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Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

On the asymptotic behavior of the solutions for a class of thermoelastic system

por

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João Pessoa - PB

Abril/2018

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sob orientação do

Prof. Dr. Flank David Morais Bezerra

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Abstract

In this work we prove the existence and regularity of the global attractors and the pullback attractors for a class of autonomous and non-autonomous thermoelastic systems, respectively, with vanishing mean value for temperature in a bounded domain with sufficiently smooth boundary in $\mathbb{R}^n$ with $n \geqslant 2$. Moreover, we prove the upper semicontinuity of the attractors with respect to the diffusion coefficients.

Palavras-chave: thermoelasticity, global attractor, pullback attractor, upper semicontinuity, regularity.

Resumo

Neste trabalho, provamos a existência e a regularidade dos atratores globais e dos atratores de pullback para uma classe de sistemas termoelásticos autônomos e não autônomos, respectivamente, com um valor médio da temperatura se anulando em um domínio limitado com fronteira suficientemente suave em $\mathbb{R}^n$ com $n \geqslant 2$. Além disso, provamos a semicontinuidade superior dos atratores em relação aos coeficientes de difusão.

Palavras-chave: termoelasticidade, atratores globais, atratores *pullback*, semicontinuidade superior, regularidade.

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"O Problema pode ser modesto, mas se ele desafiar a curiosidade e puser em jogo as faculdades inventivas, quem o resolver por seus meios, experimenta o sentimento da autoconfiança e gozará o triunfo da descoberta. Experiências tais, numa idade suscetível, poderão gerar o gosto pelo trabalho mental e deixar, por toda a vida, a sua marca na mente e no carácter."

George Pólya

Dedicatória

minha esposa...

Contents

Notations

- Ω is a domain of $\mathbb{R}^n$ when the body is in the reference state;
- u is the displacement of the body's particle over time;
- θ is temperature variation of studies body;
- f is the specific external body force;
- $\mathcal{E}$ is the internal energy;
- κ diffusion coefficient;
- β is the thermal moduli;
- $\mathcal{L}(A, B)$ is space of bounded linear transformation of A to B;
- $\mathcal{H} = (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega) = (Y^1)^n \times Y^n \times Y$;
- $Y_* = L_0^2(\Omega) = \left\{\xi \in L^2(\Omega); \int_\Omega \xi dx = 0\right\}$;
- $\mathcal{H}_* = \mathcal{H}_*^0 = (Y^1)^n \times Y^n \times Y_*$ and $\mathcal{H}_*^1 = (Y^2)^n \times (Y^1)^n \times Y_*^1$;
- Y^α is the fractional power space associated with the negative Laplacian operator subject to homogeneous Dirichlet boundary condition;
- $Y_*^\alpha = Y^\alpha \cap Y_*$;
- $\mathcal{H}^\alpha = [\mathcal{H}_*^1, \mathcal{H}_*^0]_\alpha = (Y^{1-\alpha})^n \times (Y^{-\alpha})^n \times Y_*^{-\alpha}$;
- $\Phi : L_0^2(\Omega) \to (H_0^1(\Omega))^n$ is the Bogowskĭi operator given by $\operatorname{div} \Phi(v) = v$ for all $v \in L_0^2(\Omega)$;

List of figures

Introduction

We will work with non-linear dynamical system from problems of partial differential equations with initial and boundary data associated to models related to the movement of an elastic, isotropic, limited and sufficiently smooth boundary solid which occupies the region $\Omega \subset \mathbb{R}^n$ with $n \leqslant 1$ and we will be taken into account also the influence of its temperature in its displacement. More precisely, we will be interested in obtaining information about the asymptotic behavior of two thermoelastic systems; an autonomous system

$$\begin{cases} \partial_t^2 u - \Delta u - \nabla \operatorname{div} u + \nabla\theta = f(u), & t > 0,\ x \in \Omega, \\ \partial_t \theta - \operatorname{div}(\kappa(x)\nabla\theta) + \operatorname{div} \partial_t u = 0, & t > 0,\ x \in \Omega, \end{cases}$$

subject to boundary conditions

$$u(x,t) = 0,\ \theta(x,t) = 0, t > 0,\ x \in \partial\Omega$$

on initial conditions

$$\begin{cases} u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), & x \in \Omega, \\ \theta(x,0) = \theta_0(x) & x \in \Omega, \end{cases}$$

and a non-autonomous system

$$\begin{cases} \partial_t^2 u - \Delta u - \nabla \operatorname{div} u + \beta(t)\nabla\theta = f(u), & t > s,\ x \in \Omega, \\ \partial_t \theta - \operatorname{div}(\kappa(x)\nabla\theta) + \beta(t) \operatorname{div} \partial_t u = 0, & t > s,\ x \in \Omega, \end{cases}$$

subject to boundary conditions

$$u(x,t) = 0,\ \theta(x,t) = 0, t > s,\ x \in \partial\Omega$$

on initial conditions

$$\begin{cases} u(x,s) = u_0(x),\ \partial_t u(x,s) = u_1(x), & x \in \Omega, \\ \theta(x,s) = \theta_0(x) & x \in \Omega. \end{cases}$$

In the problems above mentioned, the map f is external force, the functional parameters κ is the diffusion coefficient and β is the thermal moduli with some suitable growth conditions which will be presented below.

We recall that for a smooth vector field in some sense $u = (u_1, \ldots, u_n)$ the gradient and Laplacian of the vector field u are denoted, respectively, by

$$\nabla u = (\nabla u_1, \ldots, \nabla u_n)$$

and

$$\Delta u = (\Delta u_1, \ldots, \Delta u_n),$$

and the divergent operator of a vector field $\partial_t u$ will be denoted by

$$\operatorname{div} \partial_t u = \sum_{i=1}^{n} \partial_{x_i} \partial_t u_i.$$

The hypotheses on the non-linearity $f = (f_1, \ldots, f_n)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$. We consider f a conservative vector field (i.e., there is a scalar field α such that $f = \nabla\alpha$) with the functions f_i twice continuously differentiable and $f_i(0) = 0$, $i = 1, 2, 3, 4, ..., n$. Moreover, we also assume that for each $\nu > 0$ there exists $C_\nu > 0$ such that

$$f(\xi) \cdot \xi \leqslant \nu|\xi|^2 + C_\nu,$$

with $\cdot$ denoting the standard dot product on $\mathbb{R}^n$. We can assume that there exist $C_\eta > 0$ and $\eta \in (0, \min\{1, \lambda_1\})$ such that if

$$F(\xi) := \int_0^\xi f d\gamma,$$

then

$$F(\xi) \leqslant \frac{\eta}{2}|\xi|^2 + C_\eta,$$

where $\lambda_1 > 0$ is the first eigenvalue of the negative Laplacian operator with zero Dirichlet boundary condition, and $\int_0^\xi f d\gamma$ represents the line integral of f along a piecewise smooth curve $\gamma : [s, t] \to \mathbb{R}^n$ wich $\gamma(s) = 0$ and $\gamma(t) = \xi$, for any $\xi \in \mathbb{R}^n$ (that is, $\nabla F(\xi) = f(\xi)$, where ∇F stands for the gradient of F in the variables $\xi \in \mathbb{R}^n$).

In addition, we shall assume throughout this text that there exists a constant $C > 0$ such that for every $i = 1, \ldots, n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$|\nabla f_i(\xi)| \leqslant C\left(1 + \sum_{i=1}^{n} |\xi_i|^{\mathfrak{p}-1}\right),$$

$$|\partial_{x_i}^2 f_i(\xi)| \leqslant C,$$

for some $1 < \mathfrak{p} < \frac{n}{n-2}$, if $n > 2$; $1 < \mathfrak{p} < +\infty$ if $n = 2$.

The coefficients κ in (3.1), are real-valued continuously differentiable function defined on Ω such that there exist constants κ_0 and κ_1 with the property

$$0 < \kappa_0 \leqslant \kappa(x) \leqslant \kappa_1, \ x \in \Omega.$$

Furthermore, we assume that there are positive constants β_0 and β_1 such that

$$0 < \beta_0 \leqslant \beta(t) \leqslant \beta_1, \quad t \in \mathbb{R}.$$

When we talk asymptotic behavior we are asking ourselves about the existence and properties that the global attractor (in the autonomous case) and the pullback attractor (the non-autonomous case). In the forward dynamic (in the autonomous case) is the behaviour of solutions as $t \to \infty$. Let $S(\cdot)$ be the semigroup that come from the global solution of the autonomous problem which we are studying. The global attractor is a set $\mathcal{A}$ such that is compact, invariant by $S(\cdot)$ and attracts bounded sets under $S(\cdot)$. Now consider a non-autonomous problem with the initial data taken in the time s and the processes $U(\cdot,\cdot)$ defined by the global solution of the problem non-autonomous. The pullback dynamic is the study of the solution of the non-autonomous problem when it fix the current time and go back to history, i.e., is the behavior of solutions as $s \to -\infty$. This is translated in the definition of the pullback attractor which will be a family of sets $\mathcal{A}(\cdot)$ such that $\mathcal{A}(t)$ are compact for all $t > s$, invariant for $t > s$ by the process $U(\cdot,\cdot)$, in the sense that

$$U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t), \ t \geqslant \tau \geqslant s$$

and $\mathcal{A}(\cdot)$ is the minimal (in the sense that if there is another family $C(\cdot)$ such that pullback attract bounded, $C(t) \subset \mathcal{A}(t)$ for all $t > s$) family such that pullback attracting all bounded sets by $\mathcal{A}(\cdot)$ under $U(\cdot,\cdot)$, i.e., for all $t > s$, $\mathcal{A}(t)$ is such that any bounded set has the Hausdorff semidistance between itself and $\mathcal{A}(t)$ tends to 0 as $s \to -\infty$. By the hypotheses which we assumed in both cases, there is only one attractor. In both cases, the space

$$L^2_0(\Omega) = \left\{ v \in L^2(\Omega); \int_\Omega v dx = 0 \right\}. \tag{1}$$

will play a crucial role in our analysis.

Another result that we find as a consequence of the propositions used to demonstrate the existence of the attractors is the exponential decay of the solutions if we consider $f \equiv 0$.

As far as we know the hypotheses that we consider in this thesis were not considered in other works that seek such decay as it is commented in Section 1.2. In general the exponential decay for the thermoelasticity system is not guarantee in $\mathbb{R}^n$ with $n > 1$, such decay depends of the geometry of domain and hypotheses about u_0 and u_1 for example. In our cases, we will ask $\theta_0 \in Y_*$.

This work is organized in four chapters:

In the first one was made a summary of general knowledge that sets the problem. We do a brief justification of the emergence of the thermoelastic system equations using the conservations laws, a synthesis of the known results about decay of the thermoelastic problem to better understand what we do and the main general results of mathematical analysis which we will use throughout the text.

The second chapter is dedicated to a summary on the theory of semigroups, processes, global attractors and pullback attractors that we will use constantly. In this chapter, we will establish the relationship between semigruop and processes with problems autonomous and non-autonomous.

In chapters 3 and 4 we will reach our goal of studying the asymptotic behavior of the problems previously announced by the use of the functional

$$\mathcal{L}(u, z, \theta) = M\mathcal{E}(u, z, \theta) + \delta_1(u, z)_{(L^2(\Omega))^n} + \delta_2(\Phi, z)_{(L^2(\Omega))^n}$$

given by a modification in the natural energy of the system

$$\mathcal{E}(u, z, \theta) = \frac{1}{2}\Big(\|u\|^2_{(H^1_0(\Omega))^n} + \|z\|^2_{(L^2(\Omega))^n} + \|\theta\|^2_{L^2(\Omega)}\Big) - \int_\Omega F(u)dx.$$

where $\|\cdot\|^2_{(H^1_0(\Omega))^n} = (\cdot,\cdot)_{(H^1_0(\Omega))^n}$, $F(u) = \int_0^u f d\gamma$ with $\int_0^u f d\gamma$ represents the line integral of f along a piecewise smooth curve with initial point 0 and final point u with $u = u(x, t)$, and, δ_1, δ_2 and M are positive constants to be chosen appropriately. Such a change is given by using the Bogoviskĭi operator Φ that naturally induces an invariant subspace of $L^2(\Omega)$ that we can take θ_0. The results obtained in Chapter 3 produced an article which was accepted for publication in the Journal Colloquium Mathematicum.

Chapter 1

Preliminary

In this first chapter we wish to contextualize the problem studied by summarizing the physical origin of the problem, some results obtained and also some important results of general knowledge that will be required throughout the text. In the first section we will establish the concept of stress and strain to induce the main equations of the thermoelastic system in its most general way using law well known in the mechanics of fluids. In the second section we mentioned some articles that previously studied cases similar to the problem that we want to analyze in this text. Finally, in the last section of the chapter we have a compilation of Sobolev spaces results, PDE's, and other similarities that we will use constantly in Chapter 3 and Chapter 4, with the purpose of helping to read the text.

1.1 Deduction of the thermoelastic system

In general word we present in this section the deduction of the thermoelastic system following the references Ciarlet [16], Dafermos [17] and, Racke and Jiang [38].

Let $\mathcal{B}$ be a body occupying a region $\Omega \subset \mathbb{R}^n$ when it is not under the effect of forces of any nature and at environment temperature in any point. We will assume that Ω is a bounded domain with a smooth boundary. Thus, associate each material point of $\mathcal{B}$ with $x \in \Omega$ your position.

Considering $\varphi(x,t) \in \mathbb{R}^n$ the position and $T(x,t)$ the temperature in time $t \geqslant t_0$ of the particle in $x \in \Omega$ when the body is in the reference condition, for some t_0 fixed. We will denote by $u(x,t) = \varphi(x,t) - x$ the displacement and by $\theta(x,t) = T(x,t) - T_0$ the temperature variation, where T_0 is a conveniently chosen reference temperature. In order to establish the equations object of our study, we will formally assume that φ and T are enough differentiable. By the nature of the problem, we assume that φ is injective on Ω. We will denote $\mathcal{D}\varphi(\cdot, t)$ as the differential of $\varphi(\cdot, t)$.

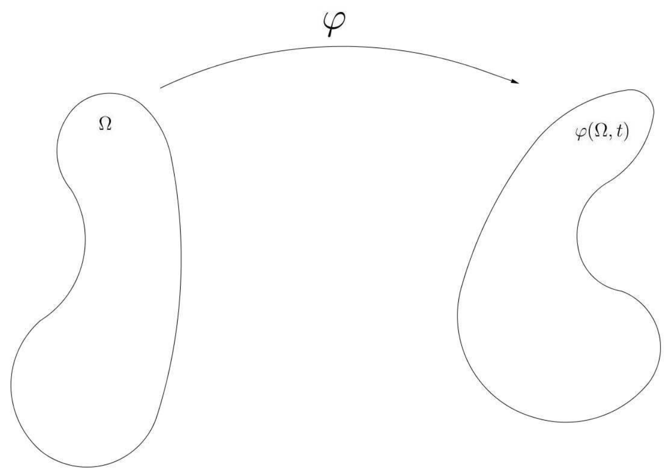

Figure 1.1: Deformation φ of the body $\mathcal{B}$

Now we will discuss the concept of **stress** on a point x in the position $\varphi(x, t)$ of the body $\mathcal{B}$ in the direction of the $\mathbf{n}$ unit vector after a deformation in time t fixed. Consider regular surface Γ with the follow proprieties:

(1.) $\varphi(x, t) \in \Gamma$, for all $x \in \Omega$;

(2.) $\mathbf{n} \in \mathbb{S}^{n-1} = \{v \in \mathbb{R}^n; \|v\| = 1\}$ is normal to Γ in $\varphi(x, t)$;

(3.) There are Ω_1 and Ω_2 subdomain of $\varphi(\Omega, t)$ such that $\varphi(\Omega, t) = \Omega_1 \cup \Omega_2$ and

$$\varphi(\Omega, t) \cap \Gamma = \Omega_1 \cap \Omega_2.$$

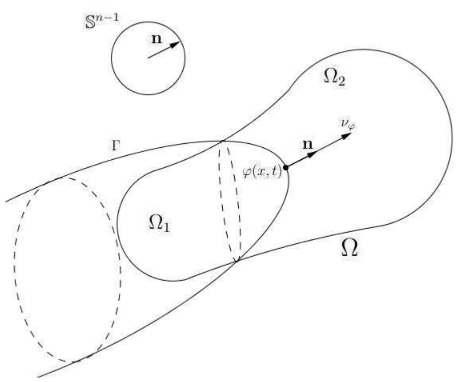

Figure 1.2: Cauchy's stress vector field

We define as the **Cauchy's stress vector field** by $\nu_\varphi : \varphi(\Omega, t) \times \mathbb{S}^{n-1} \to \mathbb{R}^n$ such that $\nu_\varphi(\varphi(x,t), \mathbf{n})$ is the force which Ω_i exert over $\varphi(x,t)$, where $-\mathbf{n}$ is normal outside of Ω_i in $\varphi(x,t)$. It can be verified that $\nu_\varphi(\varphi(x,t), \mathbf{n})$ does not depend on the choices of Γ only $\mathbf{n}$ and x. Moreover, as describe the next theorem $\nu_\varphi(\varphi(x,t), \mathbf{n})$ behaves linearly on $\mathbf{n}$.

Theorem 1.1. ***(Cauchy's Theorem)*** *Assume that for each $\mathbf{n} \in \mathbb{S}^{n-1}$ vector field $\nu_\varphi(\cdot, \mathbf{n})$ is continuously differentiable and $\nu_\varphi(\varphi(x,t), \cdot)$ is continuous for each $\varphi(x,t) \in \varphi(\Omega, t)$ with t fixed. Then exists a continuously differentiable symmetric tensor field called* ***Cauchy's stress tensor*** *define by*

$$\sigma_\varphi : \varphi(\Omega, t) \to \mathbb{M}_n$$

such that for any $\mathbf{n} \in \mathbb{S}^{n-1}$,

$$\nu_\varphi(\varphi(x,t), \mathbf{n}) = \sigma_\varphi\left(\varphi(x,t)\right)\mathbf{n}$$

where $\mathbb{M}_n$ is the set of matrices $n \times n$ of real numbers.

Proof. See Ciarlet [16, Page 62].

Recalling that the Euler variable is the way to describe the problem by taking as observation point in the object while it deforms, in other hand the Lagrange variable induces the behavior of the object by take the information in the referenced state of the object. The Cauchy's stress tensor $\sigma_\varphi(\varphi(x,t))$ is defined at the Euler variable $\varphi(x,t)$, we will use the so-called **Piola-Kirchhoff stress tensor** or **first Piola-Kirchhoff stress tensor** $\sigma(x,t)$ defined at Lagrange variable x by:

$$\sigma(x,t) := (\det \mathcal{D}\varphi(x,t))\sigma_\varphi(\varphi(x,t))\mathcal{D}\varphi(x,t)^{-T}.$$

Since in some cases it is interesting to have a symmetrical tensor and the tensor $\sigma(x,t)$ is not symmetrical in general, we have defined to meet these needs the **second Piola-Kirchhoff stress tensor** $\Sigma(x)$ by letting

$$\Sigma(x) = \mathcal{D}\varphi(x)^{-1}\sigma(x) = (\det \mathcal{D}\varphi(x))\mathcal{D}\varphi(x)^{-1}\sigma_\varphi(\varphi(x,t))\mathcal{D}\varphi(x)^{-T}.$$

The next concept we want to introduce is the **strain** which measures the deformation rate with respect to the variation of x that the body has undergone after a displacement. For any t fixed, $\varphi(\cdot, t)$ is differentiable in any point $x \in \Omega$, then for all points $x + h \in \Omega$:

$$\varphi(x+h,t) - \varphi(x,t) = \mathcal{D}\varphi(x,t)h + \mathcal{O}(|h|)$$

where $\frac{\mathcal{O}(|h|)}{|h|} \to 0$ as $h \to 0$.

The deformation is given by

$$\begin{aligned}|\varphi(x+h,t) - \varphi(x,t)|^2 = h^T\mathcal{D}\varphi^T(x,t)\mathcal{D}\varphi(x,t)h + h^T\mathcal{D}\varphi^T(x,t)\mathcal{O}(|h|) \\ + \mathcal{O}(|h|)\mathcal{D}\varphi(x,t)h + \mathcal{O}(|h|)^T\mathcal{O}(|h|).\end{aligned}$$

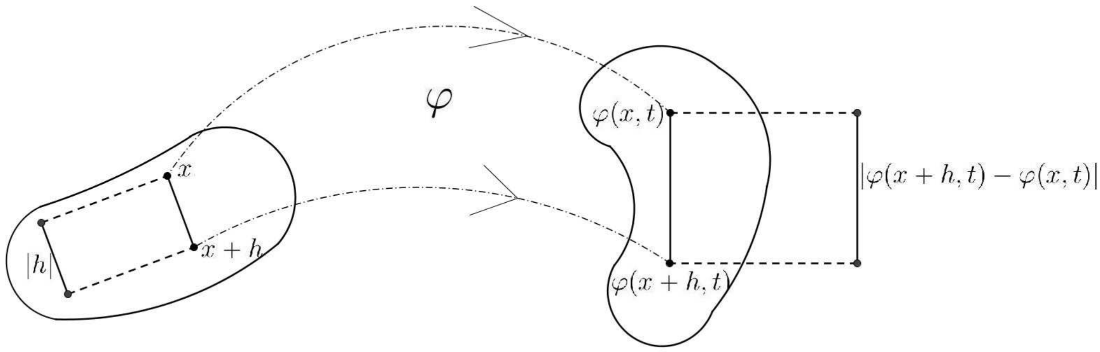

Figure 1.3: Deformation rate

The symmetric strain tensor in Euler variable is

$$\mathfrak{E}(\varphi) := \mathcal{D}\varphi^T\mathcal{D}\varphi.$$

We also can obtain that

$$\mathfrak{E}(\varphi) = \mathcal{D}\varphi^T\mathcal{D}\varphi = I + \mathcal{D}u^T + \mathcal{D}u + \mathcal{D}u^T\mathcal{D}u = I + 2E(u).$$

The **strain tensor** of a given body $\mathcal{B}$ after a displacement u is define by

$$E(u) := \frac{1}{2}(\mathcal{D}u^T + \mathcal{D}u + \mathcal{D}u^T\mathcal{D}u)$$

also called The Green-St Venant strain tensor. By assume the hypotheses of small deformations, we will be able work with the form linear of E which is

$$\mathbf{e}(u) := \frac{1}{2}(\mathcal{D}u^T + \mathcal{D}u).$$

The Duhamel-Neumann's Law witch is a generalization to the Hook's Law (which admit $\mathfrak{A}$ null), tell us that

$$E = \mathfrak{K}\sigma + \mathfrak{A}\theta$$

where $\mathfrak{K}$ is called compliance tensor and $\mathfrak{A}$ is coefficients of linear thermal expansion, and also,

$$\sigma = \mathfrak{C}E - \mathfrak{B}\theta$$

where $\mathfrak{C}$ is called stiffness tensor (also know as elastic moduli) and $\mathfrak{B}$ is know as thermal moduli. By assume the hypotheses of small deformations, we will be able to consider

$$\mathbf{e} = \mathfrak{K}\sigma + \mathfrak{A}\theta \tag{1.1}$$

and

$$\sigma = \mathfrak{C}\mathbf{e} - \mathfrak{B}\theta. \tag{1.2}$$

When tensor $\mathfrak{C}$ depends of position x we say that the material is anisotropic and when there is no dependency the material is call isotropic (for more details see Kupradze [29, Chapter 5]). In the chapters 3 and 4, we will consider the isotropic case.

The **balance of linear momentum**, in our notation, is expressed by

$$\partial_t \int_V \rho \partial_t u \, dV = \int_A \sigma \cdot \mathbf{n} \, dA + \int_V \rho f \, dV \tag{1.3}$$

where $A = \partial V$, ρ is the material density (which depends of x) and f is the specific external body force (which depends of x and t), in any $V \subset \Omega$. By using (1.2) in (1.3), we have

$$\partial_t \int_V \rho \partial_t u \, dV = \int_A (\mathfrak{C}\mathbf{e} - \mathfrak{B}\theta) \cdot \mathbf{n} \, dA + \int_V \rho f \, dV$$

thus,

$$\partial_t \int_V \rho \partial_t u \, dV = \int_A \left(\frac{1}{2} \mathfrak{C}(\mathcal{D}u^T + \mathcal{D}u) - \mathfrak{B}\theta \right) \cdot \mathbf{n} \, dA + \int_V \rho f \, dV.$$

Using Divergence Theorem and since previous identity is true for any V, we obtain the following equation

$$\rho \partial_t^2 u = \frac{1}{2} \operatorname{div} \big(\mathfrak{C}(\mathcal{D}u^T + \mathcal{D}u) \big) - \operatorname{div}(\mathfrak{B}\theta) + \rho f. \tag{1.4}$$

which is also presented as follows

$$\rho \partial_t^2 u_i = \sum_{j=1}^{n} \partial_{x_j} \left(\mathfrak{C}_{ijkl} \partial_{x_l} u_k \right) - \sum_{j=1}^{n} \partial_{x_j} (\mathfrak{B}_{ij}\theta) + \rho f_i$$

where $\frac{1}{2}\mathfrak{C}(\mathcal{D}u^T + \mathcal{D}u) = [\mathfrak{C}_{ijkl} \partial_{x_l} u_k]$ and $\mathfrak{B}\theta = [\mathfrak{B}_{ij}\theta]$.

We denote $\mathcal{E}_h$ by the Helmholtz free energy and η by the entropy (which is the quotient of the amount of heat absorbed from the body $\mathcal{B}$ by its temperature.). We can assume that[1]:

$$\mathcal{E}_h = \frac{1}{2} \mathfrak{C}_{ijkl} \partial_{x_j} u_i \partial_{x_l} u_k - \mathfrak{B}_{ij} \partial_{x_j} u_i \theta - \frac{1}{2T_0} \rho c_D \theta^2$$

where $c_D(x)$ is specific heat at the point x when $\partial_{x_j} u_i + \partial_{x_i} u_j = 0$.

Using the notation $U = \nabla u$. Thanks to Racke and Jiang [38, Chapter 1]

$$\eta(U, \theta) = -\frac{\partial \mathcal{E}_h}{\partial \theta}(U, \theta).$$

The Fourier's Law set

$$q_i = \mathbf{K}_{ij} \partial_{x_j} \theta$$

where q is the heat flux and $[\mathbf{K}_{ij}(x)]$ is the heat conduction tensor.

The conservation law of energy,

$$\rho T_0 \, \partial_t \eta = \operatorname{div}(q) + \rho c_D r$$

[1]for more details see [17].

where r is the heat source. So

$$T_0 \sum_{i=1}^{n} \mathfrak{B}_{ij} \partial_{x_j} \partial_t u_i + \rho c_D \partial_t \theta = \operatorname{div}(\mathbf{K}_{1j} \partial_{x_j} \theta, \ldots, \mathbf{K}_{nj} \partial_{x_j} \theta) + \rho c_D r, \tag{1.5}$$

or in other notation,

$$\rho c_D \partial_t \theta - \operatorname{div}(\mathbf{K} \nabla \theta) + T_0 \mathfrak{B} \mathcal{D}(\partial_t u_i) = \rho c_D r.$$

The equations (1.4) and (1.5) characterizes the thermoelastic problem. When Ω is bounded, the boundary condition

$$u = 0,\ \theta = 0 \text{ on } \partial\Omega,$$

is called of condition of **rigidly clamped and constant temperature**, and the boundary condition

$$\sigma\nu = 0,\ \nu q = 0 \text{ on } \partial\Omega,$$

is called of condition of **tracion free insulated** where ν is the outward unit normal to $\partial\Omega$.

1.2 Previous results about the thermoelasticity system

Dafermos [17] studies the well-posedness of the anisotropic thermoelastic problem

$$\begin{cases} \rho \partial_t^2 u_i - \displaystyle\sum_{j=1}^{n} \partial_{x_j} \left(\mathfrak{C}_{ijkl} \partial_{x_l} u_k \right) + \sum_{j=1}^{n} \partial_{x_j} (\mathfrak{B}_{ij} \theta) = \rho b_i \\ \rho c_D \partial_t \theta - \operatorname{div}(\mathbf{K} \nabla \theta) + T_0 \mathfrak{B} \mathcal{D}(\partial_t u_i) = \rho c_D r. \end{cases} \tag{1.6}$$

and commented that the homogeneous version of the problem (1.6) has a decay, but not necessarily exponential when we consider $n \geqslant 2$.

In the period from 1991 to 1993, several papers on the case one-dimensional obtain exponential decay rate (e.g., Henry, Perissinotto and Lopes in [26], Liu and Zheng in [33], Slemrod [40] and references therein), and the question about exponential decay rate in the case n-dimension for $n \geqslant 2$ attracted more and more attention from researchers.

About this problem, we can note that in particular the system thermoelastic

$$\begin{cases} \partial_t^2 u - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla \theta = 0, & x \in \Omega,\ t > 0, \\ \partial_t \theta - \Delta \theta + \operatorname{div} \partial_t u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{1.7}$$

subject to initial-boundary conditions

$$\begin{cases} u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x), & x \in \Omega, \\ u(t,x) = 0,\ \theta(t,x) = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

can be obtain from (1.6) (where $\mu > 0$ and $\lambda > 0$ are the Lamé coefficients) by the correct choice of tensors. In the last years the famous question of thermoelasticity theory about obtaining the necessary and sufficient conditions to ensure the exponential uniform decay of the energy of the linear thermoelastic system n-dimensional, under some geometric conditions of the domain and regularity of the vector field u this problem was solved in Amann [1], Racke, Rivera and Jiang [27], Koch [28], Kupradze [29], Lebeau and Zuazua [30], Lebeau and Zuazua [31], Liu and Zheng [33], Rivera and Shibata [35], Rivera [39], Slemrod [40] and references therein. More precisely, Lebeau and Zuazua in [30], have shown that in a smooth boundary domain in $\mathbb{R}^n$ which possesses an arbitrarily large ray of geometrical optics which is always perpendicularly reflected at the boundary, the problem not have exponential decay (see too Lebeau and Zuazua [31]). Later, Koch in his work [28] extends this result showing that the exponential decay is not possible if the domain is convex. But Rivera in 1997 study a the case when considerer the displacement divergent free for all point of the general smooth domain in the paper Rivera [39]. He got the exponetial decay rate and shows that if $P_d(u_0) \neq 0$ or $P_d(u_1) \neq 0$, then

$$\mathcal{E}(t) \geqslant \int_\Omega |P_d(u_1)|^2 + |\nabla P_d(u_0)|^2 \, dx$$

where $P_d(u)$ is a projection of u in $V_d = \{w \in H_0^1(\Omega); \operatorname{div}(w) \neq 0\}$ and $\mathcal{E}$ is the natural energy of the system (1.7). Also the work of Jiang, Riveira and Racke (1998) in [27] has verified exponential decay in the case where the initial data and domain are radially symmetric (under such hypotheses the solutions are radially symmetric and the displacement has vanishing rotation).

1.3 Embeddings and inequalities

Here, we want to enunciate some well-known theorems of sobolev immersions and differential equations, as well as useful inequalities, with the aim of easy reading and comprehension of the text.

Theorem 1.2. *(Sobolev embedding) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^m.*

(1.) If $mp < n$, then the following embedding is continuous

$$W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega), \quad \textit{where } \frac{1}{q^*} = \frac{1}{p} - \frac{m}{n}.$$

Moreover, the embedding is compact for any q, with $1 \leqslant q < q^*$.

(2.) If $mp = n$, then the following embedding is continuous and compact

$$W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega), \quad \textit{for all } 1 \leqslant q < \infty.$$

Moreover, if $p = 1$ and $m = n$, then is possible assume $q = \infty$.

Proof. See Evans [21, Section 5.6].

The next theorem is a well-know result for the weak solution of the parabolic problem which we will use in the sections about regularity of attractors.

