Gustavo da Silva Araújo

Some classical inequalities, summability of multilinear operators and strange functions

João Pessoa-PB
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Universidade Federal da Paraíba
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Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para a obtenção do título de Doutor em Matemática.

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Aos meus pais.

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Resumo

Este trabalho está dividido em três partes. Na primeira parte, investigamos o comportamento das constantes das desigualdades polinomial e multilinear de Bohnenblust–Hille e Hardy–Littlewood. Na segunda parte, mostramos um resultado ótimo de espaçabilidade para o complementar de uma classe de operadores múltiplo somantes em ℓ_p e também generalizamos um resultado relacionado a cotipo (de 2010) devido a G. Botelho, C. Michels e D. Pellegrino. Além disso, provamos novos resultados de coincidência para as classes de operadores multilineares absolutamente e múltiplo somantes (em particular, mostramos que o famoso teorema de Defant–Voigt é ótimo). Ainda na segunda parte, mostramos uma generalização das desigualdades multilineares de Bohnenblust–Hille e Hardy–Littlewood e apresentamos uma nova classe de operadores multilineares somantes, a qual recupera as classes dos operadores multilineares absolutamente e múltiplo somantes. Na terceira parte, provamos a existência de grandes estruturas algébricas dentro de certos conjuntos, como, por exemplo, a família das funções mensuráveis à Lebesgue que são sobrejetivas em um sentido forte, a família das funções reais não constantes e diferenciáveis que se anulam em um conjunto denso e a família das funções reais não contínuas e separadamente contínuas.

Palavras-chave: Desigualdade de Bohnenblust–Hille, desigualdade de Hardy–Littlewood, função contínua, função diferenciável, função mensurável, lineabilidade, operadores multilineares somantes.

Abstract

This work is divided into three parts. In the first part, we investigate the behavior of the constants of the Bohnenblust–Hille and Hardy–Littlewood polynomial and multilinear inequalities. In the second part, we show an optimal spaceability result for a set of non-multiple summing forms on ℓ_p and we also generalize a result related to cotype (from 2010) as highlighted by G. Botelho, C. Michels, and D. Pellegrino. Moreover, we prove new coincidence results for the class of absolutely and multiple summing multilinear operators (in particular, we show that the well-known Defant–Voigt theorem is optimal). Still in the second part, we show a generalization of the Bohnenblust–Hille and Hardy–Littlewood multilinear inequalities and we present a new class of summing multilinear operators, which recovers the class of absolutely and multiple summing operators. In the third part, it is proved the existence of large algebraic structures inside, among others, the family of Lebesgue measurable functions that are surjective in a strong sense, the family of non-constant differentiable real functions vanishing on dense sets, and the family of non-continuous separately continuous real functions.

Key-words: Bohnenblust–Hille inequality, continuous function, differentiable function, Hardy–Littlewood inequality, lineability, measurable function, summing multilinear operators.

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Contents

Introduction

Part I: The Bohnenblust–Hille and Hardy–Littlewood inequalities

To solve a problem posed by P.J. Daniell, Littlewood [101] proved in 1930 his famous 4/3-inequality, which asserts that

$$\left(\sum_{i,j=1}^{\infty} |T(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \, \|T\| ,$$

for every continuous bilinear form $T : c_0 \times c_0 \to \mathbb{K}$. One year later, and due to his interest in solving a long standing problem on Dirichlet series, H.F. Bohnenblust and E. Hille proved in [42] a generalization of Littlewood's 4/3 inequality to m-linear forms: there exists a (optimal) constant $B_{\mathbb{K},m}^{\text{mult}} \geq 1$ such that for all continuous m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \, \|T\| .$$

The problem was posed by H. Bohr and consisted in determining the width of the maximal strips on which a Dirichlet series can converge absolutely but non uniformly. More precisely, for a Dirichlet series $\sum_n a_n n^{-s}$, Bohr defined

$$\sigma_a = \inf \left\{ r : \sum_n a_n n^{-s} \text{ converges for } Re(s) > r \right\} ,$$

$$\sigma_u = \inf \left\{ r : \sum_n a_n n^{-s} \text{ converges uniformly in } Re\,(s) > r + \varepsilon \text{ for every } \varepsilon > 0 \right\}$$

and $S := \sup \{\sigma_a - \sigma_u\}$. Bohr's question asked for the precise value of S. The answer came from H.F. Bohnenblust and E. Hille (1931): $S = 1/2$. The main tool is the, by now, so-called Bohnenblust–Hille inequality. The precise growth of the constants $B_{\mathbb{K},m}^{\text{mult}}$ is important for applications and is nowadays a challenging problem in Mathematical Analysis. For real scalars, the estimates of $B_{\mathbb{R},m}^{\text{mult}}$ are important in Quantum Information Theory (see [106]). In the last years a series of papers related to the Bohnenblust–Hille inequality have been published and several advances were achieved (see [6, 62, 65, 68, 111, 119, 129] and the references therein). Only very recently, in [32, 111] it was shown that the constants $B_{\mathbb{K},m}^{\text{mult}}$ have a sublinear growth, which means a major change in this panorama since all previous estimates (from 1931 up to 2011) predicted an exponential growth. For real scalars, in 2014 (see [75]) it was shown that the optimal constant for $m = 2$ is $\sqrt{2}$ and in general $B_{\mathbb{R},m}^{\text{mult}} \geq 2^{1-\frac{1}{m}}$. In the case of complex scalars it is still an open problem whether the optimal constants are strictly greater than 1.

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and x^α stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$. The polynomial Bohnenblust–Hille inequality (see [6, 42] and the references therein) ensures that, given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_∞^n given by $P(x_1, ..., x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{pol}} \, \|P\| \, ,$$

for some constant $B_{\mathbb{K},m}^{\text{pol}} \geq 1$ which does not depend on n (the exponent $\frac{2m}{m+1}$ is optimal), where $\|P\| := \sup_{z \in B_{\ell_\infty^n}} |P(z)|$. The search of precise estimates of the growth of the constants $B_{\mathbb{K},m}^{\text{pol}}$ is crucial for different applications and remains an important open problem (see [32] and the references therein). For real scalars, it was shown in [56] that the hypercontractivity of $B_{\mathbb{R},m}^{\text{pol}}$ is optimal. For complex scalars the behavior of $B_{\mathbb{K},m}^{\text{pol}}$ is still unknown. Moreover, in the complex scalar case, having good estimates for $B_{\mathbb{C},m}^{\text{pol}}$ is crucial to applications in Complex Analysis and Analytic Number Theory (see [65]); for instance, the subexponentiality of the constants of the polynomial version of the Bohnenblust–Hille inequality (complex scalars case) was recenly used in [32] in order to obtain the asymptotic growth of the Bohr radius of the n-dimensional polydisk. More precisely, according to Boas and Khavinson [41], the Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right| .$$

The Bohr radius K_1 was estimated by H. Bohr, and it was later shown (independently) by M. Riesz, I. Schur and F. Wiener that $K_1 = 1/3$ (see [41, 43] and the references therein). For $n \geq 2$, exact values of K_n are unknown. In [32], the subexponentiality of the constants of the complex polynomial version of the Bohnenblust–Hille inequality was established and using this fact it was finally proved that

$$\lim_{n \to \infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1,$$

solving a challenging problem in Mathematical Analysis.

The Hardy-Littlewood inequality is a natural generalization of the Bohnenblust–Hille inequality for ℓ_p spaces. The bilinear case was proved by Hardy and Littlewood in 1934 (see [91]) and in 1981 it was extended to multilinear operators by Praciano-Pereira (see [128]). More precisely, the classical Hardy–Littlewood inequality asserts that for $0 \leq |1/\mathbf{p}| := 1/p_1 + \cdots + 1/p_m \leq 1/2$ there exists a (optimal) constant $C_{\mathbb{K},m,\mathbf{p}}^{\text{mult}} \geq 1$ such that, for all positive integers n and all continuous m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$,

$$\left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1},\dots,e_{j_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}} \|T\| .$$

When $|1/\mathbf{p}| = 0$ (or equivalently $p_1 = \cdots = p_m = \infty$) since $2m/(m+1-2|1/\mathbf{p}|) = 2m/(m+1)$, we recover the classical Bohnenblust–Hille inequality (see [42]).

When replacing ℓ_∞^n by ℓ_p^n the extension of the polynomial Bohnenblust–Hille inequality is called polynomial Hardy–Littlewood inequality. More precisely, given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_p^n, with $2m \leq p \leq \infty$, given by $P(x_1,\dots,x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, then there is a constant $C_{\mathbb{K},m,p}^{\mathrm{pol}} \geq 1$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{K},m,p}^{\mathrm{pol}} \|P\| ,$$

and $C_{\mathbb{K},m,p}^{\mathrm{pol}}$ does not depend on n, where $\|P\| := \sup_{z \in B_{\ell_p^n}} |P(z)|$.

When $p = \infty$ we recover the polynomial Bohnenblust–Hille inequality. Using the generalized Kahane–Salem–Zygmund inequality (see, for instance, [6]) we can verify that the exponents in the above inequalities are optimal. The precise estimates of the constants of the Hardy–Littlewood inequalities are unknown and even its asymptotic growth is a mystery (as it happens with the Bohnenblust–Hille inequality).

Very recently, an extended version of the Hardy–Littlewood inequality was presented in [6] (see also [73]).

Theorem 0.1 (Generalized Hardy–Littlewood inequality for $0 \leq |1/\mathbf{p}| \leq 1/2$)**.** *Let* $\mathbf{p} := (p_1,\dots,p_m) \in [1,+\infty]^m$ *such that* $0 \leq |1/\mathbf{p}| \leq 1/2$. *Let also* $\mathbf{q} := (q_1,\dots,q_m) \in [(1-|1/\mathbf{p}|)^{-1}, 2]^m$. *The following are equivalent:*

(1) There is a (optimal) constant $C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\mathrm{mult}} \geq 1$ *such that*

$$\left(\sum_{j_1=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} |T(e_{j_1},\dots,e_{j_m})|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\mathrm{mult}} \|T\|$$

for all continuous m-linear forms $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$.

(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$.

For the case $1/2 \leq |1/\mathbf{p}| < 1$ there is also a version of the multilinear Hardy–Littlewood inequality, which is an immediate consequence of Theorem 1.2 from [5] (see also [73]).

Theorem 0.2 (Hardy–Littlewood inequality for $1/2 \leq |1/\mathbf{p}| < 1$)**.** *Let* $m \geq 1$ *and* $\mathbf{p} = (p_1,\dots,p_m) \in [1,\infty]^m$ *be such that* $1/2 \leq |1/\mathbf{p}| < 1$. *Then there is a (optimal) constant* $D_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}} \geq 1$ *such that*

$$\left(\sum_{i_1,\dots,i_m=1}^{N} |T(e_{i_1},\dots,e_{i_m})|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}} \|T\|$$

for every continuous m-linear operator $T : \ell_{p_1}^N \times \cdots \times \ell_{p_m}^N \to \mathbb{K}$*. Moreover, the exponent* $(1 - |1/\mathbf{p}|)^{-1}$ *is optimal.*

In this part of the work, we investigate the behavior of the constants $C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\text{mult}}$, $D_{\mathbb{K},m,\mathbf{p}}^{\text{mult}}$ (Chapter 1) and $C_{\mathbb{K},m,p}^{\text{pol}}$ (Chapter 3). In Chapter 2 we answer, for $1 \leq p \leq m$, the question on how the Hardy–Littlewood multilinear inequalities behave if we replace the exponents $2mp/(mp + p - 2m)$ and $p/(p - m)$ by a smaller value r (see Theorem 2.1). This case ($1 \leq p \leq m$) was only explored for the case of Hilbert spaces ($p = 2$, see [47, Corollary 5.20] and [61]) and the case $p = \infty$ was explored in [57].

Part II: Summability of multilinear operators

In 1950 A. Dvoretzky and C. A. Rogers [76] solved a long standing problem in Banach Space Theory when they proved that in every infinite-dimensional Banach space there exists an unconditionally convergent series which is not absolutely convergent. This result is the answer to Problem 122 of the Scottish Book [104], addressed by S. Banach in [26, page 40]). It was the starting point of the theory of absolutely summing operators.

A. Grothendieck, in [88], presented a different proof of the Dvoretzky-Rogers theorem and his “Résumé de la théorie métrique des produits tensoriels topologiques” brought many illuminating insights to the theory of absolutely summing operators.

The notion of absolutely p-summing linear operators is credited to A. Pietsch [124] and the notion of (q, p)-summing operator is credited to B. Mitiagin and A. Pełczyński [105]. In 1968 J. Lindenstrauss and A. Pełczyński's seminal paper [100] re-wrote Grothendieck's Résumé in a more comprehensive form, putting the subject in the spotlight. In 2003 M. Matos [102] and, independently, F. Bombal, D. Pérez-García and I. Villanueva [44] introduced a more general notion of absolutely summing operators called multiple summing multilinear operators, which has gained special attention, being considered by several authors as the most important multilinear generalization of absolutely summing operators: let $1 \leq p_1, \ldots, p_m \leq q < \infty$. A bounded m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is multiple $(q; p_1, \ldots, p_m)$-summing if there exists $C_m > 0$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left\| T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}) \right\|^q \right)^{\frac{1}{q}} \leq C_m \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,p_k},$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^w(E_k)$, $k = 1, \ldots, m$. The class of all multiple $(q; p_1, \ldots, p_m)$-summing operators from $E_1 \times \cdots \times E_m$ to F will be denoted by

$$\Pi_{\text{mult}(q;p_1,\ldots,p_m)}^m(E_1, \ldots, E_m; F).$$

The roots of the subject could probably be traced back to 1930, when, as we have already said, Littlewood [101] proved his famous 4/3-inequality to solve a problem posed by P.J. Daniell. One year later, interested in solving a long standing problem on Dirichlet series, H.F. Bohnenblust and E. Hille generalized Littlewood's 4/3 inequality to m-linear forms. Using that $\mathcal{L}(c_0; E)$ is isometrically isomorphic to $\ell_1^w(E)$ (see [72]), the Bohnenblust–Hille inequality can be interpreted as the beginning of the notion of multiple summing operators, because in the modern terminology, the classical Bohnenblust–Hille inequality [42] ensures that, for all $m \geq 2$ and all Banach spaces $E_1, ..., E_m$,

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\mathrm{mult}\left(\frac{2m}{m+1};1,...,1\right)}\left(E_1, ..., E_m; \mathbb{K}\right).$$

In Chapter 4 we prove that, if $1 < s < p^*$, the set $(\mathcal{L}\left({}^m\ell_p; \mathbb{K}\right) \smallsetminus \Pi^m_{\mathrm{mult}\left(\frac{2m}{m+1};s\right)}\left({}^m\ell_p; \mathbb{K}\right)) \cup \{0\}$ contains a closed infinite-dimensional Banach space with the same dimension of $\mathcal{L}({}^m\ell_p; \mathbb{K})$. As a consequence we observe, for instance, a new optimal component of the Bohnenblust–Hille inequality: the terms 1 from the tuple $(2m/(m+1); 1, ..., 1)$ is also optimal. Moreover, we generalize a result related to cotype (from 2010) due to G. Botelho, C. Michels, and D. Pellegrino, and we investigate the optimality of coincidence results for multiple summing operators in c_0 and in the framework of absolutely summing multilinear operators. As a result, we observe that the Defant–Voigt theorem is optimal. In Chapter 5 we present a new class of summing multilinear operators, which recovers the class of absolutely (and multiple) summing operators. Moreover, we present a unified version of the Bohnenblust–Hille and the Hardy–Littlewood inequalities with partial sums which ensures that these results are in fact, corollaries of a unique yet general result.

Part III: Strange functions

Lebesgue ([99], 1904) was probably the first to show an example of a real function on the reals satisfying the rather surprising property that it takes on each real value in any nonempty open set (see also [86, 87]). The functions satisfying this property are called *everywhere surjective* (functions with even more stringent properties can be found in [80, 95]). Of course, such functions are nowhere continuous but, as we will see later, it is possible to construct a Lebesgue measurable everywhere surjective function. Entering a very different realm, in 1906 Pompeiu [126] was able to construct a nonconstant differentiable function on the reals whose derivative vanishes on a dense set. Passing to several variables, the first problem one meets related to the "minimal regularity" of functions at a elementary level is that of whether separate continuity implies continuity, the answer being given in the negative.

In this part of the thesis we will consider the families consisting of each of these kinds of functions and analyze the existence of large algebraic structures inside all these families. Nowadays the topic of lineability has had a major influence in many different areas on mathematics, from Real and Complex Analysis [29], to Set Theory [84], Operator Theory [92], and even (more recently) in Probability Theory [79]. Our main goal here is to continue with this ongoing research. We will focus on diverse lineability properties of the families $\mathcal{MES}$ (the family of Lebesgue measurable functions $\mathbb{R} \to \mathbb{R}$ that are everywhere surjective), $\mathcal{P}$ (the vector space of Pompeiu functions, i.e., the functions $f : \mathbb{R} \to \mathbb{R}$ that are differentiable and f' vanishes on a dense set in $\mathbb{R}$), $\mathcal{DP}$ (the vector space of the derivatives of Pompeiu functions) and certain subsets of discontinuous functions, completing or extending a number of known results about several strange classes of real functions.

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Preliminaries and Notation

For any function f, whenever it makes sense, we formally define $f(\infty) = \lim_{p\to\infty} f(p)$. Throughout this thesis, $E, E_1, E_2, \dots, F$ shall denote Banach spaces over $\mathbb{K}$, which shall stands for the complex $\mathbb{C}$ or real $\mathbb{R}$ fields. In addition, $\mathcal{L}(E_1, ..., E_m; F)$ stands for the Banach space of all bounded m-linear operators from $E_1 \times \cdots \times E_m$ to F under the supremum norm and when $E_1 = \cdots = E_m = E$ we denote $\mathcal{L}(E_1, ..., E_m; F)$ by $\mathcal{L}(^m E; F)$. The topological dual of E shall be denoted by E^* and for any $p \geq 1$ its conjugate is represented by p^*, i.e., $1/p + 1/p^* = 1$. For $p \in [1, \infty]$, as usual, we consider the Banach spaces of weakly and strongly p-summable sequences, respectively, as bellow:

$$\ell_p^w(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \left\| (x_j)_{j=1}^\infty \right\|_{w,p} := \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^\infty |\varphi(x_j)|^p \right)^{1/p} < \infty \right\}$$

and

$$\ell_p(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \left\| (x_j)_{j=1}^\infty \right\|_p := \left(\sum_{j=1}^\infty \|x_j\|^p \right)^{1/p} < \infty \right\}$$

(naturally, the sum $\sum$ should be replaced by the supremum if $p = \infty$). Besides, we set $X_\infty := c_0$ and $X_p := \ell_p := \ell_p(\mathbb{K})$. For a positive integer m, $\mathbf{p}$ stands for a multiple exponent $(p_1, \dots, p_m) \in [1, \infty]^m$ and

$$\left| \frac{1}{\mathbf{p}} \right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Khinchine's inequality

The real Khinchine inequality (see [72]) asserts that for any $0 < q < \infty$, there are positive constants $A_{\mathbb{R},q}$, $B_{\mathbb{R},q}$ such that, regardless of the scalar sequence $(a_j)_{j=1}^\infty$ in ℓ_2, we have

$$A_{\mathbb{R},q} \left(\sum_{j=1}^\infty |a_j|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{j=1}^\infty a_j r_j(t) \right|^q dt \right)^{\frac{1}{q}} \leq B_{\mathbb{R},q} \left(\sum_{j=1}^\infty |a_j|^2 \right)^{\frac{1}{2}},$$

where r_j are the Rademacher functions. More generally, from the above inequality together with the Minkowski inequality we know that (see [17], for instance, and the references therein)

$$
\begin{aligned}
A_{\mathbb{R},q}^{m}\left(\sum_{j_1,\ldots,j_m=1}^{\infty}|a_{j_1\cdots j_m}|^2\right)^{\frac{1}{2}} &\leq \left(\int_I\left|\sum_{j_1,\ldots,j_m=1}^{\infty}a_{j_1\cdots j_m}r_{j_1}(t_1)\cdots r_{j_m}(t_m)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq B_{\mathbb{R},q}^{m}\left(\sum_{j_1,\ldots,j_m=1}^{\infty}|a_{j_1\cdots j_m}|^2\right)^{\frac{1}{2}}, \qquad (1)
\end{aligned}
$$

where $I=[0,1]^m$ and $dt=dt_1\cdots dt_m$, for all scalar sequences $(a_{j_1\cdots j_m})_{j_1,\ldots,j_m=1}^{\infty}$ in ℓ_2. The optimal constants $A_{\mathbb{R},q}$ of the Khinchine inequality (these constants are due to U. Haagerup [90]) are:

- $A_{\mathbb{R},q}=\sqrt{2}\left(\frac{\Gamma\left(\frac{1+q}{2}\right)}{\sqrt{\pi}}\right)^{\frac{1}{q}}$ if $q>q_0\cong 1.8474$;
- $A_{\mathbb{R},q}=2^{\frac{1}{2}-\frac{1}{q}}$ if $q<q_0$.

The definition of the number q_0 above is the following: $q_0\in(1,2)$ is the unique real number with

$$\Gamma\left(\frac{q_0+1}{2}\right)=\frac{\sqrt{\pi}}{2}.$$

For complex scalars, using Steinhaus variables instead of Rademacher functions it is well known that a similar inequality holds, but with better constants (see [98, 133]). In this case the optimal constant is:

- $A_{\mathbb{C},q}=\Gamma\left(\frac{q+2}{2}\right)^{\frac{1}{q}}$ if $q\in[1,2]$.

The notation of the constant $A_{\mathbb{K},q}$ shown above will be employed throughout this thesis.

Kahane–Salem–Zygmund's inequality

Using the argument introduced in [39, Theorem 4] we present a variant of a result by Boas, that first appeared in [6, Lemma 6.1], and that is proved in [1].

Kahane–Salem–Zygmund's inequality. *Let $m,n\geq 1$, $p_1,\ldots,p_m\in[1,+\infty]^m$ and, for $p\geq 1$, define*

$$\alpha(p)=\begin{cases}\frac{1}{2}-\frac{1}{p}, & \text{if } p\geq 2;\\ 0, & \text{otherwise.}\end{cases}$$

Then there exists a m-linear map $A:\ell_{p_1}^n\times\cdots\times\ell_{p_m}^n\to\mathbb{K}$ of the form

$$A(z_1,\ldots,z_m)=\sum_{j_1,\ldots,j_m=1}^{n}\epsilon_{j_1\ldots j_m}z_{j_1}^1\cdots z_{j_m}^d$$

with $\epsilon_{j_1\cdots j_m}\in\{-1,1\}$, such that

$$\|A\|\leq C_m\cdot n^{\frac{1}{2}+\alpha(p_1)+\cdots+\alpha(p_m)} \qquad (2)$$

where $C_m=(m!)^{1-\frac{1}{\min\{\max\{p_1,\ldots,p_m\},2\}}}\sqrt{32m\log(6m)}$.

The essence of the Kahane–Salem–Zygmund inequalities probably appeared for the first time in [96], but our approach follows the lines of Boas' paper [39]. Paraphrasing Boas, the Kahane–Salem–Zygmund inequalities use probabilistic methods to construct a homogeneous polynomial (or multilinear operator) with a relatively small supremum norm but relatively large majorant function (we refer [1, Appendix B] for a more detailed study of the Kahane–Salem–Zygmund inequalities).

Minkowski's inequality

The following result is a corollary of one of the many versions of Minkowski's inequality, whose proof can be found, for instance, in [85, Corollary 5.4.2].

