Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Buchsbaum-Eisenbud complexes in a Koszul-Čech approach

por

Thiago Fiel da Costa Cabral

João Pessoa - PB

Agosto/2021

Buchsbaum-Eisenbud complexes in a Koszul-Čech approach

por

Thiago Fiel da Costa Cabral †

sob orientação do

Prof. Dr. José Naéliton Marques da Silva

e sob coorientação do

Prof. Dr. Seyed Hamid Hassanzadeh Hafshejani

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

João Pessoa - PB

Agosto/2021

†Este trabalho contou com apoio financeiro da Capes

Catalogação na publicação
Seção de Catalogação e Classificação

C117c Cabral, Thiago Fiel da Costa.
Complexos de Buchsbaum-Eisenbud em uma abordagem Koszul-Cech / Thiago Fiel da Costa Cabral. - João Pessoa, 2021.
66 f.

Orientação: José Naéliton Marques da Silva.
Coorientação: Seyed Hamid Hassanzadeh Hafshejani.
Tese (Doutorado) - UFPB/CCEN.

1. Álgebra. 2. Sequência espectral Koszu-Cech. 3. Complexos de Buchsbaum- Eisenbud. 4. Multiplicidade de Buchsbaum-Rim. I. Silva, José Naéliton Marques da. II. Hassanzadeh, Seyed Hamid Hassanzadeh. III. Título.

UFPB/BC CDU 512(043)

Elaborado por Gracilene Barbosa Figueiredo - CRB-15/794

Universidade Federal da Paraíba

Universidade Federal de Campina Grande

Programa Associado de Pós-Graduação em Matemática

Doutorado em Matemática

Área de Concentração: Álgebra

Aprovada em: 25 de Agosto de 2021

Prof. Dr. José Naéliton Marques da Silva

Prof. Dr. Seyed Hamid Hassanzadeh Hafshejani

Prof. Dr. Aron Simis

Prof. Dr. Cleto Brasileiro Miranda Neto

Prof. Dr. Laurent Buse

Prof. Dr. Victor Hugo Jorge Perez

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

João Pessoa - PB

Agosto/2021

Resumo

Neste trabalho apresentamos um estudo sobre a conhecida família de complexos de Buchsbaum-Eisenbud via a abordagem de sequência espectral de Koszul-Cech dada por Bouça e Hassanzadeh. Primeiro, construímos essa família de complexos usando a estrutura advinda da sequência espetral de Koszul-Čech e damos novas demonstrações para fatos básicos como aciclicidade e suporte das homologias. Segundo, usando a convergência de espectrais, damos uma formula para multiplicidade de Buchsbaum-Rim como o gênero aritmético (característica de Euler-Poincaré) de feixes de homologias de Koszul em um espaço projetivo sobre um esquema base Noetheriano arbitrário. Essa fórmula é uma generalização de Serre, a fórmula da multiplicidade de Hilbert-Samuel de um sistema de parâmetros para o caso da multiplicidade de Buchsbaum-Rim. Com o proposito de obter essa formula, introduzimos uma noção de função de Hilbert de um anel graduado sobre um anel de base Noetheriano arbitrário.

Palavras-chave: Sequência espectral Koszu-Cech; Complexos de Buchsbaum-Eisenbud; Multiplicidade de Buchsbaum-Rim.

Abstract

In this work we present an study of the known family of Buchsbaum-Eisenbud complexes via the approach of Koszul-Čech spectral sequences given by Bouça and Hassanzadeh. We first construct this family of complexes using the Koszul-Čech structure and give new proofs for the basic facts as acyclicity and support of the homologies. Second, via convergence of spectral sequences, we give a formula of the Buchsbaum-Rim multiplicity as the arithmetic genus (Euler-Poincaré characteristic) of Koszul homology sheaves on a projective space over an arbitrary Noetherian base scheme. This formula is a generalization of Serre, the formula for the Hilbert-Samuel multiplicity of a system of parameters to the case of Buchsbaum-Rim multiplicity. In order to obtain this formula, we introduce a notion of Hilbert function of a graded ring over an arbitrary Noetherian base ring.

Keywords: Koszul-Cech spectral sequences; Buchsbaum-Eisenbud complexes; Buchsbaum-Rim multiplicity.

Agradecimentos

Agradeço a Deus, à minha família, à minha comunidade de fé e aos meus amigos.

Agradeço aos professores envolvidos na produção e na avaliação deste trabalho de tese, em especial aos professores Hamid e Vinícius.

Agradeço a todos do Departamento de Matemática da UFPB. Aos professores Naéliton, Ricardo, Mírian, Ugo e Napóleon, pelos cursos; aos professores Cleto, Manassés, Flank, Ricardo e Damião, pelo apoio e pelo suporte nas questões burocráticas. Ao professor Roberto (in memoriam). Ao secretário Júnior. Aos alunos que se tornaram amigos, em especial Rafael.

Entre outros, agradeço a Mylena, aos amigos do Departamento de Matemática da UFPE e aos amigos da UPE.

"Se o Senhor não edificar a casa, em vão trabalham os que edificam; se o Senhor não guardar a cidade, em vão vigia a sentinela."

Salmos 127:1

Dedicatória

À minha família, à minha comunidade
de fé e aos meus amigos.

Contents

Introduction

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and dimension $d > 0$. If I is a ideal of definition of a finitely generated R-module L, then we can define $\mathrm{e}(I, L)$ the Hilbert-Samuel multiplicity of I on L as being the positive integer $d!\cdot$(the coefficient of the term of degree d in the Hilbert-Samuel polynomial of I on L), where this polynomial is obtained from the function $\nu \mapsto \ell_R(L/I^\nu L)$, and ℓ_R denotes the length over R. In their work [BR64], Buchsbaum and Rim introduced and studied a multiplicity associated to N a submodule of finite colength in a finite free R-module $G = R^g$, that is, $\ell_R(G/N) < \infty$. This multiplicity generalizes the Hilbert-Samuel multiplicity, and nowadays is called the Buchsbaum-Rim multiplicity. In more detail, the function

$$\lambda_N(\nu) = \ell_R\left(\frac{\mathcal{S}_\nu(G)}{\mathcal{R}_\nu(N)}\right)$$

is eventually a polynomial $P_N(\nu)$ of degree $d + g - 1$ [BR64, Theorems 3.1 and 3.4], where $\mathcal{S}(G)$ denotes the symmetric algebra of G and $\mathcal{R}(N)$ is the image of the induced map from $\mathcal{S}(N)$ to $\mathcal{S}(G)$. So, the Buchsbaum-Rim multiplicity of N on G, denoted by $\mathrm{br}(N)$, is the positive integer $(d+g-1)!\cdot$(the coefficient of the term of degree $d+g-1$ in the polynomial $P_N(\nu)$).

Though not surprising, it is not trivial that the Buchsbaum-Rim multiplicity has explanations in regard to the Hilbert-Samuel multiplicity. Several properties of the Hilbert-Samuel multiplicity are extended to the Buchsbaum-Rim multiplicity, for example *characterization of reduction* [KT96] and [SUV01]; *relation between multiplicity and reduction number* [BUV01]; *Lech's inequality* [NW20]. The interesting graphical computations by Jones write the Buschsbaum-Rim multiplicity in terms of Hilbert-

Samuel multiplicty [J01], which motivated the work [CLU08] using linkage theory.

While the Hilbert-Samuel multiplicity is a classical numerical invariant to study isolated singularities, the Buchsbaum-Rim multiplicity is a modern algebraic tool to study singularities of higher codimensions. The importance and geometric significance of the Buchsbaum-Rim multiplicity are due to seminal works of Gaffney [G90] [G93] in the study of Whitney equisingularities. Kleiman has also investigated the geometric meaning of Buchsbaum-Rim multiplicity and developed many aspects of the theory, e.g. [KT96], see too the projection formula and the associative formula [K17].

A well-known construct is the Koszul complex $K_\bullet(\boldsymbol{c}, L)$ of a sequence $\boldsymbol{c} = c_1, \dots, c_f$ of elements of a commutative ring R with coefficients in an R-module L. The properties concerning to the ideal generated by $\boldsymbol{c}$ and the R-module L that we can obtain from the Koszul complex make it an indispensable tool. In 1962, Eagon and Northcott generalized the Koszul complex of a sequence to one of a $g \times f$ matrix

$$\Phi = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1f} \\ c_{21} & c_{22} & \cdots & c_{2f} \\ \vdots & \vdots & \ddots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gf} \end{pmatrix}$$

with $g \leq f$, see [EN62]. Two years after, Buchsbaum and Rim defined a family of complexes associated to the matrix Φ, see [BR64]. The complex of Eagon and Northcott and the family of complexes of Buchsbaum and Rim are obtained by different way, while the first complex tries to solve the quotient of R by the ideal of the maximal minors of Φ the second considers exterior powers of the cokernel of the map defined by Φ, including $M = \text{Coker}(\Phi)$. Both admit a version with coefficients in an R-module L. In 1973, Buchsbaum and Eisenbud introduced a family of complexes $\mathcal{C}^\nu_\bullet$ associated to the matrix Φ for each integer ν, which bears their names [BE73], see also [E95], and in this same year Kirby simultaneously constructed this same family, denoted by $K_\bullet(\Phi, L, \nu)$, in a different way, which he called generalized Koszul complex [Kir73]. This new family satisfies the folowing properties:

(i) $K_\bullet(\Phi, L, 0)$ is the Eagon-Northcott complex;

(ii) $K_\bullet(\Phi, L, 1)$ is one of the Buchsbaum-Rim complexes;

(iii) when $g = 1$, $K_\bullet(\Phi, L, \nu)$ is the Koszul complex of the sequence $c_1, \ldots, c_f$, for all ν;

(iv) There is a duality, i.e., $K_\bullet(\Phi, L, \nu) \simeq K^\bullet(\Phi, f - g - \nu)$.

Back to Buchsbaum and Rim's work [BR64], one of the most difficult results was to show that the difference function of the polynomial $P_N(\nu)$ is indeed the Euler-Poincaré characteristic of its family of complexes [BR64, Theorem 4.2], and thus, they could show that the Euler-Poincaré characteristic of the Buchsbaum-Rim complex is the Buchsbaum-Rim multiplicity, if $N = \mathrm{im}(\Phi)$ with Φ a parameter matrix, that is, $\ell_R(\operatorname{Coker} \Phi) < \infty$ and $f - g + 1 = d$. In 1985, Kirby shows that the Euler-Poincaré characteristic of his generalized Koszul complex $\mathfrak{B}(\Phi)$ is the Buchsbaum-Rim multiplicity $\mathrm{br}(N)$, see [Kir85, Theorem 4].

In this thesis, which was contemplated in [BFHN21], we use the Bouça and Hassanzadeh construction of the Buchsbaum-Eisenbud complexes, which uses the Koszul-Čech spectral sequence [BHa19, Section 3], to give new proof to basic properties of these complexes. The Koszul-Čech construction is a powerful algebraic tool to prove these basic properties. Using the convergence of spectral sequences, we describe the Buchsbaum-Rim multiplicity as the Euler-Poincaré characteristic (arithmetic genus) of special sheaves on a projective space over an arbitrary Noetherian base scheme. Yet another geometric nature of this multiplicity. This is another extension of Serre's formula of Hilbert-Samuel multiplicity in which the multiplicity is described as the Euler-Poincaré characteristic of the Koszul homology, c.f. [Ser65] and [BH98, Theorem 4.7.6 and notes on page 203].

We begin this thesis with some preliminaries, in Chapther 1. There are a lot of objects which will be used in the text, for example: Koszul complexes, Čech complexes, local cohomology, sheaf cohomology, spectral sequences, etc. We will give a brief description of these objects and some results about them, to support the beginner reader.

In Chapter 2, Section 2.1 recaps the result of [BHa19] on the Koszul-Čech spectral sequence. We let $\mathfrak{B}_\bullet(\Phi, L, \nu)$ denote the complexes derived from this spectral sequence. These are the same as the generalized Koszul complex of Kirby, $K_\bullet(\Phi, L, \nu)$; and the Buchsbaum-Eisenbud complex $\mathcal{C}^\nu_\bullet$. In Section 2.2, we present basic properties, some of which (re)proved using the Koszul-Čech spectral as the natural exact sequence 2.2.3 and the useful property about the support of the homologies of $\mathfrak{B}_\bullet(\Phi, L, \nu)$ 2.2.4. Section 2.3 ends this chapter with the depth and acyclicity properties of $\mathfrak{B}_\bullet(\Phi, L, \nu)$. Lemma 2.3.1, which we (re)prove, and the support property yield the grade sensitivity 2.3.2.

In Chapter 3, we first present the definition of the Buchsbaum-Rim multiplicity and of all objects involved, the Kirby's result which relates this multiplicity with the Euler-Poincaré characeristic of the homologies of $\mathfrak{B}_\bullet(\Phi, L, \nu)$ and a dicussion about parameter module using the theory of reduction of modules. Section 3.2 introduces a formal notion of the Hilbert function in the case of graded rings over an arbitrary Noetherian base ring, whereas the classical theory of Hilbert function is over an Artinian base ring. Finally, in Section 3.3, Theorem 3.3.4 determines the relation between the Euler-Poincaré characteristic of $\mathfrak{B}_\bullet(\Phi, L, \nu)$, which is the Buchsbaum-Rim multiplicity $\mathrm{br}(\mathrm{im}(\Phi), L)$, and the Hilbert polynomials of Koszul homologies of a symmetric algebra. This Theorem is a generalization of Serre's theorem on the relation between the Hilbert-Samuel multiplicity and the length of Koszul homologies, see [Ser65] and [BH98, Theorem 4.7.5 and 4.7.4]. More explicitly:

Theorem. *Let R be a Noetherian local ring, $N = \mathrm{im}(\Phi)$ is a finite colength submodule and $H_j := H_j(\gamma)$ is the j-th Koszul homology module. Then for any integer ν*

$$P_{H_0}(\nu) - P_{H_1}(\nu) + \cdots + (-1)^f P_{H_f}(\nu) = \begin{cases} \mathrm{br}(N) & , \quad \textit{if } \Phi \textit{ is a parameter matrix,} \\ 0 & , \quad \textit{otherwise}; \end{cases}$$

where $P_{H_j}(\nu)$ is the Hilbert polynomial of H_j and γ is the ideal of $\mathcal{S}(R^g)$ such that $\mathcal{S}(R^g)/\gamma \simeq \mathcal{S}(\mathrm{Coker}(\Phi))$. Clearly, God is in the details! To establish this theorem, one needs a generalization of the concept due to Serre (which relates the difference between Hilbert function and Hilbert polynomial with local cohomology modules, see [BH98, Theorem 4.4.3] and Definition 3.2.1) for projective space over any affine scheme instead of projective space over a closed point. We give an example using *Macaulay2* of this theorem, which was motivated by the computation of Jones [J01]. Corollary 3.3.8 is a genus explanation of the Buchsbaum-Rim multiplicity, which justifies the title of [BFHN21]. We finish this thesis work with some questions, in Section 3.4.

The appendix contains two results used to prove Lemma 2.3.1 and Theorem 3.3.2. The first is a duality given by Jouanolou [Jou09] and the second is a fact about "Euler-Poincaré characteristic" of pages of a spectral sequence with finite length terms.

Chapter 1

Preliminaries

1.1 The Koszul complexes

The goal of this section is to recall the definition and results used in this work, for support the reader to understand the details of the tools and proofs in all text. We will present and comment the results, but quote all them.

In the first subsection we recall the Koszul complex. We will give some properties: self-duality, the grade sensitivity and a property that uses the concept of tensor product of exterior algebras to put the Koszul as an certain invariant. In the second subsection, we present the graded Koszul complex. In the third subsection we recall the concept of multiplicity and its relation with the Euler-Poicaré characteristic of homology modules of Koszul complex, due to Serre [Ser65]. We recommend to the reader the Section 1.6 and Section 4.7 of the Part I in [BH98], for more details of the Koszul complex and its structural properties as exterior algebra and the concept of multiplicity, respectively.

1.1.1 Definition and some properties

Let R be a commutative ring, and $\mathbf{a} = a_1, \dots, a_f \in R$ be a sequence of elements. We define the Koszul complex of the sequence $\mathbf{a}$, denoted by $K_\bullet(\mathbf{a})$, as the complex

$$0 \to K_f(\mathbf{a}) \to \cdots \to K_j(\mathbf{a}) \stackrel{\partial_j}{\to} K_{j-1}(\mathbf{a}) \to \cdots \to K_1(\mathbf{a}) \stackrel{\partial_1}{\to} K_0(\mathbf{a}) \to 0$$

where $K_j(\mathbf{a}) = R^{\binom{f}{j}}$ and the differentials

$$\partial_j(e_{i_1} \wedge \cdots \wedge e_{i_j}) \mapsto \sum_{k=1}^{j}(-1)^k a_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_j}$$

with $1 \leq i_1 < \cdots < i_j \leq f$ and in the right side $\widehat{e_{i_k}}$ means to omit e_{i_k}. For an R-module M, then

$$K_\bullet(\mathbf{a}, M) := K_\bullet(\mathbf{a}) \otimes_R M$$

is the Koszul complex with coefficients in M. We will denote $H_i(\mathbf{a}, M)$ for the i-th homology module of the Koszul complex $K_\bullet(\mathbf{a}, M)$, and denote $H^i(\mathbf{a}, M)$ for the i-th cohomology module of the dual $K^\bullet(\mathbf{a}, M) := \mathrm{Hom}_R(K_\bullet(\mathbf{a}), M)$, i.e.,

$$H_i(\mathbf{a}, M) = H_i(K_\bullet(\mathbf{a}, M));$$

$$H^i(\mathbf{a}, M) = H^i(\mathrm{Hom}_R(K_\bullet(\mathbf{a}), M)).$$

Proposition 1.1.1 (Self-duality)**.** *Let R be a commutative ring, $\mathbf{a} = a_1, \ldots, a_f \in R$ be a sequence of elements and M be an R-module. Then*

$$K_\bullet(\mathbf{a}, M) \simeq K^\bullet(\mathbf{a}, M).$$

In particular, $H_i(\mathbf{a}, M) \simeq H^{f-i}(\mathbf{a}, M)$ for all i.

Proof. [BH98, Proposition 1.6.10] □

Theorem 1.1.2 (Grade sensitive)**.** *Let R be a Noetherian ring, $\mathfrak{a} = (\mathbf{a}) = (a_1, \ldots, a_f)$ be an ideal of R and M be an R-module. If $M \neq IM$, then*

$$\mathrm{grade}(I, M) = \max\{i \in \mathbb{N}; H_{f-i}(\mathbf{a}, M) \neq 0\}.$$

Proof. [BH98, Theorem 1.6.7] □

Let M be an R-module and $M^{\otimes i}$ be the i-th tensor product of M for $i \geq 1$ and $M^{\otimes 0} = R$. We define the *tensor algebra* of M by

$$\bigotimes M = \bigoplus_{i \geq 0} M^{\otimes i},$$

and the *exterior algebra* of M by

$$\bigwedge M = (\bigotimes M)/\mathfrak{I},$$

where $\mathfrak{J}$ is the two-sided ideal generated by the elements $x \otimes x$, with $x \in M$. The multiplication in $\bigwedge M$ is denoted by $x \wedge y$ and the components is denoted by $\bigwedge^i M$, called the i-th *exterior power* of M

$$\bigwedge M = \bigoplus_i \bigwedge^i M.$$

Proposition 1.1.3. *Let R be a commutative ring, $\mathbf{a} = a_1, \ldots, a_f$ be a sequence of elements and $\mathbf{a}' = a_1, \ldots, a_g$ with $g \leq f$. Then*

$$K_\bullet(\mathbf{a}) = K_\bullet(\mathbf{a}') \otimes \bigwedge^\bullet R^{f-g},$$

here $\bigwedge^\bullet R^{f-g}$ means a complex with zero differentials. In particular, for an R-module M, we have

$$H_\bullet(\mathbf{a}, M) = H_\bullet(\mathbf{a}', M) \otimes \bigwedge^\bullet R^{f-g},$$

that is,

$$H_i(\mathbf{a}, M) = \sum_{k+j=i} H_k(\mathbf{a}', M) \otimes \bigwedge^j R^{f-g}.$$

Proof. [BH98, Proposition 1.6.21] □

1.1.2 Graded Koszul complex

Let $S = \bigoplus_{\nu \geq 0} S_{[\nu]}$ be a graded ring, and $\boldsymbol{\gamma} = \gamma_1, \ldots, \gamma_f \in S$ be a sequence of homogeneous elements, with $\deg(\gamma_j) = \alpha_j \geq 0$. We define the graded Koszul complex of the sequence $\boldsymbol{\gamma}$, denoted by $K_\bullet(\boldsymbol{\gamma})$, as the complex

$$0 \to K_f(\boldsymbol{\gamma}) \to \cdots \to K_j(\boldsymbol{\gamma}) \xrightarrow{\partial_j} K_{j-1}(\boldsymbol{\gamma}) \to \cdots \to K_1(\boldsymbol{\gamma}) \xrightarrow{\partial_0} K_0(\boldsymbol{\gamma}) \to 0$$

with the terms

$$K_j(\boldsymbol{\gamma}) = \bigoplus_{|I|=j} S(-\textstyle\sum_{k=1}^{j} \alpha_{i_k})$$

where $I = (i_1, \ldots, i_j)$, with $1 \leq i_1 < \cdots < i_j \leq f$; and the differentials

$$\partial_j(e_{i_1} \wedge \cdots \wedge e_{i_j}) \mapsto \sum_{k=1}^{j} (-1)^k \gamma_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_j}$$

where $\widehat{e_{i_k}}$ means to omit e_{i_k}. Due to the torsion on the terms in $K_\bullet(\boldsymbol{\gamma})$, the differentials are homogeneous maps, and thus, the homology modules $H_\bullet(\boldsymbol{\gamma})$ are graded S-modules.