Theorem 1.3. *(See [8, Page 340]) Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. The dual space H^* is identified with H. Let V be another Hilbert space with norm $\|\cdot\|_V$. We assume that $V \subset H$ with dense and continuous injection, so that*

$$V \subset H \subset V^*.$$

For each $T > 0$ fixed. We are considering a bilinear form $a(t;\cdot,\cdot) : V \times V \to \mathbb{R}$ for a.e $t \in [0,T]$, satisfying the following properties:

(1.) For every $u, v \in V$ the function $t \mapsto a(t;u,v)$ is meansurable;

(2.) $\|a(t;u,v)\|_H \leqslant M\|u\|_V\|v\|_V$ for a.e. $t \in [0,T]$, $\forall u, v \in V$;

(3.) $a(t;v,v) \geqslant \alpha\|v\|_V^2 - C\|v\|_H^2$ for a.e. $t \in [0,T]$, $\forall \in V$;

where α, M and C are positive constants. Given $f \in L^2(0,T;V^*)$ and $u_0 \in H$, there exists a unique function u satisfying $u \in L^2(0,T;V) \cap C([0,T];H)$,

$$\frac{du}{dt} \in L^2(0,T;V)$$

$$\left(\frac{du}{dt}(t), v\right) + a\left(t; u(t), v\right) = \left(f(t), v\right)$$

for a.e. $t \in (0,T), \forall v \in V$, and $u(0) = u_0$.

Proof. See Lions and Magenes [32].

In the next theorem is consequence of the Divergence Theorem.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary with $n > 1$. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then*

$$-\int_\Omega (\Delta u)v\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\,dS,$$

and if $u \in (H^2(\Omega))^n$ and $v \in H^1(\Omega)$, then

$$-\int_\Omega (\operatorname{div} u)v\,dx = \int_\Omega u \cdot \nabla v\,dx - \int_{\partial\Omega} uv \cdot \nu\,dS,$$

where ν is the outward unit normal to $\partial\Omega$.

Proof. See Boyer and Fabrie [7, Page 133] and Evans [21, Page 711].

Theorem 1.5. ***(Poincaré inequality)*** *If $u \in H_0^1(\Omega)$, then there is a positive constant C depending only on Ω and n such that*

$$\|u\|_{L^2(\Omega)} \leqslant \lambda_1 \|\nabla u\|_{L^2(\Omega)}, \ \ \forall\, u \in H_0^1(\Omega)$$

where λ_1 is a minimal eigenvalue of the operator associate to the Dirichlet problem of negative Laplace's equation.

Proof. See Evans [21, Page 290].

Lemma 1.6. ***(Grönwall's inequality)*** *Let $J : [0,T] \to [0,+\infty)$ be a differential function, which satisfy the following property:*

$$J'(t) \leqslant -\alpha(t)J(t) + \beta(t), \ \ \textit{for } t \in [0,T],$$

where $\alpha, \beta : [0,T] \to \mathbb{R}$ are integrable functions in $[0,T]$. Then, for any $t \in [0,T]$

$$J(t) \leqslant e^{-\int_0^t \alpha(s)ds} \left[J(0) + \int_0^t \beta(\tau) d\tau \right].$$

Proof. See Evans [21, Page 708].

Lemma 1.7. ***(Young's inequality)*** *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\epsilon > 0$. Then,*

$$ab \leqslant \epsilon a^p + \frac{(\epsilon p)^{-q/p}}{q} b^q, \ \ \forall a, b \geqslant 0.$$

Proof. See Evans [21, Page 706].

Chapter 2

Semigroups, evolution processes and attractors

In the follows we recall some concepts and definitions from theory of nonlinear semigroup and nonlinear evolution process, for more details see Babin and Vishik [3], Brezis [8], Carvalho, Langa and Robinson [13], Hale [24], Pazy [36], Vrabie [41] and reference therein.

Throughout the text of this chapter, let $(\mathfrak{M}, \mathrm{d})$ be a complete metric space and let $(X, \|\cdot\|_X)$ be a Banach space. We will denote $\mathcal{C}(\mathfrak{M})$ the set of all continuous maps from $\mathfrak{M}$ into itself and $(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}(X)})$ the space of all bounded linear operators from X into itself with the norm

$$\|T\|_{\mathcal{L}(X)} = \sup_{x \in X; \|x\|_X \leqslant 1} \|Tx\|_X.$$

2.1 Nonlinear semigroups

We begin the section giving the most simple and comprehensive definition of semigroup that is found in Babin and Vishik [3].

Definition 2.1. *A* ***nonlinear semigroup*** *is a family of maps $\{S(t); t \geqslant 0\}$ in $\mathcal{C}(\mathfrak{M})$ with the properties*

(1.) $S(0) = I$;

(2.) $S(t+s) = S(t)S(s)$, *for all* $t, s \geqslant 0$;

(3.) $[0, \infty) \times X \ni (t, x) \mapsto S(t)x \in \mathcal{C}(\mathfrak{M})$ *is continuous.*

Definition 2.2. *A semigroup $S(\cdot)$ in $\mathfrak{M}$ is called* ***semigroup of class*** C_0 *(or for simplicity* C_0***-semigroup****) if for all $x \in \mathfrak{M}$, a function $S(\cdot)x : [0, \infty) \to \mathfrak{M}$ is continuous and $S(t)$ is a map continuous for all $t \geqslant 0$. A C_0-semigroup is called* ***strongly continuous semigroup,*** *too.*

The compacity asymptotic is one of the conditions required in the theorem which we will use to prove the existence of global attrator for a semigroup, see Theorem 2.38.

Definition 2.3. *A semigroup $S(\cdot)$ in $\mathfrak{M}$ is called **asymptotically compact** if, for each sequence (t_n) such that $t_n \to \infty$ as $n \to \infty$ and for each bounded sequence (x_n) of points of $\mathfrak{M}$, the sequence $(S(t_n)x_n)$ has a subsequece which is convergent in $\mathfrak{M}$.*

The definition above is equivalent in a Banach space to say that for every bounder closed and not empty $B \subset X$, there is a compact set $K \subset X$ such that there exists $t_0 > 0$ such that $S(t)B \subset K$ for $t > t_0$.

Definition 2.4. *A semigroup $S(\cdot)$ **eventually bounded** in X if there is a $t_0 \in [0, \infty)$ such that*

$$\bigcup_{t \geqslant t_0} S(t)B$$

*is bounded in X for every bounded B, where $S(t)B = \{S(t)x \in X; x \in B\}$. Case $t_0 = 0$, we say that $S(\cdot)$ is **bounded**.*

The next result gives a sufficient condition for a semigroup to be asymptotically compact.

Theorem 2.5. *Let $S(\cdot)$ be a bounded semigroup defined in X such that for each $t \geqslant 0$, we can write*

$$S(t) = S_1(t) + S_2(t)$$

where

(1.) For every bounded set B and each $t > 0$ there exists $t_{(B,t)} \geqslant 0$ and compact set $K(B,t)$ such that $S_2(s)B \subset K(B,t)$ always that $t \geqslant s \geqslant t_{(B,t)}$;

(2.) There exists a function $g : [0, \infty) \times [0, \infty) \to \mathbb{R}$ with $g(\cdot, r)$ non-increasing for each $r > 0$, $\lim_{s \to \infty} g(s, r) = 0$ and for all $x \in X$ with $\|x\| \leqslant r$,

$$\|S_1(t)x\|_X \leqslant g(t, r).$$

Then the semigroup $S(\cdot)$ is asymptotically compact.

Proof. See Carvalho, Langa and Robinson [13, Page 42].

2.2 Linear semigroups

Now we deal with the case of $S(t)$ is a linear operator for all $t > 0$, when this happens we call $S(\cdot)$ of a linear semigroup. We will initially define a class within the linear semigroups that is more comprehensive than C_0-semigroups.

Definition 2.6. *We say that $S(\cdot) \subset \mathcal{L}(X)$ is a* ***uniformly continuous linear semigroup***

$$\lim_{t \to s} \|S(t) - S(s)\|_{\mathcal{L}(X)} = 0.$$

Definition 2.7. *The operator A is called* ***infinitesimal generator*** *of a linear semigroup $S(\cdot)$ when*

$$D(A) = \left\{ x \in X; \lim_{t \to 0^+} \frac{1}{t}(S(t)x - x) \textit{ exist} \right\}$$

and for each $x \in D(A)$ we have

$$Ax = \lim_{t \to 0^+} \frac{1}{t}(S(t)x - x).$$

If A is an infinitesimal generator of the linear semigroup $S(\cdot)$, we can say $S(\cdot)$ is generated by A.

Definition 2.8. *A semigroup $S(\cdot)$ is of* ***type*** *(M, α) if there are constants $\alpha \in \mathbb{R}$ and $M \geqslant 1$ such that*

$$\|S(t)x\|_X \leqslant M e^{\alpha t} \|x\|_X, \ \forall t \geqslant 0.$$

We say that $S(\cdot)$ is ***exponential stable*** *if it is a semigroup type (M, α) with $\alpha < 0$.*

Theorem 2.9. *If $S(\cdot)$ is a C_0-semigroup, then $S(\cdot)$ is of type (M, α).*

Proof. See Vrabie [41, Page 41] .

Definition 2.10. *Let $A : D(X) \subset X \to X$ be a closed densely defined linear operator (not necessarily bounded). The* ***resolvent set*** *of A is*

$$\rho(A) = \{\lambda \in \mathbb{C}; \lambda - A \textit{ is injective and surjective}\}.$$

The $\sigma(A) = \mathbb{C} \backslash \rho(A)$ is called ***spectrum*** *of A.*

From closed graph theorem, if $\lambda - A$ is injective and surjective, then $(\lambda - A)^{-1} \in \mathcal{L}(X)$.

Theorem 2.11. *(Hille-Yosida) Let $A : D(X) \subset X \to X$ be a linear operator, then following statement are equivalent:*

(1.) A is the infinitesimal generator of a C_0-semigroup of linear operators $S(\cdot)$ of type (M, α);

(2.) A is closed, $\overline{D(A)} = X$, $\rho(A)$ contains (α, ∞) and

$$\|(\lambda I - A)^{-n}\| \leqslant \frac{M}{(\lambda - \alpha)^n}, \textit{ for } \lambda > \alpha \textit{ and } n = 1, 2, \ldots.$$

Proof. See Pazy [36, Page 8].

Remark 2.12. Note that the liner operator A is not required to be bounded, however conditions are required for the A resolvent.

From now on, we will denote X^* as the dual of X and we remind the reader that $\langle x, x*\rangle = \langle x*, x\rangle = x^*(x)$ is the value of $x^* \in X^*$ at $x \in X$.

Definition 2.13. *Let $A : D(A) \subset X \to X$ be a linear operator. We say that A is a* ***dissipative operator*** *when for each $x \in D(A)$ there is an $x^* \in \mathfrak{F}(x)$*

$$Re\langle Ax, x*\rangle \leqslant 0,\ \forall x \in D(A)$$

where $\mathfrak{F}(x) = \{y \in X^*; (y, x) = \|x\|^2 = \|y\|^2\}$.

One of the reasons we are interested in dissipative operators in the semigroup theory is the Lumer-Phillipis's Theorem.

Theorem 2.14. *(Lumer-Phillips) Let A be a linear operator with dense domain $D(A)$ in X. The following affirmations are equivalents:*

(1.) If A is dissipative and there is a $\lambda_0 > 0$ such that $R(\lambda_0 I - A)$, the range of $\lambda_0 I - A$, is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

(2.) If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, $Re(Ax, x^*) \leqslant 0$, for every $x \in D(A)$ and every $x^* \in \mathfrak{F}(x)$.

Proof. See Vrabie [41, Page 60].

We now want to discuss how the semigroup theory is made application of the semigroup theory to solve problems involving partial differential equations. Consider an initial value problem which we can write as follows

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + A\mathbf{u} = F,\ t > 0 \\ \mathbf{u}(0) = \mathbf{u}_0 \end{cases} \tag{2.1}$$

where $-A$ is a linear operator with domain $D(A) \subset X$ is also the set in which the other conditions of the problem are satisfied (for example boundary condition) and $F \in L^1([0, T]; X)$.

Definition 2.15. *We will call $\mathbf{u} : [0, T] \to X$ a* ***classical solution*** *of the problem (2.1) if $\mathbf{u} \in C^1([0, T]; D(A))$ and it satisfies $\dfrac{d\mathbf{u}}{dt}(t) + A\mathbf{u}(t) = F(t)$ for each $t \in [0, T]$ and $\mathbf{u}(0) = \mathbf{u}_0$.*

Definition 2.16. *We will call $\mathbf{u} : [0, T] \to X$ a* ***strong solution*** *of the problem (2.1) if $\mathbf{u}$ is absolutely continuous on $[0, T]$, $\mathbf{u}' \in L^1((0, T]; X)$, $u(t) \in D(A)$ and it satisfies $\partial_t \mathbf{u}(t) + A\mathbf{u}(t) = F(t)$ for each $t \in [0, T]$ and $\mathbf{u}(0) = \mathbf{u}_0$.*

The classical solution can be call C^1**-solution**. The classical solution can be call absolutely continuous solution. A classical solution of (2.1) is a strong solution, but not conversely. Assume that $-A$ is the infinitesimal generator of a C_0-semigroup $S(\cdot)$.

Definition 2.17. *We will call* $\mathbf{u} : [0,T] \to X$ *a* ***mild solution*** *if* $\mathbf{u}$ *is defined by*

$$\mathbf{u}(t) = S(t)\mathbf{u}_0 + \int_0^t S(t-s)F(s,u(s))ds. \tag{2.2}$$

As say the next theorem, if $\mathbf{u}$ is a strong solution (or a classical solution), then $\mathbf{u}$ is a mild solution.

Theorem 2.18. *(Duhamel Principle) Each strong solution of* (2.1) *is given by* (2.2).

Proof. See Vrabie [41, Page 184].

Theorem 2.19. *If* $-A : D(A) \subset X \to X$ *is the infinitesimal generator of a* C_0*-semigroup* $S(\cdot)$*, and* F *is of class* C^1 *on* $[0,T]$*, then, for each* $\mathbf{u}_0 \in D(A)$*, the problem* (2.1) *has a unique classical solution.*

Proof. See Vrabie [41, Page 186].

Theorem 2.20. *If* $-A : D(A) \subset X \to X$ *is the infinitesimal generator of a* C_0*-semigroup* $S(\cdot)$*, and* F *is of class* C^0 *on* $[0,T]$ *and locally Lipschitz continuous in* $\mathbf{u}$ *in bounded intervals of* $[0,T]$*, then, for each* $\mathbf{u}_0 \in X$*, the problem* (2.1) *has a unique mild solution. Moreover, if* $T < \infty$ *then*

$$\lim_{t \to T} \|u(t)\| = \infty.$$

Proof. See Pazy [36, Page 186].

Definition 2.21. *Let* $R = \{z := re^{i\theta} \in \mathbb{C}; \theta \in [\theta_1, \theta_2]$ *and* $\theta_1 < 0 < \theta_2\}$*. A family of bounded linear operator* $\{S(z); z \in R\}$ *is called* ***analytic semigroup*** *on* R *if*

(1.) $z \mapsto S(z)$ *is analytic on* R*;*

(2.) $S(0) = I$ *and* $\lim_{z \to 0} S(z)x = x$*, for all* $x \in R$*;*

(3.) $S(z_1 + z_2) = S(z_1)S(z_2)$*, for all* $z_1, z_2 \in R$.

A C_0-semigroup $S(\cdot)$ is called analytic if there is an analytic semigroup $S_1(\cdot)$ on $R = \{z := re^{i\theta} \in \mathbb{C}; \theta \in [\theta_1, \theta_2]$ and $\theta_1 < 0 < \theta_2\}$ such that $[0, +\infty) \subset R$ and $S_1(t) = S(t)$. Note that this tells us that the restriction of an analytic semigroup to the nonnegative real axis is a C_0 semigroup. But the reciprocal is not true in general.

Definition 2.22. *A closed densely defined linear operator* $A : D(A) \subset X \to X$ *is* ***sectorial*** *if there exist* $\alpha \in (0, \pi/2)$*,* $a \in \mathbb{R}$ *and* $M \geqslant 1$ *such that*

(1.) $\Sigma_{a,\alpha} = \{z \in \mathbb{C}; \alpha \leqslant |\arg(z-a)| \leqslant \pi,\ \lambda \neq a\}$ *contain the resolvent set of* A*;*

(2.) $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \dfrac{M}{|\lambda - a|},\ \ \forall \lambda \in \Sigma_{a,\alpha}.$

Theorem 2.23. *If* A *is a sectorial operator, then* $-A$ *is the infinitesimal generator of an analytic semigroup* $S(\cdot)$*.*

Proof. See Henry [25, Page 21].

Definition 2.24. *A linear operator* $A : D(A) \subset X \to X$ *is* ***positive*** *with constant* $M \geqslant 1$ *if* A *is closed, densely defined in* X*,* $[0, +\infty) \subset \rho(-A)$ *and*

$$(1+s)\|(s+A)^{-1}\|_{\mathcal{L}(X)} \leqslant M, \ \ s \in \mathbb{R}^+.$$

For more details see Carbone, Nascimento, Schiabel-Silva and Silva [10], Pazy [36], Vraibe [41] and reference therein.

If A is a positive linear operator with constant M, notice that

$$\Sigma_M = \{z = z_1 + z_2 \in \mathbb{C}; |\arg z_1| \leqslant \arcsin(1/2M) \text{ and } |z_2| \leqslant 1/2M\} \subset \rho(-A).$$

Definition 2.25. *We define the* ***fractional power*** *of the positive operator* A *with exponent* $\alpha \in \mathbb{C}$ *when* $Re(\alpha) > 0$*, by the operator* $A^{-\alpha} : D(A^{-\alpha}) \subset X \to X$ *given by*

$$A^{-\alpha} = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{-\alpha}(\lambda + A)^{-1}\, d\lambda,$$

where $\Gamma \subset \Sigma_M \backslash \mathbb{R}^+$ *is a simple curve which there is a parameterization given by* $|r(t)|e^{i\beta(t)}$ *with* $\lim_{t\to\pm\infty} r(t) = \infty$ *and* $\lim_{t\to\infty} \beta(t) = -\lim_{t\to-\infty}\beta(t)$*. We assume* $D(A^0) = X$ *and* $A^0 = I$ *as definition.*

Remark 2.26. The fractional power is well defined because it does not depend on the parameterization for Γ. Moreover, $A^{-\alpha} : D(A^{-\alpha}) \subset X \to X$ is bounded. If $S(\cdot)$ is C_0-semigroup exponentially stable, then we have $D(A^{-\alpha}) = X$.

It is well-know that for $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$, $A^{-\alpha}$ is injective. Therefore, we define $A^{\alpha} = (A^{-\alpha})^{-1}$. Now, we will give the basic relations between the operators of positive and negative fractional power.

Theorem 2.27. *Let* $-A$ *be an infinitesimal generator of a* C_0*-semigroup exponentially stable* $S(\cdot)$*. Then:*

(1.) A^α *is a closed operator with domain be the range of* $A^{-\alpha}$*, for* $\alpha > 0$*.*

(2.) If $w \geqslant z > 0$*, then* $D(A^w) \hookrightarrow D(A^z) \hookrightarrow X$ *are dense.*

(3.) If $z,\ w$ and $z+w$ in $\mathbb{R}$ and $x \in D(A^u)$ where $u = \max\{z, w, z+w\}$, we get

$$A^z A^w x = A^{z+w} x.$$

Proof. See Pazy [36, Page 72].

The next two result, helps to verifies when the inclusion between spaces defined by fractional power operators are continuous, see Vrabie [41, Section 7.6].

Theorem 2.28. *Let$A$ be an infinitesimal generator of a C_0-semigroup exponentially stable $S(\cdot)$. If $\alpha \in (0,1)$, there is $C > 0$ such that, for each $x \in D(A)$ and for each $\rho > 0$, we have*

$$\|A^\alpha x\| \leqslant C(\rho^\alpha \|x\| + \rho^{\alpha-1}\|Ax\|)$$

and

$$\|A^\alpha x\| \leqslant 2C\|x\|^{1-\alpha}\|Ax\|^\alpha.$$

Theorem 2.29. *Let$A$ be an infinitesimal generator of a C_0-semigroup exponentially stable $S(\cdot)$. Let $B : D(B) \subset X \to X$ be a closed operator with $D(A^\alpha) \subset D(B)$. If $\alpha \in (0,1]$, there is $C > 0$ such that, for each $x \in D(A^\alpha)$ and for each $\rho > 0$, we have that*

$$\|Bx\| \leqslant C\|A^\alpha x\|.$$

and if $x \in D(A)$, we have that

$$\|Bx\| \leqslant C(\rho^\alpha \|x\| + \rho^{\alpha-1}\|Ax\|).$$

Theorem 2.30. *Let H be a Hilbert space and let A be a positive definite self-adjoint linear operator in H. Then A has bounded imaginary power.*

Proof. See Amann [1, Pages 164 and 157].

Proposition 2.31. *Let $A : D(A) \to X$ be a sectorial operator in a Banach X and consider a closed linear operator $B : D(B) \to X$ such that $D(A) \subset D(B) \subset X$ and B is subordinated to A according to the condition*

$$\|Bv\|_X \leqslant c\|Av\|_X + c'\|v\|_X,\ v \in X \tag{2.3}$$

If the condition 2.3 holds with $c \leqslant M_0 = \frac{1}{2(1+M)}$ and $4c'M \leqslant |\lambda|$, then the perturbed operator $A+B$ with $D(A+B) = D(A)$ is sectorial in X.

Proof. See Cholewa and Dlotko [15, Page 37].

Corollary 2.32. *Under the assumptions of Proposition 2.31 and additional requirements that both A and $A+B$ are positive operators with its fractional powers of purely imaginary exponent being bounded operators, the following equality holds:*

$$D((A+B)^\alpha) = D(A^\alpha),\ \alpha \in (0,1). \tag{2.4}$$

Proof. See Cholewa and Dlotko [15, Page 52].

2.3 Global attractors

In order to understand the definition of global attractor, we need to introduce some terminologies, for more details we refer to Carvalho, Langa and Robinson [13] and references therein.

Definition 2.33. *The Hausdorff semidistance between A and B is defined as*

$$\textbf{\textit{dist}}_H(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b).$$

Definition 2.34. *Let A and B be subsets of $\mathfrak{M}$. We say that A **attracts** B under semigroup $S(\cdot)$ if*

$$\lim_{t\to 0} \textbf{\textit{dist}}_H(S(t)B, A) = 0.$$

*When there is a bounded set $B \subset \mathfrak{M}$ which attracts each bounded set of $\mathfrak{M}$ by the semigroup $S(\cdot)$, we call $S(\cdot)$ of a **bounded dissipative semigroup**.*

Definition 2.35. *Let A and B be subsets of $\mathfrak{M}$. We say that A **absorbs** B under semigroup $S(\cdot)$ if there is $t_0 > 0$ such that*

$$S(t)B \subset A, \ \ \forall t \geqslant t_0.$$

Definition 2.36. *The set B is an **absorbing set** of $S(\cdot)$ if each bounded set $B_0 \subset \mathfrak{M}$, B absorb B_0 under $S(\cdot)$.*

Definition 2.37. *The **global attractor** of the semigroup $S(\cdot)$ is a set $\mathcal{A} \subset X$ such that*

- *$\mathcal{A}$ is compact;*
- *$\mathcal{A}$ is invariant under semigroup $S(\cdot)$;*
- *$\mathcal{A}$ attracts any bounded subsets of X under the semigroup.*

The global attractor, if it exists, is easily seen to be unique. The next result will be useful to show the existence of the global attractor.

Theorem 2.38. *If $S(\cdot)$ is bounded dissipative and asymptotically compact, then $S(\cdot)$ has a global attractor.*

Proof. See Carvalho, Langa and Robinson [13, Page 34].

Definition 2.39. *We say that the family $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ of subsets of X is **upper semicontinuitinuous** at λ_0 if*

$$\lim_{\lambda\to\lambda_0} \textbf{\textit{dist}}_H(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0.$$

2.4 Nonlinear evolution processes

We wish to deal with the non-autonomous case of the thermoelastic problem. To do so, we need to adapt the concept of semigroup for the non-autonomous case, for more details we refer to Babin and Vishik [3], Carvalho, Langa and Robinson [13], Cholewa and Dlotko [15], Hale [24] and references therein.

Definition 2.40. *A family of maps $\{U(t,s); t \geqslant s\}$ in $\mathcal{C}(\mathfrak{M})$ is a* ***nonlinear evolution process*** *if*

(1.) $U(t,t) = I$, for all $t \in \mathbb{R}$,

(2.) $U(t,s) = U(t,\tau)S(\tau,s)$, for all $t \geqslant \tau \geqslant s$,

(3.) $\{(t,s,x) \in \mathbb{R}^2 \times X; t \geqslant s\} \ni (t,s,x) \mapsto U(t,s)x \in \mathfrak{M}$ is continuous.

Consider the problem

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + A(t)\mathbf{u} = F, \ t \geqslant s \\ \mathbf{u}(s) = \mathbf{u}_0 \in Y \end{cases} \tag{2.5}$$

where $Y \subset X$ is dense, $F \in L^1([0,T];X)$ and $\{-A(t); t \in \mathbb{R}\}$ is a family of operator with domain $D(A(t)) \subset X$ for any $t \in \mathbb{R}$ under conditions sufficient to ensures the existence of a process $U(\cdot,\cdot)$ such that if $\mathbf{u}$ is a classical or strong solution for (2.5), then

$$\mathbf{u} = U(t,s)\mathbf{u}_0 + \int_s^t U(t,\tau)F(\tau,\mathbf{u})d\tau, \ t \geqslant s \tag{2.6}$$

where the process $U(\cdot,\cdot)$ is given by

$$U(t,s) = e^{-(t-s)A(s)} + \int_s^t U(t,\tau)[A(\tau) - A(s)]e^{-(\tau-s)A(s)}d\tau.$$

Definition 2.41. *We will call $\mathbf{u} : [s,T] \to X$ a* ***mild solution*** *for* (2.5) *if $\mathbf{u}$ is defined by* (2.6) *for each $\mathbf{u}_0 \in X$.*

Theorem 2.42. *Let $F : [s,T] \times X \to X$ be continuous in t on $[s,T]$ and uniformly Lipschitz continuous on X. If $-A(s)$ is the infinitesimal generator of a linear C_0-semigroup $U(\cdot,s)$ on X for each $s \in (-\infty,T]$, then every $u_0 \in X$ the initial value problem* (2.5) *has a unique mild solution $u \in C([s,T],X)$. Moreover, the mapping $X \ni \mathbf{u_0} \mapsto \mathbf{u} \in C([s,\tau];X)$ is Lipschitz continuous from X into $C([s,T],X)$.*

Proof. For a given $\mathbf{u}_0 \in X$, we define a mapping

$$G_{\mathbf{u}_0} : C(\mathbb{R},X) \to C(\mathbb{R},X)$$

by

$$(2.7) \qquad (G_{\mathbf{u}_0}\mathbf{u})(t) = U(t,s)\mathbf{u}_0 + \int_s^t U(t,\tau)F(\tau,\mathbf{u}(\tau))d\tau,\ s \leqslant t \leqslant T.$$

Denoting by $\|\mathbf{u}\|_\infty$ the norm of $\mathbf{u}$ as an element of $C((-\infty,T],X)$ it follows from the choice of F that

$$(2.8) \qquad \|(G_{\mathbf{u}_0}u)(t) - (G_{\mathbf{u}_0}v)(t)\|_X \leqslant M(s)L(t-s)\|u-v\|_\infty$$

where $M(s)$ is a bound of $\|U(t,s)\|$ on $[s,T]$. Using (2.7), (2.8) and finite induction on n it follows easily that

$$\|(G^n_{\mathbf{u}_0}u)(t) - (G^n_{\mathbf{u}_0}v)(t)\|_X \leqslant \frac{(M(s)L(t-s))^n}{n!}\|u-v\|_\infty$$

whence

$$(2.9) \qquad \|G^n_{\mathbf{u}_0}u - G^n_{\mathbf{u}_0}v\| \leqslant \frac{(M(s)LT)^n}{n!}\|u-v\|_\infty$$

For n large enough $(M(s)LT)^n/n! < 1$ for all $s \in \mathbb{R}$ and by a well known extension of the contraction principle $G_{\mathbf{u}_0}$ has a unique fixed point u in $C([s,T],X)$. This fixed point is the desired solution of the integral equation (2.6).

The uniqueness of $\mathbf{u}$ and the Lipschitz continuity of the map $\mathbf{u}_0 \to \mathbf{u}$ are consequences of the following argument. Let $\mathbf{v}$ be a mild solution of (2.5) on $[s,T]$ with the initial value $\mathbf{v}_0$. Then,

$$\begin{aligned}\|\mathbf{u}(t)-\mathbf{v}(t)\| &\leqslant \|U(t,s)\mathbf{u}_0 - U(t,s)\mathbf{v}_0\| + \int_s^t \|U(t,s)\|\|F(s,\mathbf{u}(s)) - F(s,\mathbf{v}(s))\| \\ &\leqslant M\|\mathbf{u}_0-\mathbf{v}_0\| + ML\int_s^t \|\mathbf{u}(s)-\mathbf{v}(s)\|ds,\end{aligned}$$

which implies, by Gronwall's inequality, that

$$\|\mathbf{u}(t)-\mathbf{v}(t)\| \leqslant Me^{ML(T-s)}\|\mathbf{u}_0-\mathbf{v}_0\|$$

and therefore

$$\|\mathbf{u}-\mathbf{v}\| \leqslant Me^{ML(T-s)}\|\mathbf{u}_0-\mathbf{v}_0\|$$

which yields both the uniqueness of $\mathbf{u}$ and the Lipschitz continuity of the map $\mathbf{u}_0 \mapsto \mathbf{u}$.

□

Theorem 2.43. *Let $F : \mathbb{R}\times X \to X$ be continuous in t on $\mathbb{R}$ and locally Lipschitz continuous on X. If $-A(s)$ is the infinitesimal generator of a C_0-semigroup $U(\cdot,s)$ on X for each $s \in \mathbb{R}$, then every $\mathbf{u}_0 \in X$ the initial value problem* (2.5) *has a unique mild solution $\mathbf{u} \in C([s,t_{\max}),X)$. Moreover, if $t_{\max} < \infty$ then*

$$\lim_{t\to t_{\max}} \|\mathbf{u}(t)\| = \infty.$$

Proof. We start by showing that for every $t_0 \geqslant s$, $\mathbf{u}_0 \in X$, the initial value problem (2.5) has, under the assumptions of our theorem, a unique mild solution u on an interval $[t_0, t_1]$ whose length is bounded below by

$$\delta(t_0, \|u_0\|) = \min\left\{1, \frac{\|u_0\|}{K(t_0,s)L(K(t_0,s),t_0+1) + M(t_0,s)}\right\} \tag{2.10}$$

where $L(c,t)$ is the local Lipschitz constant of F, $M(t_0,s) = \sup\{\|U(t,s)\|; s \leqslant t \leqslant t_0+1\}$, $K(t_0,s) = 2\|u_0\|M(t_0,s)$ and $N(t_0,s) = \max\{\|F(t,0)\|; s \leqslant t \leqslant t_0+1\}$. Indeed, let $t_1 = t_0 + \delta(t_0, \|u_0\|)$, the mapping $G_{\mathbf{u}_0}$ defined by (2.7) maps the ball of radius $K(t_0,s)$ centred at 0 of $C([t_0,t_1];X)$ into itself. This follows from the estimates

$$\begin{aligned}
\|(G_{\mathbf{u}_0}\mathbf{u})(t)\| &\leqslant M(t_0,s)\|\mathbf{u}_0\| + \int_{t_0}^{t} \|U(t,\tau)\| \left(\|F(\tau,\mathbf{u}(\tau)) - F(\tau,0)\| + \|F(\tau,0)\|\right) ds \\
&\leqslant M(t_0,s)\left\{\|\mathbf{u}_0\| + K(t_0,s)L(K(t_0,s),t_0+1)(t-t_0) + N(t_0,s)(t-t_0)\right\} \\
&\leqslant 2M(t_0,s)\|\mathbf{u}_0\| = K(t_0,s)
\end{aligned}$$

In this ball, G satisfies a uniform Lipschitz condition with constant $L = L\left(K(t_0,s), t_0+1\right)$ and thus as in the proof of Theorem 2.42 it has a unique fixed point $\mathbf{u}$ in the ball. This fixed point is the desired solution of (2.5) on the interval $[t_0, t_1]$.