Minkowski's inequality. *For any* $0 < p \leq q < 1$ *and for any scalar matrix* $(a_{ij})_{i,j\in\mathbb{N}}$*,*

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|a_{ij}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leq \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|a_{ij}|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}.$$

Hölder's (interpolative) inequality for mixed $\ell_{\mathbf{p}}$ spaces

The following general Hölder's inequality was presented in the classical paper [33] on mixed norms in L_p spaces. We shall now work with $L_p(\mathbb{N}) = \ell_p$, since it is the case we are interested in. We need to recall some useful multi-index notation: for a positive integer m and a non-void subset $D \subset \mathbb{N}$ we denote the set of multi-indices $\mathbf{i} = (i_1, \ldots, i_m)$, with each $i_k \in D$, by

$$\mathcal{M}(m, D) := \{\mathbf{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m;\ i_k \in D,\ k = 1, \ldots, m\} = D^m.$$

We also denote

$$\mathcal{M}(m, n) := \mathcal{M}(m, \{1, 2, \ldots, n\}).$$

For $\mathbf{p} = (p_1, \ldots, p_m) \in [1, \infty)^m$, and a Banach space E, let us consider the space

$$\ell_{\mathbf{p}}(X) := \ell_{p_1}\left(\ell_{p_2}\left(\ldots\left(\ell_{p_m}(X)\right)\ldots\right)\right),$$

namely, a vector matrix $(x_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,\mathbb{N})} \in \ell_{\mathbf{p}}(X)$ if, and only if,

$$\left(\sum_{i_1=1}^{\infty}\left(\sum_{i_2=1}^{\infty}\left(\cdots\left(\sum_{i_{m-1}=1}^{\infty}\left(\sum_{i_m=1}^{\infty}\|x_{\mathbf{i}}\|_E^{p_m}\right)^{\frac{p_{m-1}}{p_m}}\right)^{\frac{p_{m-2}}{p_{m-1}}}\cdots\right)^{\frac{p_2}{p_3}}\right)^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} < \infty.$$

When $X = \mathbb{K}$, we just write $\ell_{\mathbf{p}}$ instead of $\ell_{\mathbf{p}}(\mathbb{K})$. Also, we deal with the coordinatewise product of two scalar matrices $\mathbf{a} = (a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ and $\mathbf{b} = (b_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$, i.e.,

$$\mathbf{ab} := (a_{\mathbf{i}}b_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}.$$

The following result seems to be first observed by A. Benedek and R. Panzone (see

[33]):

Hölder's inequality for mixed $\ell_{\mathbf{p}}$ spaces. *Let m,n,N be positive integers, $\mathbf{r}\in[1,\infty)^m$ and $\mathbf{q}(1),\ldots,\mathbf{q}(N)\in[1,\infty]^m$ be such that*

$$\tfrac{1}{r_j}=\tfrac{1}{q_j(1)}+\cdots+\tfrac{1}{q_j(N)},\quad j\in\{1,2,\ldots,m\}$$

and also let $\mathbf{a}_k:=(a^k_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$, $k=1,\ldots,N$ be scalar matrices. Then

$$\left\|\prod_{k=1}^N \mathbf{a}_k\right\|_{\mathbf{r}}\leq\prod_{k=1}^N\|\mathbf{a}_k\|_{\mathbf{q}(k)}\,.$$

In particular, if each $\mathbf{q}(k)\in[1,\infty)^m$, we have

$$\left(\sum_{i_1=1}^n\left(\cdots\left(\sum_{i_m=1}^n|a^1_{\mathbf{i}}a^2_{\mathbf{i}}\ldots a^N_{\mathbf{i}}|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$
$$\leq\prod_{k=1}^N\left[\left(\sum_{i_1=1}^n\left(\cdots\left(\sum_{i_m=1}^n|a^k_{\mathbf{i}}|^{q_m(k)}\right)^{\frac{q_{m-1}(k)}{q_m(k)}}\cdots\right)^{\frac{q_1(k)}{q_2(k)}}\right)^{\frac{1}{q_1(k)}}\right].$$

Using the above result we are able to recover the interpolative inequality from [5, 6, 7, 32] (see also [4]) that we can also, in some sense, call Hölder's inequality for multiple exponents. Under the point of view of interpolation theory it is not a complicated result but, just in 2013, it began to be used in all its full strength.

For a positive real number θ, let us define $\mathbf{a}^\theta:=\left(a^\theta_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{M}(m,n)}$. It is straightforward to see that

$$\left\|\mathbf{a}^\theta\right\|_{\mathbf{q}/\theta}=\|\mathbf{a}\|^\theta_{\mathbf{q}},$$

where $\mathbf{q}/\theta:=(q_1/\theta,\ldots,q_m/\theta)$.

Hölder's interpolative inequality for multiple exponents. *Let m,n,N be positive integers and $\mathbf{r},\mathbf{q}(1),\ldots,\mathbf{q}(N)\in[1,\infty]^m$ and $\theta_1,\ldots,\theta_N\in[0,1]$ be such that $\theta_1+\cdots+\theta_N=1$ and*

$$\tfrac{1}{r_j}=\tfrac{\theta_1}{q_j(1)}+\cdots+\tfrac{\theta_N}{q_j(N)},\qquad \text{for all } j=1,\ldots,m.$$

Then, for all scalar matrix $\mathbf{a}=(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$ we have

$$\|\mathbf{a}\|_{\mathbf{r}}\leq\prod_{k=1}^N\|\mathbf{a}\|^{\theta_k}_{\mathbf{q}(k)}\,.$$

In particular, if each $\mathbf{q}(k)\in[1,\infty)^m$, the previous inequality means that

$$\left(\sum_{i_1=1}^n\left(\cdots\left(\sum_{i_m=1}^n|a_{\mathbf{i}}|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}$$

$$\leq \prod_{k=1}^{N} \left[\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} |a_{\mathbf{i}}|^{q_m(k)} \right)^{\frac{q_{m-1}(k)}{q_m(k)}} \cdots \right)^{\frac{q_1(k)}{q_2(k)}} \right)^{\frac{1}{q_1(k)}} \right]^{\theta_k} .$$

Lineability notions

A number of concepts have been coined in order to describe the algebraic size of a given set; see [21, 31, 35, 78, 89] (see also the survey paper [38] and the forthcoming book [18] for an account of lineability properties of specific subsets of vector spaces). Namely, if X is a vector space, α is a cardinal number and $A \subset X$, then A is said to be:

- *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$,
- *α-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$ (hence lineability means $\aleph_0$-lineability, where $\aleph_0 = \mathrm{card}\,(\mathbb{N})$, the cardinality of $\mathbb{N}$), and
- *maximal lineable* in X if A is $\dim\,(X)$-lineable.

If, in addition, X is a topological vector space, then A is said to be:

- *dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$ (hence dense-lineability implies lineability as soon as $\dim(X) = \infty$), and
- *maximal dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$ and $\dim\,(M) = \dim\,(X)$,
- *spaceable* in X if there is a closed infinite dimensional vector subspace M such that $M \setminus \{0\} \subset A$ (hence spaceability implies lineability), and
- *maximal spaceable* in X if A in X is spaceable and $\dim(A) = \dim(X)$.

According to [24, 28], when X is a topological vector space contained in some (linear) algebra, then A is called:

- *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.
- *densely algebrable* in X if, in addition, M can be taken dense in X.
- *α-algebrable* if there is an α-generated algebra M with $M \setminus \{0\} \subset A$.
- *strongly α-algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$ (for $\alpha = \aleph_0$, we simply say *strongly algebrable*).
- *densely strongly α-algebrable* if, in addition, the free algebra M can be taken dense in X.

Observe that strong α-algebrability $\Longrightarrow$ α-algebrability $\Longrightarrow$ α-lineability, and none of these implications can be reversed; see [38, p. 74].

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Part I

The Bohnenblust–Hille and Hardy–Littlewood inequalities

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Chapter 1

The m-linear Bohnenblust–Hille and Hardy–Littlewood inequalities

In 1931 F. Bohnenblust and E. Hille proved in [42] that there exists a (optimal) constant $B_{\mathbb{K},m}^{\text{mult}} \geq 1$ such that for all continuous m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$, and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \left\| T \right\|. \tag{1.1}$$

The precise growth of the constants $B_{\mathbb{K},m}^{\text{mult}}$ is important for many applications (see, e.g., [106]) and remains an open problem. Only very recently, in [32, 111] it was shown that the constants have a sublinear growth. For real scalars (2014, see [75]) it was shown that the optimal constant for $m = 2$ is $\sqrt{2}$ and in general $B_{\mathbb{R},m}^{\text{mult}} \geq 2^{1-\frac{1}{m}}$. In the case of complex scalars it is still an open problem whether the optimal constants are strictly grater than 1; in the polynomial case, in 2013 D. Núñez-Alarcón proved that the complex constants are strictly greater than 1 (see [108]). Even basic questions related to the constants $B_{\mathbb{K},m}^{\text{mult}}$ remain unsolved. For instance:

- Is the sequence of optimal constants $\left(B_{\mathbb{K},m}^{\text{mult}}\right)_{m=1}^{\infty}$ increasing?
- Is the sequence of optimal constants $\left(B_{\mathbb{K},m}^{\text{mult}}\right)_{m=1}^{\infty}$ bounded?
- Is $B_{\mathbb{C},m}^{\text{mult}} = 1$?

The best known estimates for the constants in (1.1), which are recently presented in [32], are ($B_{\mathbb{K},1}^{\text{mult}} = 1$ is obvious)

$$B_{\mathbb{K},m}^{\text{mult}} \leq \prod_{j=2}^{m} A_{\mathbb{K},\frac{2j-2}{j}}^{-1},$$

where $A_{\mathbb{K},(2j-2)/j}$ are the respective constants of the Khnichine inequality, i.e.,

$$B_{\mathbb{C},m}^{\text{mult}} \leq \prod_{j=2}^{m} \Gamma\left(2 - \tfrac{1}{j}\right)^{\frac{j}{2-2j}},$$

$$\begin{aligned} B^{\text{mult}}_{\mathbb{R},m} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}, && \text{for } 2 \leq m \leq 13, \\ B^{\text{mult}}_{\mathbb{R},m} &\leq 2^{\frac{446381}{55440}-\frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}}, && \text{for } m \geq 14. \end{aligned} \tag{1.2}$$

In a more friendly presentation the above formulas tell us that the growth of the constants $B^{\text{mult}}_{\mathbb{K},m}$ is sublinear since, from the above estimates it can be proved that (see [32])

$$\begin{aligned} B^{\text{mult}}_{\mathbb{C},m} &< m^{\frac{1-\gamma}{2}} < m^{0.21139}, \\ B^{\text{mult}}_{\mathbb{R},m} &< 1.3 \cdot m^{\frac{2-\log 2-\gamma}{2}} < 1.3 \cdot m^{0.36482}, \end{aligned}$$

where γ denotes the Euler–Mascheroni constant. Differently of the above estimates, all previous estimates (from 1931 up to 2011) predicted an exponential growth. It was only in 2012, with [119] (motivated by [68]), when the perspective on the subject changed entirely.

The Hardy-Littlewood inequality is a natural generalization of the Bohnenblust–Hille inequality to ℓ_p spaces. More precisely, the classical Hardy–Littlewood inequality asserts that for $0 \leq |1/\mathbf{p}| \leq 1/2$ there exists a (optimal) constant $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \geq 1$ such that, for all positive integers n and all continuous m-linear forms $T : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \|T\|. \tag{1.3}$$

Using the generalized Kahane-Salem-Zygmund inequality (2) (see [6]) one can easily verify that the exponents $2m/(m+1-2|1/\mathbf{p}|)$ are optimal. When $|1/\mathbf{p}| = 0$ (or equivalently $p_1 = \cdots = p_m = \infty$), since $2m/(m+1-2|1/\mathbf{p}|) = 2m/(m+1)$, we recover the classical Bohnenblust–Hille inequality (see [42]).

The precise estimates of the constants of the Hardy–Littlewood inequalities are unknown and even its asymptotic growth is a mystery (as it happens with the Bohnenblust–Hille inequality). The original estimates for $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$ (see [6]) were of the form

$$C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \leq \left(\sqrt{2}\right)^{m-1}. \tag{1.4}$$

Very recently an extended version of the Hardy–Littlewood inequality was presented in [6] (see also [73]).

Theorem 1.1 (Generalized Hardy–Littlewood inequality for $0 \leq |1/\mathbf{p}| \leq 1/2$)**.** *Let* $\mathbf{p} := (p_1,\ldots,p_m) \in [1,+\infty]^m$ *be such that* $0 \leq |1/\mathbf{p}| \leq 1/2$. *Let also* $\mathbf{q} := (q_1,\ldots,q_m) \in [(1-|1/\mathbf{p}|)^{-1}, 2]^m$. *The following are equivalent:*

(1) There is a (optimal) constant $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p},\mathbf{q}} \geq 1$ *such that*

$$\left(\sum_{j_1=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} |T(e_{j_1},\ldots,e_{j_m})|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C^{\text{mult}}_{\mathbb{K},m,\mathbf{p},\mathbf{q}} \|T\|$$

for all continuous m*-linear forms* $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$.

(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$.

Some particular cases of $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p},\mathbf{q}}$ will be used throughout this chapter, therefore, we will establish notations for the (optimal) constants in some special cases:

- If $p_1 = \cdots = p_m = \infty$ we recover the generalized Bohnenblust–Hille inequality and we will denote $C^{\text{mult}}_{\mathbb{K},m,(\infty,\ldots,\infty),\mathbf{q}}$ by $B^{\text{mult}}_{\mathbb{K},m,\mathbf{q}}$. Moreover, if $q_1 = \cdots = q_m = 2m/(m+1)$ we recover the classical Bohnenblust–Hille inequality and we will denote $B^{\text{mult}}_{\mathbb{K},m,(2m/(m+1),\ldots,2m/(m+1))}$ by $B^{\text{mult}}_{\mathbb{K},m}$;
- If $q_1 = \cdots = q_m = 2m/(m+1-2|1/\mathbf{p}|)$ we recover the classical Hardy–Littlewood inequality and we will denote $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p},(2m/(m+1-2|1/\mathbf{p}|),\ldots,2m/(m+1-2|1/\mathbf{p}|))}$ by $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$. Moreover, if $p_1 = \cdots = p_m = p$ we will denote $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$ by $C^{\text{mult}}_{\mathbb{K},m,p}$.

For the case $1/2 \leq |1/\mathbf{p}| < 1$ there is also a version of the multilinear Hardy–Littlewood inequality, which is an immediate consequence of Theorem 1.2 from [5] (see also [73]).

Theorem 1.2 (Hardy–Littlewood inequality for $1/2 \leq |1/\mathbf{p}| < 1$)**.** *Let $\mathbf{p} = (p_1, \ldots, p_m) \in [1,\infty]^m$ be such that $1/2 \leq |1/\mathbf{p}| < 1$. Then there is a (optimal) constant $D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \geq 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} |T(e_{i_1}, \ldots, e_{i_m})|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \|T\|$$

for every continuous m-linear operator $T : \ell^N_{p_1} \times \cdots \times \ell^N_{p_m} \to \mathbb{K}$. Moreover, the exponent $(1 - |1/\mathbf{p}|)^{-1}$ is optimal.

The best known upper bounds for the constants on the previous result are $D^{\text{mult}}_{\mathbb{R},m,\mathbf{p}} \leq \left(\sqrt{2}\right)^{m-1}$ and $D^{\text{mult}}_{\mathbb{C},m,\mathbf{p}} \leq (2/\sqrt{\pi})^{m-1}$ (see [5, 73]). We will only deal with this second case of the Hardy–Littlewood inequality in Chapters 2 and 5. Again, we will establish notations for the (optimal) constants $D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$ in some special cases:

- When $p_1 = \cdots = p_m = p$ we denote $D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$ by $D^{\text{mult}}_{\mathbb{K},m,p}$.

Our main contributions regarding the constants of the multilinear case of the Hardy–Littlewood inequality can be summarized in the following result, which is a direct consequence of the forthcomings sections 1.1 and 1.2.

Theorem 1.3. *Let $m \geq 2$ and let $\sigma_{\mathbb{R}} = \sqrt{2}$ and $\sigma_{\mathbb{C}} = 2/\sqrt{\pi}$. Then,*

(1) Let $\mathbf{q} = (q_1, ..., q_m) \in [1,2]^m$ such that $|1/\mathbf{q}| = (m+1)/2$ and $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1)$, then

$$B^{\text{mult}}_{\mathbb{K},m,\mathbf{q}} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}.$$

(2) $C^{\text{mult}}_{\mathbb{R},m,p} \geq 2^{\frac{mp+2m-2m^2-p}{mp}}$ for $2m < p \leq \infty$ and $C^{\text{mult}}_{\mathbb{R},m,2m} > 1$.

(3) (i) For $|1/\mathbf{p}| \leq 1/2$,

$$C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \leq (\sigma_{\mathbb{K}})^{2(m-1)\left|\frac{1}{\mathbf{p}}\right|} \left(B^{\text{mult}}_{\mathbb{K},m}\right)^{1-2\left|\frac{1}{\mathbf{p}}\right|}.$$

In particular, $\left(C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}\right)_{m=1}^{\infty}$ *is sublinear if* $|1/\mathbf{p}| \leq 1/m$.

(ii) For $2m^3 - 4m^2 + 2m < p \leq \infty$,

$$C^{\text{mult}}_{\mathbb{K},m,p} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}.$$

(4) Let $2m < p \leq \infty$ *and let* $\mathbf{q} := (q_1, ..., q_m) \in [p/(p-m), 2]^m$ *such that* $|1/\mathbf{q}| = (mp + p - 2m)/2p$. *If* $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1)$, *then*

$$C^{\text{mult}}_{\mathbb{K},m,p,\mathbf{q}} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}.$$

Note that, for instance, if $2m^3 - 4m^2 + 2m < p \leq \infty$, the formula of item *(3)(ii)* is not dependent on p, contrary to what happens in item *(3)(i)*, where we can see a dependence on p but, paradoxically, it is *worse* than the formula from item *(3)(ii)*. This suggests the following problems:

- Are the optimal constants of the Bohnenblust–Hille and Hardy–Littlewood inequalities the same?
- Are the optimal constants of the Hardy–Littlewood inequality independent of p (at least for large p)?

Several advances and improvements have been obtained by various authors in this context. We can highlight and summarize these findings in the following remarks:

Remark 1.4. D. Pellegrino and D.M. Serrano-Rodríguez proved in [120] the following result: if $m \geq 2$ is a positive integer, and $\mathbf{q} = (q_1, ..., q_m) \in [1,2]^m$ are such that $|1/\mathbf{q}| = (m+1)/2$, then, for $j = 1, 2$,

$$B^{\text{mult}}_{\mathbb{R},m,\mathbf{q}} \geq 2^{\frac{(m-1)(1-q_j)\widehat{q_j} + \sum\limits_{\substack{i=1\\ i\neq j}}^{m} \widehat{q_i}}{q_1 \cdots q_m}},$$

with $\widehat{q_i} = q_1 \cdots q_m / q_i$, $i = 1, ..., m$. In particular[1],

$$B^{\text{mult}}_{\mathbb{R},m,(1,2,...,2)} = B^{\text{mult}}_{\mathbb{R},m,(2,1,2,...,2)} = (\sqrt{2})^{m-1}.$$

Remark 1.5. J. Campos, W. Cavalcante, V.V. Fávaro, D. Núñez-Alarcón, D. Pellegrino and D.M. Serrano-Rodríguez proved in [55] that, for $q_m \in [1, 2]$,

$$B^{\text{mult}}_{\mathbb{R},m,\left(\frac{2(m-1)q_m}{(m+1)q_m-2},...,\frac{2(m-1)q_m}{(m+1)q_m-2},q_m\right)} \geq 2^{\frac{3q_m m - 2m - 5q_m + 4}{2q_m(m-1)}}.$$

[1]The optimal value for the constant $B^{\text{mult}}_{\mathbb{R},m,(1,2,...,2)}$ was first obtained by Daniel Pellegrino in [118].

In particular, it was possible to conclude that

$$B^{\text{mult}}_{\mathbb{R},3,(4/3,4/3,2)} = B^{\text{mult}}_{\mathbb{R},3,(4/3,8/5,8/5)} = B^{\text{mult}}_{\mathbb{R},3,(4/3,2,4/3)} = 2^{3/4}.$$

Remark 1.6. Very recently, D. Pellegrino presented[2] new lower bounds for the real case of the Hady–Littlewood inequalities, which improve the so far best known lower estimates (item (2) of the previous theorem) and provide a closed formula even for the case $p = 2m$ (see [55]). Pellegrino's approach is very interesting because even with a simple argument, he finds an overlooked connection between the Clarkson's inequalities and Hardy–Littlewood's constants which helps to find analytical lower estimates for these constants. More precisely, using Clarkson's inequalities, D. Pellegrino proved that for $m \geq 2$ and $p \geq 2m$, we have

$$C^{\text{mult}}_{\mathbb{R},m,p} \geq \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{\sup\limits_{x\in[0,1]} \frac{((1+x)^{p^*}+(1-x)^{p^*})^{1/p^*}}{(1+x^p)^{1/p}}}.$$

Remark 1.7. If $\mathbf{p} = (p, ..., p)$ in Theorem 1.3 *(3)(i)* we have the following estimate for $C^{\text{mult}}_{\mathbb{K},m,p}$ with $2m \leq p \leq \infty$:

$$C^{\text{mult}}_{\mathbb{K},m,p} \leq (\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} (B^{\text{mult}}_{\mathbb{K},m})^{\frac{p-2m}{p}}. \tag{1.5}$$

Very recently, D. Pellegrino in [113][3] proved that, for $m \geq 3$ and $2m \leq p \leq 2m^3 - 4m^2 + 2m$, we can improve (1.5) to

$$C^{\text{mult}}_{\mathbb{K},m,p} \leq (\sigma_{\mathbb{K}})^{\frac{p-2m-mp+6m^2-6m^3+2m^4}{mp(m-2)}} (B^{\text{mult}}_{\mathbb{K},m})^{(m-1)\left(\frac{2m-p+mp-2m^2}{m^2p-2mp}\right)}.$$

When $p = 2m^3 - 4m^2 + 2m$ this formula coincides with Theorem 1.3 (3)(ii) when $p \to 2m^3 - 4m^2 + 2m$.

Remark 1.8. Let $p_0 \in (1, 2)$ be the unique real number satisfying

$$\Gamma\left(\tfrac{p_0+1}{2}\right) = \tfrac{\sqrt{\pi}}{2}.$$

D. Núñez-Alarcón and D. Pellegrino in [109] found the exact value of the constant in the particular case $\mathbb{K} = \mathbb{R}$, $m = 2$, $\mathbf{q} = (p/(p-1), 2)$ and $\mathbf{p} = (p, \infty)$ with $p \geq p_0/(p_0 - 1)$. More precisely, they showed that

$$C^{\text{mult}}_{\mathbb{R},2,(p,\infty),\left(\frac{p}{p-1},2\right)} = 2^{\frac{1}{2}-\frac{1}{p}}$$

whenever $p \geq p_0/(p_0 - 1)$. For $2 < p < p_0$, they found almost optimal constants, with better precision than 4×10^{-4}.

Remark 1.9. D. Pellegrino proved in [116] that for $m \geq 3$, $2m \leq p \leq \infty$ and $\mathbf{q} := (q_1, ..., q_m) \in [p/(p-m), 2]^m$ such that $|1/\mathbf{q}| = (mp + p - 2m)/2p$ and $\max q_i \geq (2m^2 -$

[2] The original paper that D. Pellegrino presented the new lower bounds for the real case of the Hardy–Littlewood inequalities has been withdrawn by the author (see [112]). This arXiv preprint is now incorporated to [55].

[3] This arXiv preprint is now incorporated to [14].

$4m+2)/(m^2-m-1)$, we have

$$C_{\mathbb{K},m,p,\mathbf{q}}^{\mathrm{mult}} \leq (\sigma_{\mathbb{K}})^{(m-1)\left(1-\frac{(m+1)(2-\max q_i)(m-1)^2}{(m^2-m-2)\max q_i}\right)} \left(\prod_{j=2}^{m} A_{\mathbb{K},\frac{2j-2}{j}}^{-1}\right)^{\frac{(m+1)(2-\max q_i)(m-1)^2}{(m^2-m-2)\max q_i}}. \tag{1.6}$$

The estimates (1.6) behaves continuously when compared with Theorem 1.3 *(4)*.