The component $K_\bullet(\boldsymbol{\gamma})_{[\nu]}$ is a complex of S_0-modules and it is called the *strand* of the graded complex $K_\bullet(\boldsymbol{\gamma})$ on degree ν. So, for an S-module $\mathcal{N}$, we have

$$H_i(\boldsymbol{\gamma}, \mathcal{N}) = \bigoplus_{\nu\in\mathbb{Z}} H_i(K_\bullet(\boldsymbol{\gamma}, \mathcal{N})_{[\nu]})$$

that is, $H_i(\boldsymbol{\gamma}, \mathcal{N})_{[\nu]} = H_i(K_\bullet(\boldsymbol{\gamma}, \mathcal{N})_{[\nu]})$.

1.1.3 Multiplicity and Euler-Poincaré characteristic

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $M \neq 0$ a finitely generated R-module. A proper ideal $I \subset R$ is called an *ideal of definition* of M if $\ell_R(M/IM) < \infty$, where ℓ_R denotes the length over R. We define the Hilbert-Samuel function by the assignment

$$n \mapsto \ell_R(M/I^{n+1}M)$$

and it is a polynomial function of degree $\dim(M)$ for large n. So, there is a polynomial function

$$P_I(n, M) = \sum_{i=0}^{d} (-1)^i e_i \binom{n+d-i-1}{d-i}$$

for integers e_i, such that $P_I(n, M) = \ell_R(M/I^{n+1}M)$ for $n >> 0$. This polynomial fucntion is called the *Hilbert-Samuel polynomial* of M with respect to I. We define the *multiplicity* of I on M, denoted by $e(I, M)$, being the coefficient e_0 in $P_I(n, M)$.

A sequence of elements $\mathbf{a} = a_1, \ldots, a_n$ is called a *multiplicity system* for an finitely generated R- module M if $\ell_R(M/\mathfrak{a}M) < \infty$, where $\mathfrak{a}$ is the ideal generated by $\mathbf{a}$, i.e., $\mathbf{a}$ is a multiplicity system if $\mathfrak{a}$ is an ideal of definition of M. It follows that the homology modules of the Koszul complex $K_\bullet(\mathbf{a}, M)$ have finite length, i.e., $\ell_R(H^i(\mathbf{a}, M)) < \infty$ for all i, and we define the *Euler-Poincaré characteristic* of the Koszul homologies by

$$\chi(\mathbf{a}, M) = \sum_i (-1)^i \ell_R(H_i(\mathbf{a}, M))$$

Theorem 1.1.4 (Serre)**.** *Let R be a Noetherian local ring, M a finitely generated R-module, $\mathbf{a} = a_1, \ldots, a_n$ be a multiplicity system of M with $\mathfrak{a} = (\mathbf{a})$. Then*

$$\chi(\mathbf{a}, M) = \begin{cases} e(\mathfrak{a}) & , \quad \text{if } \mathbf{a} \text{ is a system of parameter,} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Proof. [BH98, Theorem 4.7.6] □

1.2 Local cohomology review

In this section, we will give a brief description of the objects and results on Local cohomology, which will be used in the next Chapter. The concept of local cohomology modules, torsion functor, ideal transform, algebraic Čech complex, etc. We recommend to the reader the book [BS13], for more details.

1.2.1 I-torsion functor and local cohomology modules

Let R be a Noetherian ring, $I \subset$ be an ideal and M be an R-module. We define the I-torsion functor $\Gamma_I : \mathfrak{Mod}(R) \to \mathfrak{Mod}(R)$, where $\mathfrak{Mod}(R)$ denotes the category of R-modules, by the assignment

$$\Gamma_I(M) = \bigcup_{n \geq 1} (0 :_M I^n),$$

that is, $\Gamma_I(M)$ is the submodule of M whose elements are anihilated by some power of the ideal I.

Fact 1. *Γ_I is a covariant functor left exact.*

The category of R-modules $\mathfrak{Mod}(R)$ is an Abelian category and it has enough injectives. The i-th right derived functor of Γ denoted by H^i_I is the i-th *local cohomology functor* with respect to I. For an R-module M, we consider $E^\bullet$ an injective resolution of M

$$0 \to E^0 \to E^1 \to \cdots$$

($H^0(E^\bullet) = M$ and $H^i(E^\bullet) = 0$), and thus, we apply the I-torsion functor to $E^\bullet$ and obtain

$$H^i_I(M) = H^i(\Gamma_I(E^\bullet))$$

which is called the i-th *local cohomology module* of M with respect to I. Notice that $\Gamma_I(M) = H^0_I(M)$.

1.2.2 Direct limits of Ext modules and Ideal transform

The I-torsion functor Γ_I can be related with a functor defined in terms of direct limits of 'Hom' modules. For an R-module M, the isomorphisms of R-modules

$$\mathrm{Hom}_R(R/I^n, M) \simeq (0 :_M I^n)$$

yield an isomorphism of functors

$$\Gamma_I(\bullet) \simeq \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(R/I^n, \bullet),$$

and since that the exactness of taking direct limits, the i-th local cohomology functor can be defined as

$$H^i_I(\bullet) \simeq \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/I^n, \bullet).$$

Due to above isomorphism, using Ext modules [BH98, Theorem 1.2.5], we have the next theorem about grade.

Theorem 1.2.1. *Let R be a Noetheiran ring, I an ideal of R and M be a finitely generated R module such that $M \neq IM$. Then*

$$\mathrm{grade}(I, M) = \min\{i \in \mathbb{N}; H^i_I(M) \neq 0\}$$

Proof. [BS13, Theorem 6.2.7] □

This new point of view of the local cohomology functors leads us to define the I-transform functor as follow. For positive integers $n \geq m$, we consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n & \longrightarrow & R & \longrightarrow & R/I^n & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & I^m & \longrightarrow & R & \longrightarrow & R/I^m & \longrightarrow & 0. \end{array}$$

Using the Horseshoe lemma [W94, Lemma 2.2.8], we obtain a commutative diagram of projective resolutions with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^n_\bullet & \longrightarrow & K^n_\bullet & \longrightarrow & P^n_\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^m_\bullet & \longrightarrow & K^m_\bullet & \longrightarrow & P^m_\bullet & \longrightarrow & 0. \end{array}$$

Applying the contravariant functor $\mathrm{Hom}_R(\bullet, M)$, where M is an R-module, we obtain a commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(P^n_\bullet, M) & \longrightarrow & \mathrm{Hom}_R(L^n_\bullet, M) & \longrightarrow & \mathrm{Hom}_R(K^n_\bullet, M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(P^n_\bullet, M) & \longrightarrow & \mathrm{Hom}_R(L^n_\bullet, M) & \longrightarrow & \mathrm{Hom}_R(K^n_\bullet, M) & \longrightarrow & 0. \end{array}$$

and it yields a commutative diagram of R-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(R/I^n, M) & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_R(I^n, M) & \longrightarrow & \mathrm{Ext}^1_R(R/I^n, M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(R/I^m, M) & \longrightarrow & M & \longrightarrow & \mathrm{Hom}_R(I^m, M) & \longrightarrow & \mathrm{Ext}^1_R(R/I^m, M) & \longrightarrow & 0. \end{array}$$

So, the last induces an exact sequence of direct systems, and taking the direct limit, we obtain the exact sequence of R-modules

$$0 \to H^0_I(M) \to M \to \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(I^n, M) \to H^1_I(M) \to 0,$$

and we define the *I-transform functor* by

$$D_I(\bullet) = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(I^n, \bullet),$$

and $D_I(M)$ is called the *ideal transform* of M with respect to I.

Theorem 1.2.2. *Let R be a Noetherian ring, I an ideal of R and M an R-module. Then*

(i) There is an exact sequence

$$0 \to \Gamma_I(M) \to M \to D_I(M) \to H^1_I(M) \to 0;$$

(ii) For $i \geq 1$, there are isomorphisms

$$\mathcal{R}^i D_I(M) \simeq H^{i+1}_I(M)$$

where the left is the i-th right derived functor of $D_I(\bullet)$.

Proof. See [BS13, Theorem 2.22.6]. □

1.2.3 Čech complexes

Let R be a commutative ring, $\mathbf{a} = a_1, \ldots, a_n \in R$ be a sequence of elements and M an R-module. We define the (algebraic) Čech complex of M with respect to $\mathbf{a}$, denoted by $C^\bullet_{\mathbf{a}}(M)$, ass follow:

$$0 \to C^0_{\mathbf{a}}(M) \overset{\check{\partial}_0}{\to} C^1_{\mathbf{a}}(M) \overset{\check{\partial}_1}{\to} C^2_{\mathbf{a}}(M) \to \cdots \to C^n_{\mathbf{a}}(M) \to 0$$

with

$$C^k_{\mathbf{a}}(M) = \bigoplus_{1\leq i_1<\cdots<i_k\leq g} M_{a_{i_1}\cdots a_{i_k}},$$

where $M_{a_{i_1}\cdots a_{i_k}}$ is the localization of M on the multiplicative set $\{(a_{i_1}\cdots a_{i_k})^l; l \geq 0\}$, and thus,

$$0 \to M \overset{\check{\partial}_0}{\to} \bigoplus_i M_{a_i} \overset{\check{\partial}_1}{\to} \bigoplus_{i<j} M_{a_i a_j} \to \cdots \to M_{a_1\cdots a_n} \to 0$$

and the diferential $\check{\partial}_{k-1} : C^{k-1}_{\mathbf{t}} \to C^k_{\mathbf{t}}$ is defined in the summands by

$$\frac{\alpha}{a_{i_1}\cdots \widehat{a_{i_l}} \cdots a_{i_k}} \in M_{a_{i_1}\cdots\widehat{a_{i_l}}\cdots a_{i_k}} \mapsto (-1)^{l-1}\frac{a_{i_l}\alpha}{a_{i_1}\cdots a_{i_k}} \in M_{a_{i_1}\cdots a_{i_k}}.$$

Notice that $H^0(C^\bullet_{\mathbf{a}}(M)) = \Gamma_{\mathfrak{a}}(M)$, where $\mathfrak{a} = (\mathbf{a})$ is the ideal of R generated by the elements $\mathbf{a} = a_1, \ldots, a_n$. Furthermore,

$$H^i(C^\bullet_{\mathbf{a}}(M)) \simeq H^i_{\mathfrak{a}}(M),$$

that is, the homologies of the Čech complex are the local cohomologies, see [BS13, Theorem 5.1.20].

The Čech complex comes from a direct limit of homology modules of Koszul complexes, i.e.,

$$\{n\}C^\bullet_{\mathbf{a}}(M) \simeq \varinjlim_{l\in\mathbb{N}} K_\bullet(\mathbf{a}^l, M),$$

where $\mathbf{a}^l = a_1^l, \ldots, a_n^l$ and $(\{n\}C^\bullet_{\mathbf{a}}(M))_i = C^{-i+n}_{\mathbf{a}}(M)$ is a shift, for more details see [BS13, Theorem 5.2.5].

Example 1.2.3. *Let $S = R[T_1, \ldots, T_n]$ be the polynomial ring in the indeterminates $T_1, \ldots, T_n$. We will describe $H^n_{\mathbf{t}}(S)$ the n-th local cohomology module of S with respect to the ideal $\mathbf{t}$.*

Consider $C^\bullet_{\mathbf{t}}(S)$ the Čech complex of S with respect to the sequence $T_1, \ldots, T_n$, more especifically, the last not zero differential

$$\begin{array}{ccccc} \bigoplus_{i=1}^n S_{T_1\cdots\widehat{T}_i\cdots T_n} & \overset{\check{\partial}}{\to} & S_{T_1\cdots T_n} & \to & 0 \\ \frac{\alpha}{(T_1\cdots\widehat{T}_i\cdots T_n)^l} & \mapsto & (-1)^i\frac{T_i^l\alpha}{(T_1\cdots T_n)^l} & & . \end{array}$$

So, $H^n_{\mathbf{t}}(S) = S_{T_1\cdots T_n}/\operatorname{Im}\check{\partial}$ an R-algebra. Firt, notice that $H^n_{\mathbf{t}}(S)$ is generated by

$$\left\{\frac{T_1^{l_1}\cdots T_n^{l_n}}{(T_1\cdots T_n)^l} + \operatorname{Im}\check{\partial}; l \geq 1 \text{ and } l_i < l, \text{ for all } i\right\}$$

as R-module. In fact, if $l = 0$, then

$$T_1^{l_1}\cdots T_n^{l_n} = \check{\partial}(T_1^{l_1}\cdots T_n^{l_n}),$$

and if $l \geq 1$ and $l_i \geq l$ for some i, then

$$\frac{T_1^{l_1}\cdots T_n^{l_n}}{(T_1\cdots T_n)^l} = \check{\partial}\left((-1)^i\frac{T_1^{l_1}\cdots T_i^{l_i-l}\cdots T_n^{l_n}}{(T_1\cdots\widehat{T}_i\cdots T_n)^l}\right)$$

in both cases, it is zero in $H^n_{\mathbf{t}}(S)$*. So, these generators form a basis to* $H^n_{\mathbf{t}}(S)$*, futhermore, it is generated by*

$$\left\{ \frac{T_1 \cdots \widehat{T_i} \cdots T_n}{T_1 \cdots T_n} + \operatorname{Im} \check{\partial} \right\}_{i=1}^{n}$$

as an R*-algebra, and thus, we have the isomorphism*

$$\begin{array}{ccc} H^n_{\mathbf{t}}(S) & \to & R[T_1^{-1}, \ldots, T_n^{-1}] \\ \frac{T_1 \cdots \widehat{T_i} \cdots T_n}{T_1 \cdots T_n} + \operatorname{Im} \check{\partial} & \mapsto & T_i^{-1} \end{array}$$

where $R[T_1^{-1}, \ldots, T_n^{-1}]$ *is the* S*-module of inverse polynomials in* $T_1, \ldots, T_n$ *over* S*, and the* S*-structure is defined by*

$$T_i(T_1^{l_1} \cdots T_n^{l_n}) = \begin{cases} T_1^{l_1} \cdots T_i^{l_i+1} \cdots T_n^{l_n} & l_i < -1 \\ 0 & l_i = 1. \end{cases}$$

1.3 Sheaf cohomology on projective scheme

The main result of this section (see Theorem 1.3.2) is to relate the sheaf cohomology with local cohomology in the projective case. We recommend to the reader the sections 1, 2, 3 and 4 of the chapter III in [Har77], and see too [ILLMMSW, Lecture 13]

1.3.1 Some definitions

Let $S = \oplus_{i \geq 0} S_i$ be a graded ring with $S_+ = \oplus_{i \geq 1} S_i$ the irrelevant graded ideal. We define the set

$$\operatorname{Proj}(S) = \{\mathfrak{P} \in \operatorname{Spec}(S); \mathfrak{P} \textit{ is graded and } \mathfrak{P} \not\supset S_+\}.$$

The subsets of the form $V(I) = \{\mathfrak{P} \in \operatorname{Proj}(S); \mathfrak{P} \supset I\}$, with I being a graded ideal, define the closed subsets on $\operatorname{Proj}(S)$ in the Zariski topology. For an homogeneous element $f \in S$, the open subset $D_+(f) = \operatorname{Proj}(S) \backslash V(f)$ is called a *basic* open subset of $\operatorname{Proj}(S)$. The below facts follow by a straightforward verification.

Fact 2. $D_+(f) \simeq \operatorname{Spec}([S_f]_0)$

Fact 3. $D_+(f) \cap D_+(g) \simeq \operatorname{Spec}([S_{fg}]_0)$

Fact 4. $\{D_+(f); f \in S$ *homogeneous*$\}$ *is an open covering for* X*.*

Fact 5. $\{D_+(f_i)\}_{i=1}^r$ *is an open covering for* X *if, and only if,* $\sqrt{(f_1, \ldots, f_r)} = S_+$*.*

The sheaf structure of $X = \text{Proj}(S)$, denoted by $\mathcal{O}_X$, is defined on the open covering $\{D_+(f)\}$, with $f \in S$ being an homogeneous element, and the assignment:

$$\Gamma(D_+(f), \mathcal{O}_X) = [S_f]_0,$$

where, $[S_f]_0$ denotes the zero-th component of the graded localization S_f. For a graded S-module M, we define the *sheafification* of M, denoted by $\tilde{M}$, as the sheaf of $\mathcal{O}_X$-modules with assignment:

$$\Gamma(D_+(f), \tilde{M}) = [M_f]_0,$$

for a homogeneous element $f \in S$. A sheaf of $\mathcal{O}_X$-modules which is a sheafification of a module is called a *quasi-coherent sheaf*, when M is finitely generated, we call *coherent sheaf*.

The shefification of $S(a)$, denoted by $\mathcal{O}_X(a)$, is called *twisted sheaf*, where $a \in \mathbb{Z}$ and $S(a)_i = S_{a+i}$ are the components of the graded S-module $S(a)$. We set $\mathcal{F}(a) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(a)$, for all sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. We define the functor from the category of quasi-coherent sheaves on X to the category of graded S-modules: For each sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X, set

$$\Gamma_*(\mathcal{F}) = \bigoplus_{a \in \mathbb{Z}} \Gamma(X, \mathcal{F}(a)).$$

Fact 6. *$\Gamma_*(\mathcal{O}_X)$ is a graded ring and $\Gamma_*(\mathcal{F})$ is a graded $\Gamma_*(\mathcal{O}_X)$-module. Furthermore, $\Gamma_*(\mathcal{F})$ is a graded S-module.*

1.3.2 Sheaf cohomology

Let $(X, \mathcal{O}_X)$ be a ringed space and

$$\Gamma(X, \bullet) : \mathfrak{Mod}(X) \to \mathfrak{Ab}$$

be the global section functor from the category of sheaves of $\mathcal{O}_X$-modules to the category of abelian groups.

Fact 7. *$\mathfrak{Mod}(X)$ is an abelian category with enough injectives. [Har77, III.1 and III.2]*

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules and $\mathcal{I}^\bullet$ an injective resolution to $\mathcal{F}$. Applying $\Gamma(X, \bullet)$, we have the complex $\Gamma(X, \mathcal{I}^\bullet)$ in $\mathfrak{Ab}$, and its cohomology groups, denoted by $H^i(X, \mathcal{F})$, are called the *cohomology groups of* $\mathcal{F}$.

1.3.3 Čech cohomology

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{U} = \{U_i\}_{i=\Lambda}$ be an open covering for X. For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we define $C^\bullet(\mathcal{U}, \mathcal{F})$ the Čech complex of $\mathcal{F}$ with respect to $\mathcal{U}$ to be

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_{|I|=k+1} \Gamma(\mathcal{U}_I, \mathcal{F})$$

where $I = (i_1, \ldots, i_{k+1})$ with $i_1 < \cdots i_{k+1}$ and $\mathcal{U}_I = D_+(x_{i_1}) \cap \cdots \cap D_+(x_{i_{k+1}})$; and $d_{k-1} : C^{k-1}(\mathcal{U}, \mathcal{F}) \to C^k(\mathcal{U}, \mathcal{F})$ is induced on summands by

$$\sigma \in \Gamma(\mathcal{U}_{I \setminus \{i_l\}}, \mathcal{F}) \mapsto (-1)^l \sigma|_{\mathcal{U}_I} \in \Gamma(\mathcal{U}_I, \mathcal{F})$$

where $I = (i_1, \ldots, i_l, \ldots, i_{k+1})$. The cohomology groups of $C^\bullet(\mathcal{U}, \mathcal{F})$, denoted by $\check{H}^k(\mathcal{U}, \mathcal{F})$, are called the *Čech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$.*

Example 1.3.1. *Let $S = R[x, y]$ be the polynomial ring, M be a graded S-module and $\mathcal{U} = \{D_+(x), D_+(y)\}$ be an open covering for $X = \operatorname{Proj}(S)$. Then, $C^\bullet(\mathcal{U}, \tilde{M})$ is*

$$0 \to \Gamma(D_+(x), \tilde{M}) \oplus \Gamma(D_+(y), \tilde{M}) \to \Gamma(D_+(x) \cap D_+(y), \tilde{M}) \to 0,$$

that is,

$$\begin{array}{ccccccc} 0 & \to & [M_x]_0 \oplus [M_y]_0 & \to & [M_{xy}]_0 & \to & 0 \\ & & \left(\frac{\alpha}{x^i}, \frac{\beta}{y^j}\right) & \mapsto & \frac{x^j\beta}{(xy)^j} - \frac{y^i\alpha}{(xy)^i} & & . \end{array}$$

Notice that $C^\bullet(\mathcal{U}, \tilde{M})[-1]$ is a subcomplex of $C^\bullet(x, y; M)$ (Algebraic Čech complex).

Fact 8. *Let $S = \oplus_{i \geq 0} S_i$ be a graded ring, $X = \operatorname{Proj}(S)$ be a scheme and $\mathcal{F}$ a quasi-coherent sheaf. Suppose that S_0 is a Noetherian ring and $S = S_0[S_1]$ with S_1 generated by $x_0, \ldots, x_n$ as S_0-module. Consider $\mathcal{U} = \{D_+(x_0), \ldots, D_+(x_n)\}$ an open covering for X. Then, for all $i \geq 0$*

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(X, \mathcal{F}).$$

Proof. Although X is not separated (S_0 is not a field), $\mathcal{U}$ is a affine open covering such that the intersections are affine, by Facts 2, 3 and 5. Therefore, we proceed in the same way as [Har77, III Theorem 4.5]. □

Theorem 1.3.2. *Let $S = \oplus_{i \geq 0} S_i$ be a graded ring such that S_0 is a Noetherian ring and $S = S_0[S_1]$ with S_1 generated by $x_0, \ldots, x_n$ as S_0-module. For each graded S-module M, we have an exact sequence*

$$0 \to H^0_{S_+}(M) \to M \to \Gamma_*(X, \tilde{M}) \to H^1_{S_+}(M) \to 0,$$

and for $i \geq 1$, the isomorphisms

$$\bigoplus_{a\in\mathbb{Z}} H^i(X, \tilde{M}(a)) \simeq H^{i+1}_{S_+}(M),$$

where $H^i_{S_+}(M)$ denotes the i-th local cohomology modules of M with respect to the ideal S_+.