From what we have just proved it follows that if $\mathbf{u}$ is a mild solution of (2.5) on the interval $[s, \tau_0]$ it can be extended to the interval $[s, \tau_0+\delta]$ with $\delta > 0$ by defining on $[\tau_0, \tau_0 + \delta]$, $\mathbf{u}(t) = \mathbf{w}(t)$ where $\mathbf{u}(t)$ is the solution of the integral equation

$$\mathbf{u}(t) = U(t,\tau_0)\mathbf{u}(\tau_0) + \int_{\tau_0}^{t} U(t,\tau)F(\tau,\mathbf{w}(\tau))d\tau, \quad \tau_0 \leqslant t \leqslant \tau_0 + \delta. \tag{2.11}$$

Moreover, δ depends only on $\|u(\tau_0)\|$, $K(\tau_0,s)$ and $N(\tau_0,s)$.

Let $[s, t_{\max})$ be the maximal interval of existence of the mild solution $\mathbf{u}$ of (2.5). If $t_{\max} < \infty$ then $\lim_{t\to t_{\max}} \|\mathbf{u}(t)\| = \infty$ since otherwise there is a sequence $t_n \to t_{\max}^{+}$ such that $\|\mathbf{u}(t_n)\| \leqslant C$ for all n. This would imply by what we have just proved that for each t_n, near enough to $t_{\max}$, $\mathbf{u}$ defined on $[s, t_n]$ can be extended to $[s, t_n+\delta]$ where $\delta > 0$ is independent of t_n and hence $\mathbf{u}$ can be extended beyond $t_{\max}$ contradicting the definition of $t_{\max}$.

To prove the uniqueness of the local mild solution $\mathbf{u}$ of (2.5) we note that if $\mathbf{v}$ is a mild solution of (2.5) then on every closed interval $[s, t_0]$ on which both $\mathbf{u}$ and $\mathbf{v}$ exist they coincide by the uniqueness argument given at the end of the proof of Theorem 2.42. Therefore, both $\mathbf{u}$ and $\mathbf{v}$ have the same $t_{\max}$ and on $[s, t_{\max})$, $\mathbf{u} = \mathbf{v}$.

□

2.5 Pullback attractors

In order to understand the definition of pullback attractor, we need to introduce some terminologies, for more details on the concept of pullback attractor we refer to Carvalho, Langa and Robinson [13] and references therein.

Definition 2.44. *Let $U(\cdot,\cdot)$ be a nonlinear evolution process. Given $t \in \mathbb{R}$, a set $K \subset X$* ***pullback attracts*** *a set D at time t under $U(\cdot,\cdot)$ if*

$$\lim_{s\to-\infty} \textit{\textbf{dist}}_H(U(t,s)D, K) = 0. \tag{2.12}$$

K pullback attracts bounded sets at time t if (2.12) *holds for each bounded subset D of X.*

A time dependent family of subset of X, $K(\cdot)$ pullback attracts bounded subsets of X under $U(\cdot,\cdot)$ if $K(t)$ pullback attracts bounded sets at time t under $U(\cdot,\cdot)$, for each $t \in \mathbb{R}$.

Definition 2.45. *A family $\{\mathcal{A}(t); t \in \mathbb{R}\}$ is the* ***pullback attractor*** *for a nonlinear evolution process $U(\cdot,\cdot)$ if*

(1.) $\mathcal{A}(t)$ is compact for each $t \in \mathbb{R}$,

(2.) $A(\cdot)$ is invariant with respect to $U(\cdot,\cdot)$,

(3.) $\mathcal{A}(\cdot)$ pullback attracts bounded subsets of X, and

(4.) if there is another family $C(\cdot)$ of closed sets that pullback attracts bounded subsets of X, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Definition 2.46. *A nonlinear evolution process $U(\cdot,\cdot)$ in $\mathfrak{M}$ is said to be* ***pullback asymptotically compact*** *if, for each $t \in \mathbb{R}$, each sequence $\{s_k\} \leqslant t$ with $s_k \to -\infty$ as $k \to \infty$, and each bounded sequence $\{x_k\} \in X$, the sequence $\{U(t,s_k)x_k\}$ has a convergent subsequence.*

Definition 2.47. *A nonlinear evolution process $U(\cdot,\cdot)$ in $\mathfrak{M}$ is said to be* ***strongly pullback bounded dissipative*** *if for each $t \in \mathbb{R}$ there is a bounded subset $B(t)$ of $\mathfrak{M}$ that pullback attracts bounded subsets of $\mathfrak{M}$ at time τ for each $\tau \leqslant t$; that is, given a bounded subset D of $\mathfrak{M}$ and $\tau \leqslant t$, $\lim_{t\to-\infty} \textit{\textbf{dist}}_H(U(\tau,s)D, B(t)) = 0$.*

The following is a result that gives sufficient conditions for the existence of attractor.

Theorem 2.48. *If a nonlinear process $U(\cdot,\cdot)$ is strongly pullback bounded dissipative and pullback asymptotically compact and $B(\cdot)$ is a family of bounded subsets of X such that, for each $t \in \mathbb{R}$, $B(t)$ pullback attracts bounded subsets of X at time τ for each $\tau \leqslant t$, then $U(\cdot,\cdot)$ has a compact pullback attractor $\mathcal{A}(\cdot)$ such that $\bigcup_{s\leqslant t} \mathcal{A}(s)$ is bounded for each $t \in \mathbb{R}$.*

Proof. See Carvalho, Langa and Robinson [13, Page 35].

Definition 2.49. *We say that $\{\mathcal{A}_\epsilon(\cdot)\}_{\epsilon\in[0,1]}$ is* ***upper semicontinuous*** *as $\epsilon \to 0$ if, for each $t \in \mathbb{R}$, the family $\{\mathcal{A}_\epsilon(t)\}_{\epsilon\in[0,1]}$ is* ***upper semicontinuous*** *as $\epsilon \to 0$.*

We suppose that we have a sequence of nonlinear processes $U_\epsilon(\cdot,\cdot)$ that converges to a limiting process $U_0(\cdot,\cdot)$ in the following sense: for each $t \in \mathbb{R}$ for each compact subset K of X and each $T > 0$,

$$\sup_{\tau\in[0,T]} \sup_{x\in K} \textit{dist}_H(U_\epsilon(t,t-\tau), U_0(t,t-\tau)x) \to 0 \text{ as } \epsilon \to 0. \tag{2.13}$$

It is therefore natural to make the standing assumption that for each $t \in \mathbb{R}$

$$\overline{\bigcup_{\epsilon\in[0,1]} \mathcal{A}_\epsilon(t)} \text{ is compact} \tag{2.14}$$

if we want prove continuity of attractors.

We have already seen that pathologies are possible when the pullback attractor is not bounded in the past. We therefore impose the additional condition that for each $t \in \mathbb{R}$,

$$\bigcup_{\epsilon\in[0,1]} \bigcup_{s<t} \mathcal{A}_\epsilon(s) \text{ is bounded.} \tag{2.15}$$

Theorem 2.50. *Let $U_\epsilon(\cdot,\cdot)$ be a sequence of nonlinear evolution processes with corresponding pullback attractors $\mathcal{A}_\epsilon(\cdot)$ for $\epsilon \in [0,1]$. Assume that*

(1.) for each $t \in \mathbb{R}$ and for each compact subset K of X and each $T > 0$,

$$\sup_{\tau\in[0,T]} \sup_{x\in K} \textit{dist}_H(U_\epsilon(t,t-\tau)x, U_0(t,t-\tau)x) \to 0 \textit{ as } \epsilon \to 0;$$

(2.) $\overline{\bigcup_{\epsilon\in[0,1]} \mathcal{A}_\epsilon(t)}$ *is compact;*

(3.) $\bigcup_{\epsilon\in[0,1]} \bigcup_{s<t} \mathcal{A}_\epsilon(s)$ *is bounded.*

Then, $\mathcal{A}_\epsilon(\cdot)$ is upper semicontinuous as $\epsilon \to 0$.

Proof. See Carvalho, Langa and Robinson [13, Page 57].

Chapter 3

Autonomous n-dimensional thermoelasticity system

In this is one of the main chapter, we aim to make a study of the asymptotic behavior, in the sense of global attractors, of the solutions of a class of n-dimensional thermoelastic systems with $n \geqslant 2$. The results presented here make up an article entitled "Attractors for a class of thermoelastic systems with vanishing mean value for temperature", which was accepted for publication in the Journal Colloquium Mathematicum.

3.1 Preliminary

We are interested in the study of asymptotic behavior of mild solutions for a multidimensional semilinear thermoelastic systems; namely, initial-boundary value problems with space dependent diffusion coefficients

$$\begin{cases} \partial_t^2 u - \Delta u - \nabla \operatorname{div} u + \nabla\theta = f(u), & x \in \Omega,\ t > 0, \\ \partial_t \theta - \operatorname{div}(\kappa(x)\nabla\theta) + \operatorname{div} \partial_t u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{3.1}$$

subject to boundary conditions

$$u(x,t) = 0,\ \theta(x,t) = 0, x \in \partial\Omega,\ t > 0, \tag{3.2}$$

and

$$\kappa(x)\nabla\theta(x,t) - \partial_t u(x,t) = 0, x \in \partial\Omega,\ t > 0, \tag{3.3}$$

on initial conditions

$$\begin{cases} u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), & x \in \Omega, \\ \theta(x,0) = \theta_0(x) & x \in \Omega. \end{cases} \tag{3.4}$$

In this problem, the map f is external force and the functional parameters κ is the diffusion coefficient with some suitable growth conditions which will be presented below.

We recall that for a smooth vector field in some sense $u = (u_1, \ldots, u_n)$ the gradient and Laplacian of the vector field u are denoted, respectively, by

$$\nabla u = (\nabla u_1, \ldots, \nabla u_n)$$

and

$$\Delta u = (\Delta u_1, \ldots, \Delta u_n),$$

and the divergent operator of a vector field $\partial_t u$ will be denoted by

$$\operatorname{div} \partial_t u = \sum_{i=1}^{n} \partial_{x_i} \partial_t u_i.$$

The hypotheses on the nonlinearity $f = (f_1, \ldots, f_n)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$. We consider f a conservative vector field (i.e., there is a scalar field α such that $f = \nabla \alpha$) with the functions f_i twice continuously differentiable and $f_i(0) = 0$, $i = 2, 3, 4$. Moreover, we also assume that for each $\nu > 0$ there exists $C_\nu > 0$ such that

$$f(\xi) \cdot \xi \leqslant \nu |\xi|^2 + C_\nu, \tag{3.5}$$

with $\cdot$ denoting the standard dot product on $\mathbb{R}^n$. Because of (3.5), we can assume that there exist $C_\eta > 0$ and $\eta \in (0, \min\{1, \lambda_1\})$ such that if

$$F(\xi) := \int_0^\xi f d\gamma,$$

then

$$F(\xi) \leqslant \frac{\eta}{2} |\xi|^2 + C_\eta, \tag{3.6}$$

where $\lambda_1 > 0$ is the first eigenvalue of the negative Laplacian operator with zero Dirichlet boundary condition, and $\int_0^\xi f d\gamma$ represents the line integral of f along a piecewise smooth curve $\gamma : [s, t] \to \mathbb{R}^n$ wich $\gamma(s) = 0$ and $\gamma(t) = \xi$, for any $\xi \in \mathbb{R}^n$ (that is, $\nabla F(\xi) = f(\xi)$, where ∇F stands for the gradient of F in the variables $\xi \in \mathbb{R}^n$).

In addition, we shall assume throughout this text that there exists a constant $C > 0$ such that for every $i = 1, \ldots, n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$\begin{aligned} &|\nabla f_i(\xi)| \leqslant C \left(1 + \sum_{i=1}^{n} |\xi_i|^{\mathfrak{p}-1} \right), \\ &|\partial_{x_i}^2 f_i(\xi)| \leqslant C, \end{aligned} \tag{3.7}$$

for some $1 < \mathfrak{p} < \frac{n}{n-2}$, if $n > 2$; $1 < \mathfrak{p} < +\infty$ if $n = 2$.

The coefficients κ in (3.1), is a real-valued continuously differentiable function defined on Ω such that there exist constants κ_0 and κ_1 with the property

$$0 < \kappa_0 \leqslant \kappa(x) \leqslant \kappa_1, \ x \in \Omega. \tag{3.8}$$

In order of better present our results, we introduce some terminology. Motivated by Lebeau and Zuazua [31] we will consider the Hilbert space $(H_0^1(\Omega))^n$ equipped with the inner product

$$(u_1, u_2)_{(H_0^1(\Omega))^n} = \int_\Omega (\nabla u_1 \nabla u_2 + \operatorname{div} u_1 \operatorname{div} u_2) dx \tag{3.9}$$

and consequently the product space

$$\mathcal{H} = (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega)$$

equipped with the inner product

$$(\mathbf{u}_1, \mathbf{u}_2)_{\mathcal{H}} = \int_\Omega \nabla u_1 \nabla u_2 dx + \int_\Omega \operatorname{div} u_1 \operatorname{div} u_2 \, dx + \int_\Omega z_1 z_2 dx + \int_\Omega \theta_1 \theta_2 dx$$

for all $\mathbf{u}_1 = (u_1, z_1, \theta_1), \mathbf{u}_2 = (u_2, z_2, \theta_2) \in \mathcal{H}$.

3.2 Well-possessedness of the problem

Let $\mathbf{u} = (u, z, \theta)$ be the state vector with $z = \partial_t u$, we rewrite (3.1)-(3.5) as an initial-value problem associated to an ordinary differential equation in the product space $\mathcal{H}$ as follows

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} + \mathbf{A}\mathbf{u} = \mathbf{F}(\mathbf{u}), \ t > 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{3.10}$$

where $\mathbf{u}_0 = (u_0, z_0, \theta_0)$, $\mathbf{A} : D(\mathbf{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear unbounded operator defined by

$$D(\mathbf{A}) = \big((H_0^1(\Omega) \cap H^2(\Omega))^n \times (H_0^1(\Omega))^n \times (H_0^1(\Omega) \cap H^2(\Omega))\big) \bigcap X,$$

and for any $(u, z, \theta) \in D(\mathbf{A})$

$$\mathbf{A}(u, z, \theta) = (-z, -\Delta u - \nabla \operatorname{div} u + \nabla\theta, -\text{div}(\kappa\nabla\theta) + \operatorname{div} z)$$

where

$$X = \{(u, z, \theta) \in \mathcal{H}; \kappa(x)\nabla\theta(x, \cdot) - \partial_t u(x, \cdot) = 0 \text{ in } L^2(\partial\Omega)\}.$$

The nonlinear term in (3.10) is defined by

$$\mathbf{F}(\mathbf{u}) = (0, f^e(u), 0),$$

where f^e denotes the Nemytskĭi operator associated with f, i.e.

$$f^e(u) = f(u(t,x)) = (f_1(u(t,x)), \dots, f_n(u(t,x)))$$

for any $t \geqslant 0, x \in \Omega$ and by simplicity of notation we also denote f^e by f.

We choose as a base space for (3.10) the product space $\mathcal{H}$, see [31] and references therein. This choice allows us may exhibit a Lyapunov functional to (3.10); namely

$$\mathcal{E}(u,z,\theta) = \frac{1}{2}\Big(\|u\|^2_{(H^1_0(\Omega))^n} + \|z\|^2_{(L^2(\Omega))^n} + \|\theta\|^2_{L^2(\Omega)}\Big) - \int_\Omega F(u)dx \tag{3.11}$$

where $\|\cdot\|^2_{(H^1_0(\Omega))^n} = (\cdot,\cdot)_{(H^1_0(\Omega))^n}$ defined in (3.9) and $F(u) = \int_0^u f d\gamma$ and $\int_0^u f d\gamma$ represents the line integral of f along a piecewise smooth curve with initial point 0 and final point u with $u = u(x,t)$, decreases along trajectories. More precisely, multiplies by $\partial_t u$ the first equation of (3.1) and second by θ, we obtain

$$\begin{cases} (\partial_t^2 u - \Delta u - \nabla \operatorname{div} u + \nabla\theta)\partial_t u = f(u)\partial_t u \\ (\partial_t\theta - \operatorname{div}(\kappa(x)\nabla\theta) + \operatorname{div}\partial_t u)\theta = 0 \end{cases}$$

by adding the two equations we obtain

$$\frac{d\mathcal{E}}{dt} = -\int_\Omega \kappa(x)|\nabla\theta|^2 dx \leqslant 0, \tag{3.12}$$

where $\mathcal{E}(t) = \mathcal{E}(u(t), z(t), \theta(t))$ for any $t \geqslant 0$.

Differentiability of the Nonlinearity

To prove the differentiability of F we first see that it is enough to prove the differentiable of f^e. Since the map F is defined from $\mathcal{H}$ into $\mathcal{H}$, its derivative DF is defined by for each $\mathbf{u} = (u,z,\theta) \in \mathcal{H}$ as follows

$$\mathcal{H} \ni \mathbf{h} = (h^1, h^2, h^3) \mapsto DF(\mathbf{u})\cdot\mathbf{h} \in \mathcal{H},$$

where

$$DF(\mathbf{u})\cdot\mathbf{h} = (0, Df^e(u)\cdot h^1, 0),$$

and

$$Df^e(u) = Df(u) = (Df_1(u), \dots, Df_n(u)),$$

according to next result.

Lemma 3.1. *If the functions f_i satisfy* (3.7)*, then there exists $C > 0$ such that for $i = 1, \dots, n$, and $u = (u_1, \dots, u_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we have that*

$$|f_i(u) - f_i(y)| \leqslant 2^{\mathfrak{p}-1} nC|u-y| \left(1 + \sum_{i=1}^n |u_i|^{\mathfrak{p}-1} + \sum_{i=1}^n |y_i|^{\mathfrak{p}-1}\right).$$

Consequently, there exists $\tilde{C} > 0$ for any $\mathbf{u}_1 = (u_1, z_1, \theta_1), \mathbf{u}_2 = (u_2, z_2, \theta_2) \in \mathcal{H}$ with $u_i = (u_{i1}, \ldots, u_{in})$ we deduce that

$$\|F(\mathbf{u}_1) - F(\mathbf{u}_2)\|_{\mathcal{H}} \leqslant \tilde{C}\|u_1 - u_2\|_{(H^1(\Omega))^n} \left(1 + \sum_{i=1}^{2}\sum_{j=1}^{n} \|u_{ij}\|_{H^1(\Omega)}^{\mathfrak{p}-1}\right). \tag{3.13}$$

Proof. Give $u = (u_1, \ldots, u_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, it follows from mean value theorem the existence of $\vartheta \in (0,1)$ such that

$$|f_i(u) - f_i(y)| \leqslant |u - y||\nabla f_i((1-\vartheta)u + \vartheta y)|$$

and by (3.7) we have that

$$\begin{aligned}
|f_i(u) - f_i(y)| &\leqslant C|u-y|\left(1 + \sum_{i=1}^{n} |(1-\vartheta)u_i + \vartheta y_i|^{\mathfrak{p}-1}\right) \\
&\leqslant 2^{\mathfrak{p}-1} nC|u-y|\left(1 + \sum_{i=1}^{n} |(1-\vartheta)u_i|^{\mathfrak{p}-1} + \sum_{i=1}^{n} |\vartheta y_i|^{\mathfrak{p}-1}\right) \\
&\leqslant 2^{\mathfrak{p}-1} nC|u-y|\left(1 + \sum_{i=1}^{n} |u_i|^{\mathfrak{p}-1} + \sum_{i=1}^{n} |y_i|^{\mathfrak{p}-1}\right).
\end{aligned}$$

Due to Hölder inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ we obtain that

$$\|F(\mathbf{u}_1) - F(\mathbf{u}_2)\|_{\mathcal{H}} = \|f(u_1) - f(u_2)\|_{(L^2(\Omega))^n},$$

where $\mathbf{u}_1 = (u_1, z_1, \theta_1), \mathbf{u}_2 = (u_2, z_2, \theta_2) \in \mathcal{H}$ and

$$\begin{aligned}
&\|f_i(u_1) - f_i(u_2)\|_{L^2(\Omega)} \\
&\leqslant 2^{\mathfrak{p}-1} nC\left[\int_\Omega |u_1 - u_2|^2 \left(1 + \sum_{i=1}^{2}\sum_{j=1}^{n} |u_{ij}|^{\mathfrak{p}-1}\right)^2 dx\right]^{\frac{1}{2}} \\
&\leqslant 2^{\mathfrak{p}-1} nC\left[\int_\Omega |u_1 - u_2|^{\frac{2n}{n-2}} dx\right]^{\frac{n-2}{2n}} \left[\int_\Omega \left(1 + \sum_{i=1}^{2}\sum_{j=1}^{n} |u_{ij}|^{\mathfrak{p}-1}\right)^n dx\right]^{\frac{1}{n}},
\end{aligned}$$

in other words,

$$\begin{aligned}
&\|f_i(u_1) - f_i(u_2)\|_{(L^2(\Omega))^n} \\
&\leqslant \tilde{C}\|u_1 - u_2\|_{(L^{\frac{2n}{n-2}}(\Omega))^n}\left(1 + \sum_{i=1}^{2}\sum_{j=1}^{n} \|u_{ij}\|_{L^{(\mathfrak{p}-1)n}(\Omega)}^{\mathfrak{p}-1}\right) \\
&\leqslant \tilde{C}\|u_1 - u_2\|_{(H^1(\Omega))^n}\left(1 + \sum_{i=1}^{2}\sum_{j=1}^{n} \|u_{ij}\|_{H^1(\Omega)}^{\mathfrak{p}-1}\right).
\end{aligned}$$

The bound in (3.13) now follows in a straightforward way from the definition of F.

□

Lemma 3.2. *If the functions f_i satisfy* (3.7)*, then the Nemytskĭi operators associated to f_i, $f_i^e : (H_0^1(\Omega))^n \to L^2(\Omega)$ are continuously differentiable and the derivative operators $Df_i^e : (H_0^1(\Omega))^n \to \mathcal{L}((H_0^1(\Omega))^n, L^2(\Omega))$ are Lipschitz continuous (in bounded subsets of $(H_0^1(\Omega))^n$). Consequently, DF is also Lipschitz continuous (in bounded subsets of $\mathcal{H}$), for $n = 3, 4$. For $n > 4$, there exists a constant $\eta \in (0, 1)$ such that*

$$\|Df_i^e(u) - Df_i^e(v)\|_{\mathcal{L}((H_0^1(\Omega))^n, L^2(\Omega))} \leqslant c\|u - v\|^{\eta}_{(H_0^1(\Omega))^n}, \quad \forall u, v \in (H_0^1(\Omega))^n.$$

Proof. For each $u \in (H_0^1(\Omega))^n$ define the map $Df_i^e(u) \in \mathcal{L}((H_0^1(\Omega))^n, L^2(\Omega))$ by

$$(Df_i^e(u) \cdot h)(x) = D(f_i)(u(x,t)) \cdot h(x). \tag{3.14}$$

First we check that this is well defined. In fact, let $u = (u_1, \ldots, u_n), h = (h_1, \ldots, h_n) \in (H_0^1(\Omega))^n$, then $u, h \in (L^{\frac{2n}{n-2}}(\Omega))^n$, and using (3.7) we get

$$\int_\Omega |Df_i(u(x,t))|^2 |h(x)|^2 dx \leqslant c^2 \int_\Omega \left(1 + \sum_{i=1}^n |u_i(x)|^{\mathfrak{p}-1}\right)^2 |h(x)|^2 dx,$$

and by Hölder inequality

$$\begin{aligned}\int_\Omega |Df_i(u(x,t))|^2 |h(x)|^2 dx &\leqslant c^2 \left\| \left(1 + \sum_{i=1} |u_i|^{\mathfrak{p}-1}\right)^2 \right\|_{L^{\frac{n}{2}}(\Omega)} \||h|^2\|_{L^{\frac{n}{n-2}}(\Omega)} \\ &\leqslant c^2 \left(1 + \sum_{i=1}^n \|u_i\|^{2(\mathfrak{p}-1)}_{L^{n(\mathfrak{p}-1)}(\Omega)}\right) \|h\|^2_{(L^{\frac{2n}{n-2}}(\Omega))^n}.\end{aligned}$$

Since $u, h \in (L^{\frac{2n}{n-2}}(\Omega))^n$ and $\mathfrak{p} < \frac{n}{n-2}$, it follows that

$$\int_\Omega |Df_i(u(x,t))|^2 |h(x)|^2 dx \leqslant c \left(1 + \sum_{i=1}^n \|u_i\|^{2(\mathfrak{p}-1)}_{H^1(\Omega)}\right) \|h\|^2_{(H^1(\Omega))^n}.$$

Hence $Df_i^e(u)h \in L^2(\Omega)$ and $Df_i^e(u) \in \mathcal{L}((H_0^1(\Omega))^n, L^2(\Omega))$. Now let us check that $Df_i^e(u)$ is indeed the Fréchet derivative of f_i^e at u. If $u = (u_1, \ldots, u_n), h = (h_1, \ldots, h_n) \in (H_0^1(\Omega))^n$ then

$$\begin{aligned}&\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|^2_{L^2(\Omega)} \\ &= \int_\Omega |f_i(u(x,t) + h(x)) - f_i(u(x,t)) - Df_i(u(x,t)) \cdot h(x)|^2 dx \\ &\leqslant \int_\Omega |D^2(f_i)(u(x,t) + \sigma(x)h(x))|^2 |h(x)|^4 dx,\end{aligned}$$

where $\sigma(x) \in (0, 1)$, for all $x \in \Omega$. Thus, using (3.7) we obtain that

$$\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|^2_{L^2(\Omega)} \leqslant c \int_\Omega |h(x)|^4 dx. \tag{3.15}$$

Case1: $n = 3$**.** It follows from (3.15) and Hölder inequality that

$$\begin{aligned}\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^2(\Omega)}^2 &\leqslant c \int_\Omega |h(x)|^4 dx \\ &\leqslant c|\Omega|^{\frac{1}{3}} \||h|^4\|_{L^{\frac{3}{2}}(\Omega)} \\ &\leqslant c|\Omega|^{\frac{1}{3}} \||h|\|_{L^6(\Omega)}^4,\end{aligned}$$

and consequently

$$\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^2(\Omega)} \leqslant c\|h\|_{(H_0^1(\Omega))^n}^2.$$

This proves the differentiability of f_i^e in u for each $u \in (H_0^1(\Omega))^n$.

Case2: $n = 4$**.** Using (3.15) we have that

$$\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^2(\Omega)}^2 \leqslant c \int_\Omega |h(x)|^4 dx = c\||h|\|_{L^4(\Omega)}^4.$$

Remember that $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ for $n = 4$, and then

$$\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^2(\Omega)} \leqslant c\|h\|_{(H_0^1(\Omega))^n}^2.$$

This proves the differentiability in this case.

Case3: $n > 4$**.** Observe that $\frac{n}{n-2} < 2 < \frac{2n}{(n-2)\mathfrak{p}}$. We have that

$$\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^{\frac{n}{n-2}}(\Omega)}^{\frac{n}{n-2}} = \int_\Omega |[D^2(f_i)(u(x,t) + \sigma(x)h(x))]h^2(x)|^{\frac{n}{n-2}} dx,$$

where $\sigma(x) \in (0,1)$, for all $x \in \Omega$. It follows from (3.7) that

$$\begin{aligned}\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^{\frac{n}{n-2}}(\Omega)}^{\frac{n}{n-2}} &= c \int_\Omega |h(x)|^{\frac{2n}{n-2}} dx \\ &\leqslant c\||h|\|_{L^{\frac{2n}{n-2}}(\Omega)}^{\frac{2n}{n-2}},\end{aligned}$$

and then

$$\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^{\frac{n}{n-2}}(\Omega)} \leqslant c\|h\|_{(H_0^1(\Omega))^n}^2. \tag{3.16}$$

By other hand, we have

$$\begin{aligned}&\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u) \cdot h\|_{L^{\frac{2n}{(n-2)\mathfrak{p}}}(\Omega)}^{\frac{2n}{(n-2)\mathfrak{p}}} \\ &= \int_\Omega |f_i(u(x,t) + h(x)) - f_i(u(x,t)) - Df_i(u(x,t)) \cdot h(x)|^{\frac{2n}{(n-2)\mathfrak{p}}} dx \\ &\leqslant \int_\Omega |D(f_i)(u(x,t) + \theta(x)h(x)) \cdot h(x) - Df_i(u(x,t)) \cdot h(x)|^{\frac{2n}{(n-2)\mathfrak{p}}} dx,\end{aligned}$$

where $\theta(x) \in (0,1)$, for all $x \in \Omega$. Using Hölder inequality with exponents $\frac{\mathfrak{p}}{\mathfrak{p}-1}$ and $\mathfrak{p}$ we have

$$
\begin{aligned}
&\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u)\cdot h\|_{L^{\frac{2n}{(n-2)\mathfrak{p}}}(\Omega)}^{\frac{2n}{(n-2)\mathfrak{p}}} \\
&\leqslant \||D(f_i)(u(x,t)+\theta(x)h(x)) - Df_i(u(x,t))|^{\frac{2n}{(n-2)\mathfrak{p}}}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}(\Omega)} \||h|^{\frac{2n}{(n-2)\mathfrak{p}}}\|_{L^{\mathfrak{p}}(\Omega)} \\
&\leqslant \||D(f_i)(u(x,t)+\theta(x)h(x)) - Df_i(u(x,t))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)}^{\frac{2n}{(n-2)\mathfrak{p}}} \||h|\|_{L^{\frac{2n}{n-2}}(\Omega)}^{\frac{2n}{(n-2)\mathfrak{p}}}
\end{aligned}
\tag{3.17}
$$

Note that

$$
\begin{aligned}
&\||D(f_i)(u(x,t)+\theta(x)h(x)) - Df_i(u(x,t))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)} \\
&\leqslant \||D(f_i)(u(x,t)+\theta(x)h(x))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)} + \||Df_i(u(x,t))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)}
\end{aligned}
$$

Using (3.7) we get

$$
\begin{aligned}
&\||D(f_i)(u(x,t)+\theta(x)h(x))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)}^{\frac{2n}{(n-2)(\mathfrak{p}-1)}} \\
&\leqslant c\int_\Omega \left(1+\sum_{i=1}^n |u_i(x)+\theta(x)h_i(x)|^{\mathfrak{p}-1}\right)^{\frac{2n}{(n-2)(\mathfrak{p}-1)}} dx \\
&\leqslant c\left(1+\sum_{i=1}^n \|u_i(x)+\theta(x)h_i(x)\|_{L^{\frac{2n}{(n-2)}}(\Omega)}^{\frac{2n}{n-2}}\right)
\end{aligned}
$$

and then

$$
\||D(f_i)(u(x,t)+\theta(x)h(x))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)}^{\frac{2n}{(n-2)(\mathfrak{p}-1)}} \leqslant c\left(1+\sum_{i=1}^n \|u_i(x)+\theta(x)h_i(x)\|_{H_0^1(\Omega)}^{\frac{2n}{n-2}}\right). \tag{3.18}
$$

In a similar way, we obtain that

$$
\||D(f_i)(u(x,t))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)}^{\frac{2n}{(n-2)(\mathfrak{p}-1)}} \leqslant c\left(1+\||u|\|_{H_0^1(\Omega)}^{\frac{2n}{n-2}}\right). \tag{3.19}
$$

Using (3.15) and (3.18) in (3.17) we have that

$$
\begin{aligned}
&\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u)\cdot h\|_{L^{\frac{2n}{(n-2)\mathfrak{p}}}(\Omega)} \\
&\leqslant k\left(1+\sum_{i=1}^n[\|u_i(x)\|_{H_0^1(\Omega)}^{\mathfrak{p}-1} + \|\theta(x)h_i(x)\|_{H_0^1(\Omega)}^{\mathfrak{p}-1}] + \|u\|_{(H_0^1(\Omega))^n}^{\mathfrak{p}-1}\right)\|h\|_{(H_0^1(\Omega))^n}.
\end{aligned}
\tag{3.20}
$$

It follows from Interpolation Theorem [22, Proposition 6.10 pg. 185], (3.16) and (3.17) that there exists $\lambda \in (0,1)$ such that

$$
\begin{aligned}
&\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u)\cdot h\|_{L^2(\Omega)} \\
&\leqslant \Big[c\|h\|^2_{(H_0^1(\Omega))^n}\Big]^{1-\lambda} \Big[k\Big(1 + \sum_{i=1}^n [\|u_i(x)\|^{\mathfrak{p}-1}_{H_0^1(\Omega)} + \|\theta(x)h_i(x)\|^{\mathfrak{p}-1}_{H_0^1(\Omega)}] \\
&\qquad\qquad + \|u\|^{\mathfrak{p}-1}_{(H_0^1(\Omega))^n}\Big)\|h\|_{(H_0^1(\Omega))^n}\Big]^{\lambda} \\
&\leqslant c\left(1 + \sum_{i=1}^n [\|u_i(x)\|^{\mathfrak{p}-1}_{H_0^1(\Omega)} + \|\theta(x)h_i(x)\|^{\mathfrak{p}-1}_{H_0^1(\Omega)}] + \|u\|^{\mathfrak{p}-1}_{(H_0^1(\Omega))^n}\right)^{\lambda} \|h\|^{2-\lambda}_{(H_0^1(\Omega))^n}.
\end{aligned}
$$

and consequently

$$\frac{\|f_i^e(u+h) - f_i^e(u) - Df_i^e(u)\cdot h\|_{L^2(\Omega)}}{\|h\|_{(H_0^1(\Omega))^n}} \longrightarrow 0$$

as $\|h\|_{(H_0^1(\Omega))^n} \to 0$. This conclude the proof that D is the Frechét derivative of f_i^e.

