre) [4, Theorem 2.2.7] *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following are equivalent:*

(i) *R is regular;*

(ii) $\operatorname{pd}_R(M) < \infty$ *for every finite R-module M;*

(iii) $\operatorname{pd}_R(k) < \infty$.

Theorem B.12 (Auslander-Buchsbaum-Nagata) [4, Theorem 2.2.19] *Let $(R, \mathfrak{m})$ be a regular local ring. Then R is a factorial domain.*

Now we will introduce another important class of rings.

Definition B.13 A local ring R is a *Gorenstein ring* if $\operatorname{id}_R(R) < \infty$. A ring is a *Gorenstein ring* if its localization at every maximal ideal is a Gorenstein local ring.

Below we clarify the position of the Gorenstein rings in the hierarchy of local rings.

Proposition B.14 [4, Proposition 3.1.20] *Let R be a local ring. Then we have the following implications:*

$$R \text{ is regular} \Rightarrow R \text{ is Gorenstein} \Rightarrow R \text{ is Cohen-Macaulay.}$$

Assume that R is local ring with residue field k. The *type* of a finite R-module M is $\mathrm{rk}_k\left(\mathrm{Ext}_R^{\mathrm{depth}_R(M)}(k, M)\right)$.

Definition B.15 Let R be a Cohen-Macaulay local ring. A maximal Cohen-Macaulay module of type 1 and of finite injective dimension is called a *canonical module of* R. In [4, Theorem 3.3.4], it is shown that a canonical module is unique up to isomorphism, and it is denoted by ω_R.

Proposition B.16 [4, Proposition 3.3.6] *Let R be a Cohen-Macaulay local ring. The following conditions are equivalent:*

(i) *R admits a canonical module;*

(ii) *R is the homomorphic image of a Gorenstein local ring.*

As an example of rings that have a canonical module, we have the class of the complete Cohen-Macaulay local rings (see [4, Corollary 3.3.8]).

Proposition B.17 [4, Proposition 3.3.10] *Let R be a Cohen-Macaulay local ring of dimension d and let D be a finite R-module. Then the following conditions are equivalent:*

(i) *D is the canonical module of R;*

(ii) *for all maximal Cohen-Macaulay R-modules M one has*

   (a) $\mathrm{Hom}_R(M, D)$ *is a maximal Cohen-Macaulay R-module,*

   (b) $\mathrm{Ext}_R^i(M, D) = 0$*, for all $i > 0$,*

   (c) *the natural homomorphism* $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, D), D)$ *is an isomorphism.*

As a immediate consequence, by Proposition B.17 with $D = M = \omega_R$, and again with $D = \omega_R$ and $M = R$, we have:

Corollary B.18 *Let R be a Cohen-Macaulay local ring with canonical module ω_R. Then ω_R is a dualizing R-module.*

Appendix C

Semiperfect rings

Horizontal linkage can be defined for arbitrary finite modules over commutative local Noetherian rings. Moreover, to have the operation $\lambda = \Omega\mathrm{Tr}$ well-defined the only condition on the ring that we need is the existence of "projective covers". Such rings are called "semiperfect rings". We refer to [1], [12] and [25] for the details of this section.

Definition C.1 (i) A submodule N of a module M is *superfluous* provided that $M = N + K$ for some submodule K only if $K = M$. In this case, we denote $N \ll M$

(ii) A morphism $f : M \to N$ of R-modules is said to be *minimal* provided that $\mathrm{Ker}(f)$ is a superfluous submodule of M, i.e. $\mathrm{Ker}(f) \ll M$.

(iii) A module M is a *projective cover* of N provided that M is projective and there exists a minimal epimorphism $M \to N$.

Definition C.2 A ring R is said to be *semiperfect* if every finite R-module has a projective cover.

Note that a ring R is semiperfect if and only if it is a finite direct product of local rings (see [25, Theorem 23.11]). In particular, every local ring is semiperfect.

Definition C.3 An exact sequence of R-modules $P_1 \xrightarrow{f} P_0 \to M \to 0$ is called a *minimal projective presentation* of M in case P_1 and P_0 are finite projective and $\mathrm{Ker}(f) \ll P_1$ and $\mathrm{Im}(f) \ll P_0$.