1.1 Lower and upper bounds for the constants of the classical Hardy–Littlewood inequality

From [32, 111] we know that $B_{\mathbb{K},m}^{\mathrm{mult}}$ has a sublinear growth. On the other hand, the best known upper bounds for the constants $C_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}}$ are $\left(\sqrt{2}\right)^{m-1}$ (see [5, 6, 73]). In this section we show that $\left(\sqrt{2}\right)^{m-1}$ can be improved to

$$\begin{aligned} C_{\mathbb{R},m,\mathbf{p}}^{\mathrm{mult}} &\leq \left(\sqrt{2}\right)^{2(m-1)\left|\frac{1}{\mathbf{p}}\right|} \left(B_{\mathbb{R},m}^{\mathrm{mult}}\right)^{1-2\left|\frac{1}{\mathbf{p}}\right|}, \\ C_{\mathbb{C},m,\mathbf{p}}^{\mathrm{mult}} &\leq \left(\tfrac{2}{\sqrt{\pi}}\right)^{2(m-1)\left|\frac{1}{\mathbf{p}}\right|} \left(B_{\mathbb{C},m}^{\mathrm{mult}}\right)^{1-2\left|\frac{1}{\mathbf{p}}\right|}. \end{aligned} \tag{1.7}$$

These estimates are better than $\left(\sqrt{2}\right)^{m-1}$ because $B_{\mathbb{K},m}^{\mathrm{mult}}$ is sublinear. Moreover, our estimates depend on $\mathbf{p}$ and m and catch more subtle information since now it is clear that the estimates improve as $|1/\mathbf{p}|$ decreases. As $|1/\mathbf{p}|$ goes to zero we note that the above estimates tend to $B_{\mathbb{K},m}^{\mathrm{mult}}$ (see (1.2)) and, for instance, if $|1/\mathbf{p}| \leq 1/m$ we conclude that $\left(C_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}}\right)_{m=1}^{\infty}$ has a sublinear growth. One of our main results in this section is the following:

Theorem 1.10. *Let $m \geq 2$ be a positive integer and $|1/\mathbf{p}| \leq 1/2$. Then, for all continuous m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n, we have*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}}\right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}} \|T\|, \tag{1.8}$$

with $C_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}}$ as in (1.7)*. In particular, $\left(C_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}}\right)_{m=1}^{\infty}$ has a sublinear growth if $|1/\mathbf{p}| \leq 1/m$.*

Remark 1.11. If $p_1 = \cdots = p_m = p$ and $2m^3 - 4m^2 + 2m < p \leq \infty$, we already have better information for $C_{\mathbb{K},m,p}^{\mathrm{mult}}$ when compared to the previous theorem (see Theorem 1.17).

Proof of Theorem 1.10. For the sake of simplicity we shall deal with the case $p_1 = \cdots = p_m = p$. The case $p = \infty$ in (1.8) is precisely the Bohnenblust–Hille inequality, so we just need to consider $2m \leq p < \infty$. Let $(2m-2)/m \leq s \leq 2$ and $\lambda_0 = 2s/(ms+s-2m+2)$. Since $(m-1)/s + 1/\lambda_0 = (m+1)/2$, from the generalized Bohnenblust–Hille inequality (see [6]) we know that there is a constant $B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)} \geq 1$ such that for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ we have, for all $i = 1, ..., m$,

$$\left(\sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_0} \right)^{\frac{1}{\lambda_0}} \leq B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)} \|T\| . \tag{1.9}$$

Above, $\sum_{\widehat{j_i}=1}^{n}$ means the sum over all j_k for all $k \neq i$. If we choose $s = 2mp/(mp + p - 2m)$, we have $\lambda_0 < s \leq 2$. The multiple exponent $(\lambda_0, s, ..., s)$ can be obtained by interpolating the multiple exponents $(1, 2, ..., 2)$ and $(2m/(m + 1), ..., 2m/(m + 1))$ with, respectively, $\theta_1 = 2(1/\lambda_0 - 1/s)$ and $\theta_2 = m(2/s - 1)$, in the sense of [6].

It is thus important to control the constants associated with the multiple exponents $(1, 2..., 2)$ and $(2m/(m + 1), ..., 2m/(m + 1))$. The exponent $(2m/(m + 1), ..., 2m/(m + 1))$ is the classical exponent of the Bohnenblust–Hille inequality and the estimate of the constant associated with $(1, 2..., 2)$ is well-known (we present the details for the sake of completeness). In fact, in general, for the exponent $(2k/(k + 1), ..., 2k/(k + 1), 2, ..., 2)$ (with $2k/(k + 1)$ repeated k times and 2 repeated $m - k$ times), using the multiple Khinchine inequality (1), we have, for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$,

$$\begin{aligned}
&\Big(\sum_{j_1,...,j_k=1}^{n} \Big(\sum_{j_{k+1},...,j_m=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^2 \Big)^{\frac{1}{2}\frac{2k}{k+1}} \Big)^{\frac{k+1}{2k}} \\
&\leq \Big(\sum_{j_1,...,j_k=1}^{n} \Big(A^{-(m-k)}_{\mathbb{K},\frac{2k}{k+1}} \Big(\int_{[0,1]^{m-k}} \Big| \sum_{j_{k+1},...,j_m=1}^{n} r_{j_{k+1}}(t_{k+1}) \cdots r_{j_m}(t_m) \\
&\qquad \times T(e_{j_1}, ..., e_{j_m}) \Big|^{\frac{2k}{k+1}} dt_{k+1} \cdots dt_m \Big)^{\frac{k+1}{2k}} \Big)^{\frac{2k}{k+1}} \Big)^{\frac{k+1}{2k}} \\
&= A^{-(m-k)}_{\mathbb{K},\frac{2k}{k+1}} \Big(\sum_{j_1,...,j_k=1}^{n} \int_{[0,1]^{m-k}} \Big| T\Big(e_{j_1}, ..., e_{j_k}, \sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1}) e_{j_{k+1}}, ..., \\
&\qquad \sum_{j_m=1}^{n} r_{j_m}(t_m) e_{j_m} \Big) \Big|^{\frac{2k}{k+1}} dt_{k+1} \cdots dt_m \Big)^{\frac{k+1}{2k}} \\
&= A^{-(m-k)}_{\mathbb{K},\frac{2k}{k+1}} \Big(\int_{[0,1]^{m-k}} \sum_{j_1,...,j_k=1}^{n} \Big| T\Big(e_{j_1}, ..., e_{j_k}, \sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1}) e_{j_{k+1}}, ..., \\
&\qquad \sum_{j_m=1}^{n} r_{j_m}(t_m) e_{j_m} \Big) \Big|^{\frac{2k}{k+1}} dt_{k+1} \cdots dt_m \Big)^{\frac{k+1}{2k}} \\
&\leq A^{-(m-k)}_{\mathbb{K},\frac{2k}{k+1}} \sup_{t_{k+1},...,t_m \in [0,1]} B^{\text{mult}}_{\mathbb{K},k} \Big\| T\Big(\cdot\, , ..., \cdot\, , \sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1}) e_{j_{k+1}}, ..., \sum_{j_m=1}^{n} r_{j_m}(t_m) e_{j_m} \Big) \Big\| \\
&= A^{-(m-k)}_{\mathbb{K},\frac{2k}{k+1}} B^{\text{mult}}_{\mathbb{K},k} \|T\| .
\end{aligned}$$

So, choosing $k = 1$, since $A_{\mathbb{K},1} = \sigma_{\mathbb{K}}^{-1}$ and $B^{\text{mult}}_{\mathbb{K},1} = 1$ we conclude that the constant associated with the multiple exponent $(1, 2, ..., 2)$ is $\sigma_{\mathbb{K}}^{m-1}$.

Therefore, the optimal constant associated with the multiple exponent $(\lambda_0, s, ..., s)$ is less than or equal to

$$\left(\sigma_{\mathbb{K}}^{m-1}\right)^{2\left(\frac{1}{\lambda_0} - \frac{1}{s}\right)} \left(B^{\text{mult}}_{\mathbb{R},m}\right)^{m\left(\frac{2}{s} - 1\right)}$$

i.e.,

$$B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,\dots,s)} \le (\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} \left(B^{\mathrm{mult}}_{\mathbb{R},m}\right)^{\frac{p-2m}{p}}. \tag{1.10}$$

More precisely, (1.9) is valid with $B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,\dots,s)}$ as above.

Let $\lambda_j = \lambda_0 p/(p-\lambda_0 j)$ for all $j = 1, \dots, m$. Note that $\lambda_m = s$ and that $(p/\lambda_j)^* = \lambda_{j+1}/\lambda_j$ for all $j = 0, \dots, m-1$. Let us suppose that $1 \le k \le m$ and that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\dots,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \le B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,\dots,s)}\|T\|$$

is true for all continuous m-linear forms $T : \ell_p^n \times \overset{k-1 \text{ times}}{\cdots} \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, \dots, m$. Let us prove that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\dots,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k}}\right)^{\frac{1}{\lambda_{k}}} \le B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,\dots,s)}\|T\|,$$

for all continuous m-linear forms $T : \ell_p^n \times \overset{k \text{ times}}{\cdots} \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, \dots, m$.

The initial case (the case $k = 0$) is precisely (1.9) with $B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,\dots,s)}$ as in (1.10). Consider

$$T \in \mathcal{L}(\ell_p^n, \overset{k \text{ times}}{\cdots}, \ell_p^n, \ell_\infty^n, \dots, \ell_\infty^n; \mathbb{K})$$

and for each $x \in B_{\ell_p^n}$ define

$$\begin{array}{rcl} T^{(x)} : \ell_p^n \times \overset{k-1 \text{ times}}{\cdots} \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n & \to & \mathbb{K} \\ (z^{(1)}, \dots, z^{(m)}) & \mapsto & T(z^{(1)}, \dots, z^{(k-1)}, xz^{(k)}, z^{(k+1)}, \dots, z^{(m)}), \end{array}$$

with $xz^{(k)} = (x_j z_j^{(k)})_{j=1}^n$. Observe that $\|T\| \ge \sup\{\|T^{(x)}\| : x \in B_{\ell_p^n}\}$. By applying the induction hypothesis to $T^{(x)}$, we obtain

$$\begin{aligned}
&\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\dots,e_{j_m})|^s |x_{j_k}|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \\
&= \left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left|T\left(e_{j_1},\dots,e_{j_{k-1}}, xe_{j_k}, e_{j_{k+1}},\dots,e_{j_m}\right)\right|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \\
&= \left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\left|T^{(x)}\left(e_{j_1},\dots,e_{j_m}\right)\right|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \\
&\le B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,\dots,s)}\|T^{(x)}\| \\
&\le B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,\dots,s)}\|T\|, \qquad (1.11)
\end{aligned}$$

for all $i = 1, \dots, m$.

We shall analyze two cases, namely, $i = k$ and $i \neq k$.

- $i = k$.

Since $(p/\lambda_{j-1})^* = \lambda_j/\lambda_{j-1}$ for all $j = 1, ..., m$, we conclude that

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \right)^{\frac{1}{\lambda_k}} \\
&= \left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_{k-1}\left(\frac{p}{\lambda_{k-1}}\right)^*} \right)^{\frac{1}{\lambda_{k-1}}\frac{1}{\left(\frac{p}{\lambda_{k-1}}\right)^*}} \\
&= \left\| \left(\left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)_{j_k=1}^{n} \right\|_{\left(\frac{p}{\lambda_{k-1}}\right)^*}^{\frac{1}{\lambda_{k-1}}} \\
&= \left(\sup_{y \in B_{\ell^n_{\frac{p}{\lambda_{k-1}}}}} \sum_{j_k=1}^{n} |y_{j_k}| \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\
&= \left(\sup_{x \in B_{\ell^n_p}} \sum_{j_k=1}^{n} |x_{j_k}|^{\lambda_{k-1}} \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\
&= \sup_{x \in B_{\ell^n_p}} \left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\
&\leq B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)} \|T\|
\end{aligned}$$

where the last inequality holds by (1.11).

- $i \neq k$.

Let us first suppose that $k \in \{1, ..., m-1\}$. It is important to note that in this case $\lambda_{k-1} < \lambda_k < s$ for all $k \in \{1, ..., m-1\}$. Denoting, for $i = 1, ..., m$,

$$S_i = \left(\sum_{\widehat{j_i}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}}$$

we get

$$\begin{aligned}
&\sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} = \sum_{j_i=1}^{n} S_i^{\lambda_k} = \sum_{j_i=1}^{n} S_i^{\lambda_k - s} S_i^s \\
&= \sum_{j_i=1}^{n} \sum_{\widehat{j_i}=1}^{n} \frac{|T(e_{j_1}, ..., e_{j_m})|^s}{S_i^{s-\lambda_k}} = \sum_{j_k=1}^{n} \sum_{\widehat{j_k}=1}^{n} \frac{|T(e_{j_1}, ..., e_{j_m})|^s}{S_i^{s-\lambda_k}}
\end{aligned}$$

$$= \sum_{j_k=1}^{n} \sum_{\widehat{j_k}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^{\frac{s(s-\lambda_k)}{s-\lambda_{k-1}}}}{S_i^{s-\lambda_k}} |T(e_{j_1},...,e_{j_m})|^{\frac{s(\lambda_k-\lambda_{k-1})}{s-\lambda_{k-1}}}.$$

Therefore, using Hölder's inequality twice (first with the exponents $(s-\lambda_{k-1})/(s-\lambda_k)$ and $(s-\lambda_{k-1})/(\lambda_k-\lambda_{k-1})$ and then next with $\lambda_k(s-\lambda_{k-1})/\lambda_{k-1}(s-\lambda_k)$ and $\lambda_k(s-\lambda_{k-1})/s(\lambda_k-\lambda_{k-1})$ we obtain

$$\begin{aligned}
&\sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} |T(e_{j_1},...,e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \\
&\leq \sum_{j_k=1}^{n} \left[\left(\sum_{\widehat{j_k}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{s-\lambda_k}{s-\lambda_{k-1}}} \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1},...,e_{j_m})|^s \right)^{\frac{\lambda_k-\lambda_{k-1}}{s-\lambda_{k-1}}} \right] \\
&\leq \left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_k}{\lambda_{k-1}}} \right)^{\frac{\lambda_{k-1}}{\lambda_k}\cdot\frac{s-\lambda_k}{s-\lambda_{k-1}}} \\
&\qquad \times \left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1},...,e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \right)^{\frac{1}{\lambda_k}\cdot\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}. \qquad (1.12)
\end{aligned}$$

We know from the case $i=k$ that

$$\left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} |T(e_{j_1},...,e_{j_m})|^s \right)^{\frac{1}{s}\lambda_k} \right)^{\frac{1}{\lambda_k}\cdot\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}} \leq \left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)} \|T\| \right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}. \quad (1.13)$$

Now we investigate the first factor of the right side in (1.12). From Hölder's inequality (with the exponents $s/(s-\lambda_{k-1})$ and s/λ_{k-1}) and (1.11) it follows that

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n} \left(\sum_{\widehat{j_k}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)^{\frac{\lambda_k}{\lambda_{k-1}}} \right)^{\frac{\lambda_{k-1}}{\lambda_k}} = \left\| \left(\sum_{\widehat{j_k}} \frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} \right)_{j_k=1}^{n} \right\|_{\left(\frac{p}{\lambda_{k-1}}\right)^*} \\
&= \sup_{y\in B_{\ell^n_{\frac{p}{\lambda_{k-1}}}}} \sum_{j_k=1}^{n} |y_{j_k}| \sum_{\widehat{j_k}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} = \sup_{x\in B_{\ell^n_p}} \sum_{j_k=1}^{n} \sum_{\widehat{j_k}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}} |x_{j_k}|^{\lambda_{k-1}} \\
&= \sup_{x\in B_{\ell^n_p}} \sum_{j_i=1}^{n} \sum_{\widehat{j_i}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^{s-\lambda_{k-1}}}{S_i^{s-\lambda_{k-1}}} |T(e_{j_1},...,e_{j_m})|^{\lambda_{k-1}} |x_{j_k}|^{\lambda_{k-1}} \\
&\leq \sup_{x\in B_{\ell^n_p}} \sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} \frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^s} \right)^{\frac{s-\lambda_{k-1}}{s}} \left(\sum_{\widehat{j_i}=1}^{n} |T(e_{j_1},...,e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s}\lambda_{k-1}} \\
&= \sup_{x\in B_{\ell^n_p}} \sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} |T(e_{j_1},...,e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s}\lambda_{k-1}} \leq \left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)} \|T\| \right)^{\lambda_{k-1}}. \qquad (1.14)
\end{aligned}$$

Replacing (1.13) and (1.14) in (1.12) we finally conclude that

$$\begin{aligned}
&\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_k} \\
&\le \left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)}\|T\|\right)^{\lambda_{k-1}\frac{s-\lambda_k}{s-\lambda_{k-1}}}\left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)}\|T\|\right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}} \\
&= \left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)}\|T\|\right)^{\lambda_k}.
\end{aligned}$$

It remains to consider $k=m$. In this case $\lambda_m = s$ and we have a simpler situation since

$$\begin{aligned}
\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_m}\right)^{\frac{1}{\lambda_m}} &= \left(\sum_{j_m=1}^{n}\left(\sum_{\widehat{j_m}=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_m}\right)^{\frac{1}{s}} \\
&\le B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)}\|T\|,
\end{aligned}$$

where the inequality is due to the case $i=k$. This concludes the proof. □

Now we will provide nontrivial lower bounds for $C^{\text{mult}}_{\mathbb{R},m,p}$. Currently, the best lower bounds for the constants of the real case of the Hardy–Littlewood inequalities can be founded in [55] (see Remark 1.6), but our next result was the first in this direction and we present the proof for the sake of completeness.

Theorem 1.12. *The optimal constants of the Hardy–Littlewood inequalities satisfies*

$$C^{\text{mult}}_{\mathbb{R},m,p} \ge 2^{\frac{mp+2m-2m^2-p}{mp}} > 1 \quad \textit{for} \quad 2m < p \le \infty,$$

and

$$C^{\text{mult}}_{\mathbb{R},m,2m} > 1.$$

Proof. Following the lines of [75], it is possible to prove that $C^{\text{mult}}_{\mathbb{R},m,p} \ge 2^{\frac{mp+2m-2m^2-p}{mp}} > 1$ for $2m < p \le \infty$, but note that when $p = 2m$ we have $2^{\frac{mp+2m-2m^2-p}{mp}} = 1$ and thus we do not have nontrivial information.

All that remains is to prove the case $p = 2m$. The next step follows the lines of [75]. For $2m \le p \le \infty$, consider $T_{2,p} : \ell_p^2 \times \ell_p^2 \to \mathbb{R}$ given by

$$(x^{(1)}, x^{(2)}) \mapsto x_1^{(1)}x_1^{(2)} + x_1^{(1)}x_2^{(2)} + x_2^{(1)}x_1^{(2)} - x_2^{(1)}x_2^{(2)}$$

and $T_{m,p} : \ell_p^{2^{m-1}} \times \cdots \times \ell_p^{2^{m-1}} \to \mathbb{R}$ given by

$$\begin{aligned}
(x^{(1)},...,x^{(m)}) \mapsto\ & (x_1^{(m)} + x_2^{(m)})T_{m-1,p}(x^{(1)},...,x^{(m)}) \\
& + (x_1^{(m)} - x_2^{(m)})T_{m-1,p}(B^{2^{m-1}}(x^{(1)}), B^{2^{m-2}}(x^{(2)}),...,B^2(x^{(m-1)})),
\end{aligned}$$

where $x^{(k)} = (x_j^{(k)})_{j=1}^{2^{m-1}} \in \ell_p^{2^{m-1}}$, $1 \le k \le m$, and B is the backward shift operator in $\ell_p^{2^{m-1}}$. Observe that

$$|T_{m,p}(x^{(1)},...,x^{(m)})|$$

$$\begin{aligned}
&\leq |x_1^{(m)} + x_2^{(m)}| \cdot |T_{m-1,p}(x^{(1)}, ..., x^{(m)})| \\
&\qquad + |x_1^{(m)} - x_2^{(m)}| \cdot |T_{m-1,p}(B^{2^{m-1}}(x^{(1)}), B^{2^{m-2}}(x^{(2)}), ..., B^2(x^{(m-1)}))| \\
&\leq \|T_{m-1,p}\|(|x_1^{(m)} + x_2^{(m)}| + |x_1^{(m)} - x_2^{(m)}|) \\
&= \|T_{m-1,p}\| 2 \max\{|x_1^{(m)}|, |x_2^{(m)}|\} \\
&\leq 2\|T_{m-1,p}\| \cdot \|x^{(m)}\|_p.
\end{aligned}$$

Therefore,

$$\|T_{m,p}\| \leq 2^{m-2}\|T_{2,p}\|. \tag{1.15}$$

Note that

$$\|T_{2,p}\| = \sup\{\|T_{2,p}^{(x^{(1)})}\| : \|x^{(1)}\|_p = 1\},$$

where $T_{2,p}^{(x^{(1)})} : \ell_p^2 \to \mathbb{R}$ is given by $x^{(2)} \mapsto T_{2,p}\left(x^{(1)}, x^{(2)}\right)$. Thus we have the operator

$$T_{2,p}^{(x^{(1)})}(x^{(2)}) = (x_1^{(1)} + x_2^{(1)})x_1^{(2)} + (x_1^{(1)} - x_2^{(1)})x_2^{(2)}.$$

Since $(\ell_p)^* = \ell_{p^*}$ for $1 \leq p < \infty$, we obtain

$$\|T_{2,p}^{(x^{(1)})}\| = \|(x_1^{(1)} + x_2^{(1)}, x_1^{(1)} - x_2^{(1)}, 0, 0, ...)\|_{p^*}.$$

Therefore

$$\|T_{2,p}\| = \sup\{(|x_1^{(1)} + x_2^{(1)}|^{p^*} + |x_1^{(1)} - x_2^{(1)}|^{p^*})^{\frac{1}{p^*}} : |x_1^{(1)}|^p + |x_2^{(1)}|^p = 1\}.$$

We can verify that it is enough to maximize the above expression when $x_1^{(1)}, x_2^{(1)} \geq 0$. Then

$$\begin{aligned}
\|T_{2,p}\| &= \sup\{((x + (1 - x^p)^{\frac{1}{p}})^{p^*} + |x - (1 - x^p)^{\frac{1}{p}}|^{p^*})^{\frac{1}{p^*}} : x \in [0, 1]\} \\
&= \max\{\sup\{f_p(x) : x \in [0, 2^{-\frac{1}{p}}]\}, \sup\{g_p(x) : x \in [2^{-\frac{1}{p}}, 1]\}\}
\end{aligned}$$

where

$$f_p(x) := ((x + (1 - x^p)^{\frac{1}{p}})^{p^*} + ((1 - x^p)^{\frac{1}{p}} - x)^{p^*})^{\frac{1}{p^*}}$$

and

$$g_p(x) := ((x + (1 - x^p)^{\frac{1}{p}})^{p^*} + (x - (1 - x^p)^{\frac{1}{p}})^{p^*})^{\frac{1}{p^*}}.$$

It is not too difficult to see that

$$\|T_{2,p}\| < 2 \tag{1.16}$$

(for instance, the precise value of $\|T_{2,4}\|$ seems to be graphically $\sqrt{3}$ (see Figure 1.1)).

From (1.15) and (1.16) we would conclude that $\|T_{m,p}\| < 2^{m-1}$. On the other hand, from Theorem 1.3 we have

$$(4^{m-1})^{\frac{mp+p-2m}{2mp}} = \left(\sum_{j_1,...,j_m=1}^{2^{m-1}} |T_{m,p}(e_{j_1}, ..., e_{j_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} < C_{\mathbb{R},m,p}^{\text{mult}} 2^{m-1}.$$

and thus

$$C_{\mathbb{R},m,p}^{\text{mult}} > \frac{(4^{m-1})^{\frac{mp+p-2m}{2mp}}}{2^{m-1}} = 2^{\frac{mp+2m-2m^2-p}{mp}} = 1,$$

as required. □

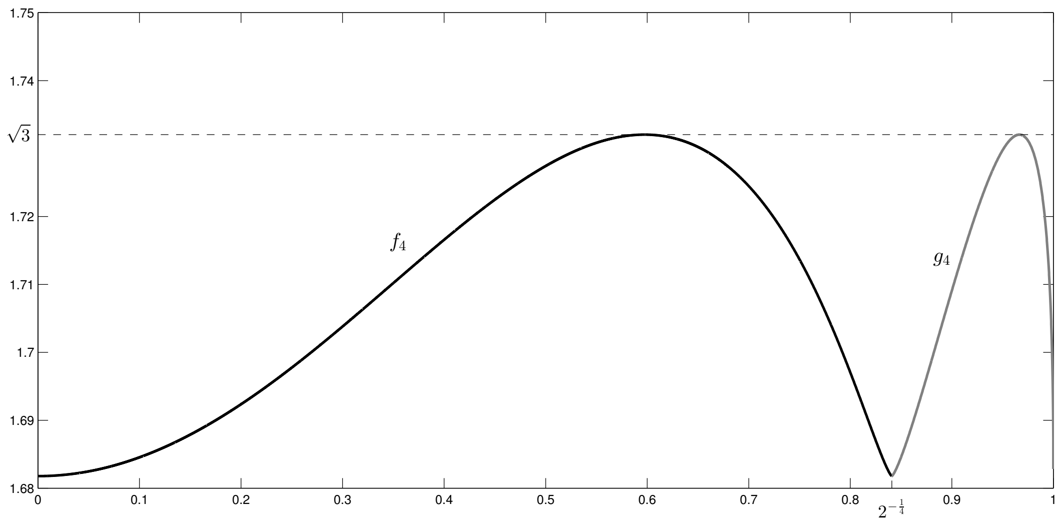

Figure 1.1: Graphs of the functions f_4 and g_4, respectively.