Now, we show that $u \mapsto Df_i^e(u)$ is Lipschitz continuous as a map from $(H_0^1(\Omega))^n$ into $\mathcal{L}((H_0^1(\Omega))^n, L^2(\Omega))$, for $n = 3, 4$.

Suppose that $n = 3$, and let $u, h \in (H_0^1(\Omega))^n$, then

$$
\begin{aligned}
&\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|^2_{L^2(\Omega)} \\
&\leqslant \int_\Omega |D(f_i)(u(x,t)) - D(f_i)(v(x,t))||h(x)|^2 dx \\
&\leqslant \int_\Omega |D^2(f_i)(u + \vartheta(x)v(x,t))|^2 |u(x,t) - v(x,t)|^2 ||h(x)|^2 dx,
\end{aligned}
$$

for some $\vartheta(x) \in (0,1)$. It follows from (3.7) that

$$
\begin{aligned}
\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|^2_{L^2(\Omega)} &\leqslant C\int_\Omega |u(x,t) - v(x,t)|^2 ||h(x)|^2 dx \\
&\leqslant C\||u-v|^2\|_{L^{\frac{3}{2}}(\Omega)} \||h|^2\|_{L^3(\Omega)} \\
&\leqslant C\||u-v|\|^2_{L^6(\Omega)} \||h|\|^2_{L^6(\Omega)} \\
&\leqslant C\|u-v\|^2_{(H_0^1(\Omega))^3} \|h\|^2_{(H_0^1(\Omega))^3}.
\end{aligned}
$$

Let $n = 4$. In a similar way, we obtain that

$$
\begin{aligned}
\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|^2_{L^2(\Omega)} &\leqslant C\int_\Omega |u(x,t) - v(x,t)|^2 ||h(x)|^2 dx \\
&\leqslant C\||u-v|\|_{L^4(\Omega)} \||h|\|^2_{L^4(\Omega)} \\
&\leqslant C\|u-v\|^2_{(H_0^1(\Omega))^4} \|h\|^2_{(H_0^1(\Omega))^4},
\end{aligned}
$$

in the last inequality we used that $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$.

Let $n > 4$. Observe that $\frac{n}{n-2} < 2 < \frac{2n}{(n-2)\mathfrak{p}}$ and then

$$L^{\frac{2n}{(n-2)\mathfrak{p}}}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{\frac{n}{n-2}}(\Omega). \tag{3.21}$$

For all $u, v, h \in (H_0^1(\Omega))^n$, we have

$$\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|_{L^{\frac{n}{n-2}}(\Omega)}^{\frac{n}{n-2}} = \int_\Omega |[D^2(f_i)(u(x)+\sigma(x)v(x))](u(x)-v(x))h(x)|^{\frac{n}{n-2}}dx,$$

where $\sigma(x) \in (0,1)$, for all $x \in \Omega$. It follows from (3.7) that

$$\begin{aligned}\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|_{L^{\frac{n}{n-2}}(\Omega)}^{\frac{n}{n-2}} &\leqslant C\int_\Omega |(u(x)-v(x))h(x)|^{\frac{n}{n-2}}dx \\ &\leqslant C\||u(x)-v(x)|^{\frac{n}{n-2}}\|_{L^2(\Omega)}\||h|^{\frac{n}{n-2}}\|_{L^2(\Omega)} \\ &\leqslant C\||u(x)-v(x)|\|_{L^{\frac{2n}{n-2}}(\Omega)}^{\frac{n}{n-2}}\||h|\|_{L^{\frac{2n}{n-2}}(\Omega)}^{\frac{n}{n-2}} \\ &\leqslant C\||u(x)-v(x)|\|_{H_0^1(\Omega)}^{\frac{n}{n-2}}\||h|\|_{H_0^1(\Omega)}^{\frac{n}{n-2}}.\end{aligned}$$

Then,

$$\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|_{L^{\frac{n}{n-2}}(\Omega)} \leqslant C\|u(x)-v(x)\|_{(H_0^1(\Omega))^n}\|h\|_{(H_0^1(\Omega))^n}. \tag{3.22}$$

Now, using Hölder inequality with exponents $\frac{\mathfrak{p}}{\mathfrak{p}-1}$ and $\mathfrak{p}$ we have

$$\begin{aligned}&\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|_{L^{\frac{2n}{(n-2)\mathfrak{p}}}(\Omega)}^{\frac{2n}{(n-2)\mathfrak{p}}} \\ &\leqslant \int_\Omega [D(f_i)(u(x,t))h(x) - D(f_i)(v(x,t))h(x)]^{\frac{2n}{(n-2)\mathfrak{p}}}dx \\ &\leqslant \||D(f_i)(u(x,t)) - D(f_i)(v(x,t))|^{\frac{2n}{(n-2)\mathfrak{p}}}\|_{L^{\frac{\mathfrak{p}}{\mathfrak{p}-1}}(\Omega)}\||h|^{\frac{2n}{(n-2)\mathfrak{p}}}\|_{L^{\mathfrak{p}}(\Omega)} \\ &\leqslant \||D(f_i)(u(x,t)) - D(f_i)(v(x,t))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)}^{\frac{2n}{(n-2)\mathfrak{p}}}\||h|\|_{L^{\frac{2n}{n-2}}(\Omega)}^{\frac{2n}{(n-2)\mathfrak{p}}},\end{aligned}$$

and then

$$\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|_{L^{\frac{2n}{(n-2)\mathfrak{p}}}(\Omega)} \leqslant \||D(f_i)(u(x,t)) - D(f_i)(v(x,t))|\|_{L^{\frac{2n}{(n-2)(\mathfrak{p}-1)}}(\Omega)}\||h|\|_{L^{\frac{2n}{n-2}}(\Omega)}.$$

It follows from (3.19) that

$$\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|_{L^{\frac{2n}{(n-2)\mathfrak{p}}}(\Omega)} \leqslant c\Big(1 + \||u|\|_{H_0^1(\Omega)}^{\mathfrak{p}-1} + \||v|\|_{H_0^1(\Omega)}^{\mathfrak{p}-1}\Big)\||h|\|_{H_0^1(\Omega)}. \tag{3.23}$$

It follows from Interpolation Theorem [22, Proposition 6.10, pg. 185], (3.22) and (3.23) that there exists $\eta \in (0,1)$ such that

$$\|Df_i^e(u)\cdot h - Df_i^e(v)\cdot h\|_{L^2(\Omega)} \leqslant c\Big(1 + \|u\|_{(H_0^1(\Omega))^n}^{\mathfrak{p}-1} + \|v\|_{(H_0^1(\Omega))^n}^{\mathfrak{p}-1}\Big)^\eta \|u-v\|_{(H_0^1(\Omega))^n}^{1-\eta}\|h\|_{(H_0^1(\Omega))^n}.$$

From this, for u, v in bounded subsets of $(H_0^1(\Omega))^n$ we obtain that there exists $\eta \in (0,1)$ such that

$$\|Df_i^e(u) - Df_i^e(v)\|_{\mathcal{L}((H_0^1(\Omega))^n, L^2(\Omega))} \leqslant C\|u-v\|_{(H_0^1(\Omega))^n}^{1-\eta}.$$

□

As we previously notice, in Dafermos [17] we can to ensure that the linear part of the problem generates a linear C_0-semigroup of contractions in $\mathcal{H}$. Since the Lemma 3.1 and the Lemma 3.2 are ensure for f, we can guarantee the local well-possessedness of the problem (3.10) thanks to the Theorem 2.21. More precisely, the next result is hold.

Theorem 3.3. *Given* $\mathbf{u}_0 = (u_0, u_1, v_0) \in \mathcal{H} = (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega)$*, the initial value problem* (3.10) *has a unique mild solution with*

$$u \in C([0, \tau_{\mathbf{u}_0}); (H_0^1(\Omega))^n) \cap C^1([0, \tau_{\mathbf{u}_0}); (L^2(\Omega))^n), \text{ and } \theta \in C([0, \tau_{\mathbf{u}_0}), L^2(\Omega)).$$

Moreover, if

$$\mathbf{u}_0 = (u_0, u_1, \theta_0) \in D(\mathbf{A}) = (H_0^1(\Omega) \cap H^2(\Omega))^n \times (H_0^1(\Omega))^n \times (H_0^1(\Omega) \cap H^2(\Omega))$$

then the following regularity property

$$u \in C([0, \tau_{\mathbf{u}_0}); (H^2(\Omega) \cap H_0^1(\Omega))^n) \cap C^1([0, \tau_{\mathbf{u}_0}); (H_0^1(\Omega))^n) \cap C^2([0, \tau_{\mathbf{u}_0}); (L^2(\Omega))^n),$$

and

$$\theta \in C([0, \tau_{\mathbf{u}_0}); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \tau_{\mathbf{u}_0}); L^2(\Omega))$$

is verified. In this case that $\mathbf{u} = (u, \partial_t u, \theta)$ *is a strong solution of* (3.10).

From this and standard ordinary differential theory via linear semigroups theory, see Pazy [36, Theorem 1.4], the problem (3.10) has a unique local solution $\mathbf{u}(t; \mathbf{u}_0)$ in $\mathcal{H}$ satisfying the initial condition $\mathbf{u}(0; \mathbf{u}_0) = \mathbf{u}_0 \in \mathcal{H}$ and defined on maximal interval of existence $[0, \tau_{\mathbf{u}_0})$.

Now we wish to prove that solutions of (3.10) are globally defined, i.e., for each $\mathbf{u}_0 = (u_0, u_1, \theta_0) \in \mathcal{H}$, $\tau_{\mathbf{u}_0} = \infty$. Thanks to Pazy [36, Theorema 1.4] and [36, Theorema 1.5], we can consider the continuously differentiable functional $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ defined by (3.11) and using the estimate (3.6) it follows that

$$\mathcal{E}(u, z, \theta) \geqslant \frac{1}{2}\Big(\|u\|^2_{(L^2(\Omega))^n} + \|z\|^2_{(L^2(\Omega))^n} + \|\theta\|^2_{L^2(\Omega)}\Big) - \frac{\eta}{2}\|u\|^2_{(L^2(\Omega))^n} - C_\eta|\Omega|,$$

and applying Poincaré inequality we obtain that

$$\begin{aligned} \mathcal{E}(u, z, \theta) &\geqslant \frac{1}{2}\Big(\|u\|^2_{(H_0^1(\Omega))^n} + \|z\|^2_{(L^2(\Omega))^n} + \|\theta\|^2_{L^2(\Omega)}\Big) - \frac{\eta}{2\lambda_1}\|u\|^2_{(H_0^1(\Omega))^n} - C_\eta|\Omega| \\ &= \frac{1}{2}\Big(1 - \frac{\eta}{\lambda_1}\Big)\|u\|^2_{(H_0^1(\Omega))^n} + \frac{1}{2}\|z\|^2_{(L^2(\Omega))^n} + \frac{1}{2}\|\theta\|^2_{L^2(\Omega)} - C_\eta|\Omega|. \end{aligned}$$

For $0 < \eta < \min\{1, \lambda_1\}$ we get

$$\|(u, z, \theta)\|^2_{\mathcal{H}} \leqslant c_1\mathcal{E}(u, z, \theta) + c_2, \tag{3.24}$$

for some $c_1 = c_1(\eta) > 0$ and $c_2 = c_2(\eta) > 0$.

Then it is clear from (3.12) that $[0, \tau_{\mathbf{u}_0}) \ni t \mapsto \mathcal{E}(u(t), \partial_t u(t), \theta(t)) \in \mathbb{R}$ is a non-increasing function. It follows from the fact that $\mathcal{E}$ is continuous and bounded in bounded subsets of $\mathcal{H}$ and from (3.12) that, given $r > 0$, there is a constant $C = C(r) > 0$ such that

$$\sup\{\|(u(t), \partial_t u(t), \theta(t))\|_{\mathcal{H}};\ \|(u_0, u_1, \theta_0)\|_{\mathcal{H}} \leqslant r, \text{ and } t \in [0, \tau_{\mathbf{u}_0})\} \leqslant C$$

This implies that for each $\mathbf{u}_0 \in \mathcal{H}$, the solution of (3.10) with $\mathbf{u}_0 = (u_0, u_1, \theta_0)$ is defined for all $t \geqslant 0$. We will write the mild solution of (3.10)

$$S(t)\mathbf{u}_0 = S_1(t)\mathbf{u}_0 + S_2(t)\mathbf{u}_0, \tag{3.25}$$

where $S_1(t)\mathbf{u}_0$ is defined as the solution of (3.10) with $\mathbf{F} \equiv 0$ and

$$S_2(t)\mathbf{u}_0 = \int_0^t S_1(t-\xi)\mathbf{F}(S(\xi)\mathbf{u}_0)d\xi,\ \forall t \geqslant 0$$

here $\mathbf{F}(u) = (0, f^e(u), 0)$, with f^e the Nemytskĭi operator to f.

3.3 Existence of global attractor

In order to study the asymptotic behavior of the system (3.1), we assume the vanishing mean value for θ_0 on Ω; that is,

$$\theta_0 \in L^2_0(\Omega) = \left\{\theta \in L^2(\Omega);\ \int_\Omega \theta(x)dx = 0\right\}. \tag{3.26}$$

where Ω is a bounded domain with sufficiently smooth boundary in $\mathbb{R}^n$, $n \geqslant 2$.

For our better knowledge, large time behavior of solution, in the sense of existence and sensitivity of global attractors, for the system (3.1) has not yet been treated in the literature if we assume (3.26) on initial data.

Since the physical interpretation of the function u as the displacement and θ as the temperature variation of a body occupying the domain $\Omega \subset \mathbb{R}^n$ is considered, we can see that the hypothesis (3.26) is natural for the problem. Let us recall that the problem (3.1) is formulated by considering a certain value $T_1 \in \mathbb{R}$, which will be the reference temperature of the environment where the body is inserted and from this define $\theta_0(x) = T_0(x) - T_1$ where $T_0(x)$ is the body temperature in $x \in \Omega$. As we are considering not an external heat source other than the environment, this hypothesis is compatible since the functions in $L^2(\Omega)$ can be written as a direct sum of the functions that satisfy (3.26) and constants functions. The condition (3.26) is necessary in our analysis to use the Bogowskĭi's operator, which is a right inverse for the divergent operator. Duran in [19] and, Duran and Muschietti in [20], we find the properties of the Bogowskiĭ operator that we will use in this work.

We will denote by

$$Y_* = L^2_0(\Omega) \tag{3.27}$$

and we will consider the problem (3.10) in the space

$$\mathcal{H}_* = (Y^1)^n \times Y^n \times Y_*.$$

equipped with usual inner product of $\mathcal{H}$.

Now we want to construct a Lyapunov functional and combine the arguments from Andrade, Silva and Ma [2], Barbosa and Ma [4], Cavalcanti, Domingos Cavalcanti and Ferreira [14], Araújo, Ma and Qin [18], Giorgi, Rivera and Pata [23] and Pokojovy [37] to prove that the nonlinear semigroup $\{S(t); t \geqslant 0\}$ has a bounded attracting set. We will apply [13, Theorem 2.23] and deduce the existence of a global attractor, to do this we need to show that the nonlinear semigroup is also asymptotically compact according to the [13, Definition 2.8].

We define the wanted functional conveniently changing the functional energy using the existence of a continuous right inverse of the divergence, which is called the Bogovskiĭ operator. Because this, we assume that Ω is star-shape domain with diameter $R_1 > 0$ with respect to a ball B_1 and the vanishing mean value for θ_0 on Ω; that is, (3.26).

Under the hypoteses about Ω, it is well know that the divergence as an operator from the Sobolev space $(H_0^1(\Omega))^n$ into the space $L_0^2(\Omega)$, it has continuous right inverse called Bogowskiĭ's operator, see e.g. [6], [9], [19], [20], [34] and [37]. Given a function $v \in L_0^2(\Omega)$, we will denote $\Phi(v) \in (H_0^1(\Omega))^n$ a solution of the problem

Since the physical interpretation of the function u as the displacement and θ as the temperature variation of a body occupying the region $\Omega \subset \mathbb{R}^n$ is considered, we can see that the hypothesis (3.26) is natural for the problem. Let us recall that the problem (3.1) is formulated by considering a certain value $T_1 \in \mathbb{R}$ which will be the reference temperature of the environment where the body is inserted and from this define $\theta_0(x) = T_0(x) - T_1$ where $T_0(x)$ is the body temperature in $x \in \Omega$. As we are considering not an external heat source other than the environment, this hypothesis is compatible since the functions in $L^2(\Omega)$ can be written as a direct sum of the functions that satisfy (3.26) and constants functions. The condition (3.26) is necessary in our analysis to use the Bogowskiĭ operator, which is a right inverse for the divergent operator. More precisely, Bogowskiĭ operator is $\Phi : L_0^2(\Omega) \to (H_0^1(\Omega))^n$ such that

$$\begin{cases} \operatorname{div}(\Phi(v)) = v \ \text{ in } \ \Omega, \\ \Phi(v) = 0 \ \text{ on } \ \partial\Omega \\ \|\Phi(v)\|_{(H_0^1(\Omega))^n} \leqslant C\|v\|_{L^2(\Omega)} \end{cases} \tag{3.28}$$

where $C > 0$ depends only on Ω, for more details see Duran [19] and, Duran and Muschietti [20]. Note that $L_0^2(\Omega)$ is a Hilbert space equiped with the inner product induced by usual inner product of $L^2(\Omega)$.

Duran and Muschietti in [20], we find the next results of the Bogowskiĭ operator that we will use in this paper. They define

$$G(x,y) = \int_0^1 \frac{1}{s^{n+1}}(x-y)\omega\left(y + \frac{x-y}{s}\right) ds,$$

for a $\omega \in C_0^\infty(\Omega)$ such that $\int_\Omega \omega dx = 1$ and proof the Lemma 3.4 and the Theorem 3.5

Lemma 3.4. *For any $\omega_1 \in C_0^\infty(\Omega)$ we define $\bar{\omega}_1 = \int_\Omega \omega(x)\omega_1(x)dx$. Then, for $y \in \Omega$ we have*

$$(\omega_1 - \bar{\omega}_1)(y) = -\int_\Omega G(x,y)\nabla\omega_1(x)dx.$$

Theorem 3.5. *Let Ω be a bounded and star-shaped with respect to a ball $B \subset \Omega$. Given $v \in L^p(\Omega)$, $1 < p < \infty$, such that $\int_\Omega v dx = 0$ define*

$$\Phi(v) = \int_\Omega G(x,y)\, v(y)dy.$$

Then

$$\Phi(v) \in (W_0^{1,p}(\Omega))^n,$$

$$\operatorname{div}(\Phi(v)) = v$$

and

$$\|\Phi(v)\|_{(W_0^{1,p}(\Omega))^n} \leqslant C\|v\|_{L^p(\Omega)}$$

where $C > 0$ depends only on Ω.

Notice that, if (u,θ) satisfy (3.1), then we have immediately that

$$\partial_t\theta = \operatorname{div}(\kappa\nabla\theta - \partial_t u).$$

for $t \geqslant 0$ and $x \in \Omega$. Besides that $\int_\Omega \partial_t\theta dx = 0$, and

$$\partial_t\theta = \partial_t \operatorname{div}(\Phi(\theta)) = \operatorname{div}(\partial_t\Phi(\theta)).$$

Therefore,

$$\operatorname{div}(\kappa\nabla\theta - \partial_t u) = \operatorname{div}(\partial_t\Phi(\theta)).$$

This leads us to think that

$$\partial_t(\Phi(\theta))(x,t) = [\kappa\nabla\theta - \partial_t u](x,t). \tag{3.29}$$

The identity (3.29) is true in $L^2(\Omega)$, because for any $\omega_1 \in C_0^\infty(\Omega)$ from Lemma 3.4 and

Theorem 3.5, we can do the following

$$
\begin{aligned}
&\int_\Omega [\partial_t \Phi(\theta) - (\kappa\nabla\theta - \partial_t u)] \cdot \nabla\omega_1(x)dx \\
&\qquad = \int_\Omega \left(\int_\Omega \partial_t\theta(y)G(x,y)dy\right)\nabla\omega_1(x)dx + \int_\Omega (\partial_t\theta\omega_1)(x)dx \\
&\qquad = \int_\Omega \partial_t\theta(y)\left(\int_\Omega G(x,y)\nabla\omega_1(x)dx\right)dy + \int_\Omega (\partial_t\theta\omega_1)(x)dx \\
&\qquad = -\int_\Omega \partial_t\theta(y)(\omega_1 - \bar{\omega}_1)(y)dy + \int_\Omega (\partial_t\theta\omega_1)(x)dx \\
&\qquad = -\int_\Omega (\partial_t\theta\omega_1)(y)dy + \int_\Omega \partial_t\theta(y)\bar{\omega}_1 dy + \int_\Omega (\partial_t\theta\omega_1)(x)dx \\
&\qquad = \bar{\omega}_1 \int_\Omega \partial_t\theta(y)dy = 0
\end{aligned}
$$

where we omit t for simplicity.

Let us consider the functional

$$
\mathcal{L}(u,z,\theta) = M\mathcal{E}(u,z,\theta) + \delta_1(u,z)_{(L^2(\Omega))^n} + \delta_2(\Phi,z)_{(L^2(\Omega))^n} \tag{3.30}
$$

where δ_1, δ_2 and M are positive constants to be chosen appropriately and Φ is define in (3.28), we obtain the following result:

Theorem 3.6. *For $M > 0$ sufficiently large, there exist constants $M_1 > 0$ and $M_2 > 0$ such that for all $t \geqslant 0$*

$$
\frac{d\mathcal{L}}{dt} \leqslant -M_1\mathcal{E}(t) + M_2, \tag{3.31}
$$

where $\mathcal{L}(t) = \mathcal{L}(u,z,\theta)$, $\mathcal{E}(t) = \mathcal{E}(u,z,\theta)$, and $(u,z,\theta) = (u(t),z(t),\theta(t))$ is the global solution of (3.1)-(3.26).

Proof. Note that

$$
\frac{d\mathcal{L}}{dt} = M\frac{d\mathcal{E}}{dt} + \delta_1\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} + \delta_2\frac{d}{dt}(\Phi,z)_{(L^2(\Omega))^n}. \tag{3.32}
$$

Thanks to (3.8), (3.12) and Poincaré inequality we have

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt}(t) &= -\int_\Omega \kappa(x)|\nabla\theta|^2dx \\
&\leqslant -\frac{\kappa_0}{2}\int_\Omega |\nabla\theta|^2dx - \frac{\kappa_0\lambda_1}{2}\int_\Omega |\theta|^2dx,
\end{aligned} \tag{3.33}
$$

where λ_1 is the first eigenvalue of negative Laplacian operator with zero Dirichlet boundary condition in Ω.

We also have

$$
\begin{aligned}
\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} &= (\partial_t u, z)_{(L^2(\Omega))^n} + (u, \partial_t z)_{(L^2(\Omega))^n} = (\partial_t u, \partial_t u)_{(L^2(\Omega))^n} + (u, \partial_t^2 u)_{(L^2(\Omega))^n} \\
&= \int_\Omega |\partial_t u|^2dx - \int_\Omega |\nabla u|^2dx - \int_\Omega |\operatorname{div} u|^2dx - \int_\Omega \nabla\theta u dx + \int_\Omega f(u)udx.
\end{aligned}
$$

To deal with the integral term, just notice that from (3.5) we have

$$\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant \int_\Omega |\partial_t u|^2 dx - \int_\Omega |\nabla u|^2 dx - \int_\Omega |\operatorname{div} u|^2 dx - \int_\Omega \nabla\theta u dx + \nu \int_\Omega |u|^2 dx + C_\nu |\Omega|$$

and again by Poincaré inequality

$$\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant \int_\Omega |\partial_t u|^2 dx - \int_\Omega |\nabla u|^2 dx - \int_\Omega |\operatorname{div} u|^2 dx - \int_\Omega \nabla\theta u dx + \frac{\nu}{\lambda_1} \int_\Omega |\nabla u|^2 dx + C_\nu |\Omega|$$

in other words

$$\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant \int_\Omega |\partial_t u|^2 dx - \frac{C_\nu}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega |\operatorname{div} u|^2 dx - \int_\Omega \nabla\theta u dx + C_\nu |\Omega|,$$

where $\nu > 0$ is chosen such that

$$C_\nu := 1 - \frac{\nu}{\lambda_1} > 0,$$

that is,

$$0 < \nu < \lambda_1.$$

Due to Young's inequality we conclude that

$$\delta_1 \frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant \delta_1 \int_\Omega |\partial_t u|^2 dx - \frac{\delta_1 C_\nu}{2} \int_\Omega |\nabla u|^2 dx - \left(\delta_1 - \frac{1}{2}\right) \int_\Omega |\operatorname{div} u|^2 dx + \frac{\delta_1^2}{2} \int_\Omega |\theta|^2 dx + \delta_1 C_\nu |\Omega|. \tag{3.34}$$

We also have

$$\frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} = (\Phi, \partial_t^2 u)_{(L^2(\Omega))^n} + (\partial_t \Phi, \partial_t u)_{(L^2(\Omega))^n},$$

and from (3.29) we obtain that

$$\frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} = \int_\Omega \Phi \Delta u dx + \int_\Omega \Phi \nabla \operatorname{div} u dx - \int_\Omega \Phi \nabla\theta dx + \int_\Omega \Phi f(u) dx + \int_\Omega \kappa \nabla\theta \partial_t u dx - \int_\Omega |\partial_t u|^2 dx$$

In other words,

$$\frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} = -\int_\Omega \nabla\Phi \nabla u dx - \int_\Omega \theta \operatorname{div} u dx + \int_\Omega |\theta|^2 dx + \int_\Omega \Phi f(u) dx + \int_\Omega \kappa(x) \nabla\theta \partial_t u dx - \int_\Omega |\partial_t u|^2 dx.$$

Using (3.8) and the Young's inequality we get for any $\epsilon > 0$,

$$\begin{aligned}
\delta_2 \frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} &\leqslant \frac{\delta_2^2}{2}\int_\Omega |\nabla\Phi|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\delta_2^2 + 2\delta_2}{2}\int_\Omega |\theta|^2 dx + \int_\Omega |\operatorname{div} u|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |f(u)|^2 dx + \frac{\delta_2^2}{2}\int_\Omega |\Phi|^2 dx + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx \\
&\quad + \left(\frac{1}{2\epsilon} - 1\right)\delta_2 \int_\Omega |\partial_t u|^2 dx \\
&\leqslant \frac{\delta_2^2 + 2\delta_2}{2}\int_\Omega |\theta|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\operatorname{div} u|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |f(u)|^2 dx + \frac{\delta_2^2}{2}\int_\Omega |\nabla\Phi|^2 dx + \frac{\delta_2^2}{2\lambda_1}\int_\Omega |\nabla\Phi|^2 dx \\
&\quad + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx + \left(\frac{1}{2\epsilon} - 1\right)\delta_2 \int_\Omega |\partial_t u|^2 dx
\end{aligned}$$

Then,

$$\begin{aligned}
\delta_2 \frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} &\leqslant \frac{1}{2}\left(\delta_2^2 + 2\delta_2 + C\delta_2^2 + \frac{C\delta_2^2}{\lambda_1}\right)\int_\Omega |\theta|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |\operatorname{div} u|^2 dx + \frac{1}{2}\int_\Omega |f(u)|^2 dx + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx \\
&\quad + \left(\frac{1}{2\epsilon} - 1\right)\delta_2 \int_\Omega |\partial_t u|^2 dx.
\end{aligned} \tag{3.35}$$

Thanks to (3.7) there exists $C > 0$ such that

$$\int_\Omega |f(u)|^2 dx \leqslant C\int_\Omega |u|^2 dx + C\sum_{i=1}^n \int_\Omega |u_i|^{2\mathfrak{p}} dx.$$

Since $1 < \mathfrak{p} < \frac{n}{n-2}$ if $n \geqslant 2$, and $1 < \mathfrak{p} < +\infty$ if $n = 2$, we see that $H^2(\Omega) \hookrightarrow L^{2\mathfrak{p}}(\Omega)$, and we obtain that

$$\begin{aligned}
\int_\Omega |f(u)|^2 dx &\leqslant C\int_\Omega |u|^2 dx + \bar{C} \\
&\leqslant \frac{\bar{C}_1}{\lambda_1}\int_\Omega |\nabla u|^2 dx + \bar{C}_2,
\end{aligned} \tag{3.36}$$

whenever $\|u\|_{(H^2(\Omega))^n} \leqslant r$ (as in Carvalho, Cholewa, Dlotko et al. [12]).