Proof. From the exact sequence of complexes (see Example 1.3.1)

$$0 \to \bigoplus_{a\in\mathbb{Z}} C^{\bullet}(\mathcal{U}, \tilde{M}(a))[-1] \to C^{\bullet}(\mathbf{x}, M) \to M \to 0,$$

we obtain a long exact sequence on homologies. Using the Fact 8, the result follows. □

1.4 Spectral sequence of a double complex

The main result of this thesis is given by the convergence of spectral sequences which comes from a double complex. The goal of this section is recall this concepts. We recommend to the reader the chapter 5 in [W94], for more details.

1.4.1 Spectral sequences

Let R be a commutative ring with identity. A *spectral sequence* is a family of R-modules and R-linear maps

$$^{\bullet}E^{\bullet\bullet} = \{^{r}E^{p,q}, {}^{r}d^{p,q} : {}^{r}E^{p,q} \to {}^{r}E^{p-r,q-r-1}\}_{r\in\mathbb{N};p,q\in\mathbb{Z}}$$

where for each $r \in \mathbb{N}$, $^{r}E^{\bullet\bullet} = \{^{r}E^{p,q}, {}^{r}d^{p,q}\}_{p,q\in\mathbb{Z}}$ is a family of complexes (called r-th page of E) such that

$$^{r+1}E^{p,q} = \frac{\ker({}^{r}d^{p,q})}{\operatorname{Im}({}^{r}d^{p+r,q+r+1})},$$

i.e., the terms of the next page in obtained by taking the homology modules of the complexes the current page. If there is an $r \in \mathbb{N}$ such that $^{s}E = {}^{r}E$ for all $s \geq r$, then we write $^{\infty}E := {}^{r}E$ and this page is called the *infinite page* of E. The family $\{^{r}f^{p,q} : {}^{r}E^{p,q} \to {}^{r}E'^{p,q}\}_{r\in\mathbb{N};p,q\in\mathbb{Z}}$ is a map of spectral sequences if ^{r+1}f is the induced map of ^{r}f on homologies of the complexes in r-th page.

A spectral sequence E is *bounded* if there are only finitely many terms different to zero on the first page. We say that a bounded spectral sequence E *converges* to family of R-modules $H_\bullet = \{H_n\}_{n\in\mathbb{Z}}$ (${}^rE \Rightarrow_r H$), if for each $n \in \mathbb{Z}$ there exists a decreasing filtration

$$0 = F_sH_n \subseteq \cdots \subseteq F_{p+1}H_n \subseteq F_pH_n \subseteq F_{p-1}H_n \subseteq \cdots \subseteq F_tH_n = H_n$$

such that

$${}^\infty E^{p,q} \simeq \frac{F_pH_{q-p}}{F_{p+1}H_{q-p}}$$

with $q - p = n$.

Let E and E' be spectral sequences converging to H and H', respectively. We say that $f : E \to E'$ is *compatible* with $h : H \to H'$ if $h(F_pH_n) \subseteq F_pH'_n$ and the diagram

$$\begin{array}{ccc} {}^\infty E^{p,q} & \xrightarrow{\simeq} & F_pH_n/F_{p+1}H_n \\ \downarrow {}^\infty f^{p,q} & & \downarrow \bar{h} \\ {}^\infty E'^{p,q} & \xrightarrow[\simeq]{} & F_pH'_n/F_{p+1}H'_n. \end{array}$$

is commutative.

1.4.2 Double complex

Let R be a commutative ring with identity. A *double complex* is a family of R-modules $M^{\bullet\bullet} = \{M^{p,q}\}_{p,q\in\mathbb{Z}}$ with two family of R-linear maps $\{d_h^{p,q} : M^{p,q} \to M^{p-1,q}\}_{p,q\in\mathbb{Z}}$ and $\{d_v^{p,q} : M^{p,q} \to M^{p,q+1}\}_{p,q\in\mathbb{Z}}$ such that $d_h \circ d_h = 0$, $d_v \circ d_v = 0$ and $d_v \circ d_h = d_h \circ d_v$

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & M^{p+1,q-1} & \longrightarrow & M^{p,q-1} & \longrightarrow & M^{p-1,q-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & M^{p+1,q} & \longrightarrow & M^{p,q} & \xrightarrow{d_h^{p,q}} & M^{p-1,q} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow d_v^{p,q} & & \downarrow & & \\ \cdots & \longrightarrow & M^{p+1,q+1} & \longrightarrow & M^{p,q+1} & \longrightarrow & M^{p-1,q+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & . \end{array}$$

We define the *total complex* of $K^{\bullet\bullet}$, denoted by $\mathrm{Tot}_\bullet(M)$, as follow:

$$\mathrm{Tot}_n(M) = \bigoplus_{q-p=n} M^{p,q}$$

and $d_n^T : \mathrm{Tot}_n(M) \to \mathrm{Tot}_{n+1}(M)$ is defined on summands by

$$x \in M^{p,q} \mapsto d_h^{p,q}(x) + (-1)^p d_v^{p,q}(x) \in M^{p-1,q} \oplus M^{p,q+1}.$$

We define the *first filtration* of $\mathrm{Tot}_\bullet(M)$, denoted by ${}^{I}F\mathrm{Tot(M)} = \{{}^{I}F^i{}_\bullet\mathrm{Tot(M)}\}_{i\in\mathbb{Z}}$, being the decreasing chain of inclusions of complexes

$$\cdots \subset {}^{I}F^{i+1}{}_\bullet\mathrm{Tot(M)} \subset {}^{I}F^{i}{}_\bullet\mathrm{Tot(M)} \subset {}^{I}F^{i-1}{}_\bullet\mathrm{Tot(M)} \subset \cdots$$

where ${}^{I}F^i{}_\bullet\mathrm{Tot(M)}$ is the complex given by

$${}^{I}F^i{}_n\mathrm{Tot(M)} = \bigoplus_{\substack{q-p=n \\ p\leq i}} M^{p,q}$$

and the differentials induced by d^T, and it commutes with the inclusions. In the same way, we define the *second filtration* of $\mathrm{Tot}_\bullet(M)$, denoted by ${}^{II}F\mathrm{Tot(M)} = \{{}^{I}F^i{}_\bullet\mathrm{Tot(M)}\}_{i\in\mathbb{Z}}$, being the increasing chain of inclusions of complexes

$$\cdots \subset {}^{II}F^{j-1}{}_\bullet\mathrm{Tot(M)} \subset {}^{II}F^{j}{}_\bullet\mathrm{Tot(M)} \subset {}^{II}F^{j+1}{}_\bullet\mathrm{Tot(M)} \subset \cdots$$

where ${}^{II}F^j{}_\bullet\mathrm{Tot(M)}$ is the complex given by

$${}^{II}F^j{}_n\mathrm{Tot(M)} = \bigoplus_{\substack{q-p=n \\ q\geq j}} M^{p,q}$$

and the differentials induced by d^T, and it commutes with the inclusions.

The first filtration of $\mathrm{Tot}_\bullet(E)$ yields a spectral sequence ${}^{r}E_{hor}$, called *horizontal* spectral sequence of M, where the first page of ${}^{r}E_{hor}$ is

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
 & \downarrow & \downarrow & \downarrow & \\
\cdots & H_{p+1}(M^{\bullet,q-1}) & H_p(M^{\bullet,q-1}) & H_{p-1}(M^{\bullet,q-1}) & \cdots \\
 & \downarrow & \downarrow & \downarrow & \\
\cdots & H_{p+1}(M^{\bullet,q}) & H_p(M^{\bullet,q}) & H_{p-1}(M^{\bullet,q}) & \cdots \\
 & \downarrow & \downarrow{\scriptstyle \bar{d}_v^{p,q}} & \downarrow & \\
\cdots & H_{p+1}(M^{\bullet,q+1}) & H_p(M^{\bullet,q+1}) & H_{p-1}(M^{\bullet,q+1}) & \cdots \\
 & \downarrow & \downarrow & \downarrow & \\
 & \vdots & \vdots & \vdots & .
\end{array}$$

i.e., we taking the homologies in the horizontal complexes and consider the induced maps in the vertical differentials. The terms on second page is

$$H_q(H_p(M^{\bullet\bullet}).$$

Analogously, we have the spectral sequence ${}^rE_{ver}$ from the second spectral sequence, where the first page is given by taking the homologies in the vertical complexes with the induced maps in the horizontal differentials, and the terms on second page is

$$H_p(H_q(M^{\bullet\bullet}).$$

Theorem 1.4.1 (Convergence)**.** *If M is bounded, i.e., there are only finitely many terms different to zero, then*

$${}^rE_{hor} \Rightarrow H_\bullet(\mathrm{Tot}(M))$$

and

$${}^rE_{ver} \Rightarrow H_\bullet(\mathrm{Tot}(M)).$$

Example 1.4.2. *Let M and N be R-modules and $P_\bullet$ and $G_\bullet$ be projective resolutions of M and N, respectively. By taking tensor product we have the double complex $P_\bullet \otimes_R G_\bullet$. From the convergence of spectral sequences of a double complex, we obtain the isomorphism*

$$\mathrm{Tor}_n^R(M, N) \simeq \mathrm{tor}_n^R(M, N)$$

where $\mathrm{Tor}_n^R(M, N) := H_n(P_\bullet \otimes_R N)$ *and* $\mathrm{tor}_n^R(M, N) := H_n(M \otimes_R Q_\bullet)$.

Chapter 2

Buchsbaum-Eisenbud complexes

2.1 Koszul-Čech spectral construction

In this first section, we will give a new construction of the Buchsbaum-Eisenbud complexes by using the Koszul-Čech spectral sequence. This construction was given by Bouça and Hassanzadeh in [BHa19] and it is similar to the construction of the residual approximation complexes, which gives the disguised residual intersection defined in [Has12], see too [HaN16]. The classical constructions are obtained by a different way [E95, Appendix 2.6][Kir73], we hope that this new aproach, using Koszul-Čech spectral sequence, gives to us new properties of the Buchsbaum-Eisenbud complexes.

Let R be a commutative ring with identity, L be an R-module and $\Phi = (c_{ij})$ be a $g \times f$-matrix over R with $f \geq g \geq 1$. The matrix Φ can be considered as an R-linear map $\varphi : R^f \to R^g$, that is, if $\{e_1, \dots, e_f\}$ is a basis for R^f and $\{x_1, \dots, x_g\}$ is a basis for R^g, then we can write $\varphi(e_j) = \sum_{i=1}^{g} c_{ij} x_i$, for all $j \in \{1, \dots, f\}$. We will denote $M := \operatorname{Coker}(\varphi)$. let $S = R[T_1, \dots, T_g]$ be the polynomial ring in the indeterminates $\mathbf{t} = T_1, \dots, T_g$ over R with standard graduation ($\deg T_i = 1$, for all i), i.e., $S \simeq \mathcal{S}(R^g)$ is the symmetric algebra of R^g.

First, let $K_\bullet(\boldsymbol{\gamma})$ be the graded Koszul complex of the sequence $\boldsymbol{\gamma} = \gamma_1, \dots, \gamma_f$ where $\gamma_j = \sum_{i=1}^{g} c_{ij} T_i$, that is, the Koszul complex of the generators of the presentation ideal of the symmetric algebra of $M = \operatorname{Coker} \varphi$. Since $\deg(\gamma_j) = 1$ for all j, then

$K_j(\boldsymbol{\gamma}) = S^{\binom{f}{j}}(-j)$, and the complex $K_\bullet(\boldsymbol{\gamma})$ is

$$0 \to S^{\binom{f}{f}}(-f) \to \cdots \to S^{\binom{f}{1}}(-1) \xrightarrow{\partial_0} S^{\binom{f}{0}} \to 0$$

for more details see Subsection 1.1.2 in the first chapter. In the construction of de Buchsbaum-Eisenbud complexes, we will consider

$$K_\bullet(\boldsymbol{\gamma}, S \otimes_R L) = K_\bullet(\boldsymbol{\gamma}) \otimes_S (S \otimes_R L)$$

the graded Koszul complex of $\boldsymbol{\gamma}$ with coefficients in the graded S-module $S \otimes_R L$.

Second, let $C^\bullet_{\mathbf{t}}$ be the (algebraic) Čech complex of the sequence $\mathbf{t} = T_1, \ldots, T_g$ over S as in the subsection 1.2.3 of the first chapter. The homology modules of this Čech complex, denoted by $H^i_{\mathbf{t}}(S)$, are called the i-th local cohomology module of S with respect to the ideal generated by $\mathbf{t}$ (See below remark), that is, the irrelevant ideal.

Remark 2.1.1. *In the text, the bold symbols will denote a sequence of elements, and sometimes, it will denote the ideal generate by this sequence, for example,* $\mathbf{t}$ *is a sequence of elements, but in* $H^i_{\mathbf{t}}(\mathcal{H})$, $\mathbf{t}$ *is the ideal generated by this sequence. The same happens with* $\boldsymbol{\gamma}$.

Therefore, we define the double complex $E^{-\bullet,-\bullet} = C^\bullet_{\mathbf{t}}(K_\bullet(\boldsymbol{\gamma}, S \otimes_R L))$

$$\begin{array}{ccccccc}
C^0_{\mathbf{t}}(K_f(\boldsymbol{\gamma}, S \otimes_R L)) & \longrightarrow \cdots \longrightarrow & C^0_{\mathbf{t}}(K_1(\boldsymbol{\gamma}, S \otimes_R L)) & \longrightarrow & C^0_{\mathbf{t}}(K_0(\boldsymbol{\gamma}, S \otimes_R L)) \\
\downarrow & & \downarrow & & \downarrow \\
C^1_{\mathbf{t}}(K_f(\boldsymbol{\gamma}, S \otimes_R L)) & \longrightarrow \cdots \longrightarrow & C^1_{\mathbf{t}}(K_1(\boldsymbol{\gamma}, S \otimes_R L)) & \longrightarrow & C^1_{\mathbf{t}}(K_0(\boldsymbol{\gamma}, S \otimes_R L)) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
C^{g-1}_{\mathbf{t}}(K_f(\gamma, S \otimes_R L)) & \longrightarrow \cdots \longrightarrow & C^{g-1}_{\mathbf{t}}(K_1(\gamma, S \otimes_R L)) & \longrightarrow & C^{g-1}_{\mathbf{t}}(K_0(\boldsymbol{\gamma}, S \otimes_R L)) \\
\downarrow & & \downarrow & & \downarrow \\
C^g_{\mathbf{t}}(K_f(\gamma, S \otimes_R L)) & \longrightarrow \cdots \longrightarrow & C^g_{\mathbf{t}}(K_1(\boldsymbol{\gamma}, S \otimes_R L)) & \longrightarrow & C^g_{\mathbf{t}}(K_0(\gamma, S \otimes_R L)).
\end{array}$$

The sign in $E^{-\bullet,-\bullet}$ says that this double complex is better viewed in third quadrant on the Cartesian plane. The differentials of E is the product tensor of the Koszul differential with the Čech differential, and the squares of E are commutative, in the natural way. The total complex of E, denoted by $\mathrm{Tot}_\bullet(E)$, is the complex with

$$\mathrm{Tot}_n(E) = \bigoplus_{j-i=n} E^{-i,-j} = \bigoplus_{j-i=n} C^j_{\mathbf{t}}(K_i(\boldsymbol{\gamma}, S \otimes_R L))$$

and the differential $\partial_n^{Tot} : \mathrm{Tot}_n(E) \to \mathrm{Tot}_n(E)$ is defined in the summands by

$$\alpha \otimes \beta \in E^{-i,-j} \mapsto \alpha \otimes \partial_i(\beta) + (-1)^i \check{\partial}_j(\alpha) \otimes \beta \in E^{-i+1,-j} \oplus E^{-i,-j-1},$$

with $E^{-i,-j} = C_{\mathbf{t}}^j \otimes_S K_i(\boldsymbol{\gamma}, S \otimes_R L)$.

For more details in the next paragraphs, we recommend to the reader [W94, Section 5.6], i.e., we will talk about spectral sequences of a double complex. There are two filtrations of the total complex, denoted by ${}^{I}F\mathrm{Tot}(E)$ and ${}^{II}F\mathrm{Tot}(E)$. The filtration ${}^{II}F\mathrm{Tot}(E)$ gives rise to a spectral sequence $\{{}^{r}E_{hor}^{-p,-q}\}$, called *horizontal spectral sequence* of E. The first page of the horizontal spectral sequence of E is

$$\begin{array}{ccccc}
C_{\mathbf{t}}^0(H_f(\boldsymbol{\gamma}, S \otimes_R L)) & \cdots & C_{\mathbf{t}}^0(H_1(\boldsymbol{\gamma}, S \otimes_R L)) & & C_{\mathbf{t}}^0(H_0(\boldsymbol{\gamma}, S \otimes_R L)) \\
\downarrow & & \downarrow & & \downarrow \\
C_{\mathbf{t}}^1(H_f(\boldsymbol{\gamma}, S \otimes_R L)) & \cdots & C_{\mathbf{t}}^1(H_1(\boldsymbol{\gamma}, S \otimes_R L)) & & C_{\mathbf{t}}^1(H_0(\boldsymbol{\gamma}, S \otimes_R L)) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
C_{\mathbf{t}}^{g-1}(H_f(\gamma, S \otimes_R L)) & \cdots & C_{\mathbf{t}}^{g-1}(H_1(\gamma, S \otimes_R L)) & & C_{\mathbf{t}}^{g-1}(H_0(\boldsymbol{\gamma}, S \otimes_R L)) \\
\downarrow & & \downarrow & & \downarrow \\
C_{\mathbf{t}}^g(H_f(\gamma, S \otimes_R L)) & \cdots & C_{\mathbf{t}}^g(H_1(\boldsymbol{\gamma}, S \otimes_R L)) & & C_{\mathbf{t}}^g(H_0(\gamma, S \otimes_R L)),
\end{array}$$

that is, we take the homologies in the horizontal complexes and consider the induced map in the vertical complexes. The terms of the second page of the horizontal spectral sequence of E, obtained by taking the homologies in the first page, is

$${}^{2}E_{hor}^{-p,-q} = H_{\mathbf{t}}^q(H_p(\boldsymbol{\gamma}, S \otimes_R L)),$$

for all $0 \leq p \leq f$ and $0 \leq q \leq g$.

The filtration ${}^{I}F\mathrm{Tot}(E)$ gives rise to a spectral sequence $\{{}^{r}E_{ver}^{-p,-q}\}$, called *vertical spectral sequence* of E. The first page of the vertical spectral sequence of E, taking the homologies in the vertical complexes and considering the induced maps in the horizontal complexes, is

$${}^{1}E_{ver}^{-\bullet,-q} = \begin{cases} 0 & , \quad q \neq g \\ 0 \to H_{\mathbf{t}}^g(K_f(\boldsymbol{\gamma}, S \otimes_R L)) \to \cdots \to H_{\mathbf{t}}^g(K_0(\boldsymbol{\gamma}, S \otimes_R L)) \to 0 & , \quad q = g \end{cases},$$

since that $T_1, \ldots, T_g$ is an $S \otimes_R L$-sequence, see Theorem 1.2.1. So, it follows that ${}^2E_{ver} = {}^\infty E_{ver}$ the infinity page, because the spectral sequence ${}^rE_{ver}$ colapses, that is, there is only one line different to zero and in the next pages the differentials are zero.

Since $H^g_{\mathfrak{t}}(\bullet) \cong \bullet \otimes_S H^g_{\mathfrak{t}}(S)$, we have

$$\begin{aligned} {}^1E^{-\bullet,-q}_{ver} &= H^g_{\mathfrak{t}}(K_\bullet(\boldsymbol{\gamma}, S \otimes_R L)) \\ &\simeq K_\bullet(\boldsymbol{\gamma}, S \otimes_R L) \otimes_S H^g_{\mathfrak{t}}(S) \\ &\simeq K_\bullet(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L)), \end{aligned}$$

that is, the Koszul complex of $\gamma_1, \ldots, \gamma_f$ with coefficients in $H^g_{\mathfrak{t}}(S \otimes_R L)$, hence

$${}^2E^{-p,-q}_{ver} = H_{-p+q+g}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L)) \simeq H_{-p+q}(\mathrm{Tot}_\bullet(E)),$$

for all n, and further, the convergence of spectral sequences says

$${}^2E^{-p,-q}_{hor} \Rightarrow H_{-p+q}(\mathrm{Tot}_n(E)).$$

Notice that the double complex E is in the category of graded S-modules, that is, the two spectral sequences are in this category, and we can look to these spectral sequence in the degrees. We have that $H^g_{\mathfrak{t}}(S) \cong R[T_1^{-1}, \ldots, T_g^{-1}]$ is the S-module of inverse polynomials [BS13, Example 13.5.3], and thus,

$$\mathrm{end}(H^g_{\mathfrak{t}}(S)) = -g,$$

where $\mathrm{end}(\mathcal{H}) = \max\{\nu \in \mathbb{Z}; \mathcal{H}_{[\nu]} \neq 0\}$, for all graded S-module $\mathcal{H}$. It follows that $({}^1E_{ver})_{[\nu]}$ is given by

$$0 \to K_f(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S\otimes_R L))_{[\nu]} \to \cdots \to K_{g+\nu+1}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S\otimes_R L))_{[\nu]} \xrightarrow{\delta_\nu} K_{g+\nu}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S\otimes_R L))_{[\nu]} \to 0$$

for $-g \leq \nu \leq f - g$, where

$$H_{g+\nu}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} = \mathrm{coker}\delta_{[\nu]}.$$

Notice that for $\nu \leq -g$, all the terms of the Koszul complex on degree ν are zero and $({}^1E_{ver})_{[\nu]} = 0$ for $\nu > f - g$.