Remark C.4 (i) When R is semiperfect, every finite R-module has a minimal presentation (see [1, p 354]), and then letting $J = J(R)$ (the Jacobson radical of R), minimality just means $\mathrm{Ker}(f) \leq JP_1$ and $\mathrm{Im}(f) \leq JP_0$.

(ii) An exact sequence of R-modules

$$\cdots \to P_n \xrightarrow{f_n} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is called a *minimal projective resolution* of M in case P_i is finite projective and $\operatorname{Ker}(f_i) \leq JP_i$ for all $i \geq 0$.

(iii) By using projective covers, it is easy to see that every finite R-module has a minimal resolution when R is semiperfect.

The minimal presentations are essentially unique as show the following result.

Lemma C.5 [1, Lemma 32.11] *If M and N have minimal projective presentations $P_1 \xrightarrow{f} P_0 \to M \to 0$ and $Q_1 \xrightarrow{g} Q_0 \to M \to 0$, then $M \cong N$ if and only if there are isomorphisms φ_1 and φ_0 making the diagram*

$$\begin{array}{ccc} P_1 & \xrightarrow{f} & P_0 \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\varphi_0} \\ Q_1 & \xrightarrow{g} & Q_0 \end{array}$$

commute.

Bibliography

[1] F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, second ed., Springer-Verlag, 1992.

[2] T. Araya, K. Iima, Remarks on torsionfreeness and its applications, Comm. Algebra **46** (2018), 191–200.

[3] M. Auslander, M. Bridger, Stable module theory, American Mathematical Society, Providence, R.I., 1969.

[4] W. Bruns, J. Herzog, Cohen-Macaulay rings, Revised edition, Cambridge University Press, 1998.

[5] W. Bruns, U. Vetter, Determinantal rings, Lecture Notes in Math., vol. 1327, Springer-Verlag, Berlin-Heidelberg-New York, 1988.

[6] L. W. Christensen, Gorenstein dimensions, Springer-Verlag Berlin, 2000.

[7] L. W. Christensen, Semi-dualizing complexes and their Auslander categories, Trans. Amer. Math. Soc. **353** (2001), 1839–1883.

[8] M. T. Dibaei, M. Gheibi, S.H. Hassanzadeh, A. Sadeghi, Linkage of modules over Cohen-Macaulay rings, J. Algebra **335** (2011), 177–187.

[9] M. T. Dibaei, A. Sadeghi, Linkage of finite Gorenstein dimension modules, J. Algebra **376** (2013), 261–278.

[10] M. T. Dibaei, A. Sadeghi, Linkage of modules and the Serre conditions, J. Pure Appl. Algebra **219** (2015), 4458–4478.

[11] M. T. Dibaei, A. Sadeghi, Linkage of modules with respect to a semidualizing module, Pacific J. Math. **294** (2018), 307–328.

[12] C. Faith, Algebra II, Ring Theory, Springer-Verlag, Berlin, 1976.

[13] E. Freitag, R. Kiehl, Algebraische Eigenschaften der lokalen Ringe in den Spitzen der Hilbertschen Modulgruppen, Invent. Math. **24** (1974), 121–148.

[14] H. B. Foxby, Gorenstein modules and related modules, Math. Scand. **31** (1972), 267–285.

[15] H.B. Foxby, Quasi-perfect modules over Cohen-Macaulay rings, Math. Nachr. **66** (1975), 103–110.

[16] Y. Geng, A generalization of the Auslander transpose and the generalized Gorenstein dimension, Czechoslovak Math. J., **63** (2013), 143–156.

[17] A. A. Gerko, On homological dimensions (Russian) Mat. Sb. **192** (2001), 79–94; translation in Sb. Math. **192** (2001), 1165–1179.

[18] E. S. Golod, G-dimension and generalized perfect ideals, Tr. Mat. Inst. Steklova **165** (1984) 62–66; English transl. in Proc. Steklov Inst. Math. **165** (1985).

[19] E. S. Golod, Uspekhi Mat. Nauk **40** (1985), 234, (Question, in Russian).

[20] R. Hartshorne, Algebraic Geometry, New York, Springer, 1977.

[21] H. Holm, P. Jørgensen, Semidualizing modules and related Gorenstein homological dimension, J. Pure Appl. Algebra **205** (2006), 423–445.

[22] Z. Y. Huang, Extension closure of relative syzygy modules, Sci China Ser A: Math, **46** (2003), 611–620.