Now combining (3.35) and (3.36) we get

$$\begin{aligned}
\delta_2 \frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} &\leqslant \frac{1}{2}\left(\delta_2^2 + 2\delta_2 + C\delta_2^2 + \frac{C\delta_2^2}{\lambda_1}\right)\int_\Omega |\theta|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx \\
&\quad + \frac{1}{2}\int_\Omega |\operatorname{div} u|^2 dx + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx + \frac{\bar{C}_1}{2\lambda_1}\int_\Omega |\nabla u|^2 dx \\
&\quad + \left(\frac{1}{2\epsilon} - 1\right)\delta_2 \int_\Omega |\partial_t u|^2 dx + \frac{\bar{C}_2}{2}.
\end{aligned} \tag{3.37}$$

Therefore, combining (3.32) with (3.33), (3.34) and (3.37) we see that

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\frac{M\kappa_0}{2}\int_\Omega |\nabla\theta|^2 dx - \frac{M\kappa_0\lambda_1}{2}\int_\Omega |\theta|^2 dx + \delta_1 \int_\Omega |\partial_t u|^2 dx - \frac{\delta_1 C_\nu}{2}\int_\Omega |\nabla u|^2 dx$$
$$- \left(\delta_1 - \frac{1}{2}\right)\int_\Omega |\operatorname{div} u|^2 dx + \frac{\delta_1^2}{2}\int_\Omega |\theta|^2 dx + \frac{1}{2}\left(\delta_2^2 + 2\delta_2 + C\delta_2^2 + \frac{C\delta_2^2}{\lambda_1}\right)\int_\Omega |\theta|^2 dx$$
$$+ \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\operatorname{div} u|^2 dx + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx + \frac{\bar{C}_1}{2\lambda_1}\int_\Omega |\nabla u|^2 dx$$
$$+ \left(\frac{1}{2\epsilon} - 1\right)\delta_2 \int_\Omega |\partial_t u|^2 dx + \frac{\bar{C}_2}{2} + \delta_1 C_\nu |\Omega|.$$

When we reorganize the previous inequality,

(3.38)
$$\frac{d}{dt}\mathcal{L}(t) \leqslant \left(\frac{\delta_2\kappa_1^2\epsilon}{2} - \frac{\kappa_0 M}{2}\right)\int_\Omega |\nabla\theta|^2 dx - \left(\frac{\delta_1 C_\nu}{2} - \frac{\bar{C}_1}{2\lambda_1} - \frac{1}{2}\right)\int_\Omega |\nabla u|^2 dx$$
$$+ \left(\frac{1}{2}\left(\delta_2^2 + 2\delta_2 + C\delta_2^2 + \frac{C\delta_2^2}{\lambda_1}\right) + \frac{\delta_1^2}{2} - \frac{\kappa_0\lambda_1 M}{2}\right)\int_\Omega |\theta|^2 dx$$
$$- (\delta_1 - 1)\int_\Omega |\operatorname{div} u|^2 dx - \left(\left(1 - \frac{1}{2\epsilon}\right)\delta_2 - \delta_1\right)\int_\Omega |\partial_t u|^2 dx + \frac{\bar{C}_2}{2} + \delta_1 C_\nu |\Omega|.$$

Now take $\epsilon > 0$ large enough to be able choose δ_1 and δ_2 such that

$$0 < \max\left\{\frac{\bar{C}_1 + \lambda_1}{\lambda_1 C_\nu}, 1\right\} < \delta_1,$$

and

$$\delta_1 < \left(1 - \frac{1}{\epsilon}\right)\delta_2.$$

Choose $M > 0$ sufficiently large too such that

$$\frac{\delta_2\kappa_1^2\epsilon}{2} - \frac{\kappa_0 M}{2} < 0 \text{ and } \frac{1}{2}\left(\delta_2^2 + 2\delta_2 + C\delta_2^2 + \frac{C\delta_2^2}{\lambda_1}\right) + \frac{\delta_1^2}{2} - \frac{\kappa_0\lambda_1 M}{2} < 0,$$

with these choices for the constants δ_1, δ_2 and M there exist $\varrho_0 > 0$ and $\varrho_1 > 0$ such that

$$\frac{d\mathcal{L}}{dt}(t) \leqslant -\varrho_0\left[\frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\text{div} u|^2 + |z|^2 + |\theta|^2\right)dx\right] + \varrho_1.$$

Finally, we observe that if $\xi \in H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, then

$$\xi|\xi|^{\mathfrak{p}} \in L^{\frac{2n}{(n-4)(\mathfrak{p}+1)}}(\Omega) \hookrightarrow L^1(\Omega), \text{ for all } 1 < \mathfrak{p} < \frac{n}{n-2},$$

and our hypothesis on f (see (3.7)) implies that

$$|f_i(\xi)| \leqslant c(1 + |\xi_1|^{\mathfrak{p}} + \cdots + |\xi_n|^{\mathfrak{p}}),\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$$

Therefore, we can find a constant $\bar{c} > 1$ such that for all $u = (u_1, \ldots, u_n) \in (H_0^1(\Omega))^n$,

$$-\int_\Omega F(u)dx \leqslant \bar{c}\|u\|^2_{(H^1(\Omega))^n}(1 + \|u_1\|^{\mathfrak{p}-1}_{H^1(\Omega)} + \cdots + \|u_n\|^{\mathfrak{p}-1}_{H^1(\Omega)}),$$

and therefore

$$-\bar{d}\int_{\Omega} F(u)dx \leqslant \|u\|^2_{(H^1(\Omega))^n}, \tag{3.39}$$

whenever $\|u\|_{(H^1_0(\Omega))^n} \leqslant r$ and considering $\bar{d} = \frac{1}{\bar{c}(1+n^{1-\mathfrak{p}}r^{\mathfrak{p}-1})} < 1$.

Thanks to (3.38) and (3.39) there exist constants $\varrho_3 > 0, M_1 > 0$ and $M_2 > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leqslant -\frac{\varrho_3}{2}\int_{\Omega}\Big(|\nabla u|^2 + |\text{div}u|^2 + |z|^2 + |\theta|^2\Big)dx - \frac{\varrho_3}{2}\int_{\Omega}\Big(|\nabla u|^2 + |\text{div}u|^2\Big)dx + \varrho_1 \\
&\leqslant -\frac{\varrho_3}{2}\int_{\Omega}\Big(|\nabla u|^2 + |\text{div}u|^2 + |z|^2 + |\theta|^2\Big)dx + \frac{\varrho_3\bar{d}}{2}\int_{\Omega} F(u)dx + \varrho_1 \\
&\leqslant -M_1\left[\frac{1}{2}\int_{\Omega}\Big(|\nabla u|^2 + |\text{div}u|^2 + |z|^2 + |\theta|^2\Big)dx - \int_{\Omega} F(u)dx\right] + M_2,
\end{aligned}$$

where $F(u) = \int_0^u f d\gamma$ and $\int_0^u f d\gamma$ represents the line integral of f along a piecewise smooth curve with initial point 0 and final point u.

Finally, from (3.11) we conclude there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$\frac{d\mathcal{L}}{dt} \leqslant -M_1\mathcal{E}(t) + M_2,$$

where $u(x,t), z = z(x,t), \theta = \theta(x,t)$. This concludes the proof of the theorem.

□

Theorem 3.7. *For $M > 0$ sufficiently large, there exists positive constants C_M, c_M, C_1 and $C_2 > 0$ such that for any $t \geqslant 0$,*

$$c_M\mathcal{E}(t) - C_1 \leqslant \mathcal{L}(t) \leqslant C_M\mathcal{E}(t) + C_2, \tag{3.40}$$

where $\mathcal{L}(t) = \mathcal{L}(u,z,\theta)$, $\mathcal{E}(t) = \mathcal{E}(u,z,\theta)$, and $(u,z,\theta) = (u(t),z(t),\theta(t))$ is the solution of (3.1)-(3.26).

Proof. In the follows we prove the two inequalities in (3.40) simultaneously, once the arguments are similar. From definition of the functional $\mathcal{L}$ and Cauchy-Schwarz inequality, for any $M > 0$ we can see that

$$M\mathcal{E}(t) - \delta_1\int_{\Omega}|u||z|dx - \delta_2\int_{\Omega}|\Phi||z|dx \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant M\mathcal{E}(t) + \delta_1\int_{\Omega}|u||z|dx + \delta_2\int_{\Omega}|\Phi||z|dx.$$

Then, it follows from Young's inequality

$$M\mathcal{E}(t) - \frac{\delta_1}{2}\|u\|^2_{(L^2(\Omega))^n} - \frac{1}{2}(\delta_1+\delta_2)\|z\|^2_{(L^2(\Omega))^n} - \frac{\delta_2}{2}\|\Phi\|^2_{(L^2(\Omega))^n} \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant M\mathcal{E}(t) + \frac{\delta_1}{2}\|u\|^2_{(L^2(\Omega))^n} + \frac{1}{2}(\delta_1+\delta_2)\|z\|^2_{(L^2(\Omega))^n} + \frac{\delta_2}{2}\|\Phi\|^2_{(L^2(\Omega))^n}.$$

Now using the Poincaré inequality, we have that

$$M\mathcal{E}(t) - \frac{\delta_1}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} - \frac{1}{2}(\delta_1+\delta_2)\|z\|^2_{(L^2(\Omega))^n} - \frac{\delta_2}{2\lambda_1}\|\Phi\|^2_{(H^1_0(\Omega))^n} \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant M\mathcal{E}(t) + \frac{\delta_1}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} + \frac{1}{2}(\delta_1+\delta_2)\|z\|^2_{(L^2(\Omega))^n} + \frac{\delta_2}{2\lambda_1}\|\Phi\|^2_{(H^1_0(\Omega))^n}.$$

From definition of the functionais E and Φ, we get

$$\begin{aligned}&\frac{1}{2}\Big[\Big(M-\frac{\delta_1}{\lambda_1}\Big)\|u\|^2_{(H^1_0(\Omega))^n} + (M-\delta_1-\delta_2)\|z\|^2_{(L^2(\Omega))^n} + \Big(M-\frac{\delta_2C^2}{\lambda_1}\Big)\|\theta\|^2_{L^2(\Omega)}\Big]\\ &\quad - M\int_\Omega F(u)dx \leqslant \mathcal{L}(t),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(t) \leqslant &\frac{1}{2}\Big[\Big(M+\frac{\delta_1}{\lambda_1}\Big)\|u\|^2_{(H^1_0(\Omega))^n} + (M+\delta_1+\delta_2)\|z\|^2_{(L^2(\Omega))^n} + \Big(M+\frac{\delta_2C^2}{\lambda_1}\Big)\|\theta\|^2_{L^2(\Omega)}\Big]\\ &+ M\int_\Omega F(u)dx.\end{aligned}$$

for some $C > 0$.

Using (3.6) we see that

$$\int_\Omega F(u)dx \leqslant \frac{\eta}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} + C_\eta|\Omega|.$$

and if we denote

$$\begin{aligned}c_1 &= \frac{\delta_1}{\lambda_1} + \delta_1 + \delta_2 + \frac{\delta_2C^2}{\lambda_1};\\ c_2 &= \frac{C_\eta\lambda_1}{\lambda_1-\eta};\\ c_3 &= \frac{2M\eta - c_1\lambda_1}{\lambda_1-\eta};\\ c_4 &= 1 - \frac{C_1(\lambda-\eta)}{2M\eta - c_1\lambda_1},\end{aligned}$$

then we conclude that

$$\begin{aligned}\mathcal{L}(t) &\geqslant \frac{1}{2}\Big[(M-c_1)\|u\|^2_{(H^1_0(\Omega))^n} + (M-c_1)\|z\|^2_{(L^2(\Omega))^n} + (M-c_1)\|\theta\|^2_{L^2(\Omega)}\Big] - M\int_\Omega F(u)dx\\ &\geqslant \frac{1}{2}\Big[M - \Big(c_1 + \frac{\eta c_2}{\lambda_1}\Big)\Big]\|(u,z,\theta)\|_{\mathcal{H}*} + \frac{\eta c_2}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} - (M-c_2)\int_\Omega F(u)dx\\ &\quad - c_2\int_\Omega F(u)dx\end{aligned}$$

and

$$\mathcal{L}(t) \leqslant \frac{1}{2}\Big[(M+c_1)\|u\|^2_{(H^1_0(\Omega))^n} + (M+c_1)\|z\|^2_{(L^2(\Omega))^n} + (M+c_1)\|\theta\|^2\Big] + M\int_\Omega F(u)dx$$
$$\leqslant \frac{1}{2}\Big[(M+c_1+c_4c_3)\|(u,z,\theta)\|_{\mathcal{H}*} - c_4c_3\|u\|^2_{(H^1_0(\Omega))^n}\Big] - (M+c_3)\int_\Omega F(u)dx$$
$$+ (2M+c_3)\int_\Omega F(u)dx,$$

where $M > 0$ is chosen sufficiently large such that $M - c_2 > 0$, $c_3 > 0$ and $c_4 > 0$, we can note that

$$c_2 = c_1 + \frac{\eta}{\lambda_1}c_2;$$
$$c_4c_3 = (2M+c_3)\frac{\eta}{\lambda_1};$$
$$c_3 = c_2 + c_4c_3.$$

Therefore by (3.6) we get

$$\frac{1}{2}\Big[M - \Big(c_1 + \frac{\eta}{\lambda_1}c_2\Big)\Big]\|(u,z,\theta)\|^2_{\mathcal{H}*} + \Big(\frac{\eta}{2\lambda_1}c_2 - \frac{\eta}{2\lambda_1}c_2\Big)\|u\|^2_{(H^1_0(\Omega))^n}$$
$$- (M-c_2)\int_\Omega F(u)dx - C_\eta c_2|\Omega| \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant \frac{1}{2}[M + (c_1 + c_3c_4)]\|(u,z,\theta)\|^2_{\mathcal{H}*} + \frac{1}{2}\Big[-c_3c_4 + (2M+c_3)\frac{\eta}{\lambda_1}\Big]\|u\|^2_{(H^1_0(\Omega))^n}$$
$$- (M+c_3)\int_\Omega F(u)dx + (2M+c_3)C_\eta|\Omega|.$$

Finally, if we define $c_M = M - c_2$, $C_1 = c_2C_\eta|\Omega|$, $C_2 = (2M+c_3)C_\eta|\Omega|$ and $C_M = M + c_3$, then

$$\frac{c_M}{2}\|(u,z,\theta)\|^2_{\mathcal{H}*} - c_M\int_\Omega F(u)dx - C_1 \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant \frac{C_M}{2}\|(u,z,\theta)\|^2_{\mathcal{H}*} - C_M\int_\Omega F(u)dx + C_2.$$

□

We have the following result as a consequence of Theorem 3.6.

Theorem 3.8. *There exists $R > 0$ such that for each bounded subset B of $\mathcal{H}_*$ there exists $t_B > 0$ with the property*

$$S(t)B \subset \mathcal{B}_{\mathcal{H}_*}(0;R),$$

for any $t \geqslant t_B$. Here, $\mathcal{B}_{\mathcal{H}_*}(0;R)$ denotes the open ball in $\mathcal{H}_*$ centered at origin of radius R.

Proof. Let B be a subset of $\mathcal{H}_*$, and let $(u,z,\theta)=(u(t),z(t),\theta(t))$ be the solution of (3.1)-(3.26) with $(u_0,u_1,\theta_0)\in B$. Using Theorem 3.6 and the second inequality of (3.40) in Theorem 3.7 we have that there exits constants $\varrho_1>0$ and $\varrho_2>0$ such that

$$\frac{d}{dt}\mathcal{L}(t)\leqslant -\varrho_1\mathcal{L}(t)+\varrho_2,$$

where $\mathcal{L}(t)=\mathcal{L}(u,z,\theta)$ for any $t\geqslant 0$.

From (1.6), we can find that

$$\mathcal{L}(t)\leqslant \mathcal{L}(0)e^{-\int_0^t\varrho_1 ds}+\int_0^t\varrho_2 e^{-\int_\tau^t\varrho_1 ds}d\tau\leqslant \mathcal{L}(0)e^{-\varrho_1 t}+\frac{\varrho_2}{\varrho_1}(1-e^{-\varrho_1 t})$$

where $\mathcal{L}(0)=\mathcal{L}(u_0,u_1,\theta_0)$, and combining with the inequalities (3.24) and (3.40) in Theorem 3.7, we get

$$\begin{aligned}\|(u,z,\theta)\|^2_{\mathcal{H}_*}&\leqslant c_1\mathcal{E}(u,z,\theta)+c_2\leqslant \frac{c_1}{c_M}\mathcal{L}(t)+\frac{C_1c_1}{c_M}+c_2\\&\leqslant\Big(\frac{c_1}{c_M}\mathcal{L}(0)-\frac{\varrho_2}{\varrho_1}\Big)e^{-\varrho_1 t}+\frac{\varrho_2c_1}{\varrho_1c_M}+\frac{C_1c_1}{c_M}+c_2,\end{aligned}$$

for some constants c_1,c_2,c_M and $C_1>0$.

Let $R_B>0$ such that $\|(u_0,u_1,\theta_0)\|^2_{\mathcal{H}_*}\leqslant R_B$, then after some calculations we conclude that there exists $t_B>0$ with

$$\|(u,z,\theta)\|^2_{\mathcal{H}_*}\leqslant 2\Big(\frac{\varrho_2c_1}{\varrho_1c_M}+\frac{C_1c_1}{c_M}+c_2\Big)\text{ for any }t\geqslant t_B.$$

□

Proposition 3.9. *There exists positive constants K and α such that*

$$\|S_1(t)\|_{\mathcal{L}(\mathcal{H}_*)}\leqslant Ke^{-\alpha t}\quad\textit{for all}\quad t\geqslant 0,$$

and $S_2(t)$ is a compact operator from $\mathcal{H}_*$ into itself for all $t>0$. In particular the nonlinear semigroup $S(\cdot)$ is asymptotically compact.

Proof. To prove the decay of $S_1(t)$, one considers the functional

$$\mathcal{L}_0(u,z,\theta)=\frac{1}{2}\Big(\|u\|^2_{(H_0^1(\Omega))^n}+\|z\|^2_{(L^2(\Omega))^n}+\|\theta\|^2_{L^2(\Omega)}\Big)+\delta_1(u,z)_{(L^2(\Omega))^n}+\delta_2(\Phi,z)_{(L^2(\Omega))^n}.$$

Thanks to Theorem 3.6 and Theorem 3.7 we have

$$\frac{d}{dt}\mathcal{L}_0(t)\leqslant-\alpha\mathcal{L}_0(t)$$

for some $\alpha>0$, where

$$\mathcal{L}_0(S_1(t)(u_0,u_1,\theta_0))\leqslant\mathcal{L}_0(S_1(t)(u_0,u_1,\theta_0))e^{-\alpha t},$$

and consequently,

$$\|S_1(t)(u_0,u_1,\theta_0)\|^2_{\mathcal{H}_*} \leqslant Ke^{-\alpha t}\|(u_0,u_1,\theta_0)\|^2_{\mathcal{H}_*}.$$

To show that $S_2(t)$ is compact, we first show that f is bounded from $(H^1_0(\Omega))^n$ into $(W^{1,r}(\Omega))^n$, with $r = \frac{2(n-1)}{(n-2)} = \frac{n}{n-2} + 1 \in (1,2)$; indeed, it follows from Lemma 3.1 that for any $u \in \mathbb{R}^n$ we have

$$|f(u)| \leqslant 2^{\mathfrak{p}-1}n^2C|u|(1+|u_1|^{\mathfrak{p}-1}+\cdots+|u_n|^{\mathfrak{p}-1}),$$

for $\mathfrak{p} < \frac{n}{n-2}$ and from (3.7),

$$\begin{aligned}
\|f(u)\|^r_{W^{1,r}(\Omega)} &= \int_\Omega (|f(u)|^r + |\nabla f(u)|^r|\nabla u|^r)dx \\
&\leqslant \int_\Omega \Big(\big(2^{\mathfrak{p}-1}n^2C|u|(1+|u_1|^{\mathfrak{p}-1}+\cdots+|u_n|^{\mathfrak{p}-1})\big)^r \\
&\quad + Cn(1+\sum_{i=1}^n |u_i|^{\mathfrak{p}-1})^r|\nabla u|^r\Big)dx \\
&\leqslant C\left(\|u\|^r_{(L^r(\Omega))^n} + \|u\|^{\mathfrak{p}r}_{(L^{\mathfrak{p}r}(\Omega))^n} + \|\nabla u\|^r_{(L^2(\Omega))^n} + \|u\|^{(\mathfrak{p}-1)r}_{(L^{\frac{2(\mathfrak{p}-1)r}{2-r}}(\Omega))^n}\|\nabla u\|^{r/2}_{(L^2(\Omega))^n}\right) \\
&\leqslant C'\left(\|u\|^{\mathfrak{p}r}_{(L^{\mathfrak{p}r}(\Omega))^n} + \|\nabla u\|^r_{(L^2(\Omega))^n} + \|u\|^{(\mathfrak{p}-1)r}_{(L^{-r}(\Omega))^n}\|\nabla u\|^{r/2}_{(L^2(\Omega))^n}\right).
\end{aligned}$$

Our choice of $r < \frac{2n}{n-2}$ implies that $\mathfrak{p}r < \frac{2n}{n-2}$ and from the embedding of $H^1_0(\Omega)$ into $L^q(\Omega)$ for $q \leqslant \frac{2n}{n-2}$ it follows that f^e is bounded from $(H^1_0(\Omega))^n$ into $(W^{1,r}(\Omega))^n$ and the latter is compactly embedded in $r > 1$. Thus, F is bounded from $\mathcal{H}_*$ into $\{0\}\times(W^{1,r}(\Omega))^n\times\{0\}$ and the latter is compactly embedded in $(H^1_0(\Omega))^n \times (W^{1,r}(\Omega))^n \times H^1_0(\Omega)$.

Now fix $t \geqslant 0$ and consider

$$S_2(t)\mathbf{u}_0 = \int_0^t S_1(\xi)F(S(\xi)\mathbf{u}_0)d\xi,\ t \geqslant 0.$$

for $\mathbf{u}_0 \in B$, where B is a bounded subset of $\mathcal{H}_*$. Since orbits of bounded subsets of $\mathcal{H}_*$ under the nonlinear semigroup $\{S(t); t \geqslant 0\}$ are bounded in $\mathcal{H}_*$, it follows that $S_2(t)$ is compact for each $t > 0$. Thus the fact of nonlinear semigroup $\{S(t); t \geqslant 0\}$ is asymptotically compact is a consequence of [13, Theorem 2.37].

□

Finally, as application of Theorem 2.38 now implies that problem (3.10) has a global attractor $\mathcal{A}$ in $\mathcal{H}_*$.

3.4 Lamé operator of linear elastostatics system

Thanks to (3.8) the linear unbounded operator with homogeneous Dirichlet boundary condition $\Lambda_\kappa : D(\Lambda_\kappa) \subset L^2(\Omega) \to L^2(\Omega)$ defined by

$$D(\Lambda_\kappa) = H^2(\Omega) \cap H^1_0(\Omega),$$

and

$$\Lambda_\kappa \phi = -\text{div}(\kappa \nabla \phi)$$

is a sectorial operator which generates exponentially decaying analytic semigroup.

Let $Y = L^2(\Omega)$ with usual inner product. Since the negative Laplacian operator subject to zero Dirichlet bounded condition is a sectorial operator in Y (see Carracedo, Alix and Sanz [11, Section 2.3] and Henry [25, Page 19]). The operator

$$\Lambda_1 : \left(Y^2\right)^n \subset Y^n \to Y^n$$

defined by

$$\Lambda_1(v) = -\Delta v = (-\Delta v_1, \ldots, -\Delta v_n)$$

is positive, self-adjoint, $-\Lambda_1$ infinitesimal generator of an analytic semigroup in Y^n.

The unbounded linear operator

$$\Lambda_2 : \left(Y^2\right)^n \subset Y^n \to Y^n$$

defined by

$$\Lambda_2(v) = -\nabla \,\text{div}\, v$$

is closed in Y^n. Using the Proposition 2.29, we have there is $C > 0$ such that

$$\|\Lambda_2 x\| \leqslant C(\rho^\alpha \|x\| + \rho^{\alpha-1}\|\Lambda_1 x\|),$$

for all $x \in D(A),\ \rho > 0$ and $\alpha \in (0, 1]$. By the Theorem 2.30, Λ_1 and $\Lambda_1 + \Lambda_2$ has bounded imaginary power for $\alpha \in (0, 1)$.

Therefore, Λ_1 and Λ_2 are in the conditions of the Proposition 2.31 and the Corollary 2.32. Hence, $\Lambda = \Lambda_1 + \Lambda_2$ is sectorial, therefore by the Theorem 2.23 Λ is infinitesimal generator of a C_0-semigroup, and

$$D((\Lambda_1 + \Lambda_2)^\alpha) = D(\Lambda_1^\alpha),\ \alpha \in (0, 1).$$

3.5 Regularity of attractors

We have that

$$(Y^n)^\alpha = D((\Lambda_1 + \Lambda_2)^\alpha),$$

$\alpha \geqslant 0$, the fractional power spaces associated with the operator $\Lambda^{\frac{\alpha}{2}}$ with the graph norm $\|\cdot\|_{(Y^n)^\alpha} = \|\Lambda^{\frac{\alpha}{2}} \cdot\|_{Y^n}$ and by $Y^\alpha = D((-\Delta)^{\frac{\alpha}{2}})$ endowed with the graph norm $\|\cdot\|_{Y^\alpha} = \|(-\Delta)^{\frac{\alpha}{2}} \cdot\|_Y$. We just verify in the previous section $(Y^n)^\alpha = (Y^\alpha)^n$ as sets, but this don't mean $\|\cdot\|_{(Y^n)^\alpha}$ is equivalent to $\|\cdot\|_{(Y^\alpha)^n}$. Through the similar argument $D((-\Delta)^{\frac{\alpha}{2}}) = D(\Lambda_\kappa^{\frac{\alpha}{2}})$, since is know $\|\cdot\|_{Y^\alpha}$ is equivalent $\|\kappa \cdot\|_Y$ thanks to (3.8).

With this notation, we have $(Y^{-\alpha})' = Y^{\alpha}$ for all $\alpha \geqslant 0$. It is of special interest the spaces

$$Y^2 = H^2(\Omega) \cap H_0^1(\Omega),\ Y^1 = H_0^1(\Omega), Y^0 = Y = L^2(\Omega) \text{ and } Y^{-1} = (Y^1)' = H^{-1}(\Omega).$$

From now on, we consider $Y_*^{\alpha} = Y^{\alpha} \cap Y_*$.

Remark 3.10. If $\mathbf{u} = (u, z, \theta)$ is a mild solution of (3.10) and $\theta_0 \in Y^{-\alpha}$, then we can ensure that $\theta \in Y^{1-\alpha}$ for t a.e. in $[0, \infty)$. Without lost of generality, we just need to show that if $\theta_0 \in L^2(\Omega)$, then $\theta \in H_0^1(\Omega)$.

Being $\mathbf{u} = (u, \partial_t u, \theta)$ a mild solution for (3.10),

$$\mathbf{u} = (u, \partial_t u, \theta) = S(t)\mathbf{u}_0 + \int_a^b S(t-s)F(\mathbf{u}(s))ds = (S_1(t), S_2(t), S_3(t))\mathbf{u}_0.$$

where $S(t)$ is a semigroup which has A as your infinitesimal generator. Notice that $F(u, \partial_t, \theta) = (0, f(u), 0)$.

Since the operator A is closed and densely defined for there is a suit $(\mathbf{u}_{0,n}) \subset D(A)$ such that $\mathbf{u}_{0,n} \to \mathbf{u}_0$ in $\mathcal{H}$. Since $\mathbf{u}_{0,n} \in D(A)$ implies that $\mathbf{u}_{0,n}$ be a classical solution, in particular

$$d_n = \left(\frac{d}{dt}S_3(t)\mathbf{u}_{0,n}, v\right) - (\kappa \nabla S_3(t)\mathbf{u}_{0,n}, \nabla v) + (\operatorname{div} \partial_t \left(S_1(t)\mathbf{u}_{0,n}\right), v) = 0,\ \forall t \geqslant 0.$$

Since

$$\operatorname{div} \partial_t \left(S_1(t)\mathbf{u}_{0,n}\right) \to \operatorname{div} \partial_t \left(S_1(t)\mathbf{u}_0\right) \text{ in } L^2(\Omega)$$

and we have $\mathbf{u}_{0,n} \to \mathbf{u}_0$ in $\mathcal{H}_*$, then $S_3(t)\mathbf{u}_{0,n} \to S_3(t)\mathbf{u}_0$ in $L^2(\Omega)$ and $d_n = d_0 = 0$.

Notice hat $\operatorname{div} \partial_t u \in L^2(\Omega) \subset H^{-1}$ with the proper identifications. Given $\bar{\theta}_0 \in L^2(\Omega)$, let $\bar{\theta} \in L^2((0,\infty); H_0^1(\Omega)) \cap C((0,\infty); L^2(\Omega)) \subset C((0,\infty); L^2(\Omega))$ be such that $\bar{\theta}(0,\cdot) = \bar{\theta}_0$ and

$$\left(\frac{d}{dt}\bar{\theta}, v\right) - \left(\kappa \nabla \bar{\theta}, \nabla v\right) + (\operatorname{div} \partial_t \left(S_1(t)\mathbf{u}_0\right), v) = 0.$$

By the Theorem 1.3, $\bar{\theta}_0$ is unique in $L^2((0,\infty); H_0^1(\Omega)) \cap C((0,\infty); L^2(\Omega))$. Therefore,

$$\bar{\theta} = S_3(t)\mathbf{u}_0,$$

because $S_3(t)\mathbf{u}_{0,n} \to S_3(t)\mathbf{u}_0$ in q. Use again the Theorem 1.3 ensure us, the following

$$S_3(t)\mathbf{u}_0 \in L^2((0,\infty); H_0^1(\Omega)) \cap C((0,\infty); L^2(\Omega)) \subset C((0,\infty); L^2(\Omega)).$$

We will also denote

$$\begin{aligned} \mathcal{H} = \mathcal{H}^0 &= (Y^1)^n \times Y^n \times Y, \\ \mathcal{H}^1 &= (Y^2)^n \times (Y^1)^n \times Y^1, \\ \mathcal{H}_* &= (Y^1)^n \times Y^n \times Y_* \end{aligned}$$

and

$$\mathcal{H}^1_* = (Y^2)^n \times (Y^1)^n \times Y^1_*,$$

all equipped with usual inner product of $(H^2(\Omega))^n \cap (H^1_0(\Omega))^n \times H^1_0(\Omega)^n \times L^2(\Omega)$.

Thanks to work as Dafermos [17] and Henry, Perissinotto and Lopes [26], the operator A is the generator of a strongly continuous semigroup of contractions on $\mathcal{H}$. Furthermore, A has a compact inverse. A partial description of the fractional power spaces $\mathcal{H}^\alpha = D(A^\alpha)$ endowed with the graph norm is given by

$$\mathcal{H}^\alpha = [\mathcal{H}^1_*, \mathcal{H}^0_*]_\alpha = (Y^{1-\alpha})^n \times (Y^{-\alpha})^n \times Y^{-\alpha}_*.$$

for $\alpha \in [0,1]$, see Amann [1, Section 2 of Chapter 1].

Now we investigate the regularity of the global attractor. As a matter of fact, we prove that $\mathcal{A}$ is a bounded subset of $\mathcal{H}^1_*$.

Theorem 3.11. *The global attractor $\mathcal{A}$ for the problem* (3.10)*, obtained in Section 3.3, lies in a more regular space than $\mathcal{H}_*$, in fact, $\mathcal{A}$ is a bounded subset of $\mathcal{H}^1_*$.*

Proof. The main idea that we will use in verifying this result is the argument of progressive increases of regularity, following Babin and Vishik in [3] (and also explored for example in Carvalho, Langa and Robinson [13, Chapter 15]). With lost of generality we will assume $\kappa = 1$ to simplify the calculations.