Now, looking to the graded Koszul complex and the torsions in the terms, we have

$$\mathrm{indeg}(K_\nu(\boldsymbol{\gamma}, S \otimes_R L)) = \nu,$$

where $\mathrm{indeg}(\mathcal{H}) = \min\{\nu \in \mathbb{Z}; \mathcal{H}_{[\nu]} \neq 0\}$, for all graded S-module $\mathcal{H}$. The strand of $K_\bullet(\boldsymbol{\gamma}, S \otimes_R L)$ in degree ν is

$$0 \to K_\nu(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} \stackrel{(\partial_\nu)_{[\nu]}}{\to} K_{\nu-1}(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} \to \cdots \to K_0(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} \to 0,$$

and thus, $H_\nu(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} = \ker(\partial_\nu)_{[\nu]}$.

In degree ν, the convergence of spectral sequences says that there exists a filtration of $\mathrm{Tot}_\bullet(E)_{[\nu]}$ such that

$$\cdots \subset F_1 \subset F_0 = (\mathrm{Tot}_\nu E)_{[\nu]} = ({}^\infty E_{ver}^{-g-\nu,-g})_{[\nu]} = \mathrm{coker}(\delta_\nu)$$

and

$$\frac{\mathrm{coker}(\delta_\nu)}{F_1} \cong ({}^\infty E_{hor}^{-\nu,0})_{[\nu]}. \tag{2.1.1}$$

So, we define the map $\tau_\nu : K_{g+\nu}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} \to K_\nu(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]}$ by the composition

$$\begin{array}{c} K_{g+\nu}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} \twoheadrightarrow \frac{\mathrm{coker}(\delta_\nu)}{F_1} \xrightarrow{\cong} ({}^\infty E_{hor}^{-\nu,0})_{[\nu]} \hookrightarrow \\ ({}^2 E_{hor}^{-\nu,0})_{[\nu]} = H^0_{\mathfrak{t}}(H_\nu(\boldsymbol{\gamma}, S \otimes_R L))_{[\nu]} \hookrightarrow H_\nu(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} \hookrightarrow K_\nu(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} \end{array} \tag{2.1.2}$$

where the epimorphisms and monomorphisms are canonical. Notice that ${}^\infty E_{hor}^{-\nu,0} \subset {}^2 E_{hor}^{-\nu,0}$, because all differentials which arrive in ${}^r E_{hor}^{-\nu,0}$ is zero, that is, ${}^{r+1} E_{hor}^{-\nu,0}$ is a kernel for all $r \geq 1$ and all ν (E is on the third quadrant).

Definition 2.1.2. *We define the family of complexes* $\mathfrak{B}(\Phi, L) = \{\mathfrak{B}_\bullet(\Phi, L, \nu); \nu \in \mathbb{Z}\}$*, where for each* ν*, the complex* $\mathfrak{B}_\bullet(\Phi, L, \nu)$ *is obtained by splicing the complexes* $K_\bullet(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]}$ *and* $K_\bullet(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]}$ *via* τ_ν*, the map defined above by the convergence of spectral sequences. So,* $\{\mathfrak{B}_i(\Phi, L, \nu), d_i\}$ *is a complex with*

$$\mathfrak{B}_i(\Phi, L, \nu) = \left\{ \begin{array}{cc} K_i(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} & , \ i \leq \nu; \\ K_{i+g-1}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} & , \ i \geq \nu, \end{array} \right.$$

and d_i *is the Koszul differential maps for* $i \neq \nu + 1$*, and* $d_{\nu+1} = \tau_\nu$

$$\mathfrak{B}_{\nu+1}(\Phi, L, \nu) = K_{g+\nu}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} \xrightarrow{\tau_\nu} K_\nu(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} = \mathfrak{B}_{\nu+1}(\Phi, L, \nu).$$

Remark 2.1.3. *The only cases where the splice occurs are the cases where $0 \leq \nu \leq f-g$, that is, the map τ_ν is not zero. For $\nu \leq -1$ or $\nu \geq f-g+1$, the map τ_ν is zero, and we have*

$$\mathfrak{B}_\bullet(\Phi, L, \nu) := \begin{cases} K_\bullet(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} & , \quad \nu > f-g \\ K_\bullet(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} & , \quad \nu < 0. \end{cases},$$

that is, the complex $\mathfrak{B}_\bullet(\Phi, L, \nu)$ is just a strand of a Koszul complex. It justifies the hypothesis $1 \leq g \leq f$, otherwise there is not splice, only strand of Koszul complexes.

Remark 2.1.4. *In all this work we will use the complexes $\mathfrak{B}_\bullet(\Phi, L, \nu)$ in this form, tensoring by an arbritary R-module L in the beginning. But we could give a construction tensoring by L in the end and we would obtain the same complexes, that is, $\mathfrak{B}_\bullet(\Phi, L, \nu) = \mathfrak{B}_\bullet(\Phi, R, \nu) \otimes_R L$.*

Notation 2.1.5. *The complexes $\mathfrak{B}_\bullet(\Phi, L, \nu)$ generalize the Koszul complex to a matrix Φ with coefficients in an R-module L, see Proposition 2.2.1. In this sense, we will denote*

$$H_i(\Phi, L, \nu) := H_i(\mathfrak{B}_\bullet(\Phi, L, \nu)),$$

for the i-th homology module of the complex $\mathfrak{B}_\bullet(\Phi, L, \nu)$.

Due to Bouça and Hassanzadeh [BHa19, Section 3.2], the family of complexes $\mathfrak{B}(\Phi) = \mathfrak{B}(\Phi, R)$, where $L = R$, is the family of the Buchsbaum-Eisenbud complexes [E95, Appendix 2.6], and for an arbitrary R-module L, we are in the Kirby approach of these complexes [Kir73].

Writing $M := \text{coker}\varphi$, we can consider the ideal $\text{Fitt}_0 M$ of R generated by the maximal minors of Φ. Notice that the complex $\mathfrak{B}_\bullet(\Phi, 0)$ is the Eagon-Northcott complex of the matrix Φ [EN62], that is,

$$0 \to \mathfrak{B}_{f-g+1}(\Phi, 0) \to \cdots \to \mathfrak{B}_1(\Phi, 0) \xrightarrow{\tau_0} \mathfrak{B}_0(\Phi, 0) \to 0 \tag{2.1.3}$$

with $\text{im}\tau_0 = \text{Fitt}_0 M$ and $H_0(\Phi, 0) = R/\text{Fitt}_0 M$, where $M = \text{Coker}(\Phi)$; and the complex $\mathfrak{B}_0(\Phi, 1)$ is a complex from the family of Buchsbaum-Rim complexes of the map φ [BR64], that is,

$$0 \to \mathfrak{B}_{f-g+1}(\Phi, 1) \to \cdots \to \mathfrak{B}_2(\Phi, 0) \xrightarrow{\tau_1} \mathfrak{B}_1(\Phi, 1) \xrightarrow{\varphi} \mathfrak{B}_0(\Phi, 1) \to 0 \tag{2.1.4}$$

with $\ker\varphi = \text{im}\tau_1$ and $H_0(\Phi, 1) = M$. In same sense, we have that the complex $\mathfrak{B}_\bullet(\Phi, \nu)$ is an approximate free resolution for the module

$$H_0(\Phi, \nu) = (S/\gamma)_{[\nu]} = \text{Sym}^\nu_R M,$$

when $\nu \geq 1$, and for the cyclic module $R/\text{Fitt}_0 M$, when $\nu = 0$.

2.2 Basic properties

The Koszul complex is defined over an R-linear map $\varphi : R^f \to R$ and we can take coefficients tensoring by an R-module. The complexes $\mathfrak{B}_\bullet(\Phi, L, \nu)$ generalize the Koszul complex for a general R-linear map of free modules $\varphi : R^f \to R^g$, with $g \leq f$.

Proposition 2.2.1. *If $g = 1$, then the complex $\mathfrak{B}_\bullet(\Phi, L, \nu)$ is the Koszul complex of φ with coefficients in the R-module L, for all ν.*

Proof. It is enough to show for $L = R$. In the case $g = 1$, $\Phi = [c_j]_{1\times f}$ is a matrix of φ. Furthermore, $S = R[T]$ is the polynomial ring in one indeterminate with coefficients in R and $H^1_{\mathfrak{t}}(S) = R[T^{-1}]$ is the inverse polynomial ring in one indeterminate with coefficients in R. Since $\boldsymbol{\gamma} = c_1T, \ldots, c_fT$, then for all ν

$$\begin{cases} K_i(\boldsymbol{\gamma}, S)_{[\nu]} = K_i(\boldsymbol{c}, R) \cdot T^{\nu-i} & , \quad 0 \leq i \leq \nu \\ K_i(\boldsymbol{\gamma}, H^1_{\mathfrak{t}}(S))_{[\nu]} = K_i(\boldsymbol{c}, R) \cdot T^{\nu-i} & , \quad \nu+1 \leq i \leq f \end{cases}$$

and from a straightforward verification, we have $(\partial_i)_{[\nu]} = \partial^{\mathbf{c}}_i$, where $\partial^{\mathbf{c}}$ is the differential of the Koszul complex of $\boldsymbol{c}$. Since that $\mathfrak{B}_i(\Phi, \nu)$ is obtained by splicing the complexes $K_\bullet(\boldsymbol{\gamma}, S)_{[\nu]}$ and $K_\bullet(\boldsymbol{\gamma}, H^1_{\mathfrak{t}}(S))_{[\nu]}$ via the map τ_ν, it is enough to show that $\tau_\nu = \partial^{\mathbf{c}}_{\nu+1}$, and it follows from [BHa19, Theorem 3.6]. □

The Koszul complex is self-dual (Proposition 1.6.10, [BH98]). There is a self-duality on the complexes $\mathfrak{B}_\bullet(\Phi, L, \nu)$ (Appendix A2.6, [E95]). For all R-module L, we set

$$\mathfrak{B}^\bullet(\Phi, L, \nu) := \mathrm{Hom}_R(\mathfrak{B}_\bullet(\Phi, \nu), L)$$

and if $L = R$, we will write $\mathfrak{B}^\bullet(\Phi, \nu)$.

Proposition 2.2.2. *The complexes $\mathfrak{B}^\bullet(\Phi, \nu)$ and $\mathfrak{B}_\bullet(\Phi, f-g-\nu)$ are isomorphic, that is, the complex $\mathfrak{B}^\bullet(\Phi, \nu)$ is dual to $\mathfrak{B}_\bullet(\Phi, f-g-\nu)$, for all ν. Moreover, for all R-module L there is an isomorphism of complexes $\mathfrak{B}^\bullet(\Phi, L, \nu) \cong \mathfrak{B}_\bullet(\Phi, L, f-g-\nu)$, for all ν.*

Proof. The multiplication $S_{[\nu]} \otimes_R H^g_{\mathfrak{t}}(S)_{[-\nu-g]} \to H^g_{\mathfrak{t}}(S)_{[-g]} \cong R$ defines a perfect pairing, and if we take the tensor by an R-module L, then $S_{[\nu]} \otimes_R H^g_{\mathfrak{t}}(S \otimes_R L)_{[-\nu-g]} \to L$ and $(S \otimes_R L)_{[\nu]} \otimes_R H^g_{\mathfrak{t}}(S)_{[-\nu-g]} \to L$ are perfect pairings too, and it yields isomorphisms

$$H^g_{\mathfrak{t}}(S \otimes_R L)_{[-\nu-g]} \cong \mathrm{Hom}_R(S_{[\nu]}, L)$$

and

$$(S \otimes_R L)_{[\nu]} \cong \mathrm{Hom}_R(H^g_{\mathfrak{t}}(S)_{[-\nu-g]}, L),$$

for all $\nu \in \mathbb{Z}$.

Notice that from the perfect pairing and known canonical isomorphisms, we obtain

$$
\begin{array}{rcl}
\operatorname{Hom}_R(K_\bullet(\boldsymbol{\gamma})_{[\nu]}, L) & = & \operatorname{Hom}_R((\wedge^\bullet S^f(-\bullet))_{[\nu]}, L) \\
& \cong & \operatorname{Hom}_R((\wedge^\bullet R^f \otimes_R S(-\bullet))_{[\nu]}, L) \\
& = & \operatorname{Hom}_R(\wedge^\bullet R^f \otimes_R S_{[\nu-\bullet]}, L) \\
& \cong & \operatorname{Hom}_R(\wedge^\bullet R^f, \operatorname{Hom}_R(S_{[\nu-\bullet]}, L)) \\
& \cong & \operatorname{Hom}_R(\wedge^\bullet R^f, H^g_{\mathfrak{t}}(S \otimes_R L)_{[-\nu+\bullet-g]}) \\
& \cong & \wedge^{f-\bullet} R^f \otimes_R H^g_{\mathfrak{t}}(S \otimes_R L)_{[-\nu+\bullet-g]} \\
& \cong & (\wedge^{f-\bullet} S^f(-f+\bullet) \otimes_S H^g_{\mathfrak{t}}(S \otimes_R L))_{[f-g-\nu]} \\
& = & K_{f-\bullet}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[f-g-\nu]},
\end{array}
\tag{2.2.1}
$$

and similarly, $\operatorname{Hom}_R(K_{\bullet+g}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S))_{[\nu]}, L) \cong K_{f-g-\bullet}(\boldsymbol{\gamma}, L)_{[f-g-\nu]}$. Since $\mathfrak{B}_\bullet(\Phi, \nu)$ is obtained by splicing the complexes $K_\bullet(\boldsymbol{\gamma})_{[\nu]}$ and $K_\bullet(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S))_{[\nu]}$, it is enough to show that the next diagram is commutative

$$
\begin{array}{ccc}
\operatorname{Hom}_R(K_\nu(\boldsymbol{\gamma})_{[\nu]}, L) & \xrightarrow{\operatorname{Hom}_R(\tau_{[\nu]}, 1_L)} & \operatorname{Hom}_R(K_{\nu+g}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S))_{[\nu]}, L) \\
\downarrow & & \downarrow \\
K_{f-\nu}(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[f-g-\nu]} & \xrightarrow{\tau_{[f-g-\nu]} \otimes_R 1_L} & K_{f-g-\nu}(\boldsymbol{\gamma}, L)_{[f-g-\nu]}.
\end{array}
$$

And this diagram is commutative due to the definition of τ (see [BHa19, Theorem 3.6]) and the canonical isomorphisms in 2.2.1. □

The Koszul complex can be constructed inductively, i.e., if $\boldsymbol{\gamma} = \gamma_1, \ldots, \gamma_f$ is a sequence of elements in S, then we have $K_\bullet(\boldsymbol{\gamma}, N) = K_\bullet(\boldsymbol{\gamma}', N) \otimes_S K_\bullet(\gamma_1, N)$ for all S-module N, where $\boldsymbol{\gamma}' = \gamma_2, \ldots, \gamma_f$, and this inductive construction yields the canonical exact sequence of Koszul complexes

$$0 \to K_\bullet(\boldsymbol{\gamma}', N) \to K_\bullet(\boldsymbol{\gamma}, N) \to K_\bullet(\boldsymbol{\gamma}', N)[-1] \to 0,$$

where $[-1]$ means the complex is moved one place left, which is split exact on the terms. For more details, consider $0 \leq n \leq f$, we have

$$0 \to K_n(\boldsymbol{\gamma}', N) \to K_n(\boldsymbol{\gamma}, N) \to K_{n-1}(\boldsymbol{\gamma}', N) \to 0,$$

where the left is the inclusion map and if $e_{i_1} \wedge \cdots \wedge e_{i_n} \in K_n(\boldsymbol{\gamma}, N)$, then the right map is given by

$$e_{i_1} \wedge \cdots \wedge e_{i_n} \mapsto \begin{cases} (-1)^n e_{i_2} \wedge \cdots \wedge e_{i_n} & , \ if \ \ i_1 = 1; \\ 0 & , \ if \ \ i_1 \neq 1. \end{cases}$$

In [Kir85], Kirby used this property to obtain the main result about Buchsbaum-Rim multiplicity the Theorem 3.1.3 in next chapter. Now, we will give a new proof for the existence of this canonical exact sequence of the complexes $\mathfrak{B}_\bullet(\Phi, L, \nu)$, which is split exact on terms, and we will see that it is due to the convergence of spectral sequence in our approach.

Proposition 2.2.3. *Let L be an R-module and ν be an integer. Then, there is an exact sequence of complexes $\boldsymbol{\mathcal{E}}_\nu : 0 \to \mathfrak{B}_\bullet(\Phi', L, \nu) \to \mathfrak{B}_\bullet(\Phi, L, \nu) \to \mathfrak{B}_\bullet(\Phi', L, \nu-1) \to 0$, which is split exact on terms.*

Proof. For $\nu < 0$ and $\nu > f-g$, $\boldsymbol{\mathcal{E}}_\nu$ is the canonical exact sequence of Koszul complexes. Supposing $f \geq g$, we must to show for $0 \leq \nu \leq f-g$, and hence, the exact sequences of complexes

$$0 \to K_\bullet(\boldsymbol{\gamma}', S \otimes_R L)_{[\nu]} \to K_\bullet(\boldsymbol{\gamma}, S \otimes_R L)_{[\nu]} \to K_\bullet(\boldsymbol{\gamma}', S \otimes_R L)_{[\nu-1]}[-1] \to 0$$

and

$$0 \to K_\bullet(\boldsymbol{\gamma}', H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} \to K_\bullet(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu]} \to K_\bullet(\boldsymbol{\gamma}', H^g_{\mathfrak{t}}(S \otimes_R L))_{[\nu-1]}[-1] \to 0$$

can be splicing by the maps $\tau'_{[\nu]}$, $\tau_{[\nu]}$ and $\tau'_{[\nu-1]}$ defined in previous section 2.1.2, respectively. Notice that the homogeneous component in the right is different to others because this exact sequence comes from the mapping cone of the multiplication by γ_1. Thus, we will show that

$$0 \to \mathfrak{B}_\bullet(\Phi', L, \nu) \to \mathfrak{B}_\bullet(\Phi, L, \nu) \to \mathfrak{B}_\bullet(\Phi', L, \nu-1) \to 0$$

is an exact sequence of complexes, for this, it's enough to show the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{B}_{\nu+1}(\Phi', L, \nu) & \longrightarrow & \mathfrak{B}_{\nu+1}(\Phi, L, \nu) & \longrightarrow & \mathfrak{B}_{\nu}(\Phi', L, \nu-1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau'_{[\nu]}} & & \downarrow{\scriptstyle \tau_{[\nu]}} & & \downarrow{\scriptstyle \tau'_{[\nu-1]}} & & \\ 0 & \longrightarrow & \mathfrak{B}_{\nu}(\Phi', L, \nu) & \longrightarrow & \mathfrak{B}_{\nu}(\Phi, L, \nu) & \longrightarrow & \mathfrak{B}_{\nu-1}(\Phi', L, \nu-1) & \longrightarrow & 0. \end{array}$$

If we denote $E'^{\bullet\bullet} = K_\bullet(\boldsymbol{\gamma}', S \otimes_R L) \otimes_S C^\bullet_{\mathfrak{t}}(S)$ and $E'^{\bullet\bullet}[-1] = K_\bullet(\boldsymbol{\gamma}', S \otimes_R L)[-1] \otimes_S C^\bullet_{\mathfrak{t}}(S)$, then there is the exact sequence of double complexes

$$0 \to E'^{\bullet\bullet} \to E^{\bullet\bullet} \to E'^{\bullet\bullet}[-1] \to 0.$$

This exact sequence induces exact sequence on total complexes

$$0 \to \mathrm{Tot}_\bullet E' \to \mathrm{Tot}_\bullet E \to \mathrm{Tot}_\bullet E'[-1] \to 0, \qquad (2.2.2)$$

and these maps are compatible with vertical and horizontal filtrations. Notice that these maps of total complexes induce maps on homologies, where the homologies of the complexes $\mathrm{Tot}_\bullet E'$, $\mathrm{Tot}_\bullet E$ and $\mathrm{Tot}_\bullet E'[-1]$ are the homologies of the complexes $K_\bullet(\boldsymbol{\gamma}', H^g_{\mathfrak{t}}(S \otimes_R L))$, $K_\bullet(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))$ and $K_\bullet(\boldsymbol{\gamma}', H^g_{\mathfrak{t}}(S \otimes_R L))[-1]$, respectively, by the convergence of the Koszul-Čech spectral sequence. So, in degree ν, the diagram below is commutative with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{B}_{\nu+1}(\Phi', L, \nu) & \longrightarrow & \mathfrak{B}_{\nu+1}(\Phi, L, \nu) & \longrightarrow & \mathfrak{B}_{\nu}(\Phi', L, \nu-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}\delta'_{[\nu]} & \longrightarrow & \mathrm{coker}\delta_{[\nu]} & \longrightarrow & \mathrm{coker}\delta'_{[\nu-1]} & \longrightarrow & 0,
\end{array}$$

where the maps δ is the same in previous section, that is, a differential of Koszul strands, and the maps on the second row are induced by the maps on the first row. Furthermore, the sequence 2.2.2 induces maps of spectral sequences compatible with the induced maps of homologies of the total complexes, that is, in degree ν

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{coker}\delta'_{[\nu]}/F'_1 & \longrightarrow & \mathrm{coker}\delta_{[\nu]}/F_1 & \longrightarrow & \mathrm{coker}\delta'_{[\nu-1]}/F'_1[-1] \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
0 & \longrightarrow & ({}^{\infty}E'^{-\nu,0}_{hor})_{[\nu]} & \longrightarrow & ({}^{\infty}E^{-\nu,0}_{hor})_{[\nu]} & \longrightarrow & ({}^{\infty}E'^{-\nu+1,0}_{hor})_{[\nu-1]}
\end{array}.$$

is a commutative diagram with exact lines, where F'_1, F_1 and $F'_1[-1]$ is given by the convergence of ${}^{r}E'_{hor}$, ${}^{r}E_{hor}$ and ${}^{r}E'_{hor}[-1]$ in same sense of 2.1.1, and it has exact line due to the modules ${}^{r}E'^{-\nu,0}_{hor}$, ${}^{r}E^{-\nu,0}_{hor}$ and ${}^{r}E'^{-\nu,0}_{hor}[-1]$ be kernel with induced maps for all r. Joining the diagrams above, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{B}_{\nu+1}(\Phi', L, \nu) & \longrightarrow & \mathfrak{B}_{\nu+1}(\Phi, L, \nu) & \longrightarrow & \mathfrak{B}_{\nu}(\Phi', L, \nu-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}\delta'_{[\nu]} & \longrightarrow & \mathrm{coker}\delta_{[\nu]} & \longrightarrow & \mathrm{coker}\delta'_{[\nu-1]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{coker}\delta'_{[\nu]}/F'_1 & \longrightarrow & \mathrm{coker}\delta_{[\nu]}/F_1 & \longrightarrow & \mathrm{coker}\delta'_{[\nu-1]}/F'_1[-1] & & \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & ({}^{\infty}E'^{-\nu,0}_{hor})_{[\nu]} & \longrightarrow & ({}^{\infty}E^{-\nu,0}_{hor})_{[\nu]} & \longrightarrow & ({}^{\infty}E'^{-\nu+1,0}_{hor})_{[\nu-1]} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{B}_{\nu}(\Phi', L, \nu) & \longrightarrow & \mathfrak{B}_{\nu}(\Phi, L, \nu) & \longrightarrow & \mathfrak{B}_{\nu-1}(\Phi', L, \nu-1) & \longrightarrow & 0.
\end{array}$$

Finally, by the definition of the maps τ's in 2.1.2, the result follows. □

The next property is about the support of the homologies of $\mathfrak{B}_\bullet(\Phi, L, \nu)$. We will give a new proof of this fact using the Koszul-Čech spectral sequence, for the classical proof see [Kir73, Theorem 1].