[23] Z. Y. Huang, ω-torsionfree modules and ω-left approximation dimension, Sci China Ser A: Math, **44** (2001), 184–192.

[24] E. Kunz, Kähler differentials, Advance Lectures in Mathematics (Friedr. Vieweg and Sohn, Braunschweig, 1986).

[25] T. Y. Lam, A First Course in Noncommutative Rings, Springer-Verlag, 1991.

[26] Y. Lequain, Differential simplicity and Complete Integral Closure, Pacific J. Math. **36** (1971), 741–751.

[27] J. Lipman, Unique factorization in complete local rings, Proc. Sympos. Pure Math., vol. **29**, Amer. Math. Soc, Providence, R. I., 1975.

[28] Z. Liu, Z. Huang, A. Xu, Gorenstein projective dimension relative to a semidualizing bimodule, Comm. Algebra **41** (2013), 1–18.

[29] A. K. Maloo, Differential simplicity and the module of derivations, Journal of Pure and Applied Algebra, **115** (1997), 81–85.

[30] A. K. Maloo, The module of derivations of regular local rings, Comm. Algebra **27** (1999), 6123–6126.

[31] A. Martsinkovsky, J.R. Strooker, Linkage of modules, J. Algebra **271** (2004), 587–626.

[32] V. Maşek, Gorenstein dimension and torsion of modules over commutative Noetherian rings, Comm. Algebra **28** (2000), 5783–5811.

[33] H. Matsumura, Commutative Ring Theory, Cambridge University Press, 1986.

[34] C. B. Miranda-Neto, Free logarithmic derivation modules over factorial domains, Math. Res. Lett. **24** (2017), 153–172.

[35] C. B. Miranda-Neto, T. S. Souza, A generalization of Maloo's theorem on freeness of derivation modules, Pacific Journal of Mathematics (to appear).

[36] C. B. Miranda-Neto, T. S. Souza, On Gorenstein dimension with respect to semidualizing modules and applications (submitted for publication).

[37] M. Salimi, S. Sather-Wagstaff, E. Tavasoli, S. Yassemi, Relative Tor functors with respect to a semidualizing module, Algebr. Represent. Theory **17** (2014), 103–120.

[38] M. Salimi, E. Tavasoli, S. Yassemi, Top local cohomology modules and Gorenstein injectivity with respect to a semidualizing module, Arch. Math (Basel) **98** (2012), 299–305.

[39] M. Salimi, E. Tavasoli, P. Moradifar, S. Yassemi, Syzygy and torsionless modules with respect to a semidualizing module, Algebr. Represent. Theory **17** (2014), 1217–1234.

[40] M. Salimi, E. Tavasoli, S. Yassemi, k-torsionless modules with finite Gorenstein dimension, Czech. Math. J. **62** (2012), 663–672.

[41] M. Salimi, E. Tavasoli, S. Yassemi, Reflexive modules with finite Gorenstein dimension with respect to a semidualizing module, Proc. Indian Acad. Sci. (Math. Sci.) **125** (2015), 21–28.

[42] A. Sadeghi, Linkage of finite G_C-dimension modules, J. Pure Appl. Algebra **221** (2017), 1344–1365.

[43] S. Sather-Wagstaff, Semidualizing modules. http://ssather.people.clemson.edu

[44] S. Sather-Wagstaff, T. Sharif, D. White, Comparison of relative cohomology theories with respect to semidualizing modules, Math. Z. **264** (2010), 571–600.

[45] S. Sather-Wagstaff, T. Sharif, D. White, Tate cohomology with respect to semidualizing modules, J. Algebra **324** (2010), 2336–2368.

[46] S. Sather-Wagstaff, S. Yassemi, Modules of finite homological dimension with respect to a semidualizing module, Arch. Math. (Basel) **93** (2009), 111–121.

[47] J. Rotman, An introduction to homological algebra, Academic Press, 1979.

[48] R. Takahashi, D. White, Homological aspects of semidualizing modules, Math. Scand. **106** (2010), 5–22.

[49] W. V. Vasconcelos, Divisor theory in module categories, Notas de Matemática, no. 14. North-Holland Publishing Company (1974).

[50] W. V. Vasconcelos, Reflexive modules over Gorenstein rings, Proc. Amer. Math. Soc. **19** (1968), 1349–1355.

[51] D. White, Gorenstein projective dimension with respect to a semidualizing module, J. Commut. Algebra **2** (2010), 111–137.