1.2 On the constants of the generalized Bohnenblust–Hille and Hardy–Littlewood inequalities

In this section, among other results, we show that for $2m^3 - 4m^2 + 2m < p \leq \infty$ the constants $C^{\text{mult}}_{\mathbb{K},m,p}$ have the exactly same upper bounds that we have now for the Bohnenblust–Hille constants (1.2). More precisely we will show that if $p > 2m^3 - 4m^2 + 2m$, then

$$\begin{aligned} C^{\text{mult}}_{\mathbb{C},m,p} &\leq \prod_{j=2}^{m} \Gamma\left(2 - \tfrac{1}{j}\right)^{\frac{j}{2-2j}}, & & \\ C^{\text{mult}}_{\mathbb{R},m,p} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}, & &\text{for } 2 \leq m \leq 13, \\ C^{\text{mult}}_{\mathbb{R},m,p} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, & &\text{for } m \geq 14. \end{aligned} \tag{1.17}$$

It is not difficult to verify that (1.17) in fact improves (1.7). However the most interesting point is that in (1.17), contrary to (1.7), we have no dependence on p in the formulas and, besides, these new estimates are precisely the best known estimates for the constants of the Bohnenblust–Hille inequality (see (1.2)).

To prove these new estimates we also improve the best known estimates for the generalized Bohnenblust–Hille inequality (see Section 1.2.1). The importance of this result (generalized Bohnenblust–Hille inequality) trancends the intrinsic mathematical novelty since, as it was recently shown (see [32]), this new approach is fundamental to improve the estimates of the constants of the classical Bohnenblust–Hille inequality. In Section 1.2.2 we use these estimates to prove new estimates for the constants of the Hardy–Littlewood inequality. In the final section (Section 1.2.3) the estimates of the previous sections (Sections 1.2.1 and 1.2.2) are used to obtain new constants for the generalized Hardy–Littlewood inequality.

1.2.1 Estimates for the constants of the generalized Bohnenblust–Hille inequality

The best known estimates for the constants $B^{\text{mult}}_{\mathbb{K},m,(q_1,\dots,q_m)}$ are presented in [5]. More precisely, for complex scalars and $1 \leq q_1 \leq \cdots \leq q_m \leq 2$, from [5] we know that, for $\mathbf{q} = (q_1, ..., q_m)$,

$$B^{\text{mult}}_{\mathbb{C},m,\mathbf{q}} \leq \left(\prod_{j=1}^{m} \Gamma\left(2 - \tfrac{1}{j}\right)^{\frac{j}{2-2j}} \right)^{2m\left(\frac{1}{q_m} - \frac{1}{2}\right)}$$
$$\times \left(\prod_{k=1}^{m-1} \left(\Gamma\left(\tfrac{3k+1}{2k+2}\right)^{\left(\frac{-k-1}{2k}\right)(m-k)} \prod_{j=1}^{k} \Gamma\left(2 - \tfrac{1}{j}\right)^{\frac{j}{2-2j}} \right)^{2k\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right)} \right). \tag{1.18}$$

In the present section we improve the above estimates for a certain family of $(q_1, ..., q_m)$. More precisely, if $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1)$, then

$$B^{\text{mult}}_{\mathbb{C},m,(q_1,\dots,q_m)} \leq \prod_{j=2}^{m} \Gamma\left(2 - \tfrac{1}{j}\right)^{\frac{j}{2-2j}}.$$

A similar result holds for real scalars. These results have a crucial importance in the next sections.

Lemma 1.13. *Let $m \geq 2$ and $i \in \{1, ..., m\}$. If $q_i \in [(2m-2)/m, 2]$ and $q = 2(m-1)q_i/((m+1)q_i - 2)$, then*

$$B^{\text{mult}}_{\mathbb{K},m,(q,\dots,q,q_i,q,\dots,q)} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}},$$

with q_i in the i-th position.

Proof. There is no loss of generality in supposing that $i = 1$. By [32, Proposition 3.1] we have, for each $k = 1, ..., m$,

$$\begin{aligned} \left(\sum_{\widehat{j_k}=1}^{n} \left(\sum_{j_k=1}^{n} \left| T\left(e_{j_1}, ..., e_{j_m}\right)\right|^2 \right)^{\frac{1}{2}\frac{2m-2}{m}} \right)^{\frac{m}{2m-2}} &\leq A^{-1}_{\mathbb{K},\frac{2m-2}{m}} B^{\text{mult}}_{\mathbb{K},m-1} \|T\| \\ &\leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}} \|T\| \end{aligned}$$

(see proof of Theorem 1.10 for details).

We define

$$\mathbf{q}_k = (\mathbf{q}_k(1), ..., \mathbf{q}_k(m)) = \left(\tfrac{2m-2}{m}, ..., \tfrac{2m-2}{m}, 2, \tfrac{2m-2}{m}, ..., \tfrac{2m-2}{m}\right),$$

where the 2 is in the k-th coordinate and take $\theta_1 = m - (2m-2)/q_1$ and $\theta_2 = \cdots = \theta_m = 2/q_1 - 1$. Recalling that $q_1 \geq (2m-2)/m$ we can see that $\theta_k \in [0,1]$ for all $k = 1, ..., m$. It can be easily checked that

$$\frac{\theta_1}{\mathbf{q}_1(1)} + \cdots + \frac{\theta_m}{\mathbf{q}_m(1)} = \frac{1}{q_1} \quad \text{and} \quad \frac{\theta_1}{\mathbf{q}_1(j)} + \cdots + \frac{\theta_m}{\mathbf{q}_m(j)} = \frac{1}{q} \text{ for } j = 2, ..., m.$$

Then a straightforward application of the Minkowski inequality (using that $(2m-2)/m < 2$) and of the generalized Hölder inequality ([33, 81]) completes the proof. □

Lemma 1.14. *Let $m \geq 2$ be a positive integer, $2m < p \leq \infty$ and $q_1, ..., q_m \in [p/(p-m), 2]$. If $|1/\mathbf{q}| = (mp + p - 2m)/2p$, then, for all $s \in (\max q_i, 2]$, the vector $\left(q_1^{-1}, ..., q_m^{-1}\right)$ belongs to the convex hull in $\mathbb{R}^m$ of $\{\sum_{k=1}^m a_{1k} e_k, ..., \sum_{k=1}^m a_{mk} e_k\}$, where $a_{jk} = s^{-1}$ if $k \neq j$ and $a_{jk} = \lambda_{m,s}^{-1}$ if $k = j$, and $\lambda_{m,s} = 2ps/(mps + ps + 2p - 2mp - 2ms)$.*

Proof. We want to prove that for $(q_1, ..., q_m) \in [p/(p-m), 2]^m$ and $s \in (\max q_i, 2]$ there are $0 < \theta_{j,s} < 1$, $j = 1, ..., m$, such that

$$\sum_{j=1}^m \theta_{j,s} = 1,$$

$$\frac{1}{q_1} = \frac{\theta_{1,s}}{\lambda_{m,s}} + \frac{\theta_{2,s}}{s} + \cdots + \frac{\theta_{m,s}}{s},$$

$$\vdots$$

$$\frac{1}{q_m} = \frac{\theta_{1,s}}{s} + \cdots + \frac{\theta_{m-1,s}}{s} + \frac{\theta_{m,s}}{\lambda_{m,s}}.$$

Observe initially that from $|1/\mathbf{q}| = (mp + p - 2m)/2p$ we have $\max q_i \geq 2mp/(mp + p - 2m)$. Note also that for all $s \in [(2mp - 2p)/(mp - 2m), 2]$ we have

$$mps + ps + 2p - 2mp - 2ms > 0 \quad \text{and} \quad \frac{p}{p-m} \leq \lambda_{m,s} \leq 2. \tag{1.19}$$

Since $s > \max q_i \geq 2mp/(mp + p - 2m) > (2mp - 2p)/(mp - 2m)$ (the last inequality is strict because we are not considering the case $p = 2m$) it follows that $\lambda_{m,s}$ is well defined for all $s \in (\max q_i, 2]$. Furthermore, for all $s > 2mp/(mp + p - 2m)$ it is possible to prove that $\lambda_{m,s} < s$. In fact, $s > 2mp/(mp + p - 2m)$ implies $mps + ps - 2ms > 2mp$ and thus adding $2p$ in both sides of this inequality we can conclude that

$$\lambda_{m,s} = \frac{2ps}{mps+ps+2p-2mp-2ms} < \frac{2ps}{2p} = s. \tag{1.20}$$

For each $j = 1, ..., m$, consider $\theta_{j,s} = \lambda_{m,s}(s - q_j)/q_j(s - \lambda_{m,s})$. Since $\sum_{j=1}^m q_j^{-1} = (mp + p - 2m)/2p$ we conclude that

$$\sum_{j=1}^m \theta_{j,s} = \sum_{j=1}^m \frac{\lambda_{m,s}(s-q_j)}{q_j(s-\lambda_{m,s})} = \frac{\lambda_{m,s}}{s-\lambda_{m,s}} \left(s \sum_{j=1}^m \frac{1}{q_j} - m \right) = 1.$$

By hypothesis $s > \max q_i \geq q_j$ for all $j = 1, ..., m$, so it follows that $\theta_{j,s} > 0$ for all $j = 1, ..., m$ and thus $0 < \theta_{j,s} < \sum_{j=1}^m \theta_{j,s} = 1$.

Finally, note that

$$\frac{\theta_{j,s}}{\lambda_{m,s}} + \frac{1-\theta_{j,s}}{s} = \frac{\frac{\lambda_{m,s}\left(s-q_j\right)}{q_j(s-\lambda_{m,s})}}{\lambda_{m,s}} + \frac{1-\frac{\lambda_{m,s}\left(s-q_j\right)}{q_j(s-\lambda_{m,s})}}{s} = \frac{1}{q_j}.$$

Therefore

$$\frac{1}{q_1} = \frac{\theta_{1,s}}{\lambda_{m,s}} + \frac{\theta_{2,s}}{s} + \cdots + \frac{\theta_{m,s}}{s},$$
$$\vdots$$
$$\frac{1}{q_m} = \frac{\theta_{1,s}}{s} + \cdots + \frac{\theta_{m-1,s}}{s} + \frac{\theta_{m,s}}{\lambda_{m,s}}$$

and the proof is done. □

Combining the two previous lemmas we have:

Theorem 1.15. *Let $m \geq 2$ be a positive integer and $\mathbf{q} = (q_1, ..., q_m) \in [1,2]^m$. If $|1/\mathbf{q}| = (m+1)/2$, and $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1)$, then*

$$B^{\text{mult}}_{\mathbb{K},m,(q_1,\ldots,q_m)} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}.$$

Proof. Let $s = (2m^2 - 4m + 2)/(m^2 - m - 1)$ and $q = (2m-2)/m$. Since $(m-1)/s + 1/q = (m+1)/2$, from Lemma 1.13 the exponents $(t_1, ..., t_m) = (s, ..., s, q), ..., (q, s, ..., s)$ are associated with

$$B^{\text{mult}}_{\mathbb{K},m,(t_1,\ldots,t_m)} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}.$$

By hypothesis $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1) = s$, then, from the previous lemma (Lemma 1.14) with $p = \infty$, the exponent $(q_1, ..., q_m)$ is the interpolation of $(2s/(ms + s + 2 - 2m), s, ..., s), ..., (s, ..., s, 2s/(ms + s + 2 - 2m))$.

Note that $2s/(ms+s+2-2m) = (2m-2)/m$ and from Lemma 1.13 they are associated with the constants

$$B^{\text{mult}}_{\mathbb{K},m,(q_1,\ldots,q_m)} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}},$$

which completes the proof. □

Corollary 1.16. *Let $m \geq 2$ be a positive integer and $\mathbf{q} = (q_1, ..., q_m) \in [1,2]^m$. If $|1/\mathbf{q}| = (m+1)/2$, and $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1)$, then*

$$\begin{aligned}
B^{\text{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)} &\leq \prod_{j=2}^{m} \Gamma\left(2 - \tfrac{1}{j}\right)^{\frac{j}{2-2j}}, \\
B^{\text{mult}}_{\mathbb{R},m,(q_1,\ldots,q_m)} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}, && \text{for } 2 \leq m \leq 13, \\
B^{\text{mult}}_{\mathbb{R},m,(q_1,\ldots,q_m)} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, && \text{for } m \geq 14.
\end{aligned}$$

The following table compares the estimate obtained for $B^{\text{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)}$ in [5] (see (1.18)) and the new and better estimate obtained in Theorem 1.15.

$m \geq 2$	$1 \leq q_1 \leq \cdots \leq q_m \leq 2$; $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2}$ and $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$	$B^{\text{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)}$ Estimates of [5] (see (1.18))	Estimates of Theorem 1.15
5	$q_1 = \cdots = q_4 = \frac{668}{401}, q_5 = 1.67$	< 1.34783	< 1.34745
10	$q_1 = \cdots = q_9 = \frac{327618}{180211}, q_{10} = 1.8201$	< 1.55231	< 1.55151
20	$q_1 = \cdots = q_{19} = \frac{14478}{7601}, q_{20} = 1.905$	< 1.79162	< 1.79137
50	$q_1 = \cdots = q_{49} = \frac{240198}{122501}, q_{50} = 1.9608$	< 2.170671	< 2.170620
100	$q_1 = \cdots = q_{99} = \frac{1960398}{990001}, q_{100} = 1.9802$	< 2.511775	< 2.511760
1000	$q_1 = \cdots = q_{999} = \frac{665334666000666}{333000000333667}$, $q_{1000} = 1.998002000002$	< 4.08463471	< 4.08463446

1.2.2 Application 1: Improving the constants of the Hardy–Littlewood inequality

The main result of this section shows that for $2m^3 - 4m^2 + 2m < p \leq \infty$ the optimal constants satisfying the Hardy–Littlewood inequality for m-linear forms in ℓ_p spaces are dominated by the best known estimates for the constants of the m-linear Bohnenblust–Hille inequality; this result improves (for $2m^3 - 4m^2 + 2m < p \leq \infty$) the best estimates we have thus far (see (1.7)), and may suggest a subtler connection between the optimal constants of those inequalities.

Theorem 1.17. *Let $m \geq 2$ be a positive integer and $2m^3 - 4m^2 + 2m < p \leq \infty$. Then, for all continuous m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n, we have*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq \left(\prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}} \right) \|T\| . \tag{1.21}$$

Proof. The case $p = \infty$ in (1.21) is precisely the Bohnenblust–Hille inequality, so we just need to consider $2m^3 - 4m^2 + 2m < p < \infty$. Let $(2m-2)/m \leq s \leq 2$ and $\lambda_{0,s} = 2s/(ms + s + 2 - 2m)$. Note that

$$ms + s + 2 - 2m > 0 \qquad \text{and} \qquad 1 \leq \lambda_{0,s} \leq 2. \tag{1.22}$$

Since $(m-1)/s + 1/\lambda_{0,s} = (m+1)/2$, from the generalized Bohnenblust–Hille inequality (see [6]) we know that there is a constant $C_m \geq 1$ such that for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ we have

$$\left(\sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_{0,s}} \right)^{\frac{1}{\lambda_{0,s}}} \leq C_m \|T\| , \tag{1.23}$$

for all $i = 1, ..., m$.

If we choose $s = 2mp/(mp + p - 2m)$ (note that this s belongs to the interval $[(2m-2)/m, 2]$), we have $s > 2m/(m+1)$ (this inequality is strict because we are considering the case $p < \infty$) and thus $\lambda_{0,s} < s$. In fact, $s > 2m/(m+1)$ implies $ms + s > 2m$ and thus adding 2 in both sides of this inequality we can conclude that

$$\lambda_{0,s} = \tfrac{2s}{(ms+s+2-2m)} < \tfrac{2s}{2} = s. \tag{1.24}$$

Since $p > 2m^3 - 4m^2 + 2m$ we conclude that $s < (2m^2 - 4m + 2)/(m^2 - m - 1)$. Thus, from Theorem 1.15, the optimal constant associated with the multiple exponent $(\lambda_{0,s}, s, s, ..., s)$ is less than or equal to

$$C_m = \prod_{j=2}^{m} A_{\mathbb{K}, \frac{2j-2}{j}}^{-1}.$$

More precisely, (1.23) is valid with C_m as above. Now the proof follows the same lines, *mutatis mutandis*, of the proof of Theorem 1.10 (see [17, Theorem 1.1]), which has its roots in the work of Praciano-Pereira [128]. □

It is simple to verify that these new estimates are better than the old ones. In fact, for complex scalars the inequality

$$\prod_{j=2}^{m} A_{\mathbb{C}, \frac{2j-2}{j}}^{-1} < \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^{m} A_{\mathbb{C}, \frac{2j-2}{j}}^{-1}\right)^{\frac{p-2m}{p}}$$

is a straightforward consequence of

$$\prod_{j=2}^{m} A_{\mathbb{C}, \frac{2j-2}{j}}^{-1} < \left(\frac{2}{\sqrt{\pi}}\right)^{m-1},$$

which is true for $m \geq 3$. The case of real scalars is analogous.

The following table compares the estimates for $C_{\mathbb{C},m,p}^{\text{mult}}$ obtained in Theorem 1.10 (see [17]) and the estimate obtained in Theorem 1.17 for $2m^3 - 4m^2 + 2m < p \leq \infty$.

$m \geq 2$	$2m^3 - 4m^2 + 2m < p \leq \infty$	$C_{\mathbb{C},m,p}^{\text{mult}}$ Estimates of Theorem 1.10	$C_{\mathbb{C},m,p}^{\text{mult}}$ Estimates of Theorem 1.17
4	$p = 73$	< 1.30433	< 1.28890
	$p = 500$	< 1.29114	
	$p = 1000$	< 1.29002	
10	$p = 1621$	< 1.56396	< 1.55151
	$p = 3000$	< 1.55822	
	$p = 5000$	< 1.55553	
50	$p = 240101$	< 2.175275	< 2.170620
	$p = 500000$	< 2.172854	
	$p = 1000000$	< 2.171737	
100	$p = 1960201$	< 2.514590	< 2.511760
	$p = 5000000$	< 2.512869	
	$p = 20000000$	< 2.512037	
1000	$p = 1996002001$	< 4.08512258	< 4.08463446
	$p = 6000000000$	< 4.08479684	
	$p = 50000000000$	< 4.08465395	

Recall that from the previous section that for $p \geq m^2$ the constants of the Hardy–Littlewood inequality have a sublinear growth. The graph 1.2 illustrates what we have

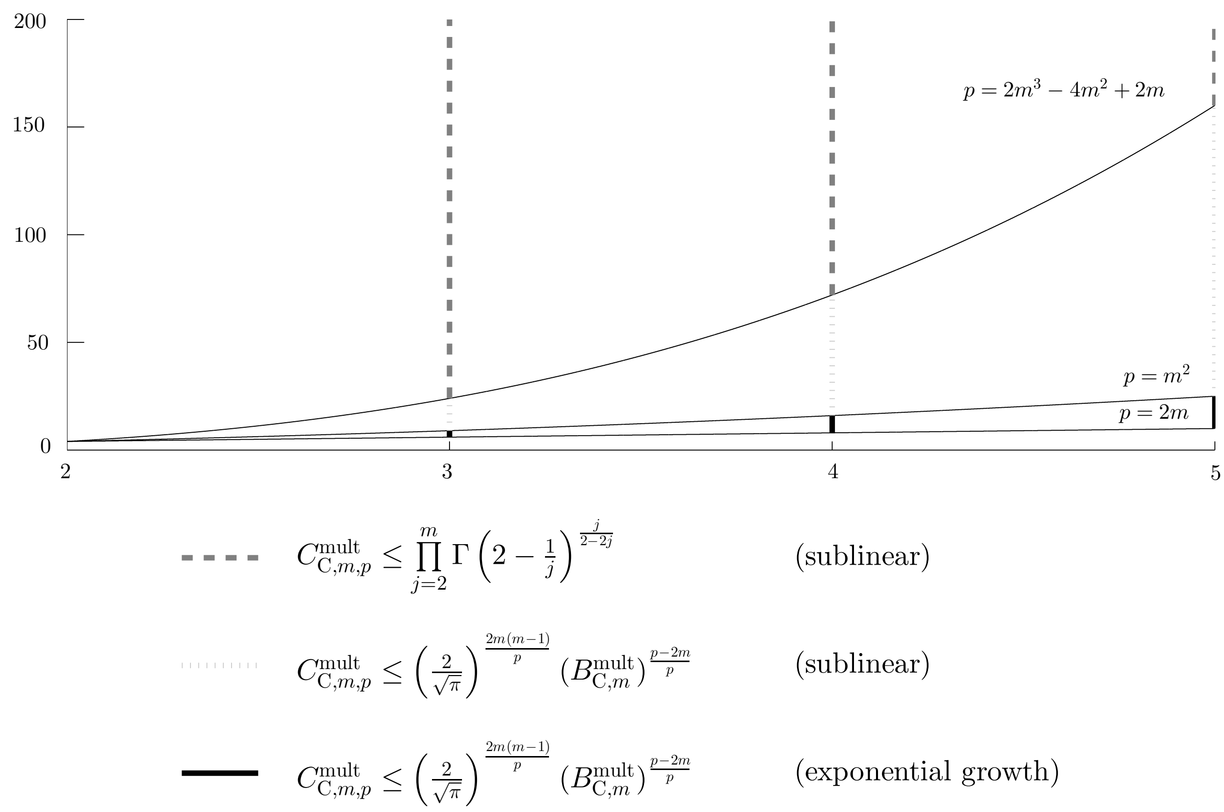

Figure 1.2: Behavior of $C^{\mathrm{mult}}_{\mathbb{C},m,p}$.

thus far, combined with Theorem 1.17.

1.2.3 Application 2: Estimates for the constants of the generalized Hardy–Littlewood inequality

The best known estimates for the constants $C^{\mathrm{mult}}_{\mathbb{K},m,p,\mathbf{q}}$ are $\left(\sqrt{2}\right)^{m-1}$ for real scalars and $(2/\sqrt{\pi})^{m-1}$ for complex scalars (see [6]). In Theorem 1.10 (see [17, Theorem 1.1]) and the previous section (see (1.17)) better constants were obtained when $q_1 = ... = q_m = 2mp/(mp + p - 2m)$. Now we extend the results from [17] to general multiple exponents. Of course the interesting case is the borderline case, i.e., $1/q_1 + \cdots + 1/q_m = (mp + p - 2m)/2p$. The proof is slightly more elaborated than the proof of Theorem 1.17 and also a bit more technical than the proof of the main result in [17].

Theorem 1.18. *Let $m \geq 2$ be a positive integer, let $2m < p \leq \infty$ and let $\mathbf{q} := (q_1, ..., q_m) \in [p/(p-m), 2]^m$ be such that $|1/\mathbf{q}| = (mp + p - 2m)/2p$. If $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1)$, then*

$$C^{\mathrm{mult}}_{\mathbb{K},m,p,\mathbf{q}} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}.$$

Proof. The arguments follow the general lines of [17], but are slightly different and due to the technicalities we present the details for the sake of clarity. Define for $s \in (\max q_i, (2m^2 - 4m + 2)/(m^2 - m - 1))$,

$$\lambda_{m,s} = \frac{2ps}{mps + ps + 2p - 2mp - 2ms}. \tag{1.25}$$

Observe that $\lambda_{m,s}$ is well defined for all $s \in (\max q_i, (2m^2 - 4m + 2)/(m^2 - m - 1))$.