Proposition 2.2.4. *Let L be a finitely generated R-module and $\nu \in \mathbb{Z}$ and $M = \mathrm{Coker}(\Phi)$. Then, for all n we have $\mathrm{Supp}_R H_n(\mathfrak{B}_\bullet(\Phi, L, \nu)) \subset \mathrm{Supp}_R(M \otimes_R L)$. In particular, there is an integer $s \geq 1$ such that $(\mathrm{Fitt}_0 M)^s H_n(\mathfrak{B}_\bullet(\Phi, L, \nu)) = 0$, for all n.*

Proof. If $\mathfrak{P} \notin \mathrm{Supp}_R L$, then obviously $H_i(\mathfrak{B}_\bullet(\Phi, L, \nu))_\mathfrak{P} = 0$. If $\mathfrak{P} \notin \mathrm{Supp}_R M$, that is, $\mathfrak{P} \notin V(\mathrm{Fitt}_0 M)$, then

$$\left(\frac{S}{\gamma}\right)_\mathfrak{P} \cong (\mathrm{Sym}_R M)_\mathfrak{P} \cong \mathrm{Sym}_{R_\mathfrak{P}} M_\mathfrak{P} = R_\mathfrak{P},$$

since $M_\mathfrak{P} = 0$. Thus, $\boldsymbol{\gamma}_\mathfrak{P} = \mathbf{t}_\mathfrak{P}$ and $H_\bullet(\boldsymbol{\gamma}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P}) \cong H_\bullet(\mathbf{t}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P}) \otimes_{S_\mathfrak{P}} \wedge^\bullet S_\mathfrak{P}^{f-g}$, by [BH98, Proposition 1.6.21]. As $(T_1, ..., T_g)$ is an $S \otimes_R L-$sequence, then $K_\bullet(\mathbf{t}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P})$ is acyclic and $H_0(\mathbf{t}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P}) = L_\mathfrak{P}$. Therefore,

$$\begin{array}{rcl} H_i(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P} & \cong & H_i(\boldsymbol{\gamma}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P}) \\ & \cong & \sum_{j=0}^i H_j(\mathbf{t}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P}) \otimes_{S_\mathfrak{P}} \wedge^{i-j} S_\mathfrak{P}^{f-g} \\ & = & L_\mathfrak{P} \otimes_{S_\mathfrak{P}} S_\mathfrak{P}^{\binom{f-g}{i}}(-i) \\ & \cong & L_\mathfrak{P}^{\binom{f-g}{i}}(-i), \end{array} \tag{2.2.3}$$

and thus, $H_i(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P}$ is $\mathbf{t}$-torsion all $0 \leq i \leq f-g$, since there is only one graded component different to zero, and $H_i(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P} = 0$ for $i \geq f-g+1$ or $i \leq -1$. So, it follows that ${}^2(E_\mathfrak{P})_{hor} = {}^\infty(E_\mathfrak{P})_{hor}$, that is, ${}^2(E_\mathfrak{P})_{hor}^{-i,0} = H_\mathbf{t}^0(H_i(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P})$ for $0 \leq i \leq f-g$, and the other terms are zero. Hence, by the convergence of spectral sequences, the vertical and horizontal spectral sequence can be related, and

$$\begin{array}{rcl} H_i(\boldsymbol{\gamma}, H_\mathbf{t}^g(S \otimes_R L))_\mathfrak{P} & \cong & H_i(\boldsymbol{\gamma}_\mathfrak{P}, H_{\mathbf{t}_\mathfrak{P}}^g(S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P})) \\ & \cong & H_{\mathbf{t}_\mathfrak{P}}^0(H_{i-g}(\boldsymbol{\gamma}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P})) \\ & = & H_{i-g}(\boldsymbol{\gamma}_\mathfrak{P}, S_\mathfrak{P} \otimes_{R_\mathfrak{P}} L_\mathfrak{P}) \\ & = & H_{i-g}(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P}, \end{array} \tag{2.2.4}$$

for all i. If $\nu \geq f-g+1$, then $H_i(\mathfrak{B}_\bullet(\Phi, L, \nu)_\mathfrak{P}) = (H_i(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P})_\nu$, and by 2.2.3, it is zero. Similarly, for $\nu \leq -1$, we have $H_i(\mathfrak{B}_\bullet(\Phi, L, \nu)_\mathfrak{P}) = (H_{i+g-1}(\boldsymbol{\gamma}, H_\mathbf{t}^g(S \otimes_R L))_\mathfrak{P})_\nu = 0$, by 2.2.3 and 2.2.4.

Now, we consider $\mathfrak{B}_\bullet(\Phi, L, \nu)$ with $0 \leq \nu \leq f-g$. So

$$H_i(\mathfrak{B}_\bullet(\Phi, L, \nu)_\mathfrak{P}) = \left\{ \begin{array}{ccl} (H_i(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P})_{[\nu]} & , & 0 \leq i \leq \nu - 1; \\ (H_{i-1+g}(\boldsymbol{\gamma}, H_\mathbf{t}^g(S \otimes_R L))_\mathfrak{P})_{[\nu]} & , & \nu + 2 \leq i \leq f-g+1. \end{array} \right. \tag{2.2.5}$$

By (2.2.3), (2.2.4) and (2.2.5), it follows $H_i(\mathfrak{B}_\bullet(\Phi, L, \nu)_\mathfrak{P}) = 0$, for $i \neq \nu, \nu+1$.

In the same way of the construction of the complexes $\mathfrak{B}_{\nu+1}(\Phi, L, \nu)$ in 2.1.2, we can consider the double complex $E_\mathfrak{P} = E \otimes_R R_\mathfrak{P}$, which is obtained by localization in the prime $\mathfrak{P}$, and we take the map $(\tau_\nu)_\mathfrak{P}$ and the module $F_1^{L_\mathfrak{P}}$, which is given by the convergence of the spectral sequence in degree ν.

There is a module $\mathfrak{F}_1^{L_\mathfrak{P}} \in \mathfrak{B}_{\nu+1}(\Phi, L, \nu)_\mathfrak{P}$ such that $F_1^{L_\mathfrak{P}} = \mathfrak{F}_1^{L_\mathfrak{P}}/\mathrm{im}(\tau_\nu)_\mathfrak{P}$, and thus $\ker(\tau_\nu)_\mathfrak{P} = \mathfrak{F}_1^{L_\mathfrak{P}}$. It follows that $H_{\nu+1}(\mathfrak{B}_\bullet(\Phi, L, \nu)_\mathfrak{P}) = F_1^{L_\mathfrak{P}}$. Notice that (2.2.4) implies $F_1^{L_\mathfrak{P}} = 0$, since $\mathrm{coker}(\delta_\nu)_\mathfrak{P} = (H_{\nu+g}(\boldsymbol{\gamma}, H_\mathfrak{t}^g(S \otimes_R L))_\mathfrak{P})_{[\nu]} \cong (H_\nu(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P})_{[\nu]}$, and therefore, $\ker(\tau_\nu)_\mathfrak{P} = \mathrm{im}(\delta_\nu)_\mathfrak{P}$ and $H_{\nu+1}(\mathfrak{B}_\bullet(\Phi, L, \nu)_\mathfrak{P}) = 0$. Finally,

$$\begin{aligned} \mathrm{im}(\tau_\nu)_\mathfrak{P} &= \mathrm{coker}(\delta_\nu)_\mathfrak{P} \\ &\cong (H_\nu(\boldsymbol{\gamma}, S \otimes_R L)_\mathfrak{P})_{[\nu]} \\ &= \ker((\partial_\nu)_\mathfrak{P})_{[\nu]}, \end{aligned}$$

and $H_\nu(\mathfrak{B}_\bullet(\Phi, L, \nu)_\mathfrak{P}) = 0$. □

Remark 2.2.5. *Kirby in [Kir73, Theorem 1] shows that the power s in the above proposition is equal to one, that is, the ideal* $\mathrm{Fitt}_0 M$ *annihilates the homologies of* $\mathfrak{B}_\bullet(\Phi, L, \nu)$*, for all integer ν. In this work, the last proposition is sufficient, see 3.3.1.*

2.3 Depth and acyclicity properties

An important property of the Koszul complex, in our case for g=1 (see the Proposition 2.2.1), is the fact that the homologies measure the grade of a finitely generated R-module in the ideal $I = \mathrm{im}\varphi$, when R is a Noetherian ring, and this grade can determines how close the Koszul complex is to being acyclic, see [BH98, Theorem 1.6.17]. In this sense, the complexes $\mathfrak{B}_\bullet(\Phi, L, \nu)$ measure the grade of a finitely generated R-module relative to the ideal $\mathrm{Fitt}_0 M$, which Kirby called of grade sensitive, and this grade calculate the acyclicity of these complexes, see Theorem A2.10 [E95] and [Kir73, Corollary 2].

Eagon and Northcott show that the fact that $\mathrm{Fitt}_0 M$ does not contain an L-regular element is equivalent to the homology $H_{f-g+1}(\mathfrak{B}_\bullet(\Phi, L, 0))$ is not zero [EN62, Proposition 1], and Buchsbaum and Rim show that the complex $\mathfrak{B}_\bullet(\Phi, L, 1)$ has the same property in [BR64, Proposition 2.3]. Finally in [Kir73, Theorem 2], Kirby shows that this property holds for all $\nu \leq f - g$. Using the Koszul-Čech spectral construction of the complexes $\mathfrak{B}_\bullet(\Phi, L, \nu)$, we will give a new proof of this fact.

Lemma 2.3.1. *Let R be a Noetherian ring, L be a finitely generated R-module and $\nu \leq f - g$. Then $H_{f-g+1}(\Phi, L, \nu) \neq 0$ if, and only if,* $\mathrm{Hom}_R(R/\mathrm{Fitt}_0 M, L) \neq 0$.

Proof. Note that for $\nu \leq f-g-1$, we have $H_{f-g+1}(\Phi, L, \nu) = H_f(\boldsymbol{\gamma}, H_\mathfrak{t}^g(S \otimes_R L))_{[\nu]}$, i.e., the ν-th component of Koszul homology of $\gamma_1, \ldots, \gamma_f$ with coefficients in $H_\mathfrak{t}^g(S \otimes_R L)$, and by Kuszul duality, it follows

$$H_{f-g+1}(\Phi, L, \nu) = \mathrm{Hom}_S(S/\boldsymbol{\gamma}, H_\mathfrak{t}^g(S \otimes_R L))(-f)_{[\nu]}.$$

From the perfect pairing given by the multiplication $S_{[i]} \otimes_R H^g_{\mathfrak{t}}(S \otimes_R L)_{[-i-g]} \to L$, we obtain the duality $\mathrm{Hom}_S(S/\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S \otimes_R L))(-f)_{[\nu]} \cong \mathrm{Hom}_R((S/\boldsymbol{\gamma})_{[f-g-\nu]}, L)$ [Jou09]. Applying the functor $\mathrm{Sym}_R(\bullet)$ to $R^f \xrightarrow{\varphi} R^g \to M \to 0$, we obtain the exact sequence

$$0 \to \mathrm{im}\varphi \cdot \mathrm{Sym}_R R^g \to \mathrm{Sym}_R R^g \to \mathrm{Sym}_R M \to 0$$

where $S \cong \mathrm{Sym}_R R^g$, and so, $S/\gamma \cong \mathrm{Sym}_R M$. Note that, $\mathrm{Supp}_R(M) = \mathrm{Supp}_R(\mathrm{Sym}^i_R M)$, for $i \geq 1$. Therefore, by [BH98, Exercise 1.2.27] and [E95, Proposition 20.7], we have for $\nu \leq f-g-1$

$$\begin{aligned} \mathrm{Ass}_R(\mathrm{Hom}_R((S/\boldsymbol{\gamma})_{[f-g-\nu]}, L)) &= \mathrm{Ass}_R(\mathrm{Hom}_R(\mathrm{Sym}^{f-g-\nu}_R M, L)) \\ &= \mathrm{Supp}_R(\mathrm{Sym}^{f-g-\nu}_R M) \cap \mathrm{Ass}_R L \\ &= \mathrm{Supp}_R M \cap \mathrm{Ass}_R L \\ &= V(R/\mathrm{Ann} M) \cap \mathrm{Ass}_R L \\ &= V(R/\mathrm{Fitt}_0 M) \cap \mathrm{Ass}_R L \\ &= \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathrm{Fitt}_0 M, L)), \end{aligned}$$

and from this, follows the result for $\nu \leq f-g-1$. By the Proposition 2.2.2, the complex $\mathfrak{B}_\bullet(\Phi, \nu)$ is dual to complex $\mathfrak{B}_\bullet(\Phi, f-g-\nu)$, and thus $H_{f-g+1}(\Phi, L, f-g) \cong \mathrm{Hom}_R(H_0(\Phi, 0), L)$, where $H_0(\Phi, 0) = R/\mathrm{Fitt}_0 M$. □

With the previous lemma, we can proof the grade sensitive of the complexes in the family $\mathfrak{B}_\bullet(\Phi)$ which have non-zero splice, see Remark 2.1.3.

Theorem 2.3.2. *Suppose that R is a Noetherian ring, $M \otimes_R L \neq 0$, with $M =$ $\mathrm{Coker}(\Phi)$, and let $\nu \leq f-g$. Then*

$$\mathrm{grade}_R(\mathrm{Fitt}_0 M, L) = \min\{i; H_{f-g+1-i}(\Phi, L, \nu) \neq 0\}.$$

Moreover, for $m = \mathrm{grade}_R(\mathrm{Fitt}_0 M, L)$,

$$H_{f-g+1-m}(\Phi, L, \nu) \simeq \mathrm{Ext}^m_R(H_0(\Phi, f-g-\nu), L).$$

Proof. We prove by using induction on $m := \min\{i; H_{f-g+1-i}(\Phi, L, \nu) \neq 0\}$.

Let $m = 0$. As we see in the course of the proof of Lemma 2.3.1, for $\nu < f-g$

$$\begin{aligned} H_{f-g+1}(\Phi, L, \nu) &= \mathrm{Hom}_S(S/\gamma, H^g_{\mathfrak{t}}(S \otimes_R L))(-f)_\nu \\ &\cong \mathrm{Hom}_R((S/\gamma)_{[f-g-\nu]}, L) \\ &= \mathrm{Hom}_R(H_0(\Phi, f-g-\nu), L), \end{aligned}$$

and for $\nu = f-g$,

$$H_{f-g+1}(\Phi, L, f-g) = \mathrm{Hom}_R(R/\mathrm{Fitt}_0 M, L) = \mathrm{Hom}_R(H_0(\Phi, 0), L).$$

According to [BH98, Proposition 1.2.3], $\mathrm{grade}_R(\mathrm{Fitt}_0 M, L) = 0$ implies $\mathrm{Hom}_R(R/\mathrm{Fitt}_0 M, L) \neq 0$, and by Lemma 2.3.1, $H_{f-g+1}(\Phi, L, \nu) \neq 0$ too.

Now, suppose that $m > 0$. According to Lemma 2.3.1, $\text{grade}_R(\text{Fitt}_0 M, L) > 0$. By Proposition 2.2.4, there exist an integer s and an L-regular element $x \in \text{Fitt}_0 M$ such that $x^s H_\bullet(\Phi, L, \nu) = x^s M = 0$. Hence, the exact sequence

$$0 \to L \xrightarrow{x^s} L \to L/x^s L \to 0,$$

yields the following short exact sequences for any i,

$$0 \to H_{f-g+1-i}(\Phi, L, \nu) \to H_{f-g+1-i}(\Phi, L/x^s L, \nu) \to H_{f-g-i}(\Phi, L, \nu) \to 0.$$

Considering the values $i < m$ the equality about grade follows. For the second assertion, we set $i = m$ and apply the induction hypothesis. We have

$$\begin{aligned} H_{f-g+1-m}(\Phi, L, \nu) &\cong H_{f-g-m}(\Phi, L/x^s L, \nu) \\ &\cong \text{Ext}_R^{m-1}(H_0(\Phi, f-g-\nu), L/x^s L) \\ &\cong \text{Ext}_R^m(H_0(\Phi, f-g-\nu), L). \end{aligned}$$

□

Remark 2.3.3. *We mention some points:*

- *The last isomorphism in the proof is the well-known Rees formula, see [BH98, Lemma 1.2.4];*
- *This theorem above proves the Eagon's classical result that*

$$\text{grade}(\text{Fitt}_0 M, L) \le f - g + 1.$$

Corollary 2.3.4. *Let R be a Noetherian ring. If $\text{grade}_R(\text{Fitt}_0 M, R) \ge f - g + 1$, then $\mathfrak{B}_\bullet(\Phi, \nu)$ is a free resolution of $R/\text{Fitt}_0 M$ for $\nu = 0$, and of $\text{Sym}_R^\nu M$ for $1 \le \nu \le f - g$. Furthermore, $f - g + 1$ is the projective dimension of $R/\text{Fitt}_0 M$ and $\text{Sym}_R^\nu M$ for $1 \le \nu \le f - g$.*

Proof. Consider $0 \le \nu \le f - g$. The complex $\mathfrak{B}_\bullet(\Phi, \nu)$ is acyclic, by the Theorem 2.3.2. Then the projective dimension of $H_0(\Phi, \nu)$ is at most $f - g + 1$. If $\mathfrak{p}$ is a prime belonging to $H_0(\Phi, \nu)$, then $\text{grade}_R(\mathfrak{p}, R) \ge f - g + 1$. So,

$$f - g + 1 \ge \text{projdim}_R H_0(\Phi, \nu) \ge \text{grade}_R(\mathfrak{p}, R) \ge f - g + 1,$$

where the second inequality is given by a theorem due to Rees [Ree57, theorem 1.2]. □

To conclude this section, we will discuss an interesting example which is known as *generic case*. Suppose R is local with maximal ideal $\mathfrak{m}$ and let $\boldsymbol{X} = (X_{ij})$ be the generic matrix of the size $g \times f$, i.e., with X_{ij} be indeterminates over R. Let $A = R[\boldsymbol{X}]_{(\mathfrak{m}, \boldsymbol{X})}$ be the localization of the polynomial ring with indeterminates X_{ij} (the terms of $\boldsymbol{X}$) over

R by the prime ideal generated by $\mathfrak{m}$ and X_{ij}, for all i, j. In [Nor63], Northcott show that $\operatorname{grade}(\operatorname{Fitt}_0(\boldsymbol{X}), A) = f - g + 1$. By the previous corollary, $\mathfrak{B}^A_\bullet(\boldsymbol{X}, \nu)$ is acyclic for all $\nu \in \{0, \ldots, f-g\}$ (The index A indicates that this complex is obtained over the ring A).

Consider the ring map

$$\begin{array}{rcl} A & \to & R \\ X_{ij} & \mapsto & c_{ij} \end{array}$$

where c_{ij}'s are the entries of the matrix Φ, we have the isomorphism $\mathfrak{B}^A_\bullet(\boldsymbol{X}, \nu) \otimes_A R \simeq \mathfrak{B}^R_\bullet(\Phi, \nu)$, since this construction commutes with base change, see Section 2.1. Hence, the homologies of $\mathfrak{B}^R_\bullet(\Phi, \nu)$ can be interpreted as Tor modules, i.e., $H_i(\Phi, \nu) \simeq \operatorname{Tor}^A_n(H_0(\boldsymbol{X}, \nu), R)$, for all n. Hayasaka and Hyry, in the proof of positivity of the partial Euler-Poincaré characteristic of $H_\bullet(\Phi, \nu)$ [HH11], used the ideas of Buchsbaum and Rim [BR65] to show that the Buchsbaum-Eisenbud homologies can be viewed as a Koszul homology, and thus, they obtained the rigidity of these complexes for $0 \leq \nu \leq f - g$, since the Koszul complex is rigid.