Let $\xi : \mathbb{R} \to \mathcal{H}_*$ be a global bounded solution of (3.10). Then, the set $\{\xi(t);\ t \in \mathbb{R}\}$ is a bounded subset of $\mathcal{H}_*$. We already know that $\mathcal{A}$ is bounded in $\mathcal{H}_*$. Hence, if $\xi(\cdot) = \mathbb{R} \to \mathcal{H}_*$ is such that $\xi(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$, then

$$\xi(t) = S_1(t)\xi(0) + \int_0^t S_1(s)F(\xi(s))ds,$$

where $S_1(\cdot)$ is defined in (3.25). Now using the decay of $S_1(t)$ in the Proposition 3.9 and letting $t \to +\infty$ it follows that

$$\xi(t) = \int_0^{+\infty} S_1(s)F(\xi(s))ds. \tag{3.41}$$

Set $(\mu_0, \mu_1, \vartheta_0) = \xi(0)$, and we consider

$$\begin{bmatrix} \mu \\ \partial_t \mu \\ \vartheta \end{bmatrix}(t) = S_2(t)\begin{bmatrix} \mu_0 \\ \mu_1 \\ \vartheta_0 \end{bmatrix} = \int_0^t S_1(s)F(S(s)\begin{bmatrix} \mu_0 \\ \mu_1 \\ \vartheta_0 \end{bmatrix})ds,$$

and note that $(\mu(\cdot), \partial_t\mu(\cdot), \vartheta(\cdot)) \in \mathcal{H}_*$ solves the system

$$\begin{cases} \partial_t^2 \mu - \Delta\mu - \nabla \operatorname{div} \mu + \nabla\vartheta = f(\mu(t;\mu_0)), & x \in \Omega,\ t > 0, \\ \partial_t\vartheta - \Delta\vartheta + \operatorname{div} \partial_t\mu = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{3.42}$$

with

$$\mu(x,0) = \partial_t\mu(x,0) = 0 \text{ and } \vartheta(x,0) = 0,\ x \in \Omega. \tag{3.43}$$

We again consider the following functional

$$\mathcal{L}_0(\mu(t), \partial_t \mu(t), \theta(t)) = \frac{1}{2}\Big(\|\mu(t)\|^2_{(H^1_0(\Omega))^n} + \|\partial_t \mu(t)\|^2_{(L^2(\Omega))^n} + \|\theta(t)\|^2_{L^2(\Omega)} \Big) \\ + \delta_1 (\mu(t), \partial_t \mu(t))_{(L^2(\Omega))^n} + \delta_2 (\Phi(t), \partial_t \mu(t))_{(L^2(\Omega))^n}$$

to estimate the solution of (3.42)-(3.43) for $(\mu_0, \mu_1, \vartheta_0)$ in a bounded subset B of $\mathcal{H}_*$. The same arguments of the proof of Theorem 3.6 to obtain (we omitted t in order to simplify the notation)

$$\frac{d\mathcal{L}_0}{dt}(\mu, \partial_t \mu, \vartheta) \leqslant -C_0 \|\nabla \mu\|^2_{(L^2(\Omega))^n} - C_1 \|\partial_t \mu\|^2_{(L^2(\Omega))^n} - C_2 \|\vartheta\|^2_{L^2(\Omega)} + C_3, \tag{3.44}$$

where C_0, C_1, C_2 and C_3 are positive constants.

From this it follows that

$$\bigcup_{0 \leqslant \tau \leqslant t} S_2(\tau) B \text{ is a bounded subset of } \mathcal{H}_*. \tag{3.45}$$

Therefore $(\varpi, \zeta) = (\partial_t \mu, \partial_t \vartheta)$ solves the system

$$\begin{cases} \partial_t^2 \varpi - \Delta \varpi - \nabla \operatorname{div} \varpi + \nabla \zeta = f'(\mu(t; \mu_0)) \varpi(t; \mu_0), & x \in \Omega,\ t > 0, \\ \partial_t \zeta - \Delta \zeta + \operatorname{div} \partial_t \varpi = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{3.46}$$

with $\varpi(0) = 0$, $\varpi_t(0) = f(\mu_0)$, and $\zeta(0) = 0$.

In order to continue with verification, we will show that $(\mu, \partial \mu, \vartheta)$ is bounded in $\mathcal{H}^1_*$, by estimate $(\varpi, \partial_t \varpi, \zeta)$ in $\mathcal{H}_*$. But solutions are not regular enough to allow this directly, that's why we will work 'towards' $\mathcal{H}_*$ by progressive increases of regularity.

Figure 3.1: scale of the fractional power spaces of Y.

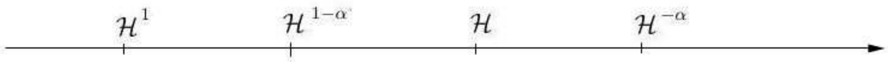

Figure 3.2: scale of the fractional power spaces of $\mathcal{H}$.

We will take $(\varpi, \partial_t \varpi, \zeta) \in \mathcal{H}^{-\alpha} = (Y^{1-\alpha})^n \times (Y^{-\alpha})^n \times Y_*^{-\alpha}$ for $\alpha \in (0,1)$ and we define

(3.47)

$$\mathcal{L}_\alpha(t) = \frac{M}{2}\Big(2\|\varpi\|^2_{(Y^{1-\alpha})^n} + \|\phi\|^2_{(Y^{-\alpha})^n} + \|\zeta\|^2_{Y^{-\alpha}} \Big) + \delta_1 (\varpi, \phi)_{(Y^{-\alpha})^n} + \delta_2 (\gamma, \phi)_{(Y_*^{-\alpha})^n},$$

where γ such that $\operatorname{div}\gamma = \zeta$.

We want to find an inequality like 3.44. Therefore, we will obtain following estimates for the terms involved in $\mathcal{L}_\alpha(t)$; First, thanks to (3.42) we get

$$\frac{d}{dt}\|\phi\|^2_{(Y^{-\alpha})^n} = 2(\Delta\varpi+\nabla\operatorname{div}\varpi,\phi)_{(Y^{-\alpha})^n}-2(\nabla\zeta,\phi)_{(Y^{-\alpha})^n}+2(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n}.$$

Because of (3.8) we have

$$\frac{d}{dt}\|\zeta\|^2_{Y^{-\alpha}} = 2(\Delta\zeta,\zeta)_{Y^{-\alpha}} - 2(\operatorname{div}\partial_t\varpi,\zeta)_{Y^{-\alpha}} \leqslant -2\|\zeta\|_{Y^{1-\alpha}} - 2(\operatorname{div}\partial_t\varpi,\zeta)_{Y^{-\alpha}}.$$

Again by (3.42) we obtain

$$\begin{aligned}\frac{d}{dt}(\varpi,\phi)_{(Y^{-\alpha})^n} &= \|\phi\|^2_{(Y^{-\alpha})^n} + (\varpi,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\varpi,\nabla\zeta)_{(Y^{-\alpha})^n}\\ &\quad + (\varpi, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}.\end{aligned}$$

Also, we see that

$$\begin{aligned}\frac{d}{dt}(\gamma,\phi)_{(Y^{-\alpha})^n} &= (\partial_t\gamma,\phi)_{(Y^{-\alpha})^n} + (\gamma,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\gamma,\nabla\zeta)_{(Y^{-\alpha})^n}\\ &\quad + (\gamma, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}.\end{aligned}$$

In this way,

$$\begin{aligned}\frac{d\mathcal{L}_\alpha}{dt} \leqslant{}& \frac{M}{2}[2(\partial_t\varpi,\varpi)_{(Y^{1-\alpha})^n} + 2(\Delta\varpi+\nabla\operatorname{div}\varpi,\phi)_{(Y^{-\alpha})^n} - 2(\nabla\zeta,\phi)_{(Y^{-\alpha})^n}\\ &+ 2(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n} - 2\|\zeta\|^2_{Y^{1-\alpha}} - 2(\operatorname{div}\partial_t\varpi,\zeta)_{Y^{-\alpha}}]\\ &+ \delta_1\Big[\|\phi\|^2_{(Y^{-\alpha})^n} + (\varpi,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\varpi,\nabla\zeta)_{(Y^{-\alpha})^n}\\ &+ (\varpi, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}\Big] + \delta_2\Big[(\partial_t\gamma,\phi)_{(Y^{-\alpha})^n} - (\gamma,\nabla\zeta)_{(Y^{-\alpha})^n}\\ &+ (\gamma,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} + (\gamma, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}\Big].\end{aligned}$$

What implies in

$$\begin{aligned}\frac{d\mathcal{L}_\alpha}{dt} \leqslant{}& M[(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n} - \|\zeta\|^2_{Y^{1-\alpha}}] + \delta_1[\|\phi\|^2_{(Y^{-\alpha})^n}\\ &+ (\varpi,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\varpi,\nabla\zeta)_{(Y^{-\alpha})^n} + (\varpi, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}]\\ &+ \delta_2[(\partial_t\gamma,\phi)_{(Y^{-\alpha})^n} + (\gamma,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} + (\zeta,\zeta)_{(Y^{-\alpha})^n}\\ &+ (\gamma, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}].\end{aligned}$$

by simplify and reorder right hand of inequality,

$$\begin{aligned}\frac{d\mathcal{L}_\alpha}{dt} \leqslant{}& M(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n} + \delta_1(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\varpi)_{(Y^{-\alpha})^n}\\ &+ \delta_2(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\gamma)_{(Y^{-\alpha})^n} - M\|\zeta\|^2_{Y^{1-\alpha}} + \delta_2\|\zeta\|^2_{Y^{-\alpha}}\\ &+ \delta_1\|\phi\|^2_{(Y^{-\alpha})^n} - \delta_1\|\varpi\|^2_{(Y^{1-\alpha})^n} + \delta_2(\partial_t\gamma,\phi)_{(Y^{-\alpha})^n}\\ &+ \delta_2(\gamma,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - \delta_1(\varpi,\nabla\zeta)_{(Y^{-\alpha})^n}.\end{aligned} \tag{3.48}$$

Next, we deal with the three terms in which it appears explicitly the nonlinearity f'. From now on, let

$$\alpha_1 := \frac{(\mathfrak{p}-1)(N-2)}{2}.$$

Note that since $\mathfrak{p} < \frac{N}{N-2}$, we obtain that $\alpha_1 < 1$.

If $\alpha \in (0, \alpha_1)$ then we can observe that

$$(f'(\mu(t;\mu_0))\varpi(t;\mu_0), g)_{(Y^{-\alpha})^n} \leqslant \|g\|_{(Y^{-\alpha})^n} \|f'(\mu(t;\mu_0))\varpi(t;\mu_0)\|_{(Y^{-\alpha})^n}$$

for $g \in \{\varphi,\ \partial_t\varphi,\ \zeta\}$ and using the embedding $(Y^{\alpha})^n \hookrightarrow (H^{2\alpha}(\Omega))^n \hookrightarrow (L^p(\Omega))^n$ (or equivalently $(L^{\frac{p}{p-1}}(\Omega))^n \hookrightarrow (Y^{-\alpha})^n$) for any $1 < p \leqslant \frac{2N}{N-2\alpha}$ and (3.7), we have that for some $c_4 > 0$

$$\begin{aligned}\|f'(\mu)\varpi\|_{(Y^{-\alpha})^n} &\leqslant c_4 \|f'(\mu)\varpi\|_{L^{\frac{2N}{N+2\alpha}}(\Omega)} \leqslant C\|\varpi(1+|\mu|^{\mathfrak{p}-1})\|_{L^{\frac{2N}{N+2\alpha}}(\Omega)} \\ &\leqslant C\|\varpi\|_{\mathcal{H}_*}\|1+|\mu|^{\mathfrak{p}-1}\|_{L^{\frac{N}{\alpha}}(\Omega)}\end{aligned}$$

and so

$$\|f'(\mu)\varpi\|^2_{(Y^{-\alpha})^n} \leqslant C^2\|\varpi\|^2_{\mathcal{H}_*}\|1+|\mu|^{\mathfrak{p}-1}\|^2_{L^{\frac{N}{\alpha}}(\Omega)}.$$

From (3.45) μ remains in a bounded subset of $\mathcal{H}_*^{\frac{1}{2}} \hookrightarrow L^{\frac{(\mathfrak{p}-1)N}{\alpha}}(\Omega)$ for any $1 < \mathfrak{p} < \frac{N-4+4\alpha}{N-4}$ and this implies that

$$\int_\Omega (1+|\mu|^{\mathfrak{p}-1})^{\frac{N}{\alpha}}\,dx \leqslant |\Omega| + \|\mu\|^{\frac{(\mathfrak{p}-1)N-\alpha}{N(\mathfrak{p}-1)}}_{L^{\frac{(\mathfrak{p}-1)N}{\alpha}}(\Omega)} \leqslant |\Omega| + c_5\|\mu\|^{\frac{(\mathfrak{p}-1)N-\alpha}{N(\mathfrak{p}-1)}}_{\mathcal{H}_*^{\frac{1}{2}}} \leqslant c_5,$$

for some $c_5 > 0$.

Therefore, there exists a positive constant C_f such that

$$\|f'(\mu)\varpi\|^2_{(Y^{-\alpha})^n} \leqslant C_f. \tag{3.49}$$

From (3.29), we have that $\partial_t\gamma = \nabla\zeta - \partial_t\varpi$, then for any $\epsilon > 0$, we have

$$\partial_t\gamma = \nabla\zeta - \partial_t\varpi,$$

then for any $\epsilon > 0$, we have

$$\delta_2(\partial_t\gamma, \phi)_{(Y^{1-\alpha})^n} \leqslant \delta_2(\nabla\zeta - \phi, \phi)_{(Y^{-\alpha})^n} \leqslant \delta_2(\nabla\zeta, \phi)_{(Y^{-\alpha})^n} - \delta_2\|\phi\|^2_{(Y^{-\alpha})^n}$$

and therefore

$$\delta_2(\partial_t\gamma, \phi)_{(Y^{1-\alpha})^n} \leqslant \frac{\delta_2}{\epsilon}\|\zeta\|^2_{Y^{1-\alpha}} + \delta_2\left(\epsilon - 1\right)\|\phi\|^2_{(Y^{-\alpha})^n}. \tag{3.50}$$

as previously discussed in Remark 3.10, $\|\zeta\|^2_{Y^{1-\alpha}} < \infty$ a.e. for $t \in [0,\infty)$.

Now we will denote

$$J_1 = \delta_2(\gamma, \Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - \delta_1(\nabla\zeta, \varpi)_{(Y^{-\alpha})^n}.$$

From $\gamma \in Y_*^{1-\alpha} \hookrightarrow Y_*^{-\alpha}$, we obtain following inequality

$$\begin{aligned} J_1 &\leqslant \frac{\delta_2}{\epsilon}\|\gamma\|^2_{(Y^{1-\alpha})^n} + \epsilon\delta_2\|\varpi\|^2_{(Y^{1-\alpha})} + \frac{\delta_1}{\epsilon}\|\zeta\|^2_{(Y^{1-\alpha})^n} + \epsilon\delta_1\|\varpi\|^2_{(Y^{-\alpha})} \\ &\leqslant \frac{\delta_2}{\epsilon}\|\gamma\|^2_{(Y^{1-\alpha})^n} + \epsilon(\delta_1+\delta_2)\|\varpi\|^2_{(Y^{1-\alpha})} + \frac{\delta_1}{\epsilon}\|\zeta\|^2_{(Y^{1-\alpha})^n} \end{aligned}$$

when we observe that $\|\gamma\|^2_{(Y^{1-\alpha})^n} \leqslant C_1\|\zeta\|^2_{(Y^{-\alpha})} \leqslant C_2\|\zeta\|^2_{(Y^{1-\alpha})}$,then

$$J_1 \leqslant \left(\frac{\delta_2 C_2}{\epsilon} + \frac{\delta_1}{\epsilon}\right)\|\zeta\|^2_{(Y^{1-\alpha})^n} + \epsilon(\delta_1+\delta_2)\|\varpi\|^2_{(Y^{1-\alpha})}. \tag{3.51}$$

Using (3.49), (3.50) and (3.51) in (3.48) we get

$$\begin{aligned} \frac{d\mathcal{L}_\alpha}{dt} &\leqslant \epsilon\|\phi\|^2_{(Y^{-\alpha})^n} + \epsilon\|\varpi\|^2_{(Y^{-\alpha})^n} + \epsilon\|\gamma\|^2_{(Y^{-\alpha})^n} + C(M,\delta_1,\delta_2) - (M-\delta_2)\|\zeta\|^2_{Y^{1-\alpha}} \\ &\quad + \delta_1\|\phi\|^2_{(Y^{-\alpha})^n} - \delta_1\|\varpi\|^2_{(Y^{1-\alpha})^n} + \left(\frac{\delta_2 C_2}{\epsilon} + \frac{\delta_1}{\epsilon}\right)\|\zeta\|^2_{Y^{1-\alpha}} \\ &\quad + \delta_2(\epsilon - 1)\|\phi\|^2_{(Y^{-\alpha})^n} + \epsilon(\delta_1+\delta_2)\|\varpi\|^2_{(Y^{1-\alpha})^n} \\ &\leqslant (-\delta_1 + \epsilon(\delta_1+\delta_2) + \epsilon)\|\varpi\|^2_{(Y^{1-\alpha})^n} + (\epsilon + \delta_1 + \delta_2\epsilon + 2\delta_2\epsilon C(\Omega) - \delta_2)\|\phi\|^2_{(Y^{-\alpha})^n} \\ &\quad + \left(\frac{\delta_2 C_3}{\epsilon} + \frac{\delta_1}{\epsilon} + \epsilon C - M + \delta_2\right)\|\zeta\|^2_{Y^{1-\alpha}} + C(M,\delta_1,\delta_2). \end{aligned}$$

Let $\epsilon > 0$ be small enough and, let $\delta_1 < \delta_2$ and $M > 0$ be large enough such that it is possible choose $\mathfrak{p}_1, \mathfrak{p}_2 > 0$ which,

$$\frac{d\mathcal{L}_\alpha}{dt} \leqslant -\mathfrak{p}_1\left(\|\varpi\|^2_{(Y^{1-\alpha})^n} + \|\phi\|^2_{(Y^{-\alpha})^n} + \|\zeta\|^2_{Y^{1-\alpha}}\right) + \mathfrak{p}_2$$

for t a.e. in $[0,\infty)$.

But $\|\zeta\|^2_{(Y^{-\alpha})^n} \leqslant \frac{C_2}{C_1}\|\zeta\|^2_{(Y^{1-\alpha})^n}$ and $\zeta \in C(0,\infty; Y^{-\alpha})$. This lead us to

$$\frac{d\mathcal{L}_\alpha}{dt} \leqslant -\mathfrak{p}_1\left(\|\varpi\|^2_{(Y^{1-\alpha})^n} + \|\phi\|^2_{(Y^{-\alpha})^n} + \|\zeta\|^2_{Y^{-\alpha}}\right) + \mathfrak{p}_2, \forall t \in [0,\infty)]$$

and from the fact that $\mathcal{A} = \{\xi(t) : \xi(\cdot) \text{ is a global bounded solution of (3.10) in } \mathcal{H}_*\}$ we obtain that

$$\mathcal{A} \text{ is bounded in } (Y^{2-\alpha_1})^n \times (Y^{1-\alpha_1})^n \times Y_*^{1-\alpha_1}. \tag{3.52}$$

Using (3.52) and restarting from with $\alpha_2 = (p+1)\alpha_1 - p < \alpha_1$ if follows that $\mathcal{A}$ is bounded in $(Y^{2-\alpha_2})^n \times (Y^{1-\alpha_2})^n \times Y_*^{1-\alpha_2}$.

How can we apply this procedure when we get $\alpha_k < \alpha_{k-1}$, we can now show that $\mathcal{A}$ is bounded in $(Y^2)^n \times (Y^1)^n \times Y_*^1$ and by the Remark 3.10 this implies in fact, that

$$\mathcal{A} \text{ is bounded in } (Y^2)^n \times (Y^1)^n \times Y_*^2.$$

□

3.6 Upper semicontinuity of the attractors

In this section we prove the upper semicontinuity of the global attractors in $\mathcal{H}^1_*$ with respect to functional parameter κ in (3.1). Let $\{\kappa_\epsilon\}_{\epsilon\in[0,1)}$ be a family of functions such that for each κ_ϵ the condition (3.8) is valid and suppose that

$$\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)} \to 0, \text{ as } \epsilon \to 0^+.$$

We observe that all previous results are also valid to the problem (3.1) with κ_ϵ instead of κ. If $\{S_\epsilon(t); t \geqslant 0\}$ denotes the evolution process associate to the problem (3.10) with global attractors $\mathcal{A}_\epsilon$ for each $\epsilon \in [0,1]$, then we have the following result.

Theorem 3.12. *The family of global attractors $\mathcal{A}_\epsilon$ is upper semicontinuous as $\epsilon \to 0^+$.*

Proof. Let $\mathbf{u}^\epsilon = S_\epsilon(t)\mathbf{u}_0$ be the solution of (3.10) with $\mathbf{u}^\epsilon = (u^\epsilon, z^\epsilon, \theta^\epsilon)$. Then we write $\mu = u^\epsilon - u^0$ and $\vartheta = \theta^\epsilon - \theta^0$.

Hence, $(\mu, \partial_t\mu, \vartheta)$ solves the following system

$$\begin{cases} \partial_t^2\mu - \Delta\mu - \nabla \operatorname{div}\mu + \beta\nabla\vartheta = f(u^\epsilon) - f(u^0), & x \in \Omega,\ t > 0, \\ \partial_t\vartheta - [\operatorname{div}(\kappa_\epsilon(x)\nabla\theta^\epsilon) - \operatorname{div}(\kappa_0(x)\nabla\theta^0)] + \beta \operatorname{div}\partial_t\mu = 0, & x \in \Omega,\ t > 0. \end{cases}$$

We be able to find

$$\int_\Omega \partial_t^2\mu^\epsilon\partial_t\mu^\epsilon dx - \int_\Omega (\Delta\mu^\epsilon + \nabla \operatorname{div}\mu^\epsilon)\partial_t\mu^\epsilon dx + \int_\Omega \nabla\vartheta^\epsilon\partial_t\mu^\epsilon dx = \int_\Omega (f(u^\epsilon) - f(u^0))\partial_t\mu^\epsilon dx,$$

by multiplies $\partial_t\mu^\epsilon$ in the first equation and,

$$\int_\Omega \partial_t\vartheta^\epsilon\vartheta^\epsilon dx - \int_\Omega [\operatorname{div}(\kappa_\epsilon(x)\nabla\theta^\epsilon) - \operatorname{div}\big(\kappa_0(x)\nabla\theta^0\big)]\vartheta^\epsilon dx + \int_\Omega \operatorname{div}\partial_t\mu^\epsilon\vartheta^\epsilon dx = 0$$

by multiplies ϑ^ϵ in the second equation.

This lead us to

$$\frac{d}{dt}\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|^2_{\mathcal{H}*} = 2\int_\Omega (f(u^\epsilon) - f(u^0))\partial_t\mu^\epsilon dx + 2\int_\Omega \operatorname{div}\big(\kappa_\epsilon(x)\nabla\theta^\epsilon - \kappa_0(x)\nabla\theta^0\big)\vartheta^\epsilon dx,$$

and in other words,

$$\frac{d}{dt}\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|^2_{\mathcal{H}*} = 2\int_\Omega (f(u^\epsilon) - f(u^0))\partial_t\mu^\epsilon dx - 2\int_\Omega \big(\kappa_\epsilon(x)\nabla\theta^\epsilon - \kappa_0(x)\nabla\theta^0\big)\nabla\vartheta^\epsilon dx.$$

Now, note that

$$\begin{aligned} &\Big|\int_\Omega \big(\kappa_\epsilon(x)\nabla\theta^\epsilon - \kappa_0(x)\nabla\theta^0\big)\nabla\vartheta^\epsilon dx\Big| \\ &\leqslant \int_\Omega \big[\kappa_\epsilon(x)(\nabla\theta^\epsilon - \nabla\theta^0) + (\kappa_\epsilon(x) - \kappa_0(x))\nabla\theta^0\big]\nabla\vartheta^\epsilon dx \\ &\leqslant \|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)}\|\theta^0\|_{Y^1_*}\|\vartheta^\epsilon\|_{Y^1} + \|\kappa_\epsilon\|_{L^\infty(\Omega)}\|\theta^\epsilon - \theta^0\|^2_{Y^1_*} \\ &\leqslant C_1\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)} + C_2\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|^2_{\mathcal{H}*}, \end{aligned}$$

where $C_1 > 0$ and $C_2 > 0$ are constants independents of ϵ thanks to Theorem 3.11.

From Lemma 3.1, we have

$$\int_\Omega (f(u^\epsilon) - f(u^0))\partial_t \mu^\epsilon dx \leqslant C_3 \|(\mu^\epsilon, \partial_t \mu^\epsilon, \vartheta^\epsilon)\|^2_{\mathcal{H}_*},$$

where $C_3 > 0$ is constant independent of ϵ.

Hence, there exist constants $C' > 0$ and $C'' > 0$ such that

$$\frac{d}{dt}\|(\mu^\epsilon, \partial_t \mu^\epsilon, \vartheta^\epsilon)\|^2_{\mathcal{H}_*} \leqslant C'\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)} + C''\|(\mu^\epsilon, \partial_t \mu^\epsilon, \vartheta^\epsilon)\|^2_{\mathcal{H}_*}.$$

and so

$$\|(\mu^\epsilon, \partial_t \mu^\epsilon, \vartheta^\epsilon)\|^2_{\mathcal{H}_*} \leqslant C' t e^{-C'' t}\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)}, \ \forall t > 0, \tag{3.53}$$

i.e., $\mathbf{u}^\epsilon \to \mathbf{u}^0$ in $\mathcal{H}_*$, as $\epsilon \to 0^+$, uniformly for t in bounded subset of the interval $[0, +\infty)$ and $\mathbf{u}_0$ in bounded subset of $\mathcal{H}_*$.

From the existence of attractor we have proved, $\mathcal{A}_{\kappa_\epsilon}$ is bounded in $\mathcal{H}_*$. Then for $\delta > 0$ given, there is a $t > 0$ large enough such that

$$\mathit{dist}_H(S_{\kappa_0}(t)\mathcal{A}_{\kappa_\epsilon}, \mathcal{A}_{\kappa_0}) < \frac{\delta}{2}, \ \forall \epsilon \in (0, 1].$$

Using (3.53), there exists $\epsilon_0 > 0$ such that

$$\sup_{\mathbf{u}^\epsilon \in \mathcal{A}_{\kappa_\epsilon}} \|S_{\kappa_\epsilon}(t)\mathbf{u}^\epsilon - S_{\kappa_0}(t)\mathbf{u}^\epsilon\|_{\mathcal{H}_*} < \frac{\delta}{2}, \ \forall \epsilon \in (0, \epsilon_0].$$

Therefore,

$$\begin{aligned}
\mathit{dist}_H(\mathcal{A}_{\kappa_\epsilon}, \mathcal{A}_{\kappa_0}) &\leqslant \mathit{dist}_H(S_{\kappa_\epsilon}(t)\mathcal{A}_{\kappa_\epsilon}, S_{\kappa_0}(t)\mathcal{A}_{\kappa_\epsilon}) + \mathit{dist}_H(S_{\kappa_0}(t)\mathcal{A}_{\kappa_\epsilon}, S_{\kappa_0}(t)\mathcal{A}_{\kappa_0}) \\
&\leqslant \sup_{\mathbf{u}^\epsilon \in \mathcal{A}_\epsilon} \inf_{\mathbf{u}^\epsilon \in \mathcal{A}_\epsilon} \|S_{\kappa_\epsilon}(t)\mathbf{u}^\epsilon - S_{\kappa_0}(t)\mathbf{u}^\epsilon\|_{\mathcal{H}_*} + \mathit{dist}_H(S_{\kappa_0}(t)\mathcal{A}_{\kappa_\epsilon}, \mathcal{A}_{\kappa_0}) \\
&\leqslant \sup_{\mathbf{u}^\epsilon \in \mathcal{A}_\epsilon} \|S_{\kappa_\epsilon}(t)\mathbf{u}^\epsilon - S_{\kappa_0}(t)\mathbf{u}^\epsilon\|_{\mathcal{H}_*} + \mathit{dist}_H(S_{\kappa_0}(t)\mathcal{A}_{\kappa_\epsilon}, \mathcal{A}_{\kappa_0}) < \delta
\end{aligned}$$

which proves the upper semicontinuity of the family of attractors with respect to the parameter ϵ.

□

Chapter 4

Nonautonomous n-dimensional thermoelasticity system

In this chapter we will be interested in checking the non-autonomous case of the thermoelastic problem. Similar to the previous chapter we will deal with the existence, regularity, and superior semicontinuity of the pullback attractor. Here we will consider $n \geqslant 2$.

4.1 Well-possessedness of nonautonomous thermoelastic system

We are interested in the study of asymptotic behavior of mild solutions for a multidimensional semilinear thermoelastic systems; namely, initial-boundary value problems with space dependent diffusion coefficients

$$
\begin{cases} \partial_t^2 u - \Delta u - \nabla \operatorname{div} u + \beta(t)\nabla\theta = f(u), & x \in \Omega,\ t > s, \\ \partial_t \theta - \operatorname{div}(\kappa(x)\nabla\theta) + \beta(t) \operatorname{div} \partial_t u = 0, & x \in \Omega,\ t > s, \end{cases} \tag{4.1}
$$

subject to initial-boundary condition

$$
\begin{cases} u(x,s) = u_0(x),\ \partial_t u(x,s) = u_1(x), & x \in \Omega, \\ \theta(x,s) = \theta_0(x) & x \in \Omega, \\ u(x,t) = 0,\ \theta(x,t) = 0, & x \in \partial\Omega,\ t > s. \\ \kappa(x)\nabla\theta(x,t) - \partial_t u(x,t) = 0, & x \in \partial\Omega, t \geqslant s \end{cases} \tag{4.2}
$$

In this problem, the map f is external force and the functional parameters κ is the diffusion coefficient with the conditions (4.1)-(4.2). Furthermore, we assume that the thermal moduli $\beta : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and there are positive constants β_0 and β_1 such that

$$
0 < \beta_0 \leqslant \beta(t) \leqslant \beta_1, \quad t \in \mathbb{R}. \tag{4.3}
$$

Let $\mathbf{u} = (u, z, \theta)$ be the state vector with $z = \partial_t u$, we rewrite (4.1)-(4.2) as ordinary differential equations in the product space $\mathcal{H}$

$$
\begin{cases} \dfrac{d\mathbf{u}}{dt} + \mathbf{A}(t)\mathbf{u} = \mathbf{F}(\mathbf{u}),\ t > s, \\ \mathbf{u}(s) = \mathbf{u}_0, \end{cases} \tag{4.4}
$$

where $\mathbf{u}_0 = (u_0, z_0, \theta_0)$, $\mathbf{A}(t) : D(\mathbf{A}(t)) \subset \mathcal{H} \to \mathcal{H}$ is a family of linear unbounded operator defined by

$$
D(\mathbf{A}(t)) = (H_0^1(\Omega) \cap H^2(\Omega))^n \times (H_0^1(\Omega))^n \times (H_0^1(\Omega) \cap H^2(\Omega)) \bigcap X,
$$

and for any $(u, z, \theta) \in D(\mathbf{A}(t))$

$$
\mathbf{A}(t)(u, z, \theta) = (-z, -\Delta u - \nabla \operatorname{div} u + \beta(t)\nabla\theta, -\mathbf{div}(\kappa\nabla\theta) + \beta(t) \operatorname{div} z)
$$

where

$$
X = \{(u, z, \theta) \in \mathcal{H}; \kappa(x)\nabla\theta(x, \cdot) - \partial_t u(x, \cdot) = 0,\ \text{for } x \in \partial\Omega\}.
$$

The nonlinear term in (4.4) is defined by

$$
\mathbf{F}(\mathbf{u}) = (0, f^e(u), 0),
$$

where f^e denotes the Nemytskĭi operator associated with f, i.e.

$$
f^e(u) = f(u(t, x)) = (f_1(u(t, x)), \dots, f_n(u(t, x)))
$$

for any $t \geqslant s, x \in \Omega$ and we have the following results about f^e which by simplicity of notation we also denote by f. We observe that the Lemma 3.1 and the Lemma 3.2 proofs for f in the previous chapter are ensure for f in here too.