In fact, as we have in (1.19) note that for all $s \in [(2mp-2p)/(mp-2m), 2]$ we have $mps + ps + 2p - 2mp - 2ms > 0$ and $p/(p-m) \leq \lambda_{m,s} \leq 2$. Since $s > \max q_i \geq 2mp/(mp+p-2m) > (2mp-2p)/(mp-2m)$ (the last inequality is strict because we are not considering the case $p = 2m$) and $(2m^2-4m+2)/(m^2-m-1) \leq 2$ it follows that $\lambda_{m,s}$ is well defined for all s.

Let us prove

$$C^{\text{mult}}_{\mathbb{K},m,p,(\lambda_{m,s},s,\ldots,s)} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}} \tag{1.26}$$

for all $s \in (\max q_i, (2m^2-4m+2)/(m^2-m-1))$. In fact, for these values of s, consider $\lambda_{0,s} = 2s/(ms+s+2-2m)$. Observe that if $p=\infty$ then $\lambda_{m,s} = \lambda_{0,s}$. Since $(m-1)/s + 1/\lambda_{0,s} = (m+1)2$, from the generalized Bohnenblust–Hille inequality (see [6]) we know that there is a constant $C_m \geq 1$ such that for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ we have, for all $i = 1, ..., m$,

$$\left(\sum_{j_i=1}^{n} \left(\sum_{\widehat{j}_i=1}^{n} \left| T\left(e_{j_1}, ..., e_{j_m}\right)\right|^s \right)^{\frac{1}{s}\lambda_{0,s}} \right)^{\frac{1}{\lambda_{0,s}}} \leq C_m \left\| T \right\|. \tag{1.27}$$

Since $2m/(m+1) \leq 2mp/(mp+p-2m) \leq \max q_i < s < (2m^2-4m+2)/(m^2-m-1)$ it is not to difficult to prove that (see (1.24)) $\lambda_{0,s} < s < (2m^2-4m+2)/(m^2-m-1)$. Since $s < (2m^2-4m+2)/(m^2-m-1)$ we conclude by Theorem 1.15 that the optimal constant associated with the multiple exponent $(\lambda_{0,s}, s, s, ..., s)$ is less than or equal to

$$\prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}.$$

More precisely, (1.27) is valid with C_m as above. Since $\lambda_{m,s} = \lambda_{0,s}$ if $p=\infty$, we have (1.26) for all $s \in (\max q_i, (2m^2-4m+2)/(m^2-m-1))$ and the proof is done for this case. For $2m < p < \infty$, let $\lambda_{j,s} = \lambda_{0,s}p/(p-\lambda_{0,s}j)$ for all $j = 1, ..., m$. Note that $\lambda_{m,s} = 2ps/(mps+ps+2p-2mp-2ms)$ and this notation is compatible with (1.25). Since $s > \max q_i \geq 2mp/(mp+p-2m) \geq 2mp/(mp+p-2j)$ for all $j = 1, ..., m$ we also observe that

$$\lambda_{j,s} < s \tag{1.28}$$

for all $j = 1, ..., m$. Moreover, observe that $(p/\lambda_{j,s})^* = \lambda_{j+1,s}/\lambda_{j,s}$ for all $j = 0, ..., m-1$. From now on, following the same steps in the proof of the Theorem 1.10, if we suppose, for $1 \leq k \leq m$, that

$$\left(\sum_{j_i=1}^{n} \left(\sum_{\widehat{j}_i=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_{k-1,s}} \right)^{\frac{1}{\lambda_{k-1,s}}} \leq C_m \|T\|$$

is true for all continuous m-linear forms $T : \ell_p^n \times \overset{k-1 \text{ times}}{\cdots} \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, ..., m$, it is possible to prove that

$$\left(\sum_{j_i=1}^{n} \left(\sum_{\widehat{j}_i=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}\lambda_{k,s}} \right)^{\frac{1}{\lambda_{k,s}}} \leq C_m \|T\|,$$

for all continuous m-linear forms $T : \ell_p^n \times \overset{k \text{ times}}{\cdots} \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, ..., m$. This allows to conclude (1.26) for all $s \in (\max q_i, (2m^2 - 4m + 2)/(m^2 - m - 1))$.

Now the proof uses a different argument from those from Theorem 1.10, since a new interpolation procedure is needed. From (1.28) we know that $\lambda_{m,s} < s$ for all $s \in (\max q_i, (2m^2 - 4m + 2)/(m^2 - m - 1))$. Therefore, using the Minkowski inequality as in [6], it is possible to obtain from (1.26) that, for all fixed $i \in \{1, ..., m\}$,

$$C^{\text{mult}}_{\mathbb{K},m,p,(s,...,s,\lambda_{m,s},s,...,s)} \leq \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}, \tag{1.29}$$

for all $s \in (\max q_i, (2m^2 - 4m + 2)/(m^2 - m - 1))$ with $\lambda_{m,s}$ in the i-th position. Finally, from Lemma 1.14 we know that $\left(q_1^{-1}, ..., q_m^{-1}\right)$ belongs to the convex hull of

$$\{(\lambda^{-1}_{m,s}, s^{-1}, \ldots, s^{-1}), \ldots, (s^{-1}, \ldots, s^{-1}, \lambda^{-1}_{m,s})\},$$

for all $s \in (\max q_i, (2m^2 - 4m + 2)/(m^2 - m - 1))$ with certain constants $\theta_{1,s}, ..., \theta_{m,s}$ and thus, from the interpolative technique from [6], we get

$$\begin{aligned} C^{\text{mult}}_{\mathbb{K},m,p,\mathbf{q}} &\leq \left(C^{\text{mult}}_{\mathbb{K},m,p,(\lambda_{m,s},s,...,s)}\right)^{\theta_{1,s}} \cdots \left(C^{\text{mult}}_{\mathbb{K},m,p,(s,...,s,\lambda_{m,s})}\right)^{\theta_{m,s}} \\ &\leq \left(\prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}\right)^{\theta_{1,s}+\cdots+\theta_{m,s}} = \prod_{j=2}^{m} A^{-1}_{\mathbb{K},\frac{2j-2}{j}}. \end{aligned}$$

□

Corollary 1.19. *Let $m \geq 2$ be a positive integer and $2m < p \leq \infty$. Let also $\mathbf{q} := (q_1, ..., q_m) \in [p/(p-m), 2]^m$ be such that $|1/\mathbf{q}| = (mp + p - 2m)/2p$. If $\max q_i < (2m^2 - 4m + 2)/(m^2 - m - 1)$, then*

$$\begin{aligned} C^{\text{mult}}_{\mathbb{C},m,p,\mathbf{q}} &\leq \prod_{j=2}^{m} \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ C^{\text{mult}}_{\mathbb{R},m,p,\mathbf{q}} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}} && \text{if } 2 \leq m \leq 13, \\ C^{\text{mult}}_{\mathbb{R},m,p,\mathbf{q}} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}} && \text{if } m \geq 14. \end{aligned}$$

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Chapter 2

Optimal Hardy–Littlewood type inequalities for m-linear forms on ℓ_p spaces with $1 \leq p \leq m$

In [47, Corollary 5.20] it is shown that in ℓ_2^n the Hardy–Littlewood multilinear inequalities has an extra power of n in its right hand side. Therefore, a natural question is:

- For $1 \leq p \leq m$, what power of n (depending on r, m, p) will appear in the right hand side of the Hardy–Littlewood multilinear inequalities if we replace the optimal exponents $2mp/(mp + p - 2m)$ and $p/(p - m)$ by a smaller value r?

This case ($1 \leq p \leq m$) was only explored for the case of Hilbert spaces ($p = 2$, see [47, Corollary 5.20] and [61]) and the case $p = \infty$ was explored in [57]. The results of this chapter answer the remaining cases of the above question (see Theorem 2.1) and extends previous results to $1 \leq p \leq m$ (c.f. [47, Corollary 5.20]).

The following theorem is the main result of this chapter and it first item recovers [47, Corollary 5.20(i)] (just make $p = 2$) and [57, Proposition 5.1].

Theorem 2.1. *Let $m \geq 2$ be a positive integer.*

(a) If $(r,p) \in ([1,2] \times [2,2m)) \cup ([1,\infty) \times [2m,\infty])$, then there is a constant $H_{\mathbb{K},m,p,r}^{\text{mult}} > 0$ (not depending on n) such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^r \right)^{\frac{1}{r}} \leq H_{\mathbb{K},m,p,r}^{\text{mult}} n^{\max\left\{ \frac{2mr+2mp-mpr-pr}{2pr}, 0 \right\}} \|T\|$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n. Moreover, the exponent $\max\{(2mr + 2mp - mpr - pr)/2pr, 0\}$ is optimal.

(b) If $(r,p) \in [2,\infty) \times (m, 2m]$, then there is a constant $H_{\mathbb{K},m,r,p}^{\text{mult}} > 0$ (not depending on n) such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^r \right)^{\frac{1}{r}} \leq H_{\mathbb{K},m,r,p}^{\text{mult}} n^{\max\left\{ \frac{p+mr-rp}{pr}, 0 \right\}} \|T\|,$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n. Moreover, the exponent $\max\{(p + mr - rp)/pr, 0\}$ *is optimal.*

Proof. Let $1 \leq q \leq r \leq \infty$ and E be a Banach space. We say that an m-linear form $S : E \times \cdots \times E \to \mathbb{K}$ is multiple $(r; q)$-summing if there is a constant $C > 0$ such that

$$\left\| (S(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}))_{j_1,...,j_m=1}^n \right\|_{\ell_r} \leq C \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^n |\varphi(x_j^{(1)})|^q \right)^{\frac{1}{q}} \cdots \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^n |\varphi(x_j^{(m)})|^q \right)^{\frac{1}{q}}$$

for all positive integers n.

(a) Let us consider first $(r, p) \in [1, 2] \times [2, 2m)$. From now on $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ is an m-linear form. Since

$$\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^n |\varphi(e_j)| = n n^{-\frac{1}{p^*}} = n^{\frac{1}{p}}$$

and since T is multiple $(2m/(m + 1); 1)$-summing (we will see in the next chapter that from the Bohnenblust–Hille inequality it is possible to prove that all continuous m-linear forms are multiple $(2m/(m + 1); 1)$-summing with constant $B_{\mathbb{K},m}^{\text{mult}}$), we conclude that

$$\left(\sum_{j_1,...,j_m=1}^n |T(e_{j_1}, ..., e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \|T\| n^{\frac{m}{p}}. \tag{2.1}$$

Therefore, if $1 \leq r < 2m/(m + 1)$, using the Hölder inequality and (2.1), we have

$$\begin{aligned}
&\left(\sum_{j_1,...,j_m=1}^n |T(e_{j_1}, ..., e_{j_m})|^r \right)^{\frac{1}{r}} \\
&\leq \left(\sum_{j_1,...,j_m=1}^n |T(e_{j_1}, ..., e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \left(\sum_{j_1,...,j_m=1}^n |1|^{\frac{2mr}{2m-rm-r}} \right)^{\frac{2m-rm-r}{2mr}} \\
&= \left(\sum_{j_1,...,j_m=1}^n |T(e_{j_1}, ..., e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} (n^m)^{\frac{2m-rm-r}{2mr}} \\
&\leq B_{\mathbb{K},m}^{\text{mult}} \|T\| n^{\frac{m}{p}} n^{\frac{2m-rm-r}{2r}} \\
&= B_{\mathbb{K},m}^{\text{mult}} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \|T\| .
\end{aligned}$$

Now we consider the case $2m/(m+1) \leq r \leq 2$. From the proof of [16, Theorem 3.2(i)] we know that, for all $2m/(m + 1) \leq r \leq 2$ and all Banach spaces E, every continuous m-linear form $S : E \times \cdots \times E \to \mathbb{K}$ is multiple $(r; 2mr/(mr + 2m - r))$-summing with constant $C_{\mathbb{K},m,2mr/(r+mr-2m)}^{\text{mult}}$. Therefore

$$\left(\sum_{j_1,...,j_m=1}^n |T(e_{j_1}, ..., e_{j_m})|^r \right)^{\frac{1}{r}}$$

$$\leq C^{\text{mult}}_{\mathbb{K},m,\frac{2mr}{r+mr-2m}} \left\|T\right\| \left[\left(\sup_{\varphi\in B_{(\ell_p^n)^*}} \sum_{j=1}^{n} \left|\varphi(e_j)\right|^{\frac{2mr}{mr+2m-r}} \right)^{\frac{mr+2m-r}{2mr}} \right]^{m}. \tag{2.2}$$

Since $1 \leq 2mr/(mr+2m-r) \leq 2m/(2m-1) = (2m)^{*} < p^{*}$, we have

$$\begin{aligned} \left(\sup_{\varphi\in B_{(\ell_p^n)^*}} \sum_{j=1}^{n} \left|\varphi\left(e_j\right)\right|^{\frac{2mr}{mr-r+2m}} \right)^{\frac{mr-r+2m}{2mr}} &= \left(n(n^{-\frac{1}{p^*}})^{\frac{2mr}{mr-r+2m}}\right)^{\frac{mr-r+2m}{2mr}} \\ &= n^{\frac{2mr+2mp-mpr-pr}{2mpr}} \end{aligned} \tag{2.3}$$

and finally, from (2.2) and (2.3), we obtain

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{r} \right)^{\frac{1}{r}} \leq C^{\text{mult}}_{\mathbb{K},m,\frac{2mr}{r+mr-2m}} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \left\|T\right\|.$$

Now we prove the optimality of the exponents. Suppose that the theorem is valid for an exponent s, i.e.,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{r} \right)^{\frac{1}{r}} \leq H^{\text{mult}}_{\mathbb{K},m,p,r} n^{s} \left\|T\right\|.$$

Since $p \geq 2$, from the generalized Kahane–Salem–Zygmund inequality (2) we have

$$n^{\frac{m}{r}} \leq C_m H^{\text{mult}}_{\mathbb{K},m,p,r} n^{s} n^{\frac{m+1}{2}-\frac{m}{p}}$$

and thus, making $n \to \infty$, we obtain $s \geq (2mr+2mp-mpr-pr)/2pr$.

The case $(r,p) \in [1, 2mp/(mp+p-2m)] \times [2m,\infty]$ is analogous. In fact, from the Hardy–Littlewood/Praciano-Pereira inequality we know that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C^{\text{mult}}_{\mathbb{K},m,p} \left\|T\right\|. \tag{2.4}$$

Therefore, from Hölder's inequality and (2.4), we have

$$\begin{aligned} &\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{r} \right)^{\frac{1}{r}} \\ &\leq \left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ &\qquad \times \left(\sum_{j_1,\ldots,j_m=1}^{n} \left|1\right|^{\frac{2mpr}{2mp+2mr-mpr-pr}} \right)^{\frac{2mp+2mr-mpr-pr}{2mpr}} \\ &\leq C^{\text{mult}}_{\mathbb{K},m,p} \left\|T\right\| (n^{m})^{\frac{2mp+2mr-mpr-pr}{2mpr}} \\ &= C^{\text{mult}}_{\mathbb{K},m,p} n^{\frac{2mp+2mr-mpr-pr}{2pr}} \left\|T\right\|. \end{aligned} \tag{2.5}$$

Since $p \geq 2m$, the optimality of the exponent is obtained *ipsis litteris* as in the previous case.

If $(r,p) \in (2mp/(mp+p-2m), \infty) \times [2m, \infty]$ we have $(2mr + 2mp - mpr - pr)/2pr < 0$ and

$$\begin{aligned}\left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1}, \dots, e_{j_m})|^r\right)^{\frac{1}{r}} &\leq \left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \\ &\leq C^{\text{mult}}_{\mathbb{K},m,p} \|T\| \\ &= C^{\text{mult}}_{\mathbb{K},m,p} \|T\| \, n^{\max\left\{\frac{2mr+2mp-mpr-pr}{2pr}, 0\right\}}.\end{aligned}$$

In this case the optimality of the exponent $\max\{(2mr + 2mp - mpr - pr)/2pr, 0\}$ is immediate, since one can easily verify that no negative exponent of n is possible.

(b) Let us first consider $(r,p) \in [2, p/(p-m)] \times (m, 2m]$. Define $q = mr/(r-1)$ and note that $q \leq 2m$ and $r = q/(q-m)$. Since $q/(q-m) = r \leq p/(p-m)$ we have $p \leq q$. Then $m < p \leq q \leq 2m$. Note that $q^* = mr/(mr+1-r)$. Since $m < q \leq 2m$, by the Hardy-Littlewood/Dimant-Sevilla-Peris inequality and using [73, Section 5] we know that every continuous m-linear form on any Banach space E is multiple $(q/(q-m); q^*)$-summing with constant $D^{\text{mult}}_{\mathbb{K},m,q}$, i.e., multiple $(r; mr/(mr+1-r))$-summing with constant $D^{\text{mult}}_{\mathbb{K},m,mr/(r-1)}$. So for $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ we have (since $q^* \leq p^*$),

$$\begin{aligned}&\left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1}, \dots, e_{j_m})|^r\right)^{\frac{1}{r}} \\ &\leq D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}} \|T\| \left[\left(\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^{n} |\varphi(e_j)|^{\frac{mr}{mr+1-r}}\right)^{\frac{mr+1-r}{mr}}\right]^m \\ &= D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}} \|T\| \left[\left(n (n^{-\frac{1}{p^*}})^{\frac{mr}{mr+1-r}}\right)^{\frac{mr+1-r}{mr}}\right]^m \\ &= D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}} \|T\| \, n^{\frac{p+mr-rp}{pr}}.\end{aligned}$$

Above, if we had tried, via Hölder's inequality, to use an argument similar to (2.5) we would obtain worse exponents.

Now we prove the optimality following the lines of [73]. Defining $R : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ by $R(x^{(1)}, \dots, x^{(m)}) = \sum_{j=1}^{n} x_j^{(1)} \cdots x_j^{(1)}$, from Hölder's inequality we can easily verify that $\|R\| \leq n^{1-\frac{m}{p}}$. So if the theorem holds for n^s, plugging the m-linear form R into the inequality we have

$$n^{\frac{1}{r}} \leq H^{\text{mult}}_{\mathbb{K},m,p,r} n^s n^{1-\frac{m}{p}}$$

and thus, by making $n \to \infty$, we obtain $s \geq (p + mr - rp)/pr$.

If $(r,p) \in (p/(p-m), \infty) \times (m, 2m]$ we have $(p + mr - rp)/pr < 0$ and

$$\begin{aligned}\left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1}, \dots, e_{j_m})|^r\right)^{\frac{1}{r}} &\leq \left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \\ &\leq D^{\text{mult}}_{\mathbb{K},m,p} \|T\|\end{aligned}$$

$$= D^{\mathrm{mult}}_{\mathbb{K},m,p}\,\|T\|\,n^{\max\left\{\frac{p+mr-rp}{pr},0\right\}}.$$

In this case the optimality of the exponent $\max\{(p+mr-rp)/pr,0\}$ is immediate, since one can easily verify that no negative exponent of n is possible. $\square$

Remark 2.2. Observing the proof of Theorem 2.1 we conclude that the optimal constant $H^{\mathrm{mult}}_{\mathbb{K},m,p,r}$ satisfies:

$$H^{\mathrm{mult}}_{\mathbb{K},m,p,r} \le \begin{cases} B^{\mathrm{mult}}_{\mathbb{K},m} & \text{if } (r,p)\in\left[1,\frac{2m}{m+1}\right]\times[2,2m),\\ C^{\mathrm{mult}}_{\mathbb{K},m,\frac{2mr}{r+mr-2m}} & \text{if } (r,p)\in\left[\frac{2m}{m+1},2\right]\times[2,2m),\\ C^{\mathrm{mult}}_{\mathbb{K},m,p} & \text{if } (r,p)\in[1,\infty)\times[2m,\infty],\\ D^{\mathrm{mult}}_{\mathbb{K},m,\frac{mr}{r-1}} & \text{if } (r,p)\in\left[2,\frac{p}{p-m}\right]\times(m,2m],\\ D^{\mathrm{mult}}_{\mathbb{K},m,p} & \text{if } (r,p)\in\left(\frac{p}{p-m},\infty\right)\times(m,2m]. \end{cases}$$

Using results of the previous chapters, we have the following estimates for the constants $H^{\mathrm{mult}}_{\mathbb{K},m,p,r}$:

$$H^{\mathrm{mult}}_{\mathbb{K},m,p,r} \le \begin{cases} \eta_{\mathbb{K},m} & \text{if } (r,p)\in\left[1,\frac{2m}{m+1}\right]\times[2,2m),\\ (\sigma_{\mathbb{K}})^{\frac{(m-1)(mr+r-2m)}{r}}(\eta_{\mathbb{K},m})^{\frac{2m-rm}{r}} & \text{if } (r,p)\in\left(\frac{2m}{m+1},2\right]\times[2,2m),\\ (\sigma_{\mathbb{K}})^{\frac{p-2m-mp+6m^2-6m^3+2m^4}{mp(m-2)}} & \\ \quad\times(\eta_{\mathbb{K},m})^{(m-1)\left(\frac{2m-p+mp-2m^2}{m^2p-2mp}\right)} & \text{if } (r,p)\in[1,\infty)\times[2m,2m^3-4m^2+2m],\\ \eta_{\mathbb{K},m} & \text{if } (r,p)\in[1,\infty)\times(2m^3-4m^2+2m,\infty],\\ (\sqrt{2})^{m-1} & \text{if } (r,p)\in[2,\infty)\times(m,2m], \end{cases}$$

where $\sigma_{\mathbb{R}}=\sqrt{2}$ and $\sigma_{\mathbb{C}}=2/\sqrt{\pi}$ and

$$\eta_{\mathbb{C},m} := \prod_{j=2}^{m}\Gamma\left(2-\tfrac{1}{j}\right)^{\frac{j}{2-2j}},$$

$$\eta_{\mathbb{R},m} := \prod_{j=2}^{m}2^{\frac{1}{2j-2}}, \qquad \text{for } m\le 13,$$

$$\eta_{\mathbb{R},m} := 2^{\frac{446381}{55440}-\frac{m}{2}}\prod_{j=14}^{m}\left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, \quad \text{for } m\ge 14.$$

Now we will obtain partial answers for the cases not covered by our main theorem, i.e., the cases $(r,p)\in[1,2]\times[1,2)$ and $(r,p)\in(2,\infty)\times[1,m]$.

Proposition 2.3. *Let $m\ge 2$ be a positive integer.*

(a) If $(r,p)\in[1,2]\times[1,2)$, then there is a constant $H^{\mathrm{mult}}_{\mathbb{K},m,p,r}>0$ such that

$$\left(\sum_{j_1,\dots,j_m=1}^{n}\left|T(e_{j_1},\dots,e_{j_m})\right|^r\right)^{\frac{1}{r}} \le H^{\mathrm{mult}}_{\mathbb{K},m,p,r}\,n^{\frac{2mr+2mp-mpr-pr}{2pr}}\,\|T\|, \tag{2.6}$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n. Moreover the optimal exponent of n is not smaller than $(2m-r)/2r$.

(b) If $(r,p) \in (2,\infty) \times [1,m]$, then there is a constant $H_{\mathbb{K},m,p,r}^{\text{mult}} > 0$ such that

$$\left(\sum_{j_1,.,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^r\right)^{\frac{1}{r}} \leq \begin{cases} H_{\mathbb{K},m,p,r}^{\text{mult}} n^{\frac{2m-p+\epsilon}{pr}} \|T\| & \text{if } p>2, \\ H_{\mathbb{K},m,p,r}^{\text{mult}} n^{\frac{2m-p}{pr}} \|T\| & \text{if } p=2, \end{cases}$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n and all $\epsilon > 0$. Moreover the optimal exponent of n is not smaller than $(2mr+2mp-mpr-pr)/2pr$ and not smaller than $(2m-r)/2r$ if $2 \leq p \leq m$. In the case $1 \leq p \leq 2$, the optimal exponent of n is not smaller than $(2m-r)/2r$.