Chapter 3

The Euler-Poincaré characteristic of $\mathfrak{B}(\Phi)$

3.1 Buchsbaum-Rim multiplicity

Let R be a Noetherian local ring of dimension d and L be a finitely generated R-module. Let N be a submodule of a free R-module G of finite rank g such that $\ell_R((G/N) \otimes_R L) < \infty$, where ℓ_R denotes the length over R. Any such module N is called a *finite colength submodule of G with respect to module L*, and for $L = R$, we only say that N is a *finite colength submodule of* G, that is, G/N has finite length. We can define the function

$$\lambda_N(\nu, L) := \ell_R\left(\frac{\mathcal{S}_\nu(G)}{\mathcal{R}_\nu(N)} \otimes_R L\right)$$

from the set of positive integers into itself, where $\mathcal{S}(G) = \oplus_{\nu \geq 0}\mathcal{S}_\nu(G)$ is the symmetric algebra of G and $\mathcal{R}(N) = \oplus_{\nu \geq 0}\mathcal{R}_\nu(N)$ is the image of the natural map $\mathcal{S}(N) \to \mathcal{S}(G)$, which is the R-subalgebra of $\mathcal{S}(G)$ generated by N. In [BR64], Buchsbaum and Rim show that $\lambda_N(\nu, L)$ is a polynomial function for sufficient large ν with degree $\dim L + g - 1$, and $\lambda_N(\nu, L)$ is called *Buchsbaum-Rim function of N with respect to L*. For $\nu >> 0$, we can write

$$\lambda_N(\nu, L) = P_N(\nu, L) = \sum_{i=0}^{d-g+1} (-1)^i e_i \binom{\nu + d + g - 2 - i}{d + g - 1 - i}$$

with integer coefficients e_i, $P_N(\nu, L)$ is called *Buchsbaum-Rim polynomial of N with respect to L*. The *Buchsbaum-Rim multiplicity* of N on L, denoted by $\mathrm{br}(N, L)$, is defined to be the coefficient e_0.

Let $\varphi : R^f \to R^g$ be a R-linear map such that $\mathrm{im}\varphi = N$, that is, N is generated by f elements. By fixing basis, we can consider Φ the representative matrix for φ and Φ is called a matrix for N. Notice that the generators of N are given by the columns of Φ. We say that Φ is a *parameter matrix* for L if (i) $\ell_R(\mathrm{Coker}\varphi \otimes_R L) < \infty$ and (ii) $f - g + 1 = \dim L$, and thus $f = \mu(N)$ the minimal number of generators of N by [BR64, Corollary 3.6]. We say that N is a *parameter module* if there is a parameter matrix Φ which is a matrix for N.

Example 3.1.1. *For the case $g = 1$, we have that $I = N \subset G = R$ is an ideal such that $l(R/I \otimes_R L) < \infty$, that is, I is an ideal of definition of L. Hence, $\mathcal{S}(G) = R[T]$ the polynomial ring in the indeterminate T over R and $\mathcal{R}(I) = R[I \cdot T]$ the Rees algebra of I. Furthermore,*

$$\frac{\mathcal{S}_\nu(G)}{\mathcal{R}_\nu(I)} = \frac{R[T]_\nu}{R[I \cdot T]_\nu} = \frac{R \cdot T^\nu}{I^\nu \cdot T^\nu} \cong \frac{R}{I^\nu}$$

and thus

$$\lambda_I(\nu, L) = \ell_R(R/I^\nu \otimes_R L) = \ell_R(L/I^\nu L)$$

that is, the Hilbert-Samuel function of I with respect to L. For $\nu >> 0$, $\lambda_I(\nu, L)$ is polynomial and we can define the Hilbert-Samuel multiplicity of I on L, denoted by $\mathrm{e}(I, L)$*, as being the coefficient* e_0 *of the polynomial*

$$\lambda_I(\nu, L) = P_I(\nu, L) = \sum_{i=0}^{d} (-1)^i e_i \binom{\nu + d - 1 - i}{d - i}.$$

The Buchsbaum-Rim multiplicity generalizes the Hilbert-Samuel multiplicity as we saw in the example above, in the same way, the concept of finite colength submodules generalize ideals of definition and the concept of parameter modules generalize parameter systems.

Remark 3.1.2. *In the example above, the Rees algebra of I is the image of natural map $\mathcal{S}(I) \to \mathcal{S}(R)$ induced by the inclusion of I in R, that is, $\mathcal{R}(I) = R[I \cdot T]$, where T is an indeterminate. In general, $\mathcal{R}(I)$ can not define the Rees algebra of I for an arbitrary embedding of I in a free R-module, see [EHU02, Example 1.1 and Theorem 1.4]. The concept of Rees algebra of a R-module N, that extends its classical definition for ideal, requires an special map $f : N \to F$, called versal map, which is defined in the following way: every maps from N to a free module factors through f; and thus, we can* *define the Rees algebra of N to being the image of $\mathcal{S}(f)$ in $\mathcal{S}(F)$, [EHU02, Proposition 1.3].*

The Hilbert-Samuel multiplicity of $I = \{c_1, \ldots, c_f\}$ on L can be expressed in terms of the Koszul homology $H_\bullet(\boldsymbol{c}, L)$, when $\boldsymbol{c}$ is a parameter system for L, that is,

$$e(I, L) = \chi(\boldsymbol{c}, L) = \sum_{i=0}^{f} (-1)^i l(H_i(\boldsymbol{c}, L))$$

where $\chi(\boldsymbol{c}, L)$ is the Euler-Poincaré characteristic of the Koszul homology, [BH98, Theorem 4.7.4]. In 1964 Buchsbaum and Rim proved a analogous result for the Buchsbaum-Rim multiplicity by using the family of the Buchsbaum-Rim complexes [BR64, Corollary 4.4], where the first complex of this family is called the Buchsbaum-Rim complex and it is part of the family of Buchsbaum-Eisenbud complexes, that is, the complex indexed by 1, see 2.1.4. Later, in 1985 Kirby proved the analogous result by using the family of Buchsbaum-Eisenbud complexes, which is of our interest in this work.

Theorem 3.1.3 ([Kir85])**.** *Let R be a Noetherian local ring, L be a finitely generated R-module and N be a finite colength in a finite free R-module G of rank g. If Φ is a matrix for N, then for all $\nu \in \mathbb{Z}$*

$$\chi(\Phi, L) := \sum_i (-1)^i \ell_R(H_i(\Phi, L, \nu)) = \begin{cases} \mathrm{br}(N, L) & , \text{ if } \Phi \text{ is a parameter matrix for } L; \\ 0 & , \text{ otherwise.} \end{cases}$$

Remark 3.1.4. *In the same way of the Euler-Poincaré characteristic of the classical Koszul homology, we use the notation $\chi(\Phi, L, \nu)$ for the Euler-Poincaré characteristic of $H_\bullet(\Phi, L, \nu)$, that is,*

$$\chi(\Phi, L, \nu) = \sum_i (-1)^i \ell_R(H_i(\Phi, L, \nu)).$$

But in the theorem above, we use $\chi(\Phi, L)$, without the index ν, because this result does not depend on it.

Example 3.1.5. *Let $R = k[x, y]_{(x,y)}$ be the localization of the polynomial ring over a field on the maximal ideal $m = (x, y)$. Consider the parameter matrix*

$$\Phi = \begin{bmatrix} x^i & 0 & -y^t \\ 0 & y^j & x^s \end{bmatrix}$$

with s, t, i, j, d, e positive integers. $N = \mathrm{im}\Phi$ is a finite colength submodule of $G = R^2$ such that $G/N \simeq I/a$, where $I = (x^s, y^t)$ and $a = (x^{s+i}, y^{t+j})$ are m-primary ideals of R. So, the Buchsbaum-Rim complex $\mathfrak{B}_\bullet(\Phi, 1)$ is given by

$$0 \longrightarrow R \xrightarrow{\Psi} R^3 \xrightarrow{\Phi} R^2 \longrightarrow 0.$$

Notice that the ideal $\mathrm{Fitt}_0 M = (x^{s+i}, y^{t+j}, x^i y^j)$ *has grade 2. By the Theorem 2.3.2, the complex* $\mathfrak{B}_\bullet(\Phi, 1)$ *is acyclic, and hence, the theorem above* 3.1.3 *implies*

$$\mathrm{br}(N) = \chi(\Phi, 1) = \ell_R(H_0(\Phi, 1)) = \ell_R(I/a) = (s+i)(t+j) - st = \mathrm{e}(a) - \mathrm{e}(I).$$

It is an interesting case studied by E. Jones in [J01], where we can relate the Buchsbaum-Rim multiplicity and Hilbert-Samuel multiplicity.

Our interest in this work is to calculate the Buchsbaum-Rim multiplicity using the Euler-Poincaré characteristic of the Buschsbaum-Eisenbud complexes, see Theorem 3.3.4. Notice that to apply the theorem above it is necessary to have a parameter matrix. To conclude this section, the next paragraphs will discuss about this problem.

In classical case, $g = 1$, the Hilbert-Samuel multiplicity of I on L can be calculated by using a reduction J of I, that is, $\mathrm{e}(J, L) = \mathrm{e}(I, L)$. Furthermore, since I is an ideal of definition of L and supposing that $k = R/m$ the residue field is infinity, then we can choose $J = (x_1, \ldots, x_t)$ being a minimal reduction of I, with $\boldsymbol{x}$ a parameter system of L, and thus, $\mathrm{e}(I, L) = \mathrm{e}(J, L) = \chi(\boldsymbol{x}, L)$, as we wanted, see [BH98, Section 4.6]. This reduction J of I is minimal and comes from the Noether normalization of the special fiber of I.

Let U be a submodule of $N \subset R^g$ with $g \geq 2$. So, $\mathcal{R}(U) \subset \mathcal{R}(N)$ is a ring extension in $\mathcal{S}(G)$. We say that U is a *reduction* of N if $\mathcal{R}(N)$ is integral over $\mathcal{R}(U)$ as rings. A minimal reduction of N is a reduction that is minimal with respect to inclusion. The *special fiber* of $\mathcal{R}(N)$ is the ring $\mathcal{F}(N) = k \otimes_R \mathcal{R}(N)$, and its Krull dimension is called the *analytic spread* of $\mathcal{R}(N)$ and is denoted by $\lambda(N)$. Now assume that the residue field k is infinite. For any reduction U of N, we have $\mu(U) \geq \lambda(N)$, where $\mu(U)$ denotes the minimal number of generators of U, and the equality holds if and only if U is minimal, [HS06, Corollary 16.4.7 and Proposition 8.3.7]. Since $\mathcal{F}(N)$ is a finitely generated standard graded algebra of dimension $\lambda = \lambda(N)$ over an infinite field k, then it admits a Noether normalization $k[y_1, \ldots, y_\lambda]$ generated by linear forms; lifting these linear forms to $x_1, \ldots, x_\lambda$ in $\mathcal{R}(N)_{[1]} = N$, we obtain a minimal reduction $U = (x_1, \ldots, x_\lambda)$ of N. If G/N has finite length, then $\mu(U) = \dim\mathcal{F}(N) = d + g - 1$ (and thus U is a parameter module) and $\mathrm{br}(U) = \mathrm{br}(N)$, see [SUV01, Theorem 5.1(c) and Corollary 5.5] and [Ree87]. We give an example using the main Theorem of this work, see 3.3.7.

3.2 Hilbert function over an Noetherian base ring

Let R be a Noetherian ring with finite dimension, L be a finitely generated R-module and $\Phi = (c_{ij})_{g\times f}$ be a matrix over R with $1 \leq g \leq f$ and $\operatorname{Coker}\Phi = M$. Consider the Koszul complex $K_\bullet(\gamma, S\otimes_R L)$ as in previous chapter, where $S = R[T_1,\ldots,T_g]$ is standard graded ring and $\gamma = \{\gamma_j = \sum_i c_{ij}T_i\}$ is a sequence of linear elements determined by Φ, and thus, its homologies are graded S-module. Notice that, $\operatorname{indeg}(H_j(\gamma, S\otimes_R L)) = j$ for all j, and

$$H_j(\gamma, S\otimes_R L)_\nu = H_j(\Phi, L, \nu)$$

for all $\nu \geq j+1$, where $H_j(\Phi, L, \nu)$ is an homology of a Buchsbaum-Eisenbud complex. If we assume that $\ell_R(M\otimes_R L) < \infty$, then by the Proposition 2.2.4 the R-modules $H_j(\gamma, S\otimes_R L)_\nu$ have finite length for all $\nu \geq j+1$, and hence, the graded S-module $H_j(\gamma, S\otimes_R L)$ have finite length components, maybe except by the j-th.

In the classical theory of Hilbert function the base ring R of the graded ring S must be an Artinian ring, but in our case, it is a general Noetherian ring. The goal of this section is to give a treatment for this case, that is, we can define the Hilbert Polynomial of a graded S-module $\mathcal{H}$ such that its components have finite length over the Noetherian base ring R eventually.

Fixing $S = R[T_1,\ldots,Tg]$ the polynomial ring in g indeterminates over a Noetherian ring R with the standard graduation and denote $X = \operatorname{Proj}(S) := \mathbb{P}^{g-1}_R$ the projective space over R of dimension $g-1$.

Definition 3.2.1. *Let $\mathcal{H}$ be a finitely generated graded S-module such that the cohomology modules $H^i(X, \tilde{\mathcal{H}}(\nu))$ are finite length R-modules for all $i \in \{0,\ldots,g-1\}$ and $\nu \in \mathbb{Z}$. We define*

$$h^i_{\mathcal{H}}(\nu) := \ell_R(H^i(X, \tilde{\mathcal{H}}(\nu)))$$

and

$$\rho_{\mathcal{H}}(\nu) := \sum_{i=0}^{g-1}(-1)^i h^i_{\mathcal{H}}(\nu).$$

Referring to Serre [BH98, Theorem 4.4.3], one may wonder if the function ρ defined in the Definition above is indeed the Hilbert polynomial.

Lemma 3.2.2. *Let $\mathcal{H}_1$,$\mathcal{H}_2$ and $\mathcal{H}_3$ be three finitely generated graded S-module such that all of the cohomology modules $H^i(X, \tilde{\mathcal{H}}_j(\nu))$ are finite length R-modules for all i,* *j and ν. If $\mathcal{H}_1$,$\mathcal{H}_2$ and $\mathcal{H}_3$ fit into a short exact sequence $0 \to \mathcal{H}_1 \to \mathcal{H}_2 \to \mathcal{H}_3 \to 0$ then for any integer ν,*

$$\rho_{\mathcal{H}_2}(\nu) = \rho_{\mathcal{H}_1}(\nu) + \rho_{\mathcal{H}_3}(\nu).$$

Using an usual long exact sequence techniques, the previous Lemma follows.

Proposition 3.2.3. *Let $\mathcal{H}$ be a finitely generated graded S-module such that the cohomology modules $H^i(X, \tilde{\mathcal{H}}(\nu))$ are finite length R-modules for all i and ν. Then the function $\rho_{\mathcal{H}} : \mathbb{N} \to \mathbb{Z}$, defined in Definition 3.2.1, is a polynomial function with eventual positive values.*

Proof. Although R is Noetherian, we will show how one can reduce the problem to Artinian local case.

Notice that there is a chain $0 \subseteq N_0 \subseteq \cdots \subseteq N_e = \mathcal{H}$ of graded submodules of $\mathcal{H}$ such that for each i, $N_{i+1}/N_i \simeq S/\mathfrak{p}_i(a_i)$ where $\mathfrak{p}_i$ is a homogeneous prime ideal of S. If one shows that the proposition holds for modules of the form $S/\mathfrak{p}$, with $\mathfrak{p}$ a homogeneous prime ideal, the result follows from Lemma 3.2.2.

Now, notice that $D_{\mathfrak{t}}(\mathcal{H}) = D_{\mathfrak{t}}(\mathcal{H}/\Gamma_{\mathfrak{t}}(\mathcal{H}))$ and $H^i_{\mathfrak{t}}(\mathcal{H}) = H^i_{\mathfrak{t}}(\mathcal{H}/\Gamma_{\mathfrak{t}}(\mathcal{H}))$ for $i \geq 2$. Consequently, $\rho_{\mathcal{H}} = \rho_{\mathcal{H}/\Gamma_{\mathfrak{t}}(\mathcal{H})}$. So that one may suppose that $\mathcal{H}$ is a $\mathfrak{t}$-torsion free S-module.

We first treat the case where $\mathfrak{p} \supseteq \mathfrak{t}$. In this case, $h^i_{\mathcal{H}}(\nu) = 0$ for all i and ν. Thus $\rho_{\mathcal{H}}$ is just the zero function. To see $h^i_{\mathcal{H}}(\nu) = 0$, we consider the ideal transform functor, $D_{\mathfrak{t}}(-)$, according to the notations in [BS13, Chapter 2]. With this setting, $H^i(X, \tilde{\mathcal{H}}(\nu)) = \mathcal{R}^i D_{\mathfrak{t}}(\mathcal{H})_\nu$.

Now, suppose that $\mathfrak{p} \not\supseteq \mathfrak{t}$. Since $\Gamma_{\mathfrak{t}}(\mathcal{H}) = 0$, $\mathcal{H}$ is a graded submodule of $D_{\mathfrak{t}}(\mathcal{H}) = \oplus_\nu H^0(X, \tilde{\mathcal{H}}(\nu))$. By our hypothesis, for each integer ν, $H^0(X, \tilde{\mathcal{H}}(\nu))$ is a finite length R-module. Hence $\mathcal{H}_\nu$ is a finite length R-module. Since $\mathcal{H}$ is a finitely generated S-module, its generators are concentrated in a finite number of graded components of $\mathcal{H}$, say $\mathcal{H}_{i_1}, \cdots, \mathcal{H}_{i_q}$. Any of $\mathcal{H}_{i_j}$ is an R-module of finite length, thus its support consists of a finite number of maximal ideals of R. Although $\mathcal{H}$ is not necessarily a finitely generated R-module, the R-support of $\mathcal{H}$ will be the union of these maximal ideals which is a finite set, say $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_c\}$.

Then $\mathcal{H}$ is annihilated by a power of $(\mathfrak{m}_1 \cdots \mathfrak{m}_c)$, say $(\mathfrak{m}_1 \cdots \mathfrak{m}_c)^k$. The change of base ring theorem for local cohomologies, [BS13, Theorem 2.2.24], shows that $h^i_{\mathcal{H}}(\nu) := \ell_R(H^i(X, \tilde{\mathcal{H}}(\nu))) = \ell_{R'}(H^i(X', \tilde{\mathcal{H}}(\nu)))$, where $R' = R/(\mathfrak{m}_1 \cdots \mathfrak{m}_c)^k$ and $X' = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$. Hence we may substitute R with R' which is an Artinian semi-local ring.

Considering the decomposition series for the finite length R'-module $H^i(X, \tilde{\mathcal{H}}(\nu))$, it is easy to see that $\ell_{R'}(H^i(X', \tilde{\mathcal{H}}(\nu))) = \sum_{j=1}^{c} \ell_{R'_{\mathfrak{m}_j}}(H^i(X', \tilde{\mathcal{H}}(\nu))_{\mathfrak{m}_j})$. Consequently, the proof of the assertion reduces to the case where R is an Artinian local ring.

For Artinian local ring R, the proof of this theorem is indeed a classical proof, see for example [BH98, Theorem 4.1.3 and Theorem 4.4.3]. We notice that since the polynomial ring S is assumed to be standard, the function $\rho_{\mathcal{H}}$ is indeed a polynomial, whereas it is a quasi-polynomial in the general case. $\square$

Remark 3.2.4. *Notice that the dimension of $\rho_{\mathcal{H}}$ is well defined, because it reduces to the Artinian case. Otherwise, we would talk about the relation between the dimension of $\widetilde{\mathcal{H}}$ as $\mathcal{O}_X$-module and the Krull dimension of $\mathcal{H}$ as a graded S-module, where S is a polynomial ring with Noetherian base ring. In [CRS20, Subsection 3.1], the authors proof to the case of Noetherian domain base ring.*

Let $\mathcal{H}$ be a finitely generated graded S-module such that $\ell_R(\mathcal{H}_\nu)$ is finite for all integer ν. In this case, an argument similar to that in the proof of the Proposition 3.2.3 shows that $\mathcal{H}$ is indeed a graded module over a ring with an Artinian base ring. So that, we may talk about Hilbert polynomial of $\mathcal{H}$ in the classical sense. We denote this function by $P_{\mathcal{H}}(\nu)$.

Lemma 3.2.5. *Let $\mathcal{H}$ be a finitely generated S-module with finite length graded components. Let $\mathcal{H}^{sat} = \mathcal{H}/\Gamma_{\mathbf{t}}(\mathcal{H})$. Then*

$$P_{\mathcal{H}}(\nu) = P_{\mathcal{H}^{sat}}(\nu)$$

for all ν.

Proof. Since $\mathcal{H}$ is a finitely generated S-module and each graded component of $\mathcal{H}$ is a finite length R-module, a similar argument to that in the proof of Proposition 3.2.3 shows that there exist maximal ideals $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_c\}$ and an integer k such that $\mathcal{H}$ is a $R/(\mathfrak{m}_1 \cdots \mathfrak{m}_c)^k$-module. Since, $\Gamma_{\mathbf{t}}(\mathcal{H})$ is a finitely generated S-module, it is annihilated by a power of $\mathbf{t}$. Thence it is annihilated by a product of maximal ideals $(\mathfrak{m}_i + \mathbf{t})$. Being a Noetherian S-module, the latter implies that it is an Artinian S-module. Therefore, the following descending chain of graded S-submodules of $\Gamma_{\mathbf{t}}(\mathcal{H})$ stops

$$\Gamma_{\mathbf{t}}(\mathcal{H})_{\geq 0} \supseteq \Gamma_{\mathbf{t}}(\mathcal{H})_{\geq 1} \supseteq \cdots .$$

The degree argument then shows that $\Gamma_{\mathbf{t}}(\mathcal{H})_\nu = 0$ for all $\nu >> 0$. Now, it follows from the exactness of the sequence

$$0 \to \Gamma_{\mathbf{t}}(\mathcal{H}) \to \mathcal{H} \to \mathcal{H}^{sat} \to 0$$

that $P_{\mathcal{H}}(\nu) = P_{\mathcal{H}^{sat}}(\nu)$ for all $\nu >> 0$. However, two polynomials are equal if they have infinitely many equal values $\square$

Proposition 3.2.6. *Let $\mathcal{H}$ be a finitely generated graded S-module such that the cohomology modules $H^i(X, \widetilde{\mathcal{H}}(\nu))$ are finite length R-modules for all i and ν. Then for all ν*

$$\rho_{\mathcal{H}}(\nu) = P_{\mathcal{H}^{sat}}(\nu)$$

where $\mathcal{H}^{sat} = \mathcal{H}/\Gamma_{\mathbf{t}}(\mathcal{H})$.