Similar to the previous case, under such circumstances, we may exhibit a Lyapunov functional $\mathcal{E}$ to (4.4) which has the same definition given in (3.11), i.e.,

$$
\mathcal{E}(u, z, \theta) = \frac{1}{2}\Big(\|u\|^2_{(H_0^1(\Omega))^n} + \|z\|^2_{(L^2(\Omega))^n} + \|\theta\|^2_{L^2(\Omega)}\Big) - \int_\Omega F(u)dx.
$$

It has already been verified in (3.12) that

$$
\frac{d\mathcal{E}}{dt} = -\int_\Omega \kappa(x)|\nabla\theta|^2 dx \leqslant 0,
$$

where $\mathcal{E}(t) = \mathcal{E}(u(t), z(t), \theta(t))$ for any $t \geqslant s$.

As previously notice Dafermos [17] ensure that the linear part of the problem (4.4) generates a strongly continuous semigroup of contraction in $\mathcal{H}$ for each $s \in \mathbb{R}$ fixed. Since the Lemma 3.1 and the Lemma 3.2 are ensure for f, we can guarantee the local well-possessedness of the problem (4.4) thank to the Theorem 2.43. More precisely, the next result is hold.

Theorem 4.1. *Given* $\mathbf{u}_0 = (u_0, u_1, v_0) \in \mathcal{H} = (H_0^1(\Omega))^n \times (L^2(\Omega))^n \times L^2(\Omega)$, *the initial value problem* (4.4) *has a unique mild solution with*

$$u \in C([s, \tau_{\mathbf{u}_0}); (H_0^1(\Omega))^n) \cap C^1([s, \tau_{\mathbf{u}_0}); (L^2(\Omega))^n), \text{ and } \theta \in C([s, \tau_{\mathbf{u}_0}), L^2(\Omega)).$$

Moreover, if

$$\mathbf{u}_0 = (u_0, u_1, \theta_0) \in D(\mathbf{A}) = (H_0^1(\Omega) \cap H^2(\Omega))^n \times (H_0^1(\Omega))^n \times (H_0^1(\Omega) \cap H^2(\Omega))$$

then the following regularity property

$$u \in C([s, \tau_{\mathbf{u}_0}); (H^2(\Omega) \cap H_0^1(\Omega))^n) \cap C^1([s, \tau_{\mathbf{u}_0}); (H_0^1(\Omega))^n) \cap C^2([s, \tau_{\mathbf{u}_0}); (L^2(\Omega))^n),$$

and

$$\theta \in C([s, \tau_{\mathbf{u}_0}); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([s, \tau_{\mathbf{u}_0}); L^2(\Omega))$$

is verified. In this case that $\mathbf{u} = (u, \partial_t u, \theta)$ *is a strong solution of* (4.4).

Now we wish to prove that solutions of (4.4) are globally defined, i.e., for each $\mathbf{u}_0 = (u_0, u_1, \theta_0) \in \mathcal{H}$, $\tau_{\mathbf{u}_0} = \infty$.

As in (3.24), for $0 < \eta < \min\{1, \lambda_1\}$ we get

$$\|(u, z, \theta)\|_{\mathcal{H}}^2 \leqslant c_1 \mathcal{E}(u, z, \theta) + c_2,$$

for some $c_1 = c_1(\eta) > 0$ and $c_2 = c_2(\eta) > 0$.

It is clear from (3.12) that $[s, \tau_{\mathbf{u}_0}) \ni t \mapsto \mathcal{E}(u(t), \partial_t u(t), \theta(t)) \in \mathbb{R}$ is a non-increasing function. It follows from the fact that $\mathcal{E}$ is continuous and bounded in bounded subsets of $\mathcal{H}$ and from (3.12) that, given $r > 0$, there is a constant $C = C(r) > 0$ such that

$$\sup\{\|(u(t), \partial_t u(t), \theta(t))\|_{\mathcal{H}};\ \|(u_0, u_1, \theta_0)\|_{\mathcal{H}} \leqslant r, \text{ and } t \in [s, \tau_{\mathbf{u}_0})\} \leqslant C.$$

This implies that for each $\mathbf{u}_0 \in \mathcal{H}$, the solution of (4.4) with $\mathbf{u}_0 = (u_0, u_1, \theta_0)$ is defined for all $t \geqslant s$. We will write the mild solution of (4.4)

$$S(t, s)\mathbf{u}_0 = S_1(t, s)\mathbf{u}_0 + S_2(t, s)\mathbf{u}_0, \tag{4.5}$$

where $S_1(t, s)\mathbf{u}_0$ is defined as the solution of (4.4) with $\mathbf{F} \equiv 0$ and

$$S_2(t, s)\mathbf{u}_0 = \int_s^t S_1(t, \xi)\mathbf{F}(S(\xi, s)\mathbf{u}_0) d\xi, \ \forall t \geqslant s$$

here $\mathbf{F}(u) = (0, f^e(u), 0)$, with f^e the Nemytskĭi operator to f.

4.2 Existence of pullback attractor

Let us consider once more the functional

$$\mathcal{L}(u,z,\theta) = M\mathcal{E}(u,z,\theta) + \delta_1(u,z)_{(L^2(\Omega))^n} + \delta_2(\Phi,z)_{(L^2(\Omega))^n}$$

where δ_1, δ_2 and M are positive constants to be chosen appropriately and Φ is define in (3.28).

Theorem 4.2. *For $M > 0$ sufficiently large, there exist constants $M_1 > 0$ and $M_2 > 0$ such that for all $t \geqslant s$*

$$\frac{d\mathcal{L}}{dt} \leqslant -M_1\mathcal{E}(t) + M_2,$$

where $\mathcal{L}(t) = \mathcal{L}(u,z,\theta)$, $\mathcal{E}(t) = \mathcal{E}(u,z,\theta)$, and $(u,z,\theta) = (u(t),z(t),\theta(t))$ is the global solution of (4.1)-(4.3).

Proof. Note that

$$\frac{d\mathcal{L}}{dt} = M\frac{d\mathcal{E}}{dt} + \delta_1\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} + \delta_2\frac{d}{dt}(\Phi,z)_{(L^2(\Omega))^n}. \tag{4.6}$$

Thanks to (3.8), (3.12) and Poincaré inequality we have

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) &= -\int_\Omega \kappa(x)|\nabla\theta|^2dx \\ &\leqslant -\frac{\kappa_0}{2}\int_\Omega |\nabla\theta|^2dx - \frac{\kappa_0\lambda_1}{2}\int_\Omega |\theta|^2dx,\end{aligned} \tag{4.7}$$

where λ_1 is the first eigenvalue of negative Laplacian operator with zero Dirichlet boundary condition in Ω.

We also have

$$\begin{aligned}\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} &= (\partial_t u,z)_{(L^2(\Omega))^n} + (u,\partial_t z)_{(L^2(\Omega))^n} = (\partial_t u,\partial_t u)_{(L^2(\Omega))^n} + (u,\partial_t^2 u)_{(L^2(\Omega))^n} \\ &= \int_\Omega |\partial_t u|^2dx - \int_\Omega |\nabla u|^2dx - \int_\Omega |\operatorname{div} u|^2dx - \int_\Omega \beta(t)\nabla\theta u dx \\ &+ \int_\Omega f(u)udx.\end{aligned}$$

To deal with the integral term, just notice that from (3.5) we have

$$\begin{aligned}\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant & \int_\Omega |\partial_t u|^2dx - \int_\Omega |\nabla u|^2dx - \int_\Omega |\operatorname{div} u|^2dx - \int_\Omega \beta(t)\nabla\theta u dx \\ &+ \nu\int_\Omega |u|^2dx + C_\nu|\Omega|\end{aligned}$$

and again by Poincaré inequality

$$\begin{aligned}\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant & \int_\Omega |\partial_t u|^2dx - \int_\Omega |\nabla u|^2dx - \int_\Omega |\operatorname{div} u|^2dx - \int_\Omega \beta(t)\nabla\theta u dx \\ &+ \frac{\nu}{\lambda_1}\int_\Omega |\nabla u|^2dx + C_\nu|\Omega|\end{aligned}$$

in other words

$$\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant \int_\Omega |\partial_t u|^2 dx - \frac{C_\nu}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega |\operatorname{div} u|^2 dx - \int_\Omega \beta(t)\nabla\theta u dx + C_\nu|\Omega|,$$

where $\nu > 0$ is chosen such that $C_\nu := 1 - \dfrac{\nu}{\lambda_1} > 0$, that is, $0 < \nu < \lambda_1$.

Due to Young's inequality we conclude that

$$\delta_1\frac{d}{dt}(u,z)_{(L^2(\Omega))^n} \leqslant \delta_1\int_\Omega |\partial_t u|^2 dx - \frac{\delta_1 C_\nu}{2}\int_\Omega |\nabla u|^2 dx - \left(\delta_1 - \frac{1}{2}\right)\int_\Omega |\operatorname{div} u|^2 dx + \frac{\delta_1^2\beta_0^2}{2}\int_\Omega |\theta|^2 dx + \delta_1 C_\nu|\Omega|. \tag{4.8}$$

We also have that

$$\frac{d}{dt}(\Phi,z)_{(L^2(\Omega))^n} = (\Phi,\partial_t^2 u)_{(L^2(\Omega))^n} + (\partial_t\Phi,\partial_t u)_{(L^2(\Omega))^n},$$

and from (3.29) we obtain that

$$\frac{d}{dt}(\Phi,z)_{(L^2(\Omega))^n} = \int_\Omega \Phi\Delta u dx + \int_\Omega \Phi\nabla\operatorname{div} u dx - \int_\Omega \Phi\beta(t)\nabla\theta dx + \int_\Omega \Phi f(u) dx + \int_\Omega \kappa\nabla\theta\partial_t u dx - \beta(t)\int_\Omega |\partial_t u|^2 dx$$

In other words, and from (3.29) we obtain that

$$\frac{d}{dt}(\Phi,z)_{(L^2(\Omega))^n} = -\int_\Omega \nabla\Phi\nabla u dx - \int_\Omega \theta\operatorname{div} u dx + \int_\Omega |\beta(t)\theta|^2 dx + \int_\Omega \Phi f(u) dx + \int_\Omega \kappa(x)\nabla\theta\partial_t u dx - \beta(t)\int_\Omega |\partial_t u|^2 dx.$$

Using (3.8) and the Young's inequality we get for any $\epsilon > 0$,

$$\begin{aligned}\delta_2\frac{d}{dt}(\Phi,z)_{(L^2(\Omega))^n} &\leqslant \frac{\delta_2^2}{2}\int_\Omega |\nabla\Phi|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \left(\frac{\delta_2^2 + 2\beta_1\delta_2}{2}\right)\int_\Omega |\theta|^2 dx + \int_\Omega |\operatorname{div} u|^2 dx \\ &\quad + \frac{1}{2}\int_\Omega |f(u)|^2 dx + \frac{\delta_2^2}{2}\int_\Omega |\Phi|^2 dx + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx \\ &\quad + \left(\frac{1}{2\epsilon} - \beta_0\right)\delta_2\int_\Omega |\partial_t u|^2 dx \\ &\leqslant \left(\frac{\delta_2^2 + 2\beta_1\delta_2}{2}\right)\int_\Omega |\theta|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\operatorname{div} u|^2 dx \\ &\quad + \frac{1}{2}\int_\Omega |f(u)|^2 dx + \frac{\delta_2^2}{2}\int_\Omega |\nabla\Phi|^2 dx + \frac{\delta_2^2}{2\lambda_1}\int_\Omega |\nabla\Phi|^2 dx \\ &\quad + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx + \left(\frac{1}{2\epsilon} - \beta_0\right)\delta_2\int_\Omega |\partial_t u|^2 dx.\end{aligned}$$

Then,

(4.9)
$$
\begin{aligned}
&\delta_2 \frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} \\
&\qquad \leqslant \frac{1}{2}\left(\delta_2^2 + 2\beta_1\delta_2 + C\delta_2^2 + \frac{C\delta_2^2}{\lambda_1}\right)\int_\Omega |\theta|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\operatorname{div} u|^2 dx \\
&\qquad\quad + \frac{1}{2}\int_\Omega |f(u)|^2 dx + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx + \left(\frac{1}{2\epsilon} - \beta_0\right)\delta_2 \int_\Omega |\partial_t u|^2 dx.
\end{aligned}
$$

Now combining (4.9) and (3.36) we get

(4.10)
$$
\begin{aligned}
&\delta_2 \frac{d}{dt}(\Phi, z)_{(L^2(\Omega))^n} \\
&\qquad \leqslant \frac{1}{2}\left(\delta_2^2 + 2\beta_1\delta_2 + C\delta_2^2 + \frac{C\delta_2^2}{\lambda_1}\right)\int_\Omega |\theta|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\operatorname{div} u|^2 dx \\
&\qquad\quad + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx + \frac{\bar{C}_1}{2\lambda_1}\int_\Omega |\nabla u|^2 dx + \left(\frac{1}{2\epsilon} - \beta_0\right)\delta_2 \int_\Omega |\partial_t u|^2 dx + \frac{\bar{C}_2\delta_2}{2}.
\end{aligned}
$$

Therefore, combining (4.6) with (4.7), (4.8) and (4.10) we see that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leqslant -\frac{M\kappa_0}{2}\int_\Omega |\nabla\theta|^2 dx - \frac{M\kappa_0\lambda_1}{2}\int_\Omega |\theta|^2 dx + \delta_1 \int_\Omega |\partial_t u|^2 dx - \frac{\delta_1 C_\nu}{2}\int_\Omega |\nabla u|^2 dx \\
&\quad - \left(\delta_1 - \frac{1}{2}\right)\int_\Omega |\operatorname{div} u|^2 dx + \frac{1}{2}\left(\delta_2^2 + 2\beta_1\delta_2 + C\left(\delta_2^2 + \frac{\delta_2^2}{\lambda_1}\right)\right)\int_\Omega |\theta|^2 dx \\
&\quad + \frac{\delta_1^2\beta_0}{2}\int_\Omega |\theta|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\operatorname{div} u|^2 dx + \frac{\delta_2\kappa_1^2\epsilon}{2}\int_\Omega |\nabla\theta|^2 dx \\
&\quad + \frac{\bar{C}_1}{2\lambda_1}\int_\Omega |\nabla u|^2 dx + \left(\frac{1}{2\epsilon} - \beta_0\right)\delta_2\int_\Omega |\partial_t u|^2 dx \\
&\quad + \frac{\bar{C}_2\delta_2}{2} + \delta_1 C_\nu |\Omega|.
\end{aligned}
$$

When we reorganize the previous inequality,

(4.11)
$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) &\leqslant \frac{1}{2}\left(\delta_2^2 + 2\beta_1\delta_2 + C\left(\delta_2^2 + \frac{\delta_2^2}{\lambda_1}\right) + \frac{\delta_1^2\beta_0}{2} - \kappa_0\lambda_1 M\right)\int_\Omega |\theta|^2 dx \\
&\quad \left(\frac{\delta_2\kappa_1^2\epsilon}{2} - \frac{\kappa_0 M}{2}\right)\int_\Omega |\nabla\theta|^2 dx - \left(\frac{\delta_1 C_\nu}{2} - \frac{\bar{C}_1}{2\lambda_1} - \frac{1}{2}\right)\int_\Omega |\nabla u|^2 dx \\
&\quad - (\delta_1 - 1)\int_\Omega |\operatorname{div} u|^2 dx - \left(\left(\beta_0 - \frac{1}{2\epsilon}\right)\delta_2 - \delta_1\right)\int_\Omega |\partial_t u|^2 dx + \frac{\bar{C}_2}{2} + \delta_1 C_\nu |\Omega|.
\end{aligned}
$$

Now take $\epsilon > 0$ large enough to be able choose δ_1 and δ_2 such that

$$
0 < \max\left\{\frac{\bar{C}_1 + \lambda_1}{\lambda_1 C_\nu}, 1\right\} < \delta_1,
$$

and

$$\delta_1 < \left(\beta_0 - \frac{1}{\epsilon}\right)\delta_2.$$

Choose $M > 0$ sufficiently large too such that

$$\frac{\delta_2\kappa_1^2\epsilon}{2} - \frac{\kappa_0 M}{2} < 0 \text{ and } \delta_2^2 + 2\beta_1\delta_2 + C\left(\delta_2^2 + \frac{\delta_2^2}{\lambda_1}\right) + \frac{\delta_1^2\beta_0}{2} - \kappa_0\lambda_1 M < 0,$$

with these choices for the constants δ_1, δ_2 and M there exist $\varrho_0 > 0$ and $\varrho_1 > 0$ such that

$$\frac{d\mathcal{L}}{dt}(t) \leqslant -\varrho_0\left[\frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\text{div}u|^2 + |z|^2 + |\theta|^2\right)dx\right] + \varrho_1.$$

Thanks to (4.11) and (3.39) there exist constants $\varrho_3 > 0, M_1 > 0$ and $M_2 > 0$ such that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leqslant -\frac{\varrho_3}{2}\int_\Omega \left(|\nabla u|^2 + |\text{div}u|^2 + |z|^2 + |\theta|^2\right)dx - \frac{\varrho_3}{2}\int_\Omega \left(|\nabla u|^2 + |\text{div}u|^2\right)dx + \varrho_1 \\ &\leqslant -\frac{\varrho_3}{2}\int_\Omega \left(|\nabla u|^2 + |\text{div}u|^2 + |z|^2 + |\theta|^2\right)dx + \frac{\varrho_3\bar{d}}{2}\int_\Omega F(u)dx + \varrho_1 \\ &\leqslant -M_1\left[\frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\text{div}u|^2 + |z|^2 + |\theta|^2\right)dx - \int_\Omega F(u)dx\right] + M_2,\end{aligned}$$

where $F(u) = \int_0^u f d\gamma$ and $\int_0^u f d\gamma$ represents the line integral of f along a piecewise smooth curve with initial point 0 and final point u.

Finally, from (3.11) we conclude there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$\frac{d\mathcal{L}}{dt} \leqslant -M_1\mathcal{E}(t) + M_2,$$

where $u(x,t), z = z(x,t), \theta = \theta(x,t)$. This concludes the proof of the theorem.

□

Theorem 4.3. *For $M > 0$ sufficiently large, there exist positive constants C_M, c_M, C_1 and $C_2 > 0$ such that for any $t \geqslant s$,*

$$c_M\mathcal{E}(t) - C_1 \leqslant \mathcal{L}(t) \leqslant C_M\mathcal{E}(t) + C_2, \tag{4.12}$$

where $\mathcal{L}(t) = \mathcal{L}(u,z,\theta)$, $\mathcal{E}(t) = \mathcal{E}(u,z,\theta)$, and $(u,z,\theta) = (u(t),z(t),\theta(t))$ is the solution of (4.1)-(4.2).

Proof. In the following, we prove the two inequalities in (4.12) simultaneously, once the arguments are similar. From definition of the functional $\mathcal{L}$ and Cauchy-Schwarz inequality, for any $M > 0$ we can see that

$$M\mathcal{E}(t) - \delta_1\int_\Omega |u||z|dx - \delta_2\int_\Omega |\Phi||z|dx \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant M\mathcal{E}(t) + \delta_1 \int_\Omega |u||z|dx + \delta_2 \int_\Omega |\Phi||z|dx.$$

Then, it follows from Young's inequality

$$M\mathcal{E}(t) - \frac{\delta_1}{2}\|u\|^2_{(L^2(\Omega))^n} - \frac{1}{2}(\delta_1 + \delta_2)\|z\|^2_{(L^2(\Omega))^n} - \frac{\delta_2}{2}\|\Phi\|^2_{(L^2(\Omega))^n} \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant M\mathcal{E}(t) + \frac{\delta_1}{2}\|u\|^2_{(L^2(\Omega))^n} + \frac{1}{2}(\delta_1 + \delta_2)\|z\|^2_{(L^2(\Omega))^n} + \frac{\delta_2}{2}\|\Phi\|^2_{(L^2(\Omega))^n}.$$

Now using the Poincaré inequality, we have that

$$M\mathcal{E}(t) - \frac{\delta_1}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} - \frac{1}{2}(\delta_1 + \delta_2)\|z\|^2_{(L^2(\Omega))^n} - \frac{\delta_2}{2\lambda_1}\|\Phi\|^2_{(H^1_0(\Omega))^n} \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant M\mathcal{E}(t) + \frac{\delta_1}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} + \frac{1}{2}(\delta_1 + \delta_2)\|z\|^2_{(L^2(\Omega))^n} + \frac{\delta_2}{2\lambda_1}\|\Phi\|^2_{(H^1_0(\Omega))^n}.$$

From definition of the functionais E and Φ, we get

$$\frac{1}{2}\Big[\Big(M - \frac{\delta_1}{\lambda_1}\Big)\|u\|^2_{(H^1_0(\Omega))^n} + (M - \delta_1 - \delta_2)\|z\|^2_{(L^2(\Omega))^n} + \Big(M - \frac{\delta_2 C^2}{\lambda_1}\Big)\|\theta\|^2_{L^2(\Omega)}\Big] - M\int_\Omega F(u)dx \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant \frac{1}{2}\Big[\Big(M + \frac{\delta_1}{\lambda_1}\Big)\|u\|^2_{(H^1_0(\Omega))^n} + (M + \delta_1 + \delta_2)\|z\|^2_{(L^2(\Omega))^n} + \Big(M + \frac{\delta_2 C^2}{\lambda_1}\Big)\|\theta\|^2_{L^2(\Omega)}\Big] + M\int_\Omega F(u)dx.$$

for some $C > 0$.

Using (3.6) we see that

$$\int_\Omega F(u)dx \leqslant \frac{\eta}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} + C_\eta|\Omega|.$$

and if we denote

$$\begin{aligned} c_1 &= \frac{\delta_1}{\lambda_1} + \delta_1 + \delta_2 + \frac{\delta_2 C^2}{\lambda_1}; \\ c_2 &= \frac{C_\eta \lambda_1}{\lambda_1 - \eta}; \\ c_3 &= \frac{2M\eta - c_1\lambda_1}{\lambda_1 - \eta}; \\ c_4 &= 1 - \frac{C_1(\lambda - \eta)}{2M\eta - c_1\lambda_1}, \end{aligned}$$

then we conclude that

$$\begin{aligned}\mathcal{L}(t) &\geqslant \frac{1}{2}\Big[(M-c_1)\|u\|^2_{(H^1_0(\Omega))^n} + (M-c_1)\|z\|^2_{(L^2(\Omega))^n} + (M-c_1)\|\theta\|^2_{L^2(\Omega)}\Big] - M\int_\Omega F(u)dx \\ &\geqslant \frac{1}{2}\Big[M - \Big(c_1 + \frac{\eta c_2}{\lambda_1}\Big)\Big]\|(u,z,\theta)\|_{\mathcal{H}*} + \frac{\eta c_2}{2\lambda_1}\|u\|^2_{(H^1_0(\Omega))^n} - (M-c_2)\int_\Omega F(u)dx \\ &- c_2\int_\Omega F(u)dx\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(t) &\leqslant \frac{1}{2}\Big[(M+c_1)\|u\|^2_{(H^1_0(\Omega))^n} + (M+c_1)\|z\|^2_{(L^2(\Omega))^n} + (M+c_1)\|\theta\|^2\Big] + M\int_\Omega F(u)dx \\ &\leqslant \frac{1}{2}\Big[(M+c_1+c_4c_3)\|(u,z,\theta)\|_{\mathcal{H}*} - c_4c_3\|u\|^2_{(H^1_0(\Omega))^n}\Big] - (M+c_3)\int_\Omega F(u)dx \\ &+ (2M+c_3)\int_\Omega F(u)dx,\end{aligned}$$

where $M > 0$ is chosen sufficiently large such that $M - c_2 > 0$, $c_3 > 0$ and $c_4 > 0$, we can note that

$$\begin{aligned}c_2 &= c_1 + \frac{\eta}{\lambda_1}c_2; \\ c_4c_3 &= (2M+c_3)\frac{\eta}{\lambda_1}; \\ c_3 &= c_2 + c_4c_3.\end{aligned}$$

Therefore by (3.6) we get

$$\begin{aligned}&\frac{1}{2}\Big[M - \Big(c_1 + \frac{\eta}{\lambda_1}c_2\Big)\Big]\|(u,z,\theta)\|^2_{\mathcal{H}*} + \Big(\frac{\eta}{2\lambda_1}c_2 - \frac{\eta}{2\lambda_1}c_2\Big)\|u\|^2_{(H^1_0(\Omega))^n} \\ &- (M-c_2)\int_\Omega F(u)dx - C_\eta c_2|\Omega| \leqslant \mathcal{L}(t),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(t) &\leqslant \frac{1}{2}[M + (c_1 + c_3c_4)]\|(u,z,\theta)\|^2_{\mathcal{H}*} + \frac{1}{2}\Big[-c_3c_4 + (2M+c_3)\frac{\eta}{\lambda_1}\Big]\|u\|^2_{(H^1_0(\Omega))^n} \\ &- (M+c_3)\int_\Omega F(u)dx + (2M+c_3)C_\eta|\Omega|.\end{aligned}$$

Finally, if we define $c_M = M - c_2$, $C_1 = c_2C_\eta|\Omega|$, $C_2 = (2M+c_3)C_\eta|\Omega|$ and $C_M = M + c_3$, then

$$\frac{c_M}{2}\|(u,z,\theta)\|^2_{\mathcal{H}*} - c_M\int_\Omega F(u)dx - C_1 \leqslant \mathcal{L}(t),$$

and

$$\mathcal{L}(t) \leqslant \frac{C_M}{2}\|(u,z,\theta)\|^2_{\mathcal{H}*} - C_M\int_\Omega F(u)dx + C_2.$$

□

We have the following result as a consequence of Theorem 4.2.

Theorem 4.4. *There exists $R > 0$ such that for each bounded subset B of $\mathcal{H}_*$ there exists $t_B > s$ with the property*

$$S(t,s)B \subset \mathcal{B}_{\mathcal{H}_*}(0;R),$$

for any $t \geqslant t_B$. Here, $\mathcal{B}_{\mathcal{H}_*}(0;R)$ denotes the open ball in $\mathcal{H}_*$ centered at origin and of radius R.

Proof. Let B be a subset of $\mathcal{H}_*$, and let $(u,z,\theta) = (u(t), z(t), \theta(t))$ be the solution of (4.1)-(3.26) with $(u_0, u_1, \theta_0) \in B$. Using Theorem 4.2 and the second inequality of (4.12) in Theorem 4.3 we have there exit constant $\varrho_1 > 0$ and $\varrho_2 > 0$ such that

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\varrho_1 \mathcal{L}(t) + \varrho_2,$$

where $\mathcal{L}(t) = \mathcal{L}(u,z,\theta)$ for any $t \geqslant s$.

From (1.6), we can find that

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)e^{-\int_0^t \varrho_1 ds} + \int_0^t \varrho_2 e^{-\int_\tau^t \varrho_1 ds} d\tau \leqslant \mathcal{L}(0)e^{-\varrho_1 t} + \frac{\varrho_2}{\varrho_1}(1 - e^{-\varrho_1 t})$$

where $\mathcal{L}(0) = \mathcal{L}(u_0, u_1, \theta_0)$, and combining with the inequalities (3.24) and (4.12) in Theorem 4.3, we get

$$\begin{aligned} \|(u,z,\theta)\|^2_{\mathcal{H}_*} &\leqslant c_1 \mathcal{E}(u,z,\theta) + c_2 \leqslant \frac{c_1}{c_M}\mathcal{L}(t) + \frac{C_1 c_1}{c_M} + c_2 \\ &\leqslant \Big(\frac{c_1}{c_M}\mathcal{L}(0) - \frac{\varrho_2}{\varrho_1}\Big)e^{-\varrho_1 t} + \frac{\varrho_2 c_1}{\varrho_1 c_M} + \frac{C_1 c_1}{c_M} + c_2, \end{aligned}$$

for some constants c_1, c_2, c_M and $C_1 > 0$.

Let $R_B > 0$ such that $\|(u_0, u_1, \theta_0)\|^2_{\mathcal{H}_*} \leqslant R_B$, then after some calculations we conclude that there exists $t_B > 0$ with

$$\|(u,z,\theta)\|^2_{\mathcal{H}_*} \leqslant 2\Big(\frac{\varrho_2 c_1}{\varrho_1 c_M} + \frac{C_1 c_1}{c_M} + c_2\Big)$$

for any $t \geqslant t_B$.

□

Proposition 4.5. *There exists positive constants K and α such that*

$$\|S_1(t,s)\|_{\mathcal{L}(\mathcal{H}_*)} \leqslant K e^{-\alpha(t-s)} \quad \textit{for all} \quad t \geqslant s,$$

and $S_2(t,s)$ is a compact operator from $\mathcal{H}_*$ into itself for all $t > s$. In particular the nonlinear process $S(\cdot,\cdot)$ is pullback asymptotically compact.

Proof.

To prove the decay of $S_1(t,s)$, one considers the functional

$$\mathcal{L}_0(u,z,\theta)=\frac{1}{2}\Big(\|u\|^2_{(H_0^1(\Omega))^n}+\|z\|^2_{(L^2(\Omega))^n}+\|\theta\|^2_{L^2(\Omega)}\Big)+\delta_1(u,z)_{(L^2(\Omega))^n}+\delta_2(\Phi,z)_{(L^2(\Omega))^n}.$$

Thanks to Theorem 4.2 and Theorem 4.3 we have

$$\frac{d}{dt}\mathcal{L}_0(t)\leqslant-\alpha\mathcal{L}_0(t)$$

for some $\alpha>0$, where

$$\mathcal{L}_0(S_1(t,s)(u_0,u_1,\theta_0))\leqslant\left[\mathcal{L}_0(S_1(t,s)(u_0,u_1,\theta_0))+\mathcal{L}_0(S_1(s,s)(u_0,u_1,\theta_0))\right]e^{-\alpha(t-s)},$$

and consequently,

$$\|S_1(t,s)(u_0,u_1,\theta_0)\|^2_{\mathcal{H}*}\leqslant Ke^{-\alpha t}\|(u_0,u_1,\theta_0)\|_{\mathcal{H}*}.$$

To show that $S_2(t,s)$ is compact, we first show that f is bounded from $(H_0^1(\Omega))^n$ into $(W^{1,r}(\Omega))^n$, with $r=\frac{2(n-1)}{(n-2)}=\frac{n}{n-2}+1\in(1,2)$; indeed, it follows from Lemma 3.1 that for any $u\in\mathbb{R}^n$ we have

$$|f(u)|\leqslant 2^{\mathfrak{p}-1}n^2C|u|(1+|u_1|^{\mathfrak{p}-1}+\cdots+|u_n|^{\mathfrak{p}-1}),$$

for $\mathfrak{p}<\frac{n}{n-2}$ and from (3.7),

$$\begin{aligned}\|f(u)\|^r_{W^{1,r}(\Omega)}&=\int_\Omega(|f(u)|^r+|\nabla f(u)|^r|\nabla u|^r)dx\\&\leqslant\int_\Omega\Big(\big(2^{\mathfrak{p}-1}n^2C|u|(1+|u_1|^{\mathfrak{p}-1}+\cdots+|u_n|^{\mathfrak{p}-1})\big)^r\\&\quad+Cn(1+\sum_{i=1}^n|u_i|^{\mathfrak{p}-1})^r|\nabla u|^r\Big)dx\\&\leqslant C\left(\|u\|^r_{(L^r(\Omega))^n}+\|u\|^{\mathfrak{p}r}_{(L^{\mathfrak{p}r}(\Omega))^n}+\|\nabla u\|^r_{(L^2(\Omega))^n}+\|u\|^{(\mathfrak{p}-1)r}_{(L^{\frac{2(\mathfrak{p}-1)r}{2-r}}(\Omega))^n}\|\nabla u\|^{r/2}_{(L^2(\Omega))^n}\right)\\&\leqslant C'\left(\|u\|^{\mathfrak{p}r}_{(L^{\mathfrak{p}r}(\Omega))^n}+\|\nabla u\|^r_{(L^2(\Omega))^n}+\|u\|^{(\mathfrak{p}-1)r}_{(L^{-r}(\Omega))^n}\|\nabla u\|^{r/2}_{(L^2(\Omega))^n}\right).\end{aligned}$$

Our choice of $r<\frac{2n}{n-2}$ implies that $\mathfrak{p}r<\frac{2n}{n-2}$ and from the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ for $q\leqslant\frac{2n}{n-2}$ it follows that f^e is bounded from $(H_0^1(\Omega))^n$ into $(W^{1,r}(\Omega))^n$ and the latter is compactly embedded in $r>1$. Thus, F is bounded from $\mathcal{H}_*$ into $\{0\}\times(W^{1,r}(\Omega))^n\times\{0\}$ and the latter is compactly embedded in $(H_0^1(\Omega))^n\times(W^{1,r}(\Omega))^n\times H_0^1(\Omega)$.