Proof. (a) The proof of (2.6) is the same of the proof of Theorem 2.1*(a)*. The estimate for the bound of the optimal exponent also uses the generalized Kahane–Salem–Zygmund inequality (2). Since $p \leq 2$ we have

$$n^{\frac{m}{r}} \leq C_m H_{\mathbb{K},m,p,r}^{\text{mult}} n^s n^{\frac{1}{2}}$$

and thus, by making $n \to \infty$, $s \geq \frac{2m-r}{2r}$.

(b) Let $\delta = 0$ if $p = 2$ and $\delta > 0$ if $p > 2$. First note that every continuous m-linear form on ℓ_p spaces is obviously multiple $(\infty; p^* - \delta)$-summing and also multiple $(2; 2m/(2m-1))$-summing (this is a consequence of the Hardy–Littlewood inequality and [73, Section 5]). Using [47, Proposition 4.3] we conclude that every continuous m-linear form on ℓ_p spaces is multiple $(r; mpr/(2m+mpr-mr-p+\epsilon))$-summing for all $\epsilon > 0$ (and $\epsilon = 0$ if $p = 2$). Therefore, there exist $H_{\mathbb{K},m,p,r}^{\text{mult}} > 0$ such that

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^r\right)^{\frac{1}{r}} \\
&\leq H_{\mathbb{K},m,p,r}^{\text{mult}} \left[\left(n(n^{-\frac{1}{p^*}})^{\frac{mpr}{2m+mpr-mr-p+\epsilon}}\right)^{\frac{2m+mpr-mr-p+\epsilon}{mpr}}\right]^m \|T\| \\
&= H_{\mathbb{K},m,p,r}^{\text{mult}} n^{\frac{2m+mpr-mr-p+\epsilon}{pr}} (n^{\frac{1}{p}-1})^m \|T\| \\
&= H_{\mathbb{K},m,p,r}^{\text{mult}} n^{\frac{2m-p+\epsilon}{pr}} \|T\|.
\end{aligned}$$

The bounds for the optimal exponents are obtained via the generalized Kahane–Salem–Zygmund inequality (2) as in the previous cases. □

Remark 2.4. Item *(b)* of the Proposition 2.3 with $p = 2$ recovers [47, Corollary 5.20(ii)].

We believe that the remaining cases (those in which we do not have achieved the optimality of the exponents) are interesting for further investigation[1] trying to have a full panorama, covering all cases with optimal estimates.

[1]Daniel Galicer informed us that already have some progresses in this context. In cooperation with Martìn Mansilla and Santiago Muro they already have found the optimality of the exponent if $1 \leq r \leq 2$ and $1 \leq p \leq 2$ or if $r > 2$ and $p = 1, 2, m$. Furthermore, if $r > 2$ and $2 < p < m$ they have better lower bounds for the exponents and they believe that $\epsilon > 0$ appearing in Proposition 2.3 may be deleted.

Chapter 3

On the polynomial Bohnenblust–Hille and Hardy–Littlewood inequalities

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and x^α stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$. The polynomial Bohnenblust–Hille inequality (see [6, 42] and the references therein) ensures that, given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_∞^n given by $P(x_1, ..., x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\mathrm{pol}} \|P\|, \tag{3.1}$$

for some constant $B_{\mathbb{K},m}^{\mathrm{pol}} \geq 1$ which does not depend on n (the exponent $2m/(m+1)$ is optimal), where $\|P\| := \sup_{z \in B_{\ell_\infty^n}} |P(z)|$. The search of precise estimates of the growth of the constants $B_{\mathbb{K},m}^{\mathrm{pol}}$ is fundamental for different applications and remains an important open problem (see [32] and the references therein).

For real scalars it was shown in [56, Theorem 2.2] that

$$(1.17)^m \leq B_{\mathbb{R},m}^{\mathrm{pol}} \leq C(\epsilon)\,(2+\epsilon)^m,$$

where $C(\epsilon)\,(2+\epsilon)^m$ means that given $\epsilon > 0$, there is a constant $C(\epsilon) > 0$ such that $B_{\mathbb{R},m}^{\mathrm{pol}} \leq C(\epsilon)\,(2+\epsilon)^m$ for all m. In other words, this means that for real scalars the hypercontractivity of $B_{\mathbb{R},m}^{\mathrm{pol}}$ is optimal.

For complex scalars the behavior of $B_{\mathbb{K},m}^{\mathrm{pol}}$ is still unknown. The best information we have thus far about $B_{\mathbb{C},m}^{\mathrm{pol}}$ are due to D. Núñez-Alarcón [108] (lower bounds) and F. Bayart, D. Pellegrino and J.B. Seoane-Sepúlveda [32] (upper bounds)

$$B_{\mathbb{C},m}^{\mathrm{pol}} \geq \begin{cases} \left(1 + \frac{1}{2^{m-1}}\right)^{\frac{1}{4}}, & \text{for } m \text{ even;} \\ \left(1 + \frac{1}{2^{m-1}}\right)^{\frac{m-1}{4m}}, & \text{for } m \text{ odd;} \end{cases}$$
$$B_{\mathbb{C},m}^{\mathrm{pol}} \leq C(\epsilon)\,(1+\epsilon)^m.$$

The following diagram shows the evolution of the estimates of $B_{\mathbb{K},m}^{\mathrm{pol}}$ for complex scalars.

Authors	Year	Estimate
Bohnenblust and Hille	1931, [42] (*Ann.Math.*)	$B_{\mathbb{C},m}^{\text{pol}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$
Defant, Frerick, Ortega-Cerdá, Ounaïes, and Seip	2011, [65] (*Ann.Math.*)	$B_{\mathbb{C},m}^{\text{pol}} \leq \left(1+\frac{1}{m-1}\right)^{m-1} \sqrt{m} \left(\sqrt{2}\right)^{m-1}$
Bayart, Pellegrino, and Seoane-Sepúlveda	2014, [32] (*Adv.Math.*)	$B_{\mathbb{C},m}^{\text{pol}} \leq C(\epsilon)\,(1+\epsilon)^m$

When replacing ℓ_∞^n by ℓ_p^n the extension of the polynomial Bohnenblust–Hille inequality is called polynomial Hardy–Littlewood inequality and the optimal exponents are $2mp/(mp+p-2m)$ for $2m \leq p \leq \infty$. More precisely, given positive integers $m \geq 2$ and $n \geq 1$, as a consequence of the multilinear Hardy–Littlewood inequality (see [5, 73]), if P is a homogeneous polynomial of degree m on ℓ_p^n with $2m \leq p \leq \infty$ given by $P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, then there is a constant $C_{\mathbb{K},m,p}^{\text{pol}} \geq 1$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leq C_{\mathbb{K},m,p}^{\text{pol}} \|P\|, \tag{3.2}$$

and $C_{\mathbb{K},m,p}^{\text{pol}}$ does not depend on n, where $\|P\| := \sup_{z \in B_{\ell_p^n}} |P(z)|$. Using the generalized Kahane–Salem–Zygmund inequality (2) (see, for instance, [6]) we can verify that the exponents $2mp/(mp+p-2m)$ are optimal for $2m \leq p \leq \infty$. When $p = \infty$, since $2mp/(mp+p-2m) = 2m/(m+1)$, we recover the polynomial Bohnenblust–Hille inequality.

As in the multilinear case, for $m < p < 2m$ there is also a version of the polynomial Hardy–Littlewood inequality (see [73]): given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_p^n with $m < p < 2m$ given by $P(x_1, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha$, then there is a (optimal) constant $D_{\mathbb{K},m,p}^{\text{pol}} \geq 1$ (not depending on n) such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \leq D_{\mathbb{K},m,p}^{\text{pol}} \|P\| \tag{3.3}$$

and the exponents $p/(p-m)$ are optimal.

In this chapter we look for upper and lower estimates for $C_{\mathbb{K},m,p}^{\text{pol}}$ and $D_{\mathbb{K},m,p}^{\text{pol}}$. Our main contributions regarding the constants of the polynomial Hardy–Littlewood inequality can be summarized in the following result (in this chapter we will only present the proof of the items *(1)(ii)* and *(3)*. For details of other results see [10]):

Theorem 3.1. *Let $m \geq 2$.*

(1) Let $2m \leq p \leq \infty$.

(i) If $\mathbb{K} = \mathbb{R}$, then $C_{\mathbb{R},m,p}^{\text{pol}} \geq 2^{\frac{m^2p+10m-p-6m^2-4}{4mp}} \geq \left(\sqrt[16]{2}\right)^m$.

(ii) If $\mathbb{K} = \mathbb{C}$, then

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \begin{cases} 2^{\frac{m}{p}}, & \text{for } m \text{ even,} \\ 2^{\frac{m-1}{p}}, & \text{for } m \text{ odd.} \end{cases}$$

(2) For $2m \leq p \leq \infty$,

$$C^{\text{pol}}_{\mathbb{K},m,p} \leq C^{\text{mult}}_{\mathbb{K},m,p} \frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}}.$$

(3) For $m < p < 2m$,

$$D^{\text{pol}}_{\mathbb{C},m,p} \geq \begin{cases} 2^{\frac{m}{p}} & \text{for } m \text{ even}, \\ 2^{\frac{m-1}{p}} & \text{for } m \text{ odd}. \end{cases}$$

(4) For $m < p < 2m$,

$$D^{\text{pol}}_{\mathbb{K},m,p} \leq D^{\text{mult}}_{\mathbb{K},m,p} \frac{m^m}{(m!)^{\frac{p-m}{p}}}.$$

Remark 3.2. Trying to find a certain pattern in the behavior of the constants of the Bohnenblust–Hille and Hady–Littlewood inequalities, we define $B^{\text{pol}}_{\mathbb{K},m}(n)$, $C^{\text{pol}}_{\mathbb{K},m,p}(n)$ and $D^{\text{pol}}_{\mathbb{K},m,p}(n)$ as the best (meaning *smallest*) value of the constants appearing in (3.1), (3.2) and (3.3), respectively, for $n \in \mathbb{N}$ fixed. A number of papers related to these particular cases are being produced and we can summarize the main findings of these papers as follows:

- $B^{\text{pol}}_{\mathbb{C},2}(2) = \sqrt[4]{3/2}$;
- $B^{\text{pol}}_{\mathbb{R},2}(2) = (2t_0^{4/3} + (2\sqrt{t_0 - t_0^2})^{4/3})^{3/4}$, with $t_0 = (2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16)/36$;
- $B^{\text{pol}}_{\mathbb{R},3}(2) \geq 2.5525$, $B^{\text{pol}}_{\mathbb{R},5}(2) \geq 6.83591$, $B^{\text{pol}}_{\mathbb{R},6}(2) \geq 10.7809$, $B^{\text{pol}}_{\mathbb{R},7}(2) \geq 19.96308$, $B^{\text{pol}}_{\mathbb{R},8}(2) \geq 33.36323$, $B^{\text{pol}}_{\mathbb{R},10}(2) \geq 90.35556$, $B^{\text{pol}}_{\mathbb{R},600}(2) \geq (1.65171)^{600}$, $B^{\text{pol}}_{\mathbb{R},602}(2) \geq (1.61725)^{602}$;
- For $4 \leq p \leq \infty$,

$$C^{\text{pol}}_{\mathbb{R},2,p}(2) = \max_{\alpha \in [0,1]} \left[2 \left| \frac{2\alpha^p - 1}{\alpha^2 + (1-\alpha^p)^{\frac{2}{p}}} \right|^{\frac{4p}{3p-4}} + \left(2\alpha (1-\alpha^p)^{\frac{1}{p}} \frac{\alpha^{p-2} + (1-\alpha^p)^{\frac{p-2}{p}}}{\alpha^2 + (1-\alpha^p)^{\frac{2}{p}}} \right)^{\frac{4p}{3p-4}} \right]^{\frac{3p-4}{4p}};$$

- $C^{\text{pol}}_{\mathbb{R},2,4}(2) = D^{\text{pol}}_{\mathbb{R},2,4}(2) = \sqrt{2}$;
- $C^{\text{pol}}_{\mathbb{R},3,6}(2) = D^{\text{pol}}_{\mathbb{R},3,6}(2) \geq 2.236067$, $C^{\text{pol}}_{\mathbb{R},5,10}(2) = D^{\text{pol}}_{\mathbb{R},5,10}(2) \geq 6.236014$, $C^{\text{pol}}_{\mathbb{R},6,12}(2) = D^{\text{pol}}_{\mathbb{R},6,12}(2) \geq 10.636287$, $C^{\text{pol}}_{\mathbb{R},7,14}(2) = D^{\text{pol}}_{\mathbb{R},7,14}(2) \geq 18.095148$, $C^{\text{pol}}_{\mathbb{R},8,16}(2) = D^{\text{pol}}_{\mathbb{R},8,16}(2) \geq 31.727174$, $C^{\text{pol}}_{\mathbb{R},10,20}(2) = D^{\text{pol}}_{\mathbb{R},10,20}(2) \geq 91.640152$.
- For $2 < p \leq 4$, $D^{\text{pol}}_{\mathbb{R},2,p}(2) = 2^{\frac{2}{p}}$.

See [11, 57, 59, 60, 94].

3.1 Lower bounds for the complex polynomial Hardy–Littlewood inequality

In this section, we provide nontrivial lower bounds for the constants of the complex case of the polynomial Hardy–Littlewood inequality. More precisely we prove that, for

$m \geq 2$ and $2m \leq p < \infty$, $C_{\mathbb{C},m,p}^{\text{pol}} \geq 2^{m/p}$ for m even, and $C_{\mathbb{C},m,p}^{\text{pol}} \geq 2^{(m-1)/p}$ for m odd. For instance, $\sqrt{2} \leq C_{\mathbb{C},2,4}^{\text{pol}} \leq 3.1915$.

Let $m \geq 2$ be an even positive integer and let $p \geq 2m$. Consider the 2-homogeneous polynomials $Q_2 : \ell_p^2 \to \mathbb{C}$ and $\widetilde{Q_2} : \ell_\infty^2 \to \mathbb{C}$ both given by $(z_1, z_2) \mapsto z_1^2 - z_2^2 + cz_1z_2$, $c \in \mathbb{R}$. We know from [22, 56] that $\|\widetilde{Q_2}\| = (4+c^2)^{\frac{1}{2}}$. If we follow the lines of [108] and we define the m-homogeneous polynomial $Q_m : \ell_p^m \to \mathbb{C}$ by $Q_m(z_1, ..., z_m) = z_3 \ldots z_m Q_2(z_1, z_2)$ we obtain

$$\|Q_m\| \leq 2^{-\frac{m-2}{p}} \|Q_2\| \leq 2^{-\frac{m-2}{p}} \|\widetilde{Q_2}\| = 2^{-\frac{m-2}{p}} \left(4+c^2\right)^{\frac{1}{2}},$$

where we use the obvious inequality $\|Q_2\| \leq \|\widetilde{Q_2}\|$. Therefore, for $m \geq 2$ even and $c \in \mathbb{R}$, from the polynomial Hardy–Littlewood inequality it follows that

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left(2+|c|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}}(4+c^2)^{\frac{1}{2}}}.$$

If

$$c > \left(\frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}},$$

it is not too difficult to prove that

$$2^{-\frac{m-2}{p}} \left(4+c^2\right)^{\frac{1}{2}} < \left(\left(2^{\frac{mp+p-2m}{2mp}}\right)^2 + c^2\right)^{\frac{1}{2}},$$

i.e.,

$$2^{-\frac{m-2}{p}} \left(4+c^2\right)^{\frac{1}{2}} < \left\|\left(2^{\frac{mp+p-2m}{2mp}}, c\right)\right\|_2.$$

Since $2mp/(mp+p-2m) \leq 2$, we know that $\ell_{\frac{2mp}{mp+p-2m}} \subset \ell_2$ and $\|\cdot\|_2 \leq \|\cdot\|_{\frac{2mp}{mp+p-2m}}$. Therefore, for all

$$c > \left(\frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}},$$

we have

$$\begin{aligned} 2^{-\frac{m-2}{p}} (4+c^2)^{\frac{1}{2}} &< \left\|\left(2^{\frac{mp+p-2m}{2mp}}, c\right)\right\|_2 \\ &\leq \left\|\left(2^{\frac{mp+p-2m}{2mp}}, c\right)\right\|_{\frac{2mp}{mp+p-2m}} = \left(2 + c^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \end{aligned}$$

from which we conclude that

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left(2+c^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}}(4+c^2)^{\frac{1}{2}}} > 1.$$

If $m \geq 3$ is odd, since $\|Q_m\| \leq \|Q_{m-1}\|$, then we have $\|Q_m\| \leq 2^{-\frac{m-3}{p}} (4+c^2)^{\frac{1}{2}}$ and thus we can now proceed analogously to the even case and finally conclude that for

$$c > \left(\frac{2^{\frac{2p+6-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-6}{p}}}\right)^{\frac{1}{2}}$$

we get

$$C^{\mathrm{pol}}_{\mathbb{C},m,p} \geq \frac{\left(2+c^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-3}{p}}(4+c^2)^{\frac{1}{2}}} > 1.$$

So we have:

Proposition 3.3. *Let $m \geq 2$ be a positive integer and let $p \geq 2m$. Then, for every $\epsilon > 0$,*

$$C^{\mathrm{pol}}_{\mathbb{C},m,p} \geq \frac{\left(2+\left(\left(\frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}}\left(4+\left(\left(\frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^2\right)^{\frac{1}{2}}} > 1 \quad \text{if } m \text{ is even}$$

and

$$C^{\mathrm{pol}}_{\mathbb{C},m,p} \geq \frac{\left(2+\left(\left(\frac{2^{\frac{2p+6-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-6}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-3}{p}}\left(4+\left(\left(\frac{2^{\frac{2p+6-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-6}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^2\right)^{\frac{1}{2}}} > 1 \quad \text{if } m \text{ is odd.}$$

However, we have another approach to the problem, which is surprisingly simpler than the above approach and still seems to give best (bigger) lower bounds for the constants of the polynomial Hardy–Littlewood inequality (even for the case $m < p < 2m$).

Theorem 3.4. *Let $m \geq 2$ be a positive integer and let $m < p \leq \infty$.*

(i) If $2m \leq p \leq \infty$, we have

$$C^{\mathrm{pol}}_{\mathbb{C},m,p} \geq \begin{cases} 2^{\frac{m}{p}}, & \text{for } m \text{ even,} \\ 2^{\frac{m-1}{p}}, & \text{for } m \text{ odd.} \end{cases}$$

(ii) If $m < p \leq 2m$, we have

$$D^{\mathrm{pol}}_{\mathbb{C},m,p} \geq \begin{cases} 2^{\frac{m}{p}}, & \text{for } m \text{ even,} \\ 2^{\frac{m-1}{p}}, & \text{for } m \text{ odd.} \end{cases}$$

Proof. Let $m \geq 2$ be a positive integer and let $p \geq 2m$. Consider $P_2 : \ell_p^2 \to \mathbb{C}$ the 2-homogeneous polynomial given by $z \mapsto z_1 z_2$. Observe that

$$\|P_2\| = \sup_{|z_1|^p+|z_2|^p=1} |z_1 z_2| = \sup_{|z|\leq 1} |z| \left(1-|z|^p\right)^{\frac{1}{p}} = 2^{-\frac{2}{p}}.$$

More generally, if $m \geq 2$ is even and P_m is the m-homogeneous polynomial given by $z \mapsto z_1 \cdots z_m$, then $\|P_m\| \leq 2^{-m/p}$. Therefore, from the polynomial Hardy–Littlewood inequality we know that

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \frac{\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{\|P_m\|} \geq \frac{1}{2^{-\frac{m}{p}}} = 2^{\frac{m}{p}}.$$

If $m \geq 3$ is odd, we define again the m-homogeneous polynomial P_m given by $z \mapsto z_1 \cdots z_m$ and since $\|P_m\| \leq \|P_{m-1}\|$, then we have $\|P_m\| \leq 2^{-(m-1)/p}$ and thus

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \frac{1}{2^{-\frac{m-1}{p}}} = 2^{\frac{m-1}{p}}.$$

With the same arguments used for the case $2m \leq p \leq \infty$, we obtain the similar estimate *(3)* of Theorem 3.1 for the case $m < p < 2m$. □

The estimates of Proposition 3.3 seems to become better when ϵ grows (this seems to be a clear sign that we should avoid the terms z_1^2 and z_2^2 in our approach). Making $\epsilon \to \infty$ in Theorem 3.3 we obtain

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \begin{cases} 2^{\frac{m-2}{p}} & \text{for } m \text{ even;} \\ 2^{\frac{m-3}{p}} & \text{for } m \text{ odd,} \end{cases}$$

which are slightly worse than the estimates from Theorem 3.4.

3.2 The complex polynomial Hardy–Littlewood inequality: Upper estimates

In this section, let us use the following notation: $S_{\ell_p^n}$ denotes the unit sphere on ℓ_p^n if $p < \infty$, and $S_{\ell_\infty^n}$ denotes the n-dimensional torus. More precisely: for $p \in (0, \infty)$

$$S_{\ell_p^n} := \left\{ z = (z_1, ..., z_n) \in \mathbb{C}^n : \|z\|_{\ell_p^n} = 1 \right\}$$

and

$$S_{\ell_\infty^n} := \mathbb{T}^n = \{ z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_i| = 1 \}.$$

Let μ^n be the normalized Lebesgue measure on the respective set. The following lemma is a particular instance ($1 \leq p = s \leq 2$ and $q = 2$) of the Khinchin-Steinhaus polynomial inequalities (for polynomials homogeneous or not) and $p \leq q$.

Lemma 3.5. *Let $1 \leq s \leq 2$. For every m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$ with values in $\mathbb{C}$, we have*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^2 \right)^{\frac{1}{2}} \leq \left(\tfrac{2}{s}\right)^{\frac{m}{2}} \left(\int_{\mathbb{T}^n} |P(z)|^s \, d\mu^n(z) \right)^{\frac{1}{s}}.$$

When $n = 1$ a result due to F.B. Weissler (see [135]) asserts that the optimal constant for the general case is $\sqrt{2/s}$. In the n-dimensional case the best constant for m-homogeneous polynomials is $(\sqrt{2/s})^m$ (see also [30]).

For $m \in [2, \infty]$ let us define $p_0(m)$ as the infimum of the values of $p \in [2m, \infty]$ such that for all $1 \leq s \leq 2p/(p-2)$ there is a $K_{s,p} > 0$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \leq K_{s,p}^m \left(\int_{S_{\ell_p^n}} |P(z)|^s \, d\mu^n(z) \right)^{\frac{1}{s}}, \tag{3.4}$$

for all positive integers n and all m-homogeneous polynomials $P : \mathbb{C}^n \to \mathbb{C}$. From Lemma 3.5 we know that this definition makes sense, since from this lemma it follows that (3.4) is valid for $p = \infty$. We conjecture that $p_0(m) \leq m^2$. If it is true that $p_0(m) < \infty$, it is possible to prove the following new estimate for $C_{\mathbb{C},m,p}^{\mathrm{pol}}$ (see [10]): for $m \in [2, \infty]$ and $1 \leq k \leq m-1$, if $p_0(m-k) < p \leq \infty$ (and $p = \infty$ if $p_0(m-k) = \infty$) then, for every m-homogeneous polynomial $P : \ell_p^n \to \mathbb{C}$, defined by $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$, we have

$$\begin{aligned} &\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\ &\leq K_{\frac{2kp}{kp+p-2k},p}^{m-k} \cdot \frac{m^m}{(m-k)^{m-k}} \cdot \left(\frac{(m-k)!}{m!} \right)^{\frac{p-2}{2p}} \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2k(k-1)}{p}} \cdot \left(B_{\mathbb{C},k}^{\mathrm{mult}} \right)^{\frac{p-2k}{p}} \|P\|, \end{aligned}$$

where $B_{\mathbb{C},k}^{\mathrm{mult}}$ is the optimal constant of the multilinear Bohnenblust–Hille inequality associated with k-linear forms.