The proof follows from the Serre's vanishing theorems [BS13, Theorem 16.1.5(ii) and Corollary 16.1.6(iii)]. We notice that, for all ν, $\mathcal{H}_\nu^{sat}$ is of finite length, since it is a subset of $H^0(X, \tilde{\mathcal{H}}(\nu))$. For large enough ν, $\rho_{\mathcal{H}^{sat}}(\nu) = \ell(\mathcal{H}_\nu^{sat}) = P_{\mathcal{H}^{sat}}(\nu)$. The equality $\rho_{\mathcal{H}^{sat}}(\nu) = P_{\mathcal{H}^{sat}}(\nu)$ for all ν follows from the fact that $\rho_{\mathcal{H}^{sat}}(\nu)$ and $P_{\mathcal{H}^{sat}}(\nu)$ are both polynomials.

Discussion 3.2.7. *Going back to the discussion before Lemma 3.2.5, one may wish to find in the literature, a generalization of the theory of Hilbert function for finitely generated S-module $\mathcal{H}$ for which only eventual values of $\ell_R(\mathcal{H}_\nu)$ are finite. In the Scheme theoretic point of view of projective varieties, this fact is what researchers indeed deal with. However from the commutative algebra point of view, the issue is in the intervention of the saturation part; as we did in Proposition 3.2.6. Thenceforth for a finitely generated S-module $\mathcal{H}$ such that $\ell_R(\mathcal{H}_\nu)$ are finite for all $\nu \geq \nu_0$, we use the notation of Hilbert polynomial $P_{\mathcal{H}}(\nu) := P_{\mathcal{H}_{\geq \nu_0}}(\nu)$.*

3.3 A genus formula for the Buchsbaum-Rim multiplicity

Finally, we reach the main section of this thesis work. In the first chapter, we defined the family of the Buchsbaum-Eisenbud complexes $\mathfrak{B}_\bullet(\Phi, L)$ using the vertical spectral sequence coming from the double complex $E^{\bullet,\bullet} = K_\bullet(\gamma, S \otimes_R L) \otimes_S C_{\mathfrak{t}}^\bullet$. We used the convergence of spectral sequence to write the Euler characteristic $\chi(\Phi, L, \nu)$ in terms of the horizontal spectral sequence of $E^{\bullet,\bullet}$. Recall that the terms on second page of the horizontal spectral sequence of $E^{\bullet,\bullet}$ in degree ν are given by

$$H_{\mathfrak{t}}^q(H_p(\gamma, S \otimes_R L))_{[\nu]},$$

and using the results of the previous section, supposing $\mathrm{Coker}(\Phi) \otimes_R L$ of finite length, we will can write $\chi(\Phi, L, \nu)$ in terms of the Hilbert polynomials of the Koszul homologies.

The next Lemma is an important case where the conditions of Lemma 3.2.3 hold.

Lemma 3.3.1. *Let R be a Noetherian ring and suppose that $M \otimes_R L$ is a finite length R-module. For $p = 0, \cdots, f$, let $H_p = H_p(\gamma, S \otimes_R L)$ be the Koszul homology modules with sheafification $\widetilde{H_p}$. Then the R-modules*

$$H^q(X, \widetilde{H_p}(\nu))$$

are of finite length for all q and ν.

Proof. The terms on the second page of the horizontal spectral sequence of third quadrant double complex $E^{\bullet,\bullet} = K_\bullet(\gamma; S) \otimes_S C^\bullet_{\mathbf{t}} \otimes_R L$ in degree ν are $H^q_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu$, for $0 \leq p \leq f$ and $0 \leq q \leq g$. (We refer to Section 1.1, for the required properties and notations related to this spectral sequence.)

First, notice that $H^q_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu$ is of finite length for $q \geq 1$. In fact, if $\mathfrak{P} \notin \mathrm{Supp}_R(M \otimes_R L)$, then either $M_{\mathfrak{P}} = 0$ or $L_{\mathfrak{P}} = 0$. The latter, clearly, implies that $H^q_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_{\mathfrak{P}} = 0$. In the former case, the map

$$\Phi_{\mathfrak{P}} : R^f_{\mathfrak{P}} \to R^g_{\mathfrak{P}}$$

is surjective. Hence the ideal generated by γ is the same as the ideal generated by $\mathbf{t}$. This fact implies that $H_p(\gamma, S \otimes_R L)_{\mathfrak{P}}$ is $\mathbf{t}_{\mathfrak{P}}$-torsion for all $\mathfrak{P}$, and thus, $H^q_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_{\mathfrak{P}} = 0$ for all $q \geq 1$.

Therefore in any degree ν, $H^q_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu$ is a finitely generated R-module whose support is contained in the support of $M \otimes_R L$. The latter consists of maximal ideals; so that

$$H^q_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu \text{ is of finite length for any } q \geq 1. \tag{3.3.1}$$

Notice that $H^q(X, \widetilde{H_p(\gamma, S \otimes_R L)}(\nu)) = H^{q+1}_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu$ for $q \geq 1$.

It remains to show that $D_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu$ is a finite length R-module for all ν. We study three cases.

Case 1. If $\nu \geq f - g + 1$. In this case, $H_p(\gamma, S \otimes_R L)_\nu = H_p(\mathfrak{B}_\bullet(\Phi, L, \nu))$ for all p, according to the structure of $\mathfrak{B}_\bullet(\Phi, L, \nu)$ which is explained in (2.1.3). Proposition 2.2.4 then shows that these homology modules have finite length.

Case 2. If $\nu \leq -1$. $H_p(\gamma, S \otimes_R L)_\nu$ is a subquotient of $\Lambda^p(S^f(-1) \otimes_R L)_\nu$ for all p. The latter is zero be degree discussion.

Case 3. If $0 \leq \nu \leq f - g$, we consider three other cases

Case 3.1. If $p > \nu$, then $H_p(\gamma, S \otimes_R L)_\nu$ is a subquotient of $\Lambda^p(S^f(-1) \otimes_R L)_\nu$ for all p. The latter is zero be degree discussion.

Case 3.2. If $p < \nu$, then $H_p(\gamma, S \otimes_R L)_\nu = H_p(\mathfrak{B}_\bullet(\Phi, L, \nu))$, according to (2.1.3), which is of finite length by Proposition 2.2.4.

In all of the above cases, $D_{\mathbf{t}}(H_\nu(\gamma, S \otimes_R L))_\nu$ is a finite length R-module by regarding the exact sequence in conjunction with (3.3.1)

$$0 \to H^0_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu \to H_p(\gamma, S \otimes_R L)_\nu \to D_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu \to H^1_{\mathbf{t}}(H_p(\gamma, S \otimes_R L))_\nu \to 0. \tag{3.3.2}$$

Case 3.3. $p = \nu$. Based on the structure of $\mathfrak{B}_\bullet(\Phi, L, \nu)$, $H_\nu(\mathfrak{B}_\bullet(\Phi, L, \nu)) = \mathrm{Ker}(\partial_\nu)_\nu / \mathrm{Im}(\tau_\nu)$, where $\mathrm{Ker}(\partial_\nu)_\nu = H_\nu(\gamma, S \otimes_R L)_\nu$. As well, $\mathrm{Im}(\tau_\nu) = {}^\infty E^{-\nu,0}_{hor}$ by (2.1.2).

Therefore, we have

$$\frac{H_\nu(\gamma, S \otimes_R L)_\nu}{{}^\infty E^{-\nu,0}_{hor}} = H_\nu(\mathfrak{B}_\bullet(\Phi, L, \nu)) \tag{3.3.3}$$

The latter is of finite length, according to Proposition 2.2.4.

Finally, the finiteness of $D_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu$ follows from the exactness of the following natural sequence

$$0 \to \frac{H^0_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu}{{}^\infty E^{-\nu,0}_{hor}} \to \frac{H_\nu(\gamma, S\otimes_R L)_\nu}{{}^\infty E^{-\nu,0}_{hor}} \to D_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu \to H^1_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu \to 0. \tag{3.3.4}$$

□

We are now ready to present and prove the following main property of $\chi(\Phi, L)$.

Theorem 3.3.2. *Let R be a Noetherian ring and suppose that* $\operatorname{Coker}(\Phi)\otimes_R L$ *is a finite length R-module. Let $\rho_j(\nu) := \rho_{H_j(\gamma,S\otimes_R L)}(\nu)$ be the ρ function defined in Definition 3.2.1 for j-th Koszul homology module $H_j(\gamma, S\otimes_R L)$. Then, for all integer ν*

$$\chi(\Phi, L, \nu) = \sum_{j=0}^{f}(-1)^j\rho_j(\nu).$$

Proof. The proof is a deep analysis of the horizontal spectral sequence of $(E^{\bullet,\bullet})_\nu = (K_\bullet(\gamma, S\otimes_R L)\otimes_S C^\bullet_{\mathbf{t}})_\nu$.

Due to Lemma 2.2.4, the modules $({}^2E^{-j,-q}_{hor})_\nu = H^q_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu$ have finite length for $q \geq 2$ and any j. Furthermore, the epimorphism

$$D_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu \to H^1_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu \to 0,$$

implies that $H^1_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu$ has finite length for every j.

We need to look into $H^0_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu$.

For $\nu \geq f-g+1$, $H_j(\gamma, S\otimes_R L)_\nu = H_j(\mathfrak{B}_\bullet(\Phi, L, \nu))$ is of finite length by the same reason as Case 1 in the proof of Lemma 2.2.4. So that $H^0_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu \subseteq H_j(\gamma, S\otimes_R L)_\nu$ is of finite length. For $\nu \leq -1$, $H_j(\gamma, S\otimes_R L)_\nu$ is a subquotient of $\Lambda^j(S^f(-1)\otimes_R L)_\nu$ for all j. The latter is zero by degree discussion, so that $H^0_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu = 0$. If $0 \leq \nu \leq f-g$, since

$$H_j(\gamma, S\otimes_R L)_\nu = \begin{cases} H_j(\mathfrak{B}_\bullet(\Phi, L, \nu)) & , \quad j<\nu; \\ 0 & , \quad j>\nu \end{cases}, \tag{3.3.5}$$

and $H^0_{\mathbf{t}}(H_j(\gamma, S\otimes_R L)) \subseteq H_j(\gamma, S\otimes_R L)$ Proposition 2.2.4 yields that $H^0_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))$ has finite length for $j \neq \nu$.

When $0 \leq \nu \leq f-g$ the R-module $H^0_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu$ is not necessarily of finite length. However, as we see in the proof of the Lemma 2.2.4(3.3.3)

$$\frac{H_\nu(\gamma, S\otimes_R L)_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu} = H_\nu(\mathfrak{B}_\bullet(\Phi, L, \nu)) \tag{3.3.6}$$

is of finite length.

Unless otherwise stated, suppose that $0 \leq \nu \leq f-g$. So far we see that every terms in the spectral sequence $(E^{\bullet,\bullet})_\nu$ except $(E^{-\nu,0})_\nu$, is of finite length. In order to relate the lengths of the homologies of $\mathfrak{B}_\bullet(\Phi, L, \nu)$ to the length of the terms of $({}^2E^{\bullet,\bullet}_{hor})_\nu$, we define a new spectral sequence ${}^rG^{\bullet,\bullet}$ which is equal to $({}^rE^{\bullet,\bullet}_{hor})_\nu$ for $(-j,-q) \neq (-\nu, 0)$ with the same differentials, and for $(-j,-q) = (-\nu, 0)$

$$ {}^rG^{-\nu,0} := \frac{({}^rE^{-\nu,0}_{hor})_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu} $$

with the induced differentials. Hence the induced differentials are just the same maps as they were in $({}^rE^{\bullet,\bullet})_\nu$ however some parts of their kernels are already killed.

The advantage of ${}^rG^{\bullet,\bullet}$ is that its terms on the second page have finite length. We only need to notice that

$$ {}^2G^{-\nu,0} = \frac{H^0_{\mathfrak{t}}(H_\nu(\gamma, S\otimes_R L))_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu} \subseteq \frac{H_\nu(\gamma, S\otimes_R L)_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu} = H_\nu(\mathfrak{B}_\bullet(\Phi, L, \nu)) $$

which is of finite length by Proposition 2.2.4. Since ${}^\infty G^{-\nu,0} = 0$, the spectral sequence ${}^rG^{\bullet,\bullet}$ converges to $\mathfrak{H}_\bullet$, where

$$ \mathfrak{H}_j = \begin{cases} H_{j+g}(\gamma, H^g_{\mathfrak{t}}(S\otimes_R L))_\nu & , \ \nu+1 \leq j \leq f-g \\ F_1 & , \ j = \nu \\ 0 & , \ otherwise. \end{cases} $$

Here, F_1 is the module defined in equation (2.1.1) which is given by the convergence of ${}^rE^{\bullet,\bullet}_{hor}$. Notice that

$$ \mathfrak{H}_j = H_{j+1}(\mathfrak{B}_\bullet(\Phi, L, \nu)) \ \text{ for } \ \nu+1 \leq j \leq f-g \ \text{ and } \ \mathfrak{H}_\nu = F_1 = H_{\nu+1}(\mathfrak{B}_\bullet(\Phi, L, \nu)) \tag{3.3.7} $$

in the same way as the proof of Proposition 2.2.4. Therefore Proposition 2.2.4 implies that all terms of $\mathfrak{H}_\bullet$ have finite length.

Applying Proposition A.2.2, it follows that

$$ \sum_{j=\nu}^{f-g} (-1)^j \ell(\mathfrak{H}_j) = \sum_j \sum_q (-1)^{j+q} \ell({}^2G^{-j,-q}_{hor}). $$

According to equation (3.3.7), we change the indices on the left side,

$$ -\sum_{j=\nu+1}^{f-g+1} (-1)^j \ell(H_j(\mathfrak{B}_\bullet(\Phi, L, \nu))) = $$

$$ \sum_j \sum_{q\geq 2} (-1)^{j+q} \ell(H^q_{\mathfrak{t}}(H_j(\gamma, S\otimes_R L))_\nu) + $$

$$ \sum_{j\neq\nu} (-1)^j \{\ell(H^0_{\mathfrak{t}}(H_j(\gamma, S\otimes_R L))_\nu) - \ell(H^1_{\mathfrak{t}}(H_j(\gamma, S\otimes_R L))_\nu)\} + \tag{3.3.8} $$

$$(-1)^{\nu}\left\{l\left(\frac{H^0_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu}\right) - \ell(H^1_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu)\right\}. \tag{3.3.9}$$

For $j \neq \nu$, we consider the following exact sequence where all terms have finite length

$$0 \to H^0_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu \to H_j(\gamma, S\otimes_R L)_\nu \to D_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu \to H^1_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu \to 0 \tag{3.3.10}$$

We have

$$\ell(H^0_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu) - \ell(H^1_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu) = \ell(H_j(\gamma, S\otimes_R L)_\nu) - \ell(D_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu). \tag{3.3.11}$$

For $j = \nu$, we consider the sequence

$$0 \to \frac{H^0_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu} \to \frac{H_\nu(\gamma, S\otimes_R L)_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu} \to D_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu \to H^1_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu \to 0, \tag{3.3.12}$$

which is exact by a straightforward verification. Thus

$$\ell\left(\frac{H^0_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu}\right) - \ell(H^1_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu) = \ell\left(\frac{H_\nu(\gamma, S\otimes_R L)_\nu}{({}^\infty E^{-\nu,0}_{hor})_\nu}\right) - \ell(D_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu). \tag{3.3.13}$$

Now, plugging (3.3.11) and (3.3.13) in (3.3.8) and (3.3.9), respectively, we have

$$\begin{gathered}
-\sum_{j=\nu+1}^{f-g+1}(-1)^j\ell(H_j(\mathfrak{B}_\bullet(\Phi, L, \nu))) = \\
\sum_j\sum_{q\geq 2}(-1)^{j+q}\ell(H^q_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu) + \\
\sum_{j\neq\nu}(-1)^j\{\ell(H_j(\gamma, S\otimes_R L)_\nu) - \ell(D_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu)\} + \\
(-1)^{\nu}\left\{l\left(\frac{H_\nu(\gamma, S\otimes_R L)_\nu}{{}^\infty E^{-\nu,0}_{hor}}\right) - \ell(D_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu)\right\}.
\end{gathered}$$

Plugging (3.3.5) and (3.3.6) in the last two lines, we have

$$\begin{gathered}
-\sum_{j=\nu+1}^{f-g+1}(-1)^j\ell(H_j(\mathfrak{B}_\bullet(\Phi, L, \nu))) = \\
\sum_j\sum_{q\geq 2}(-1)^{j+q}\ell(H^q_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu) + \\
\sum_{j\neq\nu}(-1)^j(\ell(H_j(\mathfrak{B}_\bullet(\Phi, L, \nu))) - \ell(D_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu)) + \\
(-1)^{\nu}(\ell(H_\nu(\mathfrak{B}_\bullet(\Phi, L, \nu))) - \ell(D_{\mathbf{t}}(H_\nu(\gamma, S\otimes_R L))_\nu)).
\end{gathered}$$

Finally, writing $H^q(X, \widetilde{H_j}(\gamma, S\otimes_R L)(\nu)) = \mathcal{R}^q D_{\mathbf{t}}(H_j(\gamma, S\otimes_R L))_\nu$, we obtain the equality

$$\chi(\Phi, L, \nu) = \sum_j (-1)^j \rho_j(\nu).$$

For $\nu \geq f - g + 1$ or $\nu \leq -1$, all the terms of the spectral sequence $({}^rE_{hor}^{\bullet\bullet})_\nu$ are all of finite length. Thus, without introducing the spectral sequence ${}^rG^{\bullet,\bullet}$, the computation of $\sum_{j=\nu}^{f-g}(-1)^j \ell(\mathfrak{H}_j)$ shows the asserted equality.

□

Remark 3.3.3. *The proof of Theorem 3.3.2 would be essentially the computational parts in (3.3.8) and (3.3.9), if $({}^2E_{hor}^{-j,-q})_\nu$ were of finite length for all j and q. By the way, this desire is true except for $j = \nu, q = 0$. So that, we had to verify the details thoroughly.*

The next Theorem shows how the Buchsbaum-Rim multiplicity is expressed as the alternating sum of Hilbert polynomials.

Theorem 3.3.4. *Let R be Noetherian,* $\mathrm{Coker}(\Phi) \otimes_R L$ *a finite length R-module, $H_j := H_j(\gamma, S \otimes_R L)$ the j-th Koszul homology module, $\Gamma_{\mathbf{t}}(H_j)$ the $\mathbf{t}$-torsion part of H_j, and $H_j^{sat} := H_j/\Gamma_{\mathbf{t}}(H_j)$. Then for any integer ν,*

$$\chi(\Phi, L, \nu) = P_{H_0^{sat}}(\nu) - P_{H_1^{sat}}(\nu) + \cdots + (-1)^f P_{H_f^{sat}}(\nu).$$

In particular, if R is local, then

$$\sum_{i=0}^{f}(-1)^i P_{H_i^{sat}}(\nu) = \begin{cases} \mathrm{br}(\Phi, L) & , \quad \textit{if } \Phi \textit{ is a parameter matrix ;} \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

Proof. The proof follows by combining Theorem 3.3.2, Lemma 3.2.6 and the Theorem 3.1.3. □

Remark 3.3.5. *Notice that for large ν the Hilbert polynomial is equal to Hilbert function, and thus, the terms in the alternating sum of these polynomials above is exactly the terms in $\chi(\Phi, L)$.*

To show the importance of Theorem 3.3.4, we mention how this theorem generalizes the Serre's celebrated theorem about the Hilbert-Samuel multiplicity [Ser65], e.g. [BH98, 4.7.6].