Now fix $t\geqslant s$ and consider

$$S_2(t,s)\mathbf{u}_0=\int_0^t S_1(\xi,s)F(S(\xi,s)\mathbf{u}_0)d\xi,\ t\geqslant s.$$

for $\mathbf{u}_0\in B$, where B is a bounded subset of $\mathcal{H}_*$. Since orbits of bounded subsets of $\mathcal{H}_*$ under the nonlinear process $\{S(t,s);t\geqslant s\}$ are bounded in $\mathcal{H}_*$, it follows that $S_2(t,s)$ is compact for each $t>s$. Thus the fact of nonlinear process $\{S(t,s);t\geqslant s\}$ is asymptotically compact is a consequence of [13, Theorem 2.37].

□

Finally, as application of [13, Theorem 2.23], we conclude that (4.4) has a pullback attractor $\{\mathcal{A}(t); t \in \mathbb{R}\}$ in $\mathcal{H}_*$.

4.3 Regularity of attractors

We will use the same notation here as that exposed in the Section 3.5 to investigate the regularity of the pullback attractor. As a matter of fact, we prove like in the previous case that $\bigcup_{t\geqslant s}\mathcal{A}(t)$ is a bounded subset of $\mathcal{H}^1_*$.

Theorem 4.6. *The pullback attractor $\mathcal{A}(\cdot)$ for the problem* (4.4)*, obtained in Section 4.2, lies in a more regular space than $\mathcal{H}_*$, in fact, $\bigcup_{t\geqslant s}\mathcal{A}(t)$ is a bounded subset of $\mathcal{H}^1_*$.*

Proof. Without lost of generality, we will assume $\kappa = 1$ to simplify the calculations in this proof. Let $\xi : \mathbb{R} \to \mathcal{H}_*$ be a pullback bounded solution of (4.4). Then, the set $\{\xi(t);\ t \in \mathbb{R}\}$ is a bounded subset of $\mathcal{H}_*$.

We already know that $\mathcal{A}(t)$ is bounded in $\mathcal{H}_*$. Hence, if $\xi : \mathbb{R} \to \mathcal{H}_*$ is such that $\xi(t) \in \mathcal{A}(t)$ for all $t \in \mathbb{R}$, then

$$\xi(t) = S_1(t,s)\xi(s) + \int_s^t S_1(\tau,s)F(\xi(\tau,s))d\tau,$$

where $S_1(\cdot,\cdot)$ and $S_2(\cdot,\cdot)$ is defined in (4.5). Now using the decay of $S_1(t,s)$ in the Proposition 4.5 and letting $t \to +\infty$ it follows that

$$\xi(t) = \int_s^{+\infty} S_1(\tau,s)F(\xi(\tau,s))d\tau. \tag{4.13}$$

Set $(\mu_0, \mu_1, \vartheta_0) = \xi(s)$, and we consider

$$\begin{bmatrix} \mu \\ \partial_t\mu \\ \vartheta \end{bmatrix}(t) = S_2(t)\begin{bmatrix} \mu_0 \\ \mu_1 \\ \vartheta_0 \end{bmatrix} = \int_0^t S_1(s)F(S(s)\begin{bmatrix} \mu_0 \\ \mu_1 \\ \vartheta_0 \end{bmatrix})ds,$$

and note that $(\mu(\cdot), \partial_t\mu(\cdot), \vartheta(\cdot)) \in \mathcal{H}_*$ solves the system

$$\begin{cases} \partial_t^2\mu - \Delta\mu - \nabla\operatorname{div}\mu + \beta(t)\nabla\vartheta = f(\mu(t;\mu_0)), & x \in \Omega,\ t > s, \\ \partial_t\vartheta - \Delta\vartheta + \beta(t)\operatorname{div}\partial_t\mu = 0, & x \in \Omega,\ t > s, \end{cases} \tag{4.14}$$

with

$$\mu(x,s) = \partial_t\mu(x,s) = 0 \text{ and } \vartheta(x,s) = 0,\ x \in \Omega. \tag{4.15}$$

We again consider the following functional

$$\begin{aligned}\mathcal{L}_0(\mu(t), \partial_t\mu(t), \theta(t)) = \frac{1}{2}\Big(&\|\mu(t)\|^2_{(H^1_0(\Omega))^n} + \|\partial_t\mu(t)\|^2_{(L^2(\Omega))^n} + \|\theta(t)\|^2_{L^2(\Omega)}\Big) \\ &+ \delta_1(\mu(t), \partial_t\mu(t))_{(L^2(\Omega))^n} + \delta_2(\Phi(t), \partial_t\mu(t))_{(L^2(\Omega))^n}\end{aligned}$$

to estimate the solution of (4.14)-(4.15) for $(\mu_0, \mu_1, \vartheta_0)$ in a bounded subset B of $\mathcal{H}_*$. The same arguments of the proof of Theorem 4.2 to obtain (we omitted t in order to simplify the notation)

$$\frac{d\mathcal{L}_0}{dt}(\mu, \partial_t\mu, \vartheta) \leqslant -C_0\|\nabla\mu\|^2_{(L^2(\Omega))^n} - C_1\|\partial_t\mu\|^2_{(L^2(\Omega))^n} - C_2\|\vartheta\|^2_{L^2(\Omega)} + C_3, \tag{4.16}$$

where C_0, C_1, C_2 and C_3 are positive constants.

From this it follows that

$$\bigcup_{s\leqslant\tau\leqslant t} S_2(\tau, s)B \text{ is a bounded subset of } \mathcal{H}_*. \tag{4.17}$$

Therefore $(\varpi, \zeta) = (\partial_t\mu, \partial_t\vartheta)$ solves the system

$$\begin{cases} \partial_t^2\varpi - \Delta\varpi - \nabla \operatorname{div}\varpi + \beta(t)\nabla\zeta = f'(\mu(t;\mu_0))\varpi(t;\mu_0), & x \in \Omega,\ t > s, \\ \partial_t\zeta - \Delta\zeta + \beta(t)\operatorname{div}\partial_t\varpi = 0, & x \in \Omega,\ t > s, \end{cases} \tag{4.18}$$

with $\varpi(s) = 0$, $\varpi_t(s) = f(\mu_0)$, and $\zeta(s) = 0$.

In order to continue with verification, we will show that $(\mu, \partial\mu, \vartheta)$ is a bounded solution in $\mathcal{H}^1_*$, by estimate $(\varpi, \partial_t\varpi, \zeta)$ in $\mathcal{H}_*$. But thus solutions are not regular enough to allow this directly, that's why we will work 'towards' $\mathcal{H}_*$ by progressive increases of regularity.

We will take $(\varpi, \partial_t\varpi, \zeta) \in \mathcal{H}^{-\alpha}$ and we define

(4.19)

$$\mathcal{L}_\alpha(t) = \frac{M}{2}\Big(2\|\varpi\|^2_{(Y^{1-\alpha})^n} + \|\phi\|^2_{(Y^{-\alpha})^n} + \|\zeta\|^2_{Y^{-\alpha}}\Big) + \delta_1(\varpi, \phi)_{(Y^{-\alpha})^n} + \delta_2(\gamma, \phi)_{(Y_*^{-\alpha})^n},$$

where γ such that $\operatorname{div}\gamma = \zeta$.

We want to find an inequality like (4.3). Therefore, we will obtain followings estimates for the terms involved in $\mathcal{L}_\alpha(t)$; First, thanks to (4.14) we get

$$\frac{d}{dt}\|\phi\|^2_{(Y^{-\alpha})^n} = 2(\Delta\varpi + \nabla\operatorname{div}\varpi, \phi)_{(Y^{-\alpha})^n} - 2(\beta(t)\nabla\zeta, \phi)_{(Y^{-\alpha})^n} + 2(f'(\mu(t;\mu_0))\varpi(t;\mu_0), \phi)_{(Y^{-\alpha})^n}.$$

Because of (3.8) we have that

$$\frac{d}{dt}\|\zeta\|^2_{Y^{-\alpha}} = 2(\Delta\zeta, \zeta)_{Y^{-\alpha}} - 2(\beta(t)\operatorname{div}\partial_t\varpi, \zeta)_{Y^{-\alpha}} \leqslant -2\|\zeta\|_{Y^{1-\alpha}} - 2\beta(t)(\operatorname{div}\partial_t\varpi, \zeta)_{Y^{-\alpha}}.$$

Again by (4.14) we obtain that

$$\begin{aligned} \frac{d}{dt}(\varpi, \phi)_{(Y^{-\alpha})^n} = \|\phi\|^2_{(Y^{-\alpha})^n} + (\varpi, \Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\varpi, \beta(t)\nabla\zeta)_{(Y^{-\alpha})^n} \\ + (\varpi, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}. \end{aligned}$$

Also, we see that

$$\begin{aligned} \frac{d}{dt}(\gamma, \phi)_{(Y^{-\alpha})^n} = (\partial_t\gamma, \phi)_{(Y^{-\alpha})^n} + (\gamma, \Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\gamma, \beta(t)\nabla\zeta)_{(Y^{-\alpha})^n} \\ + (\gamma, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}. \end{aligned}$$

In this way,

$$
\begin{aligned}
\frac{d\mathcal{L}_\alpha}{dt} \leqslant \frac{M}{2}&[2(\partial_t\varpi,\varpi)_{(Y^{1-\alpha})^n} + 2(\Delta\varpi + \nabla \operatorname{div}\varpi, \phi)_{(Y^{-\alpha})^n} - 2(\beta(t)\nabla\zeta,\phi)_{(Y^{-\alpha})^n} \\
&+ 2(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n} - 2\|\zeta\|^2_{Y^{1-\alpha}} - 2(\beta(t)\operatorname{div}\partial_t\varpi,\zeta)_{Y^{-\alpha}}] \\
&+ \delta_1\Big[\|\phi\|^2_{(Y^{-\alpha})^n} + (\varpi,\Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\varpi,\beta(t)\nabla\zeta)_{(Y^{-\alpha})^n} \\
&+ (\varpi, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}\Big] + \delta_2\Big[(\partial_t\gamma,\phi)_{(Y^{-\alpha})^n} - (\gamma,\beta(t)\nabla\zeta)_{(Y^{-\alpha})^n} \\
&+ (\gamma,\Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} + (\gamma, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}\Big].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{d\mathcal{L}_\alpha}{dt} \leqslant \frac{M}{2}&[-2(\partial_t\varpi,\varpi)_{(Y^{1-\alpha})^n} + 2(\Lambda^{\frac{1}{2}}(\varpi + \operatorname{div}\varpi), \Lambda^{\frac{1}{2}}\phi)_{(Y^{-\alpha})^n} \\
&+ 2(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n} - 2\|\zeta\|^2_{Y^{1-\alpha}}] + \delta_1[\|\phi\|^2_{(Y^{-\alpha})^n} \\
&+ (\varpi,\Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\varpi,\beta(t)\nabla\zeta)_{(Y^{-\alpha})^n} + (\varpi, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}] \\
&+ \delta_2[(\partial_t\gamma,\phi)_{(Y^{-\alpha})^n} + (\gamma,\Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} + \beta(t)(\zeta,\zeta)_{(Y^{-\alpha})^n} \\
&+ (\gamma, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}]
\end{aligned}
$$

in other words,

$$
\begin{aligned}
\frac{d\mathcal{L}_\alpha}{dt} \leqslant M&[(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n} - \|\zeta\|^2_{Y^{1-\alpha}}] + \delta_1[\|\phi\|^2_{(Y^{-\alpha})^n} \\
&+ (\varpi,\Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - (\varpi,\beta(t)\nabla\zeta)_{(Y^{-\alpha})^n} + (\varpi, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}] \\
&+ \delta_2[(\partial_t\gamma,\phi)_{(Y^{-\alpha})^n} + (\gamma,\Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} + \beta(t)(\zeta,\zeta)_{(Y^{-\alpha})^n} \\
&+ (\gamma, f'(\mu(t;\mu_0))\varpi(t;\mu_0))_{(Y^{-\alpha})^n}].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d\mathcal{L}_\alpha}{dt} \leqslant\ & M(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\phi)_{(Y^{-\alpha})^n} + \delta_1(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\varpi)_{(Y^{-\alpha})^n} \\
&+ \delta_2(f'(\mu(t;\mu_0))\varpi(t;\mu_0),\gamma)_{(Y^{-\alpha})^n} - M\|\zeta\|^2_{Y^{1-\alpha}} + \delta_2\beta(t)\|\zeta\|^2_{Y^{-\alpha}} \\
&+ \delta_1\|\phi\|^2_{(Y^{-\alpha})^n} - \delta_1\|\varpi\|^2_{(Y^{1-\alpha})^n} + \delta_2(\partial_t\gamma,\phi)_{(Y^{-\alpha})^n} \\
&+ \delta_2(\gamma,\Delta\varpi + \nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - \delta_1\beta(t)(\varpi,\nabla\zeta)_{(Y^{-\alpha})^n}.
\end{aligned}
\tag{4.20}
$$

Next, we deal with the three terms in which it appears explicitly the nonlinearity f'. From now on, let

$$\alpha_1 := \frac{(\mathfrak{p}-1)(N-2)}{2}.$$

Note that since $\mathfrak{p} < \frac{N}{N-2}$, we obtain that $\alpha_1 < 1$.

If $\alpha \in (0,\alpha_1)$ then we can observe that

$$(f'(\mu(t;\mu_0))\varpi(t;\mu_0), g)_{(Y^{-\alpha})^n} \leqslant \|g\|_{(Y^{-\alpha})^n}\|f'(\mu(t;\mu_0))\varpi(t;\mu_0)\|_{(Y^{-\alpha})^n}$$

for $g \in \{\varphi,\ \partial_t\varphi,\ \zeta\}$ and using the embedding $(Y^{\alpha})^n \hookrightarrow (H^{2\alpha}(\Omega))^n \hookrightarrow (L^p(\Omega))^n$ (or equivalently $\left(L^{\frac{p}{p-1}}(\Omega))^n \hookrightarrow (Y^{-\alpha})^n\right)$ for any $1 < p \leqslant \frac{2N}{N-2\alpha}$ and (3.7), we have that for some $c_4 > 0$

$$\begin{aligned}\|f'(\mu)\varpi\|_{(Y^{-\alpha})^n} &\leqslant c_4\|f'(\mu)\varpi\|_{L^{\frac{2N}{N+2\alpha}}(\Omega)} \leqslant C\|\varpi(1+|\mu|^{\mathfrak{p}-1})\|_{L^{\frac{2N}{N+2\alpha}}(\Omega)}\\ &\leqslant C\|\varpi\|_{\mathcal{H}_*}\|1+|\mu|^{\mathfrak{p}-1}\|_{L^{\frac{N}{\alpha}}(\Omega)}\end{aligned}$$

and so

$$\|f'(\mu)\varpi\|^2_{(Y^{-\alpha})^n} \leqslant C^2\|\varpi\|^2_{\mathcal{H}_*}\|1+|\mu|^{\mathfrak{p}-1}\|^2_{L^{\frac{N}{\alpha}}(\Omega)}.$$

From (4.15) μ remains in a bounded subset of $\mathcal{H}_*^{\frac{1}{2}} \hookrightarrow L^{\frac{(\mathfrak{p}-1)N}{\alpha}}(\Omega)$ for any $1 < \mathfrak{p} < \frac{N-4+4\alpha}{N-4}$ and this implies that

$$\int_\Omega (1+|\mu|^{\mathfrak{p}-1})^{\frac{N}{\alpha}}dx \leqslant |\Omega| + \|\mu\|^{\frac{(\mathfrak{p}-1)N-\alpha}{N(\mathfrak{p}-1)}}_{L^{\frac{(\mathfrak{p}-1)N}{\alpha}}(\Omega)} \leqslant |\Omega| + c_5\|\mu\|^{\frac{(\mathfrak{p}-1)N-\alpha}{N(\mathfrak{p}-1)}}_{\mathcal{H}_*^{\frac{1}{2}}} \leqslant c_5,$$

for some $c_5 > 0$.

Therefore, there exists a positive constant C_f such that

$$\|f'(\mu)\varpi\|^2_{(Y^{-\alpha})^n} \leqslant C_f. \tag{4.21}$$

From (3.29),

$$\partial_t\gamma = \nabla\zeta - \beta(t)\partial_t\varpi,$$

then $\forall\epsilon > 0$, we have

$$\delta_2(\partial_t\gamma,\phi)_{(Y^{1-\alpha})^n} \leqslant \delta_2(\nabla\zeta-\beta(t)\phi,\phi)_{(Y^{-\alpha})^n} \leqslant \delta_2(\nabla\zeta,\phi)_{(Y^{-\alpha})^n} - \delta_2\beta(t)\|\phi\|^2_{(Y^{-\alpha})^n}$$

and therefore

$$\delta_2(\partial_t\gamma,\phi)_{(Y^{1-\alpha})^n} \leqslant \frac{\delta_2}{\epsilon}\|\zeta\|^2_{Y^{1-\alpha}} + \delta_2\left(\epsilon-\beta(t)\right)\|\phi\|^2_{(Y^{-\alpha})^n}. \tag{4.22}$$

as previously discussed in Remark 3.10, $\|\zeta\|^2_{Y^{1-\alpha}} < \infty$ a.e. for $t \in \mathbb{R}$.

Now we will denote

$$J_1 = \delta_2(\gamma,\Delta\varpi+\nabla\operatorname{div}\varpi)_{(Y^{-\alpha})^n} - \delta_1\beta(t)(\nabla\zeta,\varpi)_{(Y^{-\alpha})^n}.$$

From $\gamma \in Y_*^{1-\alpha} \hookrightarrow Y_*^{-\alpha}$, we obtain following inequality

$$\begin{aligned}J_1 &\leqslant \frac{\delta_2}{\epsilon}\|\gamma\|^2_{(Y^{1-\alpha})^n} + \epsilon\delta_2\|\varpi\|^2_{(Y^{1-\alpha})} + \frac{\delta_1\beta_1^2}{\epsilon}\|\zeta\|^2_{(Y^{1-\alpha})^n} + \epsilon\delta_1\|\varpi\|^2_{(Y^{-\alpha})}\\ &\leqslant \frac{\delta_2}{\epsilon}\|\gamma\|^2_{(Y^{1-\alpha})^n} + \epsilon(\delta_1+\delta_2)\|\varpi\|^2_{(Y^{1-\alpha})} + \frac{\delta_1\beta_1^2}{\epsilon}\|\zeta\|^2_{(Y^{1-\alpha})^n}\end{aligned}$$

when we observe that $\|\gamma\|^2_{(Y^{1-\alpha})^n} \leqslant C_1\|\zeta\|^2_{Y^{-\alpha}} \leqslant C_2\|\zeta\|^2_{Y^{1-\alpha}}$, then

$$J_1 \leqslant \left(\frac{\delta_2C_2}{\epsilon} + \frac{\delta_1\beta_1^2}{\epsilon}\right)\|\zeta\|^2_{(Y^{1-\alpha})^n} + \epsilon(\delta_1+\delta_2)\|\varpi\|^2_{Y^{1-\alpha}}. \tag{4.23}$$

Using (4.21), (4.22) and (3.51) in (4.20) we get

$$\begin{aligned}\frac{d\mathcal{L}_\alpha}{dt} &\leqslant \epsilon\|\phi\|^2_{(Y^{-\alpha})^n} + \epsilon\|\varpi\|^2_{(Y^{-\alpha})^n} + \epsilon\|\gamma\|^2_{(Y^{-\alpha})^n} + C(M,\delta_1,\delta_2) - (M-\delta_2)\|\zeta\|^2_{Y^{1-\alpha}} \\ &\quad + \delta_1\|\phi\|^2_{(Y^{-\alpha})^n} - \delta_1\|\varpi\|^2_{(Y^{1-\alpha})^n} + \left(\frac{\delta_2 C_2}{\epsilon} + \frac{\delta_1\beta_1^2}{\epsilon}\right)\|\zeta\|^2_{Y^{1-\alpha}} \\ &\quad + \delta_2\left(\epsilon - \beta(t)\right)\|\phi\|^2_{(Y^{-\alpha})^n} + \epsilon(\delta_1+\delta_2)\|\varpi\|^2_{(Y^{1-\alpha})^n} \\ &\leqslant (-\delta_1 + \epsilon(\delta_1+\delta_2)+\epsilon)\|\varpi\|^2_{(Y^{1-\alpha})^n} + \left(\epsilon + \delta_1 + \delta_2\epsilon + 2\delta_2\epsilon C(\Omega) - \delta_2\beta_0\right)\|\phi\|^2_{(Y^{-\alpha})^n} \\ &\quad + \left(\frac{\delta_2 C_3}{\epsilon} + \frac{\delta_1\beta_1^2}{\epsilon} + \epsilon C - M + \delta_2\right)\|\zeta\|^2_{Y^{1-\alpha}} + C(M,\delta_1,\delta_2).\end{aligned}$$

Let $\epsilon > 0$ be small enough and, let $\delta_1 < \delta_2$ and $M > 0$ be large enough such that it is possible choose $\mathfrak{p}_1,\ \mathfrak{p}_2 > 0$ which,

$$\frac{d\mathcal{L}_\alpha}{dt} \leqslant -\mathfrak{p}_1\left(\|\varpi\|^2_{(Y^{1-\alpha})^n} + \|\phi\|^2_{(Y^{-\alpha})^n} + \|\zeta\|^2_{Y^{1-\alpha}}\right) + \mathfrak{p}_2$$

for t a.e. in $[0,\infty)$.

But $\|\zeta\|^2_{(Y^{-\alpha})^n} \leqslant \frac{C_2}{C_1}\|\zeta\|^2_{(Y^{1-\alpha})^n}$ and $\zeta \in C(0,\infty;Y^{-\alpha})$. This lead us to

$$\frac{d\mathcal{L}_\alpha}{dt} \leqslant -\mathfrak{p}_1\left(\|\varpi\|^2_{(Y^{1-\alpha})^n} + \|\phi\|^2_{(Y^{-\alpha})^n} + \|\zeta\|^2_{Y^{-\alpha}}\right) + \mathfrak{p}_2, \forall t \in \mathbb{R}$$

and from the fact that $\mathcal{A}(t) = \{\xi(\tau) : \xi(\cdot)$ is a pullback bounded solution of (4.4) in $\mathcal{H}_*\}$ we obtain that

$$\bigcup_{t\geqslant s}\mathcal{A}(t) \text{ is bounded in } (Y^{2-\alpha_1})^n \times (Y^{1-\alpha_1})^n \times Y_*^{1-\alpha_1}. \tag{4.24}$$

Using (4.24) and restarting from with $\alpha_2 = (p+1)\alpha_1 - p < \alpha_1$ if follows that $\mathcal{A}$ is bounded in $(Y^{2-\alpha_2})^n \times (Y^{1-\alpha_2})^n \times Y_*^{1-\alpha_2}$.

Iterating this procedure a finite number of times, we can now show that $\mathcal{A}$ is bounded in $(Y^2)^n \times (Y^1)^n \times Y_*^1$ and by the Remark 3.10 this implies in fact, that

$$\bigcup_{t\geqslant s}\mathcal{A}(t) \text{ is bounded in } (Y^2)^n \times (Y^1)^n \times Y_*^2.$$

□

4.4 Upper semicontinuity of the attractors

In this section we prove the upper semicontinuity of the pullback attractors in $\mathcal{H}_*^1$ with respect to functional parameter κ in (4.1). Let $\{\kappa_\epsilon\}_{\epsilon\in[0,1)}$ be a family of functions such that for each κ_ϵ the condition (3.8) is valid and suppose that

$$\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)} \to 0, \text{ as } \epsilon \to 0.$$

We observe that all previous results are also valid to the problem (4.1) with κ_ϵ instead of κ. If $\{S_\epsilon(t,s); t \geqslant s\}$ denotes the evolution process associate to the problem (4.4) with pullback attractors $\mathcal{A}_\epsilon$ for each $\epsilon \in [0,1]$, then we have the following result.

Theorem 4.7. *The family of pullback attractors $\mathcal{A}_\epsilon(t)$ is upper semicontinuous at $\epsilon = 0^+$.*

Proof. Let $\mathbf{u}^\epsilon = S_\epsilon(t,s)\mathbf{u}_0$ be the solution of (4.4) with $\mathbf{u}^\epsilon = (u^\epsilon, z^\epsilon, \theta^\epsilon)$. Then we write $\mu = u^\epsilon - u^0$ and $\vartheta = \theta^\epsilon - \theta^0$, then $(\mu, \partial_t\mu, \vartheta)$ solves the following system

$$\begin{cases} \partial_t^2\mu - \Delta\mu - \nabla \operatorname{div}\mu + \beta(t)\nabla\vartheta = f(u^\epsilon) - f(u^0), & x \in \Omega,\ t > s, \\ \partial_t\vartheta - [\operatorname{div}(\kappa_\epsilon(x)\nabla\theta^\epsilon) - \operatorname{div}(\kappa_0(x)\nabla\theta^0)] + \beta(t)\operatorname{div}\partial_t\mu = 0, & x \in \Omega,\ t > s. \end{cases}$$

We be able to find

$$\int_\Omega \partial_t^2\mu^\epsilon\partial_t\mu^\epsilon dx - \int_\Omega (\Delta\mu^\epsilon + \nabla \operatorname{div}\mu^\epsilon)\partial_t\mu^\epsilon dx + \int_\Omega \beta(t)\nabla\vartheta^\epsilon\partial_t\mu^\epsilon dx = \int_\Omega (f(u^\epsilon) - f(u^0))\partial_t\mu^\epsilon dx,$$

by multiplies $\partial_t\mu^\epsilon$ in the first equation and,

$$\int_\Omega \partial_t\vartheta^\epsilon\vartheta^\epsilon dx - \int_\Omega [\operatorname{div}(\kappa_\epsilon(x)\nabla\theta^\epsilon) - \operatorname{div}\big(\kappa_0(x)\nabla\theta^0\big)]\vartheta^\epsilon dx + \beta(t)\int_\Omega \operatorname{div}\partial_t\mu^\epsilon\vartheta^\epsilon dx = 0$$

by multiplies ϑ^ϵ in the second equation.

This lead us to

$$\frac{d}{dt}\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|_{\mathcal{H}*}^2 = 2\int_\Omega (f(u^\epsilon) - f(u^0))\partial_t\mu^\epsilon dx + 2\int_\Omega \operatorname{div}\big(\kappa_\epsilon(x)\nabla\theta^\epsilon - \kappa_0(x)\nabla\theta^0\big)\,\vartheta^\epsilon dx,$$

and in other words,

$$\frac{d}{dt}\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|_{\mathcal{H}*}^2 = 2\int_\Omega (f(u^\epsilon) - f(u^0))\partial_t\mu^\epsilon dx - 2\int_\Omega \big(\kappa_\epsilon(x)\nabla\theta^\epsilon - \kappa_0(x)\nabla\theta^0\big)\,\nabla\vartheta^\epsilon dx.$$

Now, note that

$$\begin{aligned} &\Big|\int_\Omega \big(\kappa_\epsilon(x)\nabla\theta^\epsilon - \kappa_0(x)\nabla\theta^0\big)\,\nabla\vartheta^\epsilon dx\Big| \\ &\leqslant \int_\Omega \big[\kappa_\epsilon(x)(\nabla\theta^\epsilon - \nabla\theta^0) + (\kappa_\epsilon(x) - \kappa_0(x))\nabla\theta^0\big]\,\nabla\vartheta^\epsilon dx \\ &\leqslant \|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)}\|\theta^0\|_{Y_*^1}\|\vartheta^\epsilon\|_{Y^1} + \|\kappa_\epsilon\|_{L^\infty(\Omega)}\|\theta^\epsilon - \theta^0\|_{Y_*^1}^2 \\ &\leqslant C_1\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)} + C_2\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|_{\mathcal{H}*}^2, \end{aligned}$$

where $C_1 > 0$ and $C_2 > 0$ are constants independents of ϵ thanks to Theorem (4.6).

From Lemma 3.1, we have

$$\int_\Omega (f(u^\epsilon) - f(u^0))\partial_t\mu^\epsilon dx \leqslant C_3\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|_{\mathcal{H}*}^2,$$

where $C_3 > 0$ is constant independent of ϵ.

Hence, there exist constants $C' > 0$ and $C'' > 0$ such that

$$\frac{d}{dt}\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|_{\mathcal{H}*}^2 \leqslant C'\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)} + C''\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|_{\mathcal{H}*}^2.$$

and so

$$\|(\mu^\epsilon, \partial_t\mu^\epsilon, \vartheta^\epsilon)\|_{\mathcal{H}_*}^2 \leqslant C' t e^{-C'' t}\|\kappa_\epsilon - \kappa_0\|_{L^\infty(\Omega)}, \ \forall t > s, \tag{4.25}$$

i.e., $\mathbf{u}^\epsilon \to \mathbf{u}^0$ in $\mathcal{H}_*$, as $\epsilon \to 0^+$, uniformly for t in bounded subset of the interval $[s, \infty)$ and $\mathbf{u}_0$ in bounded subset of $\mathcal{H}_*$.

From [13, Theorem 2.20], $\bigcup_{s\leqslant t} \mathcal{A}_{\kappa_\epsilon}(s)$ is bounded in Y. Then for $\delta > 0$ given, there is $\tau \in (-\infty, t]$ such that

$$\mathrm{dist}(S_{\kappa_0}(t,\tau)\mathcal{A}_{\kappa_\epsilon}(\tau), \mathcal{A}_{\kappa_0}(t)) \leqslant \mathrm{dist}(S_{\kappa_0}(t,\tau)\bigcup_{s\leqslant t}\mathcal{A}_{\kappa_\epsilon}(s), \mathcal{A}_{\kappa_0}(t)) < \frac{\delta}{2}, \ \forall \epsilon \in [0,1].$$

Using (4.25), there exists $\epsilon_0 > 0$ such that

$$\sup_{u_\epsilon \in \mathcal{A}_{\kappa_\epsilon}(t)} \|S_{\kappa_\epsilon}(t,\tau)u_\epsilon - S_{\kappa_0}(t,\tau)u_\epsilon\| < \frac{\delta}{2}$$

for any $\epsilon < \epsilon_0$. Therefore,

$$\begin{aligned}
&\mathrm{dist}(\mathcal{A}_{\kappa_\epsilon}(t), \mathcal{A}_{\kappa_0}(t)) \\
&\leqslant \mathrm{dist}(S_{\kappa_\epsilon}(t,\tau)\mathcal{A}_{\kappa_\epsilon}(\tau), S_{\kappa_0}(t,\tau)\mathcal{A}_{\kappa_\epsilon}(\tau)) + \mathrm{dist}(S_{\kappa_0}(t,\tau)\mathcal{A}_{\kappa_\epsilon}(\tau), S_{\kappa_0}(t,\tau)\mathcal{A}_{\kappa_0}(\tau)) \\
&\leqslant \sup_{u_\epsilon \in \mathcal{A}_{\kappa_\epsilon}(t)} \|S_{\kappa_\epsilon}(t,\tau)u_\epsilon - S_{\kappa_0}(t,\tau)u_\epsilon\| + \mathrm{dist}(S_{\kappa_0}(t,\tau)\mathcal{A}_{\kappa_\epsilon}(\tau), \mathcal{A}_{\kappa_0}(t)) < \delta
\end{aligned}$$

which proves the upper semicontinuity of the family of attractors at $\epsilon = 0$.

□

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