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Part II

Summability of multilinear operators

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Chapter 4

Maximal spaceability and optimal estimates for summing multilinear operators

If $1 \leq p \leq q < \infty$, we say that a continuous linear operator $T : E \to F$ is (q,p)-summing if $(T(x_j))_{j=1}^{\infty} \in \ell_q(F)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. The class of (q,p)-summing linear operators from E to F will be represented by $\Pi_{(q;p)}(E,F)$. An equivalent formulation asserts that $T : E \to F$ is (q,p)-summing if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{\infty} \|T(x_j)\|^q \right)^{1/q} \leq C \left\| (x_j)_{j=1}^{\infty} \right\|_{w,p},$$

for all $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. The above inequality can also be replaced by: there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{n} \|T(x_j)\|^q \right)^{1/q} \leq C \left\| (x_j)_{j=1}^{n} \right\|_{w,p},$$

for all $x_1, \ldots, x_n \in E$ and all positive integers n. The infimum of all C that satisfy the above inequalities defines a norm, denoted by $\pi_{(q;p)}(T)$, and $\left(\Pi_{(q;p)}(E,F), \pi_{(q;p)}(\cdot)\right)$ is a Banach space.

More generally, we can define:

Definition 4.1. *For $\mathbf{p} = (p_1, ..., p_m) \in [1, +\infty)^m$ and $1/q \leq \sum_{j=1}^{m} 1/p_j$ recall that a continuous m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is absolutely $(q; \mathbf{p})$-summing if there is a $C > 0$ such that*

$$\left(\sum_{j=1}^{n} \left\| T(x_j^{(1)}, ..., x_j^{(m)}) \right\|^q \right)^{\frac{1}{q}} \leq C \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{n} \right\|_{w,p_k}$$

for all positive integers n and all $(x_j^{(k)})_{j=1}^{n} \in E_k$, $k = 1, ..., m$.

- We represent the class of all absolutely $(q; \mathbf{p})$-summing operators from $E_1, ..., E_m$ to F by $\Pi_{\mathrm{as}(q;\mathbf{p})}^{m}(E_1, ..., E_m; F)$;
- When $p_1 = \cdots = p_m = p$, we denote $\Pi_{\mathrm{as}(q;\mathbf{p})}^{m}(E_1, \ldots, E_m; F)$ by

$$\Pi_{\mathrm{as}(q;p)}^{m}(E_1, \ldots, E_m; F).$$

The infimum over all C as above defines a norm on $\Pi_{\mathrm{as}(q;\mathbf{p})}^{m}(E_1,...,E_m;F)$, which we denote by $\pi_{\mathrm{as}(q;\mathbf{p})}(T)$ (or $\pi_{\mathrm{as}(q;p)}(T)$ if $p_1=\cdots=p_m=p$).

In 2003 Matos [102] and, independently, Bombal, Pérez-García and Villanueva [44] introduced the notion of multiple summing multilinear operators.

Definition 4.2 (Multiple summing operators [44, 102])**.** *Let $\mathbf{p}=(p_1,...,p_m)\in[1,+\infty)^m$ and $1\leq q<\infty$ such that $1\leq p_1,\ldots,p_m\leq q<\infty$. A bounded m-linear operator $T:E_1\times\cdots\times E_m\to F$ is multiple $(q;\mathbf{p})$-summing if there exists $C_m>0$ such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty}\left\|T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)})\right\|^q\right)^{\frac{1}{q}}\leq C_m\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,p_k} \tag{4.1}$$

for every $(x_j^{(k)})_{j=1}^{\infty}\in\ell_{p_k}^{w}(E_k)$, $k=1,\ldots,m$.

- The class of all multiple $(q;\mathbf{p})$-summing operators from $E_1\times\cdots\times E_m$ to F will be denoted by $\Pi_{\mathrm{mult}(q;\mathbf{p})}^{m}(E_1,\ldots,E_m;F)$.
- When $\mathbf{p}=(p,\ldots,p)$ we write $\Pi_{\mathrm{mult}(q;p)}^{m}(E_1,...,E_m;F)$ instead of

$$\Pi_{\mathrm{mult}(q;\mathbf{p})}^{m}(E_1,...,E_m;F).$$

The infimum over all C_m satisfying (4.1) defines a norm in $\Pi_{\mathrm{mult}(q;\mathbf{p})}^{m}(E_1,\ldots,E_m;F)$, which is denoted by $\pi_{\mathrm{mult}(q;\mathbf{p})}(T)$ (or $\pi_{\mathrm{mult}(q;p)}(T)$ if $p_1=\cdots=p_m=p$).

Using that $\mathcal{L}(c_0;E)$ is isometrically isomorphic to $\ell_1^w(E)$ (see [72]), Bohnenblust–Hille's inequality can be re-written as:

Theorem 4.3 (Bohnenblust–Hille *re-written* [122])**.** *If $m\geq 2$ is a positive integer and $T\in\mathcal{L}(E_1,\ldots,E_m;\mathbb{K})$, then*

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty}\left|T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq B_{\mathbb{K},m}^{\mathrm{mult}}\|T\|\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,1} \tag{4.2}$$

for every $(x_j^{(k)})_{j=1}^{\infty}\in\ell_1^w(E_k)$, $k=1,\ldots,m$ and $j=1,\ldots,N$, where $B_{\mathbb{K},m}^{\mathrm{mult}}$ is the optimal constant of the classical Bohnenblust–Hille inequality.

Proof. Let $T\in\mathcal{L}(E_1,\ldots,E_m;\mathbb{K})$ and let $(x_j^{(k)})_{j=1}^{\infty}\in\ell_1^w(E_k)$, $k=1,\ldots,m$. From [72, Prop. 2.2.] we have the boundedness of the linear operator $u_k:c_0\to E_k$ such that $u_k(e_j)=x_j^{(k)}$ and $\|u_k\|=\|(x_j^{(k)})_{j=1}^{\infty}\|_{w,1}$ for each $k=1,\ldots,m$. Thus, $S:c_0\times\cdots\times c_0\to\mathbb{K}$ defined by $S(y_1,\ldots,y_m)=T(u_1(y_1),\ldots,u_m(y_m))$ is a bounded m-linear operator and $\|S\|\leq\|T\|\|u_1\|\cdots\|u_m\|$. Therefore,

$$\begin{aligned}\left(\sum_{j_1,\ldots,j_m=1}^{\infty}\left|T\left(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)}\right)\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} &= \left(\sum_{j_1,\ldots,j_m=1}^{\infty}\left|S\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\\ &\leq B_{\mathbb{K},m}^{\mathrm{mult}}\|S\|\leq B_{\mathbb{K},m}^{\mathrm{mult}}\|T\|\prod_{k=1}^{m}\|u_k\|\\ &= B_{\mathbb{K},m}^{\mathrm{mult}}\|T\|\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,1},\end{aligned}$$

as required. □

In this sense, the Bohnenblust–Hille theorem (1.1) can be seen as the beginning of the notion of multiple summing operators, that is, in the modern terminology, the classical Bohnenblust–Hille inequality [42] ensures that, for all $m \geq 2$ and all Banach spaces $E_1, ..., E_m$,

$$\mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}\left(\frac{2m}{m+1};1\right)}(E_1, ..., E_m; \mathbb{K}).$$

4.1 Maximal spaceability and multiple summability

In this section we are interested in estimating the size of the set of non-multiple summing (and non-absolutely summing) multilinear operators. For this task we use the notion of spaceability.

Definition 4.4. *For a given Banach space E, a subset $A \subset E$ is spaceable if $A \cup \{0\}$ contains a closed infinite-dimensional subspace V of E. When* $\dim V = \dim E$*, A is called maximal spaceable.*

For details on spaceability and the related notion of lineability we refer to [21, 37, 58] and the references therein. The next result will be useful to our purpose (see [77, Theorem 5.6 and its reformulation] and [97]).

Lemma 4.5 (Drewnowski, 1984)**.** *Let X and Z be Banach spaces and $T : Z \to X$ a continuous linear operator with range $Y = T(Z)$ not closed. Then the complement $X \smallsetminus Y$ is spaceable.*

From now on $\mathfrak{c}$ denotes the cardinality of the continuum.

Proposition 4.6. *Let $E_1, ..., E_m$ be separable Banach spaces. Then,*

$$\dim \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}) = \mathfrak{c}.$$

Proof. From [46, Remark 2.5] we know that $\dim \mathcal{L}(E_1, ..., E_m; \mathbb{K}) \geq \mathfrak{c}$. Since $E_1, ..., E_m$ are separable, let $\omega_j \subseteq E_j$, $j = 1, ..., m$, be a countable, dense subset of E_j and let γ be a basis of $\mathcal{L}(E_1, ..., E_m; \mathbb{K})$. Define

$$\begin{array}{rcccl} g & : & \gamma & \to & \mathbb{K}^{\omega_1 \times \cdots \times \omega_m} \\ & & T & \mapsto & T|_{\omega_1 \times \cdots \times \omega_m}, \end{array}$$

with $\mathbb{K}^{\omega_1 \times \cdots \times \omega_m}$ the set of all functions from $\omega_1 \times \cdots \times \omega_m$ to $\mathbb{K}$. Observe that g is injective. Indeed, let $S, T \in \gamma$ such that $g(S) = g(T)$, i.e., $S|_{\omega_1 \times \cdots \times \omega_m} = T|_{\omega_1 \times \cdots \times \omega_m}$. Given $x \in E_1 \times \cdots \times E_m$, since $\omega_1 \times \cdots \times \omega_m$ is dense on $E_1 \times \cdots \times E_m$, there exist $(x_n)_{n=1}^{\infty} \subset \omega_1 \times \cdots \times \omega_m$ with $\lim_{n\to\infty} x_n = x$. Since S and T are continuous, it follows that

$$S(x) = S(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(x).$$

Thus $S = T$ and hence g is injective, as required. Therefore,

$$\dim \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \mathrm{card}(\gamma) \leq \mathrm{card}(\mathbb{K}^{\omega_1 \times \cdots \times \omega_m}) = \mathrm{card}(\mathbb{K}^{\mathbb{N}}) = \mathfrak{c},$$

where $\mathbb{K}^{\mathbb{N}}$ is the set of all functions from $\mathbb{N}$ to $\mathbb{K}$. □

Corollary 4.7. $\dim(\mathcal{L}(^m\ell_p;\mathbb{K}))=\mathfrak{c}$.

Before we introduce the next result, it is important to note that:

Remark 4.8. Let $1\le s\le r<\infty$ and let $E_1,...,E_m,F$ be Banach spaces with $\dim E_j<\infty$ for all $j=1,...,m$. Then

$$\mathcal{L}(E_1,...,E_m;F)=\Pi^m_{\mathrm{mult}(r;s)}(E_1,...,E_m;F).$$

In fact, since $s\le r$ we have $\ell_s\subseteq\ell_r$ and $\|\cdot\|_r\le\|\cdot\|_s$. Since E_j has finite dimension for all $j=1,...,m$, it follows that $\ell_s^w(E_j)=\ell_s(E_j)$ for all $j=1,...,m$. Thus, consider $T\in\mathcal{L}(E_1,...,E_m;F)$, $n\in\mathbb{N}$ and $(x^{(k)}_{j_k})_{j_k=1}^n\in\ell_s^w(E_k)$, $k=1,...,m$, and observe that

$$\begin{aligned}
\left(\sum_{j_1,...,j_m=1}^n\left\|T(x^{(1)}_{j_1},...,x^{(m)}_{j_m})\right\|^r\right)^{\frac{1}{r}} &= \left\|\left(\left\|T(x^{(1)}_{j_1},...,x^{(m)}_{j_m})\right\|\right)^n_{j_1,...,j_m=1}\right\|_r\\
\le\left\|\left(\left\|T(x^{(1)}_{j_1},...,x^{(m)}_{j_m})\right\|\right)^n_{j_1,...,j_m=1}\right\|_s &= \left(\sum_{j_1,...,j_m=1}^n\left\|T(x^{(1)}_{j_1},...,x^{(m)}_{j_m})\right\|^s\right)^{\frac{1}{s}}\\
\le\|T\|\left(\sum_{j_1,...,j_m=1}^n\|x^{(1)}_{j_1}\|^s\cdots\|x^{(m)}_{j_m}\|^s\right)^{\frac{1}{s}} &= \|T\|\left(\sum_{j_1=1}^n\|x^{(1)}_{j_1}\|^s\right)^{\frac{1}{s}}\cdots\left(\sum_{j_m=1}^n\|x^{(m)}_{j_m}\|^s\right)^{\frac{1}{s}}\\
=\|T\|\prod_{k=1}^m\left\|(x^{(k)}_{j_k})^n_{j_k=1}\right\|_s &= \|T\|\prod_{k=1}^m\left\|(x^{(k)}_{j_k})^n_{j_k=1}\right\|_{w,s},
\end{aligned}$$

i.e., $T\in\Pi^m_{\mathrm{mult}(r;s)}(E_1,...,E_m;F)$.

Theorem 4.9. *Let $m\ge1$, $p\in[2,\infty)$. If $1\le s<p^*$ and $r<2ms/(s+2m-ms)$ then $\mathcal{L}\left(^m\ell_p;\mathbb{K}\right)\smallsetminus\Pi^m_{\mathrm{mult}(r;s)}\left(^m\ell_p;\mathbb{K}\right)$ is maximal spaceable in $\mathcal{L}\left(^m\ell_p;\mathbb{K}\right)$.*

Proof. We consider the case of complex scalars. The case of real scalars is obtained from the complex case via a standard complexification argument (see [47]). An extended version of the Kahane–Salem–Zygmund inequality (see (2) and [6, Lemma 6.1]) asserts that, if $m,n\ge1$ and $p\in[2,\infty]$, there exists a m-linear map $A_n:\ell_p^n\times\cdots\times\ell_p^n\to\mathbb{K}$ of the form

$$A_n(z^{(1)},\ldots,z^{(m)})=\textstyle\sum_{j_1,...,j_m=1}^n\pm z^{(1)}_{j_1}\cdots z^{(m)}_{j_m}\tag{4.3}$$

such that $\|A_n\|\le C_mn^{(mp+p-2m)/2p}$ for certain constant $C_m>0$.

Let $\beta:=(p+s-ps)/ps$. Observe that $s<p^*$ implies $\beta>0$. From the previous remark (Remark 4.8) we have

$$\left(\sum_{j_1,...,j_m=1}^n\left|A_n\left(\frac{e_{j_1}}{j_1^\beta},...,\frac{e_{j_m}}{j_m^\beta}\right)\right|^r\right)^{\frac{1}{r}}\le\pi_{\mathrm{mult}(r;s)}\left(A_n\right)\left\|\left(\frac{e_j}{j^\beta}\right)^n_{j=1}\right\|^m_{w,s},$$

i.e.,

$$\left(\sum_{j_1,...,j_m=1}^n\left|\frac{1}{j_1^\beta\cdots j_m^\beta}\right|^r\right)^{\frac{1}{r}}\le\pi_{\mathrm{mult}(r;s)}\left(A_n\right)\left\|\left(\frac{e_j}{j^\beta}\right)^n_{j=1}\right\|^m_{w,s}.\tag{4.4}$$

Let us investigate separately the both sides of (4.4). On the one hand,

$$\begin{aligned}\left(\sum_{j_1,\ldots,j_m=1}^{n}\left|\frac{1}{j_1^{\beta}\cdots j_m^{\beta}}\right|^{r}\right)^{\frac{1}{r}} &= \left(\sum_{j_1=1}^{n}\cdots\sum_{j_m=1}^{n}\left|\frac{1}{j_1^{\beta}\cdots j_m^{\beta}}\right|^{r}\right)^{\frac{1}{r}}\\ &= \left(\sum_{j_1=1}^{n}\frac{1}{j_1^{r\beta}}\right)^{\frac{1}{r}}\cdots\left(\sum_{j_m=1}^{n}\frac{1}{j_m^{r\beta}}\right)^{\frac{1}{r}} = \left(\sum_{j=1}^{n}\frac{1}{j^{r\beta}}\right)^{\frac{m}{r}}. \qquad (4.5)\end{aligned}$$

On the other hand, for $n \geq 2$, since $\beta s + s/p^{*} = 1$, we obtain

$$\begin{aligned}\left\|\left(\frac{e_j}{j^{\beta}}\right)_{j=1}^{n}\right\|_{w,s} &= \sup_{\varphi\in B_{\ell_p^*}}\left(\sum_{j=1}^{n}\left|\varphi\left(\frac{e_j}{j^{\beta}}\right)\right|^{s}\right)^{\frac{1}{s}} = \sup_{\varphi\in B_{\ell_{p^*}}}\left(\sum_{j=1}^{n}|\varphi_j|^{s}\frac{1}{j^{\beta s}}\right)^{\frac{1}{s}}\\ &\leq \left(\left(\sum_{j=1}^{n}|\varphi_j|^{p^*}\right)^{\frac{s}{p^*}}\left(\sum_{j=1}^{n}\frac{1}{j}\right)^{\beta s}\right)^{\frac{1}{s}} \leq \left(\sum_{j=1}^{n}\frac{1}{j}\right)^{\beta}\\ &= \left(1+\sum_{j=2}^{n}\inf\left\{\tfrac{1}{x} : x\in[j-1,j]\right\}\right)^{\beta} < \left(1+\int_1^n \tfrac{1}{x}dx\right)^{\beta}\\ &= (1+\log n)^{\beta}. \qquad (4.6)\end{aligned}$$

Hence, replacing (4.5) and (4.6) in (4.4), we have

$$\left(\sum_{j=1}^{n}\frac{1}{j^{r\beta}}\right)^{\frac{m}{r}} < \pi_{\mathrm{mult}(r;s)}\left(A_n\right)(1+\log n)^{m\beta}$$

and consequently (since $\sum_{j=1}^{n} 1/j^{r\beta} \geq \sum_{j=1}^{n} 1/n^{r\beta} = n^{1-r\beta}$)

$$\left(n^{1-r\beta}\right)^{\frac{m}{r}} < \pi_{\mathrm{mult}(r;s)}\left(A_n\right)(1+\log n)^{m\beta}.$$

Since $\|A_n\| \leq C_m n^{(mp+p-2m)/2p}$, we have

$$\frac{\pi_{\mathrm{mult}(r;s)}(A_n)}{\|A_n\|} > \frac{n^{\frac{m}{r}-\left(\frac{p+s-ps}{ps}\right)m}}{(1+\log n)^{m\beta}C_m n^{\frac{mp+p-2m}{2p}}} = \frac{n^{\frac{m}{r}+\frac{m}{2}-\frac{m}{s}-\frac{1}{2}}}{C_m(1+\log n)^{r\beta}}.$$

Using that $r < 2ms/(s+2m-ms)$ we get $m/r + m/2 - m/s - 1/2 > 0$. Therefore, by making $n \to \infty$, it follows that

$$\lim_{n\to\infty}\frac{\pi_{\mathrm{mult}(r;s)}(A_n)}{\|A_n\|} = \infty. \qquad (4.7)$$

Using the above limit, let us prove that $\Pi^{m}_{\mathrm{mult}(r;s)}\left({}^{m}\ell_p;\mathbb{K}\right)$ is not closed in $\mathcal{L}\left({}^{m}\ell_p;\mathbb{K}\right)$. In fact, suppose (contrary to our claim) that $\Pi^{m}_{\mathrm{mult}(r;s)}\left({}^{m}\ell_p;\mathbb{K}\right)$ is closed in $\mathcal{L}\left({}^{m}\ell_p;\mathbb{K}\right)$. Then $\left(\Pi^{m}_{\mathrm{mult}(r;s)}\left({}^{m}\ell_p;\mathbb{K}\right), \|\cdot\|\right)$ is a Banach space and, since $\|\cdot\| \leq \pi_{\mathrm{mult}(r;s)}(\cdot)$ (see Proposition 5.3), we conclude that id : $\left(\Pi^{m}_{\mathrm{mult}(r;s)}\left({}^{m}\ell_p;\mathbb{K}\right), \pi_{\mathrm{mult}(r;s)}(\cdot)\right) \to \left(\Pi^{m}_{\mathrm{mult}(r;s)}\left({}^{m}\ell_p;\mathbb{K}\right), \|\cdot\|\right)$ given by $T \mapsto T$ is continuous. Thus by the Open Mapping Theorem (see [53, Corollary

2.7]) we conclude that id^{-1} is also continuous and thus there exists $C > 0$ such that $\pi_{\mathrm{mult}(r;s)}(\cdot) \leq C\|\cdot\|$, contrary to (4.7).

Therefore, from Lemma 4.5 we conclude that $\mathcal{L}\left({}^m\ell_p;\mathbb{K}\right) \smallsetminus \Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right)$ is spaceable. It remains to prove the maximal spaceability. From Corollary 4.7 we know that $\dim\left(\mathcal{L}({}^m\ell_p;\mathbb{K})\right) = \mathfrak{c}$. Thus, if $V \subseteq (\mathcal{L}({}^m\ell_p;\mathbb{K}) \smallsetminus \Pi^m_{\mathrm{mult}(r;s)}({}^m\ell_p;\mathbb{K})) \cup \{0\}$ is a closed infinite-dimensional subspace of $\mathcal{L}({}^m\ell_p;\mathbb{K})$, we have $\dim(V) \leq \mathfrak{c}$. Since V is a Banach space, we also have $\dim(V) \geq \mathfrak{c}$ (see [46, Remark 2.5]). Thus, by the Cantor–Bernstein–Schröeder Theorem, it follows that $\dim(V) = \mathfrak{c}$ and the proof is done. □

Remark 4.10. It is interesting to mention that it was not necessary to suppose the Continuum Hypothesis. In fact, the proof given in [46, Remark 2.5], for instance, of the fact that the dimension of every infinite-dimensional Banach space is, at least, $\mathfrak{c}$ does not depends on the Continuum Hypothesis.

4.2 Some consequences

Here we show some consequences of the results of the previous section. For instance, we observe a new optimality component of the Bohnenblust–Hille inequality: the term 1 from the pair $(2m/(m+1);1)$ is also optimal.

The following result is a simple consequence of Theorem 4.9.

Corollary 4.11. *Let $m \geq 2$ and $r \in [2m/(m+1), 2]$. Then*

$$\sup\left\{s : \mathcal{L}({}^m\ell_p;\mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}({}^m\ell_p;\mathbb{K})\right\} \leq \tfrac{2mr}{mr+2m-r}$$

for all $2 \leq p < 2mr/(r+mr-2m)$.

Proof. Since $2m/(m+1) \leq r \leq 2 < 2m$, it follows that $1 \leq 2mr/(mr+2m-r)$ and $2 < 2mr/(r+mr-2m)$. Note that $s > 2mr/(mr+2m-r)$ implies $r < 2ms/(s+2m-ms)$. Therefore, for $2 \leq p < \frac{2mr}{r+mr-2m}$, from Theorem 4.9 we know that

$$\mathcal{L}\left({}^m\ell_p;\mathbb{K}\right) \smallsetminus \Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right)$$

is spaceable for all $2mr/(mr+2m-r) < s < p^*$ (note that $p < 2mr/(r+mr-2m)$ implies $p^* > 2mr/(mr+2m-r)$). In particular, for $2 \leq p < 2mr/(r+mr-2m)$,

$$\sup\left\{s : \mathcal{L}({}^m\ell_p;\mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}({}^m\ell_p;\mathbb{K})\right\} \leq \tfrac{2mr}{mr+2m-r}.$$

□

This corollary together with Theorem 4.9 ensure that, for $r \in [2m/(m+1), 2]$ and $2 \leq p < 2mr/(r+mr-2m)$,

$$\sup\left\{s : \mathcal{L}({}^m\ell_p;\mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}({}^m\ell_p;\mathbb{K})\right\} = \tfrac{2mr}{mr+2m-r}.$$

When $p = 2$ the expression above recovers the optimality of [47, Theorem 5.14] in the case of m-linear operators on $\ell_2 \times \cdots \times \ell_2$.

In 2010 G. Botelho, C. Michels and D. Pellegrino [47] have shown that for $m \geq 1$ and Banach spaces $E_1, ..., E_m$ of cotype 2,

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\mathrm{mult}\left(2; \frac{2m}{2m-1}\right)}\left(E_1, ..., E_m; \mathbb{K}\right),$$

whereas for Banach spaces of cotype $k > 2$,

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\mathrm{mult}\left(2; \frac{km}{km-1} - \epsilon\right)}\left(E_1, ..., E_m; \mathbb{K}\right)$$

for all sufficiently small $\epsilon > 0$. We now remark that it is not necessary to make any assumptions on the Banach spaces $E_1, ..., E_m$ and $2m/(2m-1)$ holds in all cases. Given $k > 2$, in [117, page 194] it is said that it is not known if $s = km/(km-1)$ is attained or not in

$$\sup\{s : \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}(2;s)}(E_1, ..., E_m; \mathbb{K}) \text{ for all} E_j \text{ of cotype } k\} \geq \tfrac{km}{km-1}.$$

The fact that $2m/(2m-1)$ can replace $km/(km-1)$ in all cases ensures that $s = km/(km-1)$ is not attained and thus refines the estimate of [117, Corollary 3.1], which can be improved to

$$\begin{aligned}&\sup\{s : \mathcal{L}(E_1,..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}(2;s)}(E_1, ..., E_m; \mathbb{K}) \text{ for all } E_j \text{ of cotype } k\} \\ &\qquad \in \left[\tfrac{2m}{2m-1}, \tfrac{2km}{2km+k-2m}\right]\end{aligned}$$

if $k > 2$ and $m \geq k$ is a positive integer.

More precisely we prove the following more general result. Let us remark that part *(i)* of the theorem bellow can be also derived from [5, 73], although it is not explicitly written in the aforementioned papers:

Theorem 4.12. *Let $m \geq 2$ and let $r \in [2m/(m+1), \infty)$. Then the optimal s such that*

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\mathrm{mult}(r;s)}\left(E_1, ..., E_m; \mathbb{K}\right).$$

for all Banach spaces $E_1, ..., E_m$ is:

(i) $\frac{2mr}{mr+2m-r}$ *if* $r \in \left[\frac{2m}{m+1}, 2\right]$;

(ii) $\frac{mr}{mr+1-r}$ *if* $r \in (2, \infty)$.

Proof. *(i)* For $1 \leq q < \infty$, let $X_q = \ell_q$ and let us define $X_\infty = c_0$. Consider $q := 2mr/(r + mr - 2m)$. Since $r \in [2m/(m+1), 2]$ we have $q \in [2m, \infty]$. Since $m/q \leq 1/2$ and $r = 2m/(m+1-2m/q)$, from the multilinear Hardy–Littlewood inequality there is a constant $C \geq 1$ such that

$$\left(\sum_{j_1,...,j_m=1}^{\infty} \left|A\left(e_{j_1}, ..., e_{j_m}\right)\right|^r\right)^{\frac{1}{r}} \leq C \left\|A\right\|,$$

for all continuous m-linear operators $A : X_q \times \cdots \times X_q \to \mathbb{K}$. Let $T \in \mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right)$ and $(x_j^{(k)})_{j=1}^{\infty} \in \ell^w_{q^*}(E_k)$, $k = 1, ..., m$. Now we use a standard argument (see [5]) to lift the result from X_q to arbitrary Banach spaces. From [72, Proposition 2.2] there is a continuous linear operator $u_k : X_q \to E_k$ so that $u_k(e_{j_k}) = x_{j_k}^{(k)}$ and $\|u_k\| = \|(x_j^{(k)})_{j=1}^{\infty}\|_{w,q^*}$ for all $k = 1, ..., m$. Therefore, $S : X_q \times \cdots \times X_q \to \mathbb{K}$ defined by $S(y_1, ..., y_m) = T(u_1(y_1), ..., u_m(y_m))$ is m-linear, continuous and

$$\|S\| \leq \|T\| \prod_{k=1}^{m} \|u_k\| = \|T\| \prod_{k=1}^{m} \left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,q^*}.$$

Hence

$$\left(\sum_{j_1,...,j_m=1}^{\infty} \left|T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}\right)\right|^r\right)^{\frac{1}{r}} \leq C\|T\| \prod_{k=1}^{m} \left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,q^*}$$

and, since $q^* = 2mr/(mr + 2m - r)$, the last inequality proves that, for all $m \geq 2$ and $r \in [2m/(m + 1), 2]$,

$$\mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}\left(r; \frac{2mr}{mr+2m-r}\right)}(E_1, ..., E_m; \mathbb{K}).$$

Now let us prove the optimality. From what we have just proved, for $r \in [2m/(m + 1), 2]$, we have

$$\begin{aligned} U_{m,r} &:= \sup\left\{s : \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(E_1, ..., E_m; \mathbb{K}) \text{ for all Banach spaces } E_j\right\} \\ &\geq \tfrac{2mr}{mr+2m-r}. \end{aligned}$$

From Corollary 4.11 we have, for $2 \leq p < 2mr/(r + mr - 2m)$,

$$\sup\left\{s : \mathcal{L}(^m\ell_p; \mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(^m\ell_p; \mathbb{K})\right\} \leq \tfrac{2mr}{mr+2m-r}.$$

Therefore,

$$U_{m,r} \leq \sup\left\{s : \mathcal{L}(^m\ell_p; \mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(^m\ell_p; \mathbb{K})\right\} \leq \tfrac{2mr}{mr+2m-r}.$$

and we conclude that $U_{m,r} = 2mr/(mr + 2m - r)$.

(ii) Given $r > 2$ consider $m < p < 2m$ such that $r = p/(p - m)$. In this case, $p = mr/(r - 1)$ and $p^* = mr/(mr + 1 - r)$. From [73, Proposition 4.1] we know that

$$\Pi_{\mathrm{mult}(r; \frac{mr}{mr+1-r})}(\ell_p, ..., \ell_p; \mathbb{K}) = \mathcal{L}(\ell_p, ..., \ell_p; \mathbb{K}) \tag{4.8}$$

and the result is optimal, i.e., $r = p/(p - m)$ cannot be improved. If $s > p^*$ let $\varepsilon > 0$ and $q \in (m, 2m)$ be such that $q^* = p^* + \varepsilon < s$. Since $m < q < 2m$, from [73, Proposition 4.1] we have

$$\Pi_{\mathrm{mult}(\frac{q}{q-m}; q^*)}(\ell_q, ..., \ell_q; \mathbb{K}) = \mathcal{L}(\ell_q, ..., \ell_q; \mathbb{K})$$

and $q/(q - m)$ is optimal. Since $q/(q - m) > p/(p - m)$ we conclude that

$$\Pi_{\mathrm{mult}(\frac{p}{p-m}; q^*)}(\ell_q, ..., \ell_q; \mathbb{K}) \neq \mathcal{L}(\ell_q, ..., \ell_q; \mathbb{K}),$$

and, a fortiori,

$$\Pi_{\mathrm{mult}(r;s)}(\ell_q, ..., \ell_q; \mathbb{K}) \neq \mathcal{L}(\ell_q, ..., \ell_q; \mathbb{K}).$$

and the proof is done. □

The following graph (Figure 4.1) illustrates for which $(r,s) \in [1,\infty) \times [1,r]$ we have

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\text{mult}(r;s)}\left(E_1, ..., E_m; \mathbb{K}\right).$$

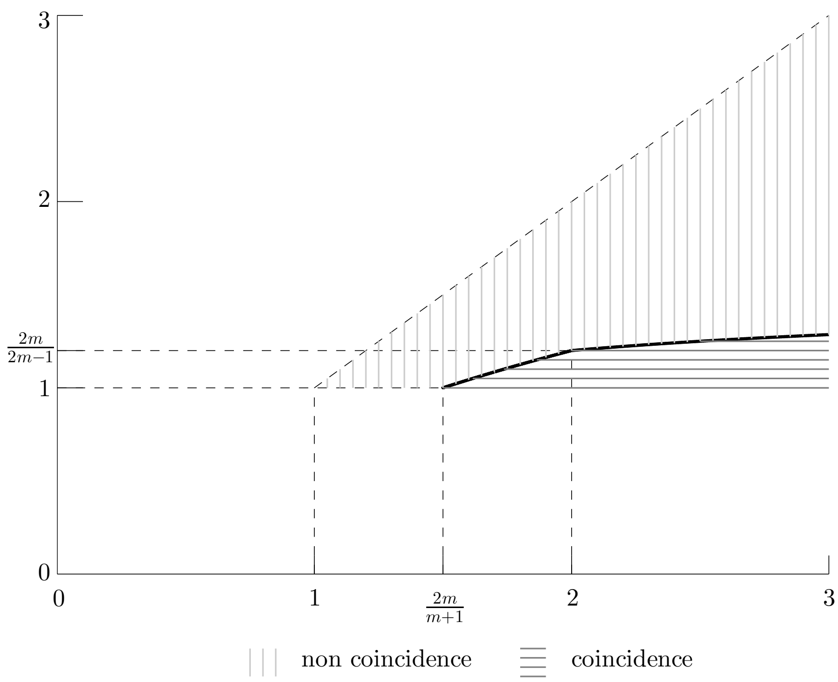

Figure 4.1: Areas of coincidence for $\Pi^m_{\text{mult}(r;s)}\left(E_1, ..., E_m; \mathbb{K}\right)$, $(r,s) \in [1,\infty) \times [1,r]$.

The table below details the results of coincidence and non-coincidence in the "boundaries" of Figure 4.1. We can clearly see that the only case that remains open is the case $(r;s)$ with $r > 2$ and $2m/(2m-1) < s \leq mr/(mr+1-r)$.

$r \geq 1$	$s = r$	non-coincidence
$1 \leq r < \frac{2m}{m+1}$	$s = 1$	non-coincidence
$\frac{2m}{m+1} \leq r \leq 2$	$s = \frac{2mr}{mr+2m-r}$	coincidence
$r \geq \frac{2m}{m+1}$	$s = 1$	coincidence
$r > 2$	$s = \frac{mr}{mr+1-r}$	coincidence

4.3 Multiple $(r;s)$-summing forms in c_0 and ℓ_∞ spaces

From standard localization procedures, coincidence results for c_0 and ℓ_∞ are the same; so we will restrict our attention to c_0. It is well known that $\Pi^m_{\text{mult}(r;s)}\left({}^m c_0; \mathbb{K}\right) = \mathcal{L}\left({}^m c_0; \mathbb{K}\right)$ whenever $r \geq s \geq 2$ (see [47]). When $s = 1$, as a consequence of the Bohnenblust–Hille inequality, we also know that the equality holds if and only if $s \geq 2m/(m+1)$. The next result encompasses essentially all possible cases:

Proposition 4.13. *If $s \in [1,\infty)$ then*

$$\inf\left\{r : \Pi^m_{\text{mult}(r;s)}\left({}^m c_0; \mathbb{K}\right) = \mathcal{L}\left({}^m c_0; \mathbb{K}\right)\right\} = \begin{cases} \frac{2m}{m+1} & \textit{if } 1 \leq s \leq \frac{2m}{m+1}, \\ s & \textit{if } s \geq \frac{2m}{m+1}. \end{cases}$$

Proof. The case $r \geq s \geq 2$ is immediate (see [47, Corollary 4.10]). The Bohnenblust–Hille inequality assures that when $s = 1$ the best choice for r is $2m/(m+1)$. So, it is obvious that for $1 \leq s \leq 2m/(m+1)$ the best value for r is not smaller than $2m/(m+1)$. More precisely,

$$\Pi_{\mathrm{mult}(r;s)}\left({}^m c_0;\mathbb{K}\right) \neq \mathcal{L}\left({}^m c_0;\mathbb{K}\right)$$

whenever $(r,s) \in [1, 2m/(m+1)) \times [1, 2m/(m+1)]$ and $r \geq s$. An adaptation of deep result due to Pisier [125] to multiple summing operators (see [123, Theorem 3.16] or [47, Lemma 5.2]) combined with the coincidence result for $(r;s) = (2m/(m+1);1)$ tells us that we also have

$$\mathcal{L}({}^m c_0;\mathbb{K}) = \Pi_{\mathrm{mult}\left(\frac{2m}{m+1};s\right)}({}^m c_0;\mathbb{K}) \tag{4.9}$$

for all $1 < s < \frac{2m}{m+1}$. The remaining case $(r;s)$ with $2m/(m+1) < s < 2$ follows from an interpolation procedure in the lines of [47, Proposition 4.3]. More precisely, given $2m/(m+1) < r < 2$ and $0 < \delta < (r(2-\theta)-2)/(2-\theta)$, where $\theta = (mr+r-2m)/r$, consider

$$\epsilon = \tfrac{2m}{m+1} - \tfrac{2(1-\theta)(r-\delta)}{2-\theta(r-\delta)}.$$

Note that $1 < 2(1-\theta)(r-\delta)/(2-\theta(r-\delta)) < 2m/(m+1)$ and thus $2m/(m+1) - \epsilon = 2(1-\theta)(r-\delta)/(2-\theta(r-\delta)) \in (1, 2m/(m+1))$. By (4.9) we know that

$$\mathcal{L}({}^m c_0;\mathbb{K}) = \Pi_{\mathrm{mult}\left(\frac{2m}{m+1};\frac{2m}{m+1}-\epsilon\right)}({}^m c_0;\mathbb{K}) \tag{4.10}$$

and by [44, Theorem 3.1] we have

$$\mathcal{L}({}^m c_0;\mathbb{K}) = \Pi_{\mathrm{mult}(2;2)}({}^m c_0;\mathbb{K}). \tag{4.11}$$

Since

$$\tfrac{1}{r} = \tfrac{\theta}{2} + \tfrac{1-\theta}{\frac{2m}{m+1}} \qquad \text{and} \qquad \tfrac{1}{r-\delta} = \tfrac{\theta}{2} + \tfrac{1-\theta}{\frac{2m}{m+1}-\epsilon},$$

from (4.10) and (4.11) and invoking [47, Proposition 4.3] we conclude that $\mathcal{L}({}^m c_0;\mathbb{K}) = \Pi_{\mathrm{mult}(r;r-\delta)}({}^m c_0;\mathbb{K})$. □

The following graph (Figure 4.2) illustrates for which $(r,s) \in [1,\infty) \times [1,r]$ we have

$$\mathcal{L}\left({}^m c_0;\mathbb{K}\right) = \Pi^m_{\mathrm{mult}(r;s)}\left({}^m c_0;\mathbb{K}\right).$$

The table below details the results of coincidence and non-coincidence in the "boundaries" of Figure 4.2.

$1 \leq r < \frac{2m}{m+1}$	$s = 1$	non-coincidence
$r = \frac{2m}{m+1}$	$1 \leq s < \frac{2m}{m+1}$	coincidence
$r \geq \frac{2m}{m+1}$	$s = 1$	coincidence
$1 \leq r < \frac{2m}{m+1}$	$s = r$	non-coincidence
$\frac{2m}{m+1} \leq r < 2$	$s = r$	unknown
$r \geq 2$	$s = r$	coincidence

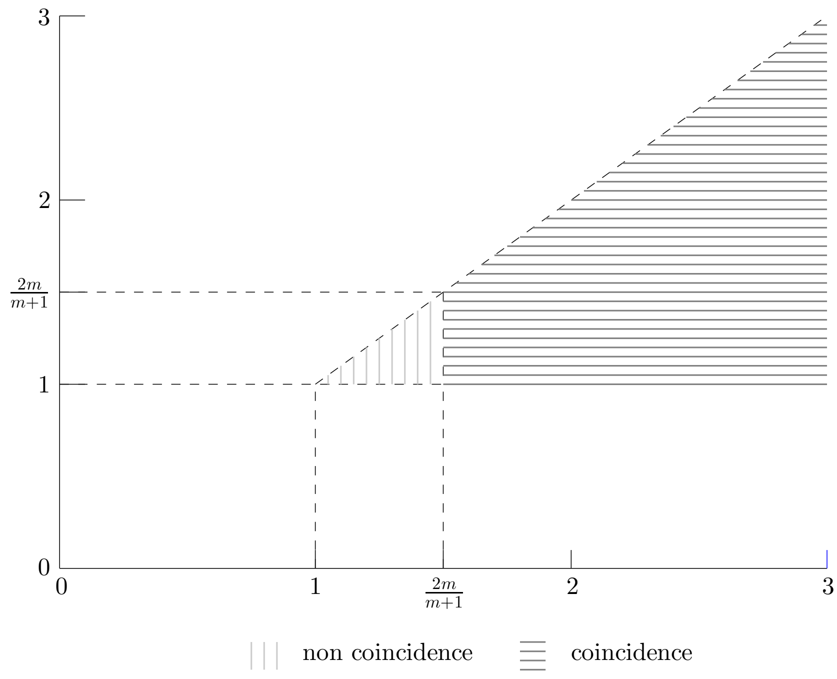

Figure 4.2: Areas of coincidence for $\Pi^m_{\mathrm{mult}(r;s)}({}^m c_0;\mathbb{K})$, $(r,s) \in [1,\infty) \times [1,r]$.

We notice the only case that remains open is the case $(r;s)$ with $2m/(m+1) \leq r < 2$ and $s = r$.

4.4 Absolutely summing multilinear operators

In this section we investigate the optimality of coincidence results within the framework of absolutely summing multilinear operators and, as consequence, we observe that the Defant–Voigt Theorem (see [8, Theorem 3.10], [23, Theorem 3], [49, Corollary 3.2] and [127] for a very interesting approach) is optimal.

Theorem 4.14 (Defant–Voigt)**.** *For all Banach spaces $E_1, ..., E_m$,*

$$\Pi^m_{\mathrm{as}(1;1)}(E_1, ..., E_m;\mathbb{K}) = \mathcal{L}(E_1, ..., E_m;\mathbb{K})$$

Combining the Defant–Voigt Theorem and a canonical inclusion theorem (see [52, Proposition 2.1] and [103, Proposition 3.5]) we conclude that, for $r, s \geq 1$ and $s \leq mr/(mr + 1 - r)$, we have

$$\Pi^m_{\mathrm{as}(r;s)}(E_1, ..., E_m;\mathbb{K}) = \mathcal{L}(E_1, ..., E_m;\mathbb{K})$$

for all $E_1, ..., E_m$. From [137, Proposition 1] it is possible to prove that for $r > 1$ and $r/(mr + 1 - r) \leq t < r$,

$$\Pi^m_{\mathrm{as}(t;\frac{mr}{mr+1-r})}(E_1, ..., E_m;\mathbb{K}) \neq \mathcal{L}(E_1, ..., E_m;\mathbb{K})$$

for some choices of $E_1, ..., E_m$. In fact (repeating an argument used in the proof of Theorem 4.12), given $r > 1$, consider $p > m$ such that $p/(p - m) = r$ and observe that in this case

$mr/(mr+1-r) = p^*$ and thus we just need to prove that for all $p^*/m \le t < p/(p-m)$,

$$\Pi^m_{\mathrm{as}(t;p^*)}(E_1,...,E_m;\mathbb{K}) \neq \mathcal{L}(E_1,...,E_m;\mathbb{K}).$$

From [137, Proposition 1] we know that if $p > m$ and $p^*/m \le t < p/(p-m)$, then there is a continuous m-linear form ϕ such that $\phi \notin \Pi^m_{\mathrm{as}(t;p^*)}(E_1,...,E_m;\mathbb{K})$, i.e.,

$$\Pi^m_{\mathrm{as}(t;p^*)}(E_1,...,E_m;\mathbb{K}) \neq \mathcal{L}(E_1,...,E_m;\mathbb{K}).$$

All these pieces of information provide Figure 4.3, which illustrates for which $(r,s) \in [1,\infty) \times [1,mr]$ we have

$$\mathcal{L}\left(E_1,...,E_m;\mathbb{K}\right) = \Pi^m_{\mathrm{as}(r;s)}\left(E_1,...,E_m;\mathbb{K}\right).$$

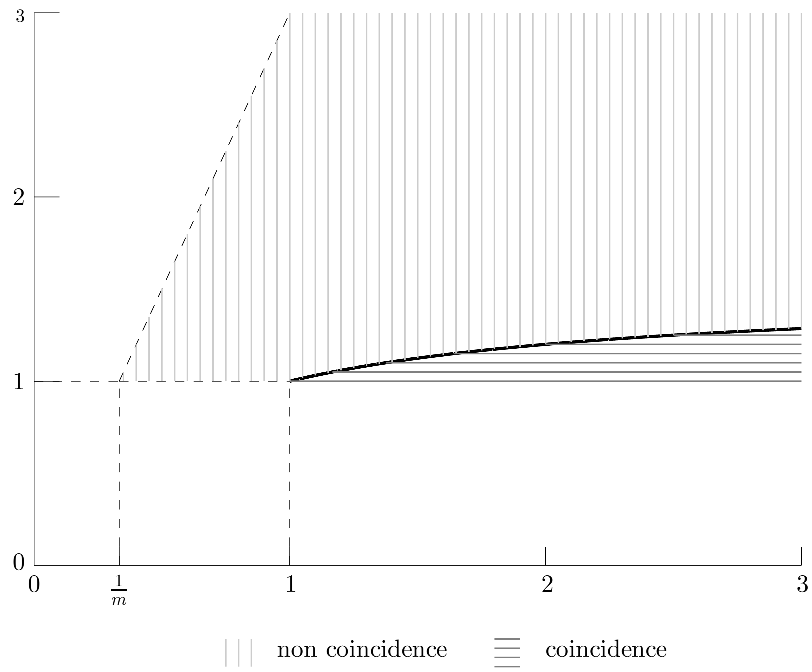

Figure 4.3: Areas of coincidence for $\Pi^m_{\mathrm{as}(r;s)}\left(E_1,...,E_m;\mathbb{K}\right)$, $(r,s) \in [1,\infty) \times [1,mr]$.

The table below details the results of coincidence and non-coincidence in the "boundaries" of Figure 4.3. The only possible open situation is the case $(r;s)$ with $s = 1$ and $r < 1$, which we answer in the next lines.

$\frac{1}{m} \le r < 1$	$s = 1$	not known
$r > \frac{1}{m}$	$s = mr$	non-coincidence
$r \ge 1$	$s = 1$	coincidence
$r \ge 1$	$s = \frac{mr}{mr+1-r}$	coincidence

Theorem 4.15. *The Defant–Voigt Theorem is optimal. More precisely, if $m \ge 1$ is a positive integer, then*

$$\min\left\{ r : \begin{array}{l} \mathcal{L}(E_1,...,E_m;\mathbb{K}) = \Pi^m_{\mathrm{as}(r;1)}(E_1,...,E_m;\mathbb{K}) \textit{ for all} \\ \textit{infinite-dimentional Banach spaces } E_j \end{array} \right\} = 1.$$

Proof. The equality holds for $r = 1$; this is the so called Defant–Voigt Theorem. It remains to prove that the equality does not hold for $r < 1$. This is simple; we just need to choose $E_j = c_0$ for all j and suppose that

$$\mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{as}(r;1)}(E_1, ..., E_m; \mathbb{K}). \tag{4.12}$$

For all positive integers n, consider the m-linear forms $T_n : c_0 \times \cdots \times c_0 \to \mathbb{K}$ defined by $T_n(x^{(1)}, ..., x^{(m)}) = \sum_{j=1}^n x_j^{(1)} \cdots x_j^{(m)}$. Then it is plain that $\|T_n\| = n$ and, from (4.12) and from the Open Mapping Theorem for F-spaces (see [130, Corollary 2.12]), there is a $C \geq 1$ such that

$$\left(\sum_{j=1}^n |T_n(e_j, ..., e_j)|^r \right)^{\frac{1}{r}} \leq C \|T_n\| \prod_{k=1}^m \sup_{\varphi \in B_{E_k^*}} \sum_{j=1}^n |\varphi(e_j)| = Cn,$$

i.e., $n^{1/r} \leq Cn$. Since n is arbitrary, we conclude that $r \geq 1$. □

This simple proposition ensures that the zone defined by $r < 1$ and $s = 1$ in the Figure 4.3 is a non-coincidence zone,i.e., the Defant–Voigt Theorem is optimal. Therefore, we can construct a new table for the results of coincidence and non-coincidence in the "boundaries" of Figure 4.3:

$\frac{1}{m} \leq r < 1$	$s = 1$	non-coincidence
$r \geq \frac{1}{m}$	$s = mr$	non-coincidence
$r \geq 1$	$s = 1$	coincidence
$r \geq 1$	$s = \frac{mr}{mr+1-r}$	coincidence

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Chapter 5

A unified theory and consequences

Our main purpose here is to present a new class of summing multilinear operators, which recovers the class of absolutely and multiple summing operators.

5.1 Multiple summing operators with multiple exponents

For $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty)^m$,we shall consider the space

$$\ell_{\mathbf{p}}(E) := \ell_{p_1}\left(\ell_{p_2}\left(\cdots\left