Corollary 3.3.6. (Serre) *Let R be a Noetherian local ring, $I = (c_1, \ldots, c_f)$ an ideal of definition of a finitely generated R-module L and $H_j := H_j(\boldsymbol{c}; L)$ the j-th homology of the Koszul complex of $\boldsymbol{c} = c_1, \ldots, c_f$ with coefficients in L. Then*

$$\sum_{i=0}^{f}(-1)^i \chi(\boldsymbol{c}, L) = \begin{cases} \mathrm{e}(\boldsymbol{c}, L) & , \quad \textit{if } \boldsymbol{c} \textit{ is a parameter system ;} \\ 0 & , \quad \textit{otherwise.} \end{cases} .$$

Proof. In Theorem 3.3.4 we set $g = 1$, $\Phi = (c_1, \ldots, c_f)$ and $\mathrm{Coker}(Phi) = R/I$. For any ν, the complex $\mathfrak{B}_\bullet(\Phi, L, \nu)$ is isomorphic to the Koszul complex $K_\bullet(\boldsymbol{c}; L)$. $H_j(\gamma, S)_\nu \cong H_j(\boldsymbol{c}; L)$ and $\dim_X(\mathrm{Supp}(H_j(\gamma, S)) = 0$ for all j. Hence $P_{H_j^{sat}}(\nu) = \ell(H_j)$ for all j and ν, that is, a constant polynomial. We, as well, notice that $\mathrm{br}(M) = e(I, R)$ in this case [BR64]. $\square$

Example 3.3.7. *Let $R = \mathbb{Q}[x, y]$ be the polynomial ring over the field of rational numbers and let*

$$\Phi = \begin{pmatrix} x & 0 & -y \\ 0 & y2 & x^3 \end{pmatrix}$$

be a parameter matrix. Using Macaulay2, *we calculate $P_{H_i^{sat}}(\nu)$ by the following way:*

(i) Define the polynomial ring $S = R[T_1, T_2]$ with standard graduation, i.e.,

$$\mathrm{S = QQ[x, y, T_1, T_2, Degree => \{0, 0, 1, 1\}];}$$

(ii) After calculate the Koszul homologies of γ, we calculate the saturation, for example

$$\mathrm{H_0^{sat} = (HH\_0\ \ K)/saturate(0\_(HH\_0\ \ K), ideal(T_1, T_2))}$$

where K is the Koszul complex of γ;

(iii) So, we can take the resolution of H_i^{sat}, look to the first differential and calculate, by our hands, the matrix which defines H_0^{sat} in degree ν, with the help of betti table. In our example for H_0^{sat} in degree 0*, we have the matrix*

$$\begin{pmatrix} x & 0 & 0 & y^2 & 0 & 0 & 0 & -y & 0 & 0 & x^3y & 0 & 0 & 0 & x^4 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & y^2 & 0 & 0 & x^3 & -y & 0 & 0 & x^3y & 0 & 0 & 0 & x^4 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & y^2 & 0 & 0 & x^3 & -y & 0 & 0 & x^3y & 0 & 0 & 0 & x^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 & x^3 & 0 & 0 & 0 & x^3y & 0 & 0 & 0 & x^4 \end{pmatrix};$$

(iv) Finally, we compute the degree *of the cokernel of this matrix.*

So, we obtain

$$P_{H_0^{sat}} - P_{H_1^{sat}} + P_{H_2^{sat}} - P_{H_3^{sat}} = 10 - 1 + 0 - 0 = 9 = \mathrm{br}(\Phi).$$

This example comes from the E. Jones work [J01], when $U = \mathrm{im}\Phi$ is a submodule of the finite colength submodule $N = (xT_1, yrT_2, x^2yT_2, x^3T_2 - yT_1) \subset R^2$ such that $R^2/N \simeq I/a$ with $I = (x^3, y)$ and $a = (x^4, y^3, x^2y^2)$ being m-primary ideals of R. In this case [J01, Theorem 5], U is a minimal reduction of N (see the discussion in the end of Section 2.1) and

$$\mathrm{br}(U) = \mathrm{br}(N) = \mathrm{e}(a) - \mathrm{e}(I).$$

We define the Euler characteristic [1] of a coherent sheaf $\mathcal{F}$ on X relative to the Noetherian affine scheme $Y = \mathrm{Spec}(R)$ to be the following integer (in the case it is finite)

$$\chi(X, \mathcal{F}) = \sum_{j=0} (-1)^j \ell_R(H^j(X, \mathcal{F})). \tag{3.3.14}$$

In spacial cases $\chi(X, \mathcal{F})$ relates the degree of $\mathcal{F}$ with the genus of X. Our last result is a (Arithmetic) genus explanation of the Buchsbaum-Rim multiplicity, [Har77, III ex 5.3].

Corollary 3.3.8. *Let R be a Noetheiran local ring,* $\mathrm{Coker}(\Phi)$ *be a finite length R-module and $H_j := H_j(\gamma, S)$ be the j-th Koszul homology module with sheafification $\widetilde{H_j}$. Then*

$$\sum_{j=0}^{f} (-1)^j \chi(X, \widetilde{H_j}) = \begin{cases} \mathrm{br}(\Phi) & , \quad \textit{if } \Phi \textit{ is a parameter matrix ;} \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

Proof. In Theorem 3.3.2, put $\nu = 0$ and use the definition (3.3.14). □

3.4 Comments and questions

With the same notations of Theorem 3.3.4, for $j \geq 0$ and any integer ν, we can define the generalized partial Euler-Poincaré characteristic

$$\chi_j = \sum_{i \geq j} (-1)^{i-j} P_{H_i^{sat}}(\nu) = P_{H_j^{sat}}(\nu) - P_{H_{j+1}^{sat}}(\nu) + \cdots + (f-j) P_{H_f^{sat}}(\nu)$$

of L with respect to Φ. If R is local and Φ is a parameter matrix, then the $\chi_0 = \mathrm{br}(\Phi, L) > 0$. Hence, we have the first question:

Question 3.4.1. *Is the partial Euler-Poincaré characteristic χ_j non-negative, for any $j \geq 0$ and any integer ν?*

In [HH11], Hayasaka and Hyry show that the partial Euler-Poincaré characteristic of the Buchsbaum-Eisenbud complexes is non-negative for all ν. Therefore, the answer to the question above is Yes for large ν, see Remark 3.3.5, and so, this question is for initial values of ν.

The Koszul-Čech construction of the Buchsbaum-Eisenbud complexes arises in [BHa19] with the purpose to study the disguised residual intersection, which is the

[1] This definition is the same as [Har77, III ex 5.1] in which R is a field.

zeroth homology of the zeroth residual approximation complex, see [HaN16]. This complex was firstly obtained in [Has12] and the construction is the same of the Koszul-Čech construction of the Buchsbaum-Eisenbud complexes if we change the Koszul complex $K_\bullet(\gamma, S \otimes_R L)$ by the subcomplex $D^L_\bullet = \mathrm{Tot}(K_\bullet \otimes_S \mathcal{Z}^L_\bullet)$, see [BHa19, Section 4], where the $\mathcal{Z}^L_\bullet$ is an approximation complex which uses the Koszul cycles. The terms of the residual approximation complexes are not free R-modules as in the complexes of this thesis work.

Question 3.4.2. *Do the residual approximation complexes have similar properties as the Buchsbaum-Eisenbud complexes? For example:*

(i) Duality 2.2.2, since that there is a duality on Koszul cycles, see [CNT19, Proposition 2.2];

(ii) Support of the homologies 2.2.4;

(iii) Acyclicity and depth 2.3.1 and 2.3.2;

(iv) The characteristic formula 3.3.2, and its algebraic or geometric meaning.

Appendix A

Some results

A.1 A natural duality

We used the next result to proof the Proposition 2.6, in the first chapter. This is a duality result from the Jouanolou's works [Jou09], which is in the second section.

Let R be a commutative ring with identity and let $S = R[T_1, \dots, T_g]$ be the polynomial ring in g indeterminates over the ring R with $\deg(T_i) = \alpha_i > 0$ for all $i \in \{1, \dots, g\}$. Let $\gamma_1, \dots, \gamma_f$ be homogeneous polynomials in S with degrees $\beta_1, \dots \beta_f$, respectively; that is

$$\gamma_j = \sum_{\substack{l_1,\dots,l_g \geq 0 \\ \sum_{i=1}^{g} l_i \alpha_i = \beta_j}} c_{j,l} T_1^{l_1} \cdots T_g^{l_g} \in S_{[\beta_j]}.$$

Consider $K_\bullet(\boldsymbol{\gamma}, S)$ the Koszul complex of the sequence $\boldsymbol{\gamma} = \gamma_1, \dots, \gamma_f$ over S, and $C_{\mathfrak{t}}^\bullet(S)$ the Cech complex of the sequence $\mathfrak{t} = T_1, \dots, T_g$ over S.

Proposition A.1.1. *There is an isomorphism*

$$H_f(\boldsymbol{\gamma}, H_{\mathfrak{t}}^g(S)) \left(-\sum_{j=1}^{f} \beta_j \right)_{[\nu]} \cong \operatorname{Hom}_R((S/\gamma)_{[\delta-\nu]}, R)$$

for all $\nu \in \mathbb{Z}$, *where* $\delta = \sum_{j=1}^{f} \beta_j - \sum_{i=1}^{g} \alpha_i$.

Proof. For all $\nu \in \mathbb{Z}$, we have a canonical perfect pairing between R-modules

$$S_{\left[-\nu-\sum_{i=1}^{g} \alpha_i\right]} \otimes_R H_{\mathfrak{t}}^g(S)_{[\nu]} \to H_{\mathfrak{t}}^g(S)_{\left[-\sum_{i=1}^{g} \alpha_i\right]} \cong R,$$

which is induced by multiplication, and this pairing yields a natural isomorphism $H_{\mathfrak{t}}^g(S)_{[\nu]} \xrightarrow{\cong} \operatorname{Hom}_R(S_{\left[-\nu-\sum_{i=1}^{g} \alpha_i\right]}, R)$, where it maps $a \in H_{\mathfrak{t}}^g(S)_{[\nu]}$ to an homomorphism

$\psi_a : S_{\left[-\nu-\sum_{i=1}^{g}\alpha_i\right]} \to R$ which is given by $\psi_a(b) = ab$, for all $b \in S_{\left[-\nu-\sum_{i=1}^{g}\alpha_i\right]}$. Thus, if we consider the S-structure of $R = S/\mathfrak{t}$, we have the isomorphism

$$\begin{aligned} H^g_{\mathfrak{t}}(S) &= \oplus_\nu H^g_{\mathfrak{t}}(S)_{[\nu]} \\ &\cong \oplus_\nu \mathrm{Hom}_R(S_{\left[-\nu-\sum_{i=1}^{g}\alpha_i\right]}, R) \\ &\cong \oplus_\nu \mathrm{Hom}_R(S\left(-\nu-\textstyle\sum_{i=1}^{g}\alpha_i\right), R) \\ &= \mathrm{Hom}_S(S\left(-\textstyle\sum_{i=1}^{g}\alpha_i\right), R) \end{aligned}$$

and therefore

$$\begin{aligned} H_f(\boldsymbol{\gamma}, H^g_{\mathfrak{t}}(S))\left(-\textstyle\sum_{j=1}^{f}\beta_j\right)_{[\nu]} &= \mathrm{Hom}_S(S/\gamma, H^g_{\mathfrak{t}}(S))\left(-\textstyle\sum_{j=1}^{f}\beta_j\right)_{[\nu]} \\ &\cong \mathrm{Hom}_S(S/\gamma, \mathrm{Hom}_S(S\left(-\textstyle\sum_{i=1}^{g}\alpha_i\right), R))\left(-\textstyle\sum_{j=1}^{f}\beta_j\right)_{[\nu]} \\ &= \mathrm{Hom}_S(S/\gamma, \mathrm{Hom}_S(S, R))(-\delta)_{[\nu]} \\ &\cong \mathrm{Hom}_S((S/\gamma)(\delta), R)_{[\nu]} \\ &\cong \mathrm{Hom}_R((S/\gamma)_{[\delta-\nu]}, R). \end{aligned}$$

□

A.2 Spectral sequence in finite length

Lemma A.2.1. *Let $C_\bullet : 0 \to C_k \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$ be a complex of finite length modules. Then*

$$\sum_{i=0}^{k}(-1)^i\ell(C_i) = \sum_{i=0}^{k}(-1)^i\ell(H_i(C_\bullet)).$$

Proof. We will use induction on k, the length of the complex $C_\bullet$. If $k = 0$, then the complex is $0 \to C_o \to 0$ and $C_0 = H_0(C_\bullet)$. Suppose $k > 0$. We can decompose the complex $C_\bullet$ in two pieces

$$0 \to C_k \to \cdots \to C_2 \xrightarrow{d_2} \mathrm{ker} d_1 \to 0$$

and

$$0 \to \mathrm{im} d_2 \to C_1 \to C_0 \to 0$$

By the induction hypothesis, we have

$$\ell(\mathrm{ker} d_1) - \sum_{i=2}^{k}(-1)^i\ell(C_i) = -\sum_{i=1}^{k}(-1)^i\ell(H_i(C_\bullet))$$

and

$$\ell(C_0) - \ell(C_1) + \ell(\mathrm{im} d_2) = \ell(H_0(C_\bullet)) - \ell(H_1(C_\bullet))$$

From the canonical exact sequence $0 \to \mathrm{im} d_2 \to \mathrm{ker} d_1 \to H_1(C_\bullet) \to 0$, we have $\ell(H_1(C_\bullet)) = \ell(\mathrm{ker} d_1) - \ell(\mathrm{im} d_2)$, and subtracting the two above

$$\ell(C_0) - \ell(C_1) + \ell(\mathrm{im} d_2) - \left(\ell(\mathrm{ker} d_1) - \sum_{i=2}^{k}(-1)^i\ell(C_i)\right) =$$

$$\ell(H_0(C_\bullet)) - \ell(H_1(C_\bullet)) - \left(- \sum_{i=1}^{k} (-1)^i \ell(H_i(C_\bullet)) \right)$$

and

$$\sum_{i=0}^{k} (-1)^i \ell(C_i) = \sum_{i=0}^{k} (-1)^i \ell(H_i(C_\bullet))$$

□

Proposition A.2.2. *Let ${}^rE \Rightarrow_r H$ be a convergent spectral sequence. Suppose that for some r, ${}^rE^{pq}$ is finite length for all p, q. Then for all $s \geq r$*

$$\sum_n (-1)^n \ell(H_n) = \sum_n (-1)^n \left(\sum_{p+q=n} \ell({}^sE^{pq}) \right)$$

Proof. For a given point (p, q) fixed, there is a complex $\cdots \to {}^sE^{pq} \overset{{}^sd^{pq}}{\to} {}^sE^{p+s,q-(s-1)} \to \cdots$ in the s-th page of the spectral ${}^rE$ passing over this point, where $s \geq r$. The terms on s-th page are sub-quocients of terms on r-th page, so they have finite length. Thus, by the lemma 3.1

$$\begin{aligned} \textstyle\sum_i (-1)^{p+is+q-i(s-1)} \ell({}^sE^{p+is,q-i(s-1)}) &= (-1)^{p+q} \textstyle\sum_i (-1)^i \ell({}^sE^{p+is,q-i(s-1)}) \\ &= (-1)^{p+q} \textstyle\sum_i (-1)^i \ell({}^{s+1}E^{p+is,q-i(s-1)}) \\ &= \textstyle\sum_i (-1)^{p+is+q-i(s-1)} \ell({}^{s+1}E^{p+is,q-i(s-1)}). \end{aligned}$$

This shows that, for all $s \geq r$

$$\sum_{p,q} (-1)^{p+q} \ell({}^sE^{pq}) = \sum_{p,q} (-1)^{p+q} \ell({}^{s+1}E^{pq}),$$

i.e., the alternating sum is over all the page. In particular,

$$\sum_{p,q} (-1)^{p+q} \ell({}^sE^{pq}) = \sum_{p,q} (-1)^{p+q} \ell({}^\infty E^{pq}).$$

The convergence of the spectral sequence gives the sum $\ell(H_n) = \sum_{p+q=n}^{\infty} E^{pq}$, and so

$$\begin{aligned} \textstyle\sum_n (-1)^n \ell(H_n) &= \textstyle\sum_n (-1)^n \left(\sum_{p+q=n} \ell({}^\infty E^{pq}) \right) \\ &= \textstyle\sum_{p,q} (-1)^{p+q} \ell({}^\infty E^{pq}) \\ &= \textstyle\sum_{p,q} (-1)^{p+q} \ell({}^sE^{pq}) \\ &= \textstyle\sum_n (-1)^n \left(\sum_{p+q=n} \ell({}^sE^{pq}) \right). \end{aligned}$$

□

Bibliography

[BFHN21] V. Bouça, T. Fiel, S. H. Hassanzadeh and J.Naeliton, *A genus explanation of Buchsbaum-Rim complex*, arXiv: 2104.09698v2 [math.AC] 21 Apr 2020. 3, 4

[BHa19] V. Bouça, S. H. Hassanzadeh *Residual intersections are Koszul-Fitting ideals*, Compositio Math. 155 (2019), 2150-2179. 3, 20, 25, 26, 27, 49, 50

[BE73] D. A. Buchsbaum and D. Eisenbud, *Remarks on ideals and resolutions*, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 193-204. Academic Press, London, 1973. 2

[BR64] D. A. Buchsbaum and D. S. Rim, *A generalized Koszul complex. II. Depth and multiplicity*, Trans. Amer. Math. Soc. 111 (1964), 19724. 1, 2, 3, 25, 31, 35, 36, 37, 48

[BR65] D. A. Buchsbaum and D. S. Rim, *A generalized Koszul complex. III. A Remark on Generic Acyclicity*, Proc. Amer. Math. Soc. 16 (1965), 55558. 34

[BUV01] J. Brenann, B. Ulrich, V. Vasconcelos *The Buchsbaum–Rim Polynomial of a Module*, Journal of Algebra 241, 379–392 (2001). 1

[BS13] M. Brodmann, R. Sharp, *Local Cohomology An Algebraic Introduction with Geometric Applications*, revised version, Cambridge University Press, Cambridge, 2013. 9, 10, 11, 12, 23, 40, 42

[BH98] W. Bruns, J. Herzog, *Cohen–Macaulay Rings*, revised version, Cambridge University Press, Cambridge, 1998. 3, 4, 5, 6, 7, 8, 10, 26, 30, 31, 32, 33, 37, 38, 39, 40, 47

[CRS20] Y. Cid-Ruiz, A. Simis, *Degree of rational maps and specialization*, International Mathematics Reaserch notices, Vol. 00, N.0 (2020), pp. 1-52 41

[CLU08] C.-Y J. Chan, J.-C Liu, B. Ulrich, *Buchsbaum-Rim multiplicities as Hilbert-Samuel multiplicities*, J. Algebra 319 (2008), no. 11, 441345. 2

[CNT19] M. Chardin; J. Naeliton; Q. H. Tran, *Cohen-Macaulayness and canonical module of residual intersections*, Trans. Amer. Math. Soc. 371 (2019), 1601-1630. 50

[EN62] J. A. Eagon, D. G. Northcott, *Ideals defined by matrices and a certain complex associated with them*. Proc. Roy. Soc. London Ser. **A 269** (1962), 188-204. 2, 25, 31

[E95] D. Eisenbud, *Commutative Algebra with a view toward algebraic geometry*, Graduate Texts in Math. **150**, Springer, New York, 1995. 2, 20, 25, 26, 31, 32

[EHU02] D. Eisenbud, C. Huneke and B. Ulrich, *What is the Rees algebra of a module?*, Proc. Amer. MAth. Soc. **131** (2002), 701-708. 36, 37

[G90] T. Gaffney *Multiplicities and equisingularity of ICIS germs*, manuscript, Spring 1990. 2

[G92] T. Gaffney *Polar multiplicities and equisingularity of map germs*, Topology 32 (1993), no. 1, 185–223.

[G93] T. Gaffney, *Integral closure of modules and Whitney equisingularity*, Invent. Math. 107 (1992), no. 2, 301–322. 2

[Har77] R. HARTSHORNE, *Algebraic geometry, Graduate Texts in Mathematics* 52 (Springer, New York, 1977). 13, 14, 15, 49

[Has12] S. H. Hassanzadeh, *Cohen-Macaulayness residual intersections and their Castelnuovo-Mumford regularity*, Trans. Amer. Math. Soc. 364 (2012), 6371-6394. 20, 50

[HaN16] S. H. Hassanzadeh, J. Naeliton, *Residual intersections and the annihilator of Koszul homologies*, Algebra Number Theory 10 (2016), 737-770. 20, 50

[HH11] F. Hayasaka and E. Hyry, *On the Buchsbaum-Rim function of a parameter module*, J. Algebra 327 (2011), 307-315. 34, 49

[HS06] C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Math. Soc. Lecture Note Series, **336**. Cambridge University Press, Cambridge, 2006. 38

[ILLMMSW] S. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, E. Miller, A. K. Singh, U. Walther, *Twenty-Four Hours of Local Cohomology*, vol. 87 of Graduate Studies in Mathematics. American Mathematical Society, Providence, Rhode Island, nov 2007. 13

[J01] E. Jones, *Computations of Buchsbaum-Rim multiplicities*, Journal of Pure and Applied Algebra 162 (2001) 37–52. 2, 4, 38, 48

[Jou09] J.P. Jouanolou, *An explicit duality for quasi-homogeneous ideals*, J. Symbolic Comput. (2007). 4, 32, 51

[K99] S. Kleiman, *Equisingularity, multiplicity, and dependence*, Commutative algebra and algebraic geometry (Ferrara), 21125, Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999.

[K17] S. Kleiman, *Two formulas for the BR multiplicity*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 63 (2017), no. 1, 14758. 2

[KT96] S. Kleiman and A. Thorup, *Mixed Buchsbaum-Rim multiplicities*, Amer. J. Math. 118 (1996), no. 3, 52969. 1, 2

[Kir73] D. Kirby, *A sequence of complexes associated a matrix* , J. London Math. Soc. (2) 7 (1974) 523–530. 2, 20, 25, 29, 31

[Kir85] D. Kirby, On the Buchsbaum-Rim multiplicity associated with a matrix, J. London Math. Soc. (2) 32 (1985), no. 1, 57-61. 3, 28, 37

[LR92] A. Lago and A. Rodicio, *Generalized Koszul complexes and Hochshild (co)-homology of complete intersections*, Invent. Math (1992), 107, 433–446.

[NW20] V. Nguyen and K. Walters, *Lech's inequality for Buchsbaum-Rim multiplicity and mixed multiplicity*, arXiv:1912.01073, 2020. 1

[Nor63] D. G. Northcott, *Some remarks on the theory of ideals defined by matrices*, Quart. J. Math. Oxford Ser. (2) 14 (1963), 193-204. 34

[Ree57] D. Rees, *The grade of an ideal or module*, Proc. Camb. Phil. Soc, 53 (1957) 28-42. 33

[Ree87] D. Rees, *Reduction of modules*, Math. Proc. Camb. Phil. Soc, 101 (1987) 431-450. 38

[Ser65] J.-P. Serre. *Algèbre locale. Multiplicités* LNM 11, Springer, 1965. 3, 4, 5, 47

[SUV01] A. Simis, B. Ulrich and W.V. Vasconcelos, *Codimension, multiplicity and integral extensions*, Math. Proc. Cambridge Philos. Soc. 130 (2001), 237–257. 1, 38

[W94] C. A. Weibel, *An introduction to homological algebra* Cambridge University Press, 1994. 10, 16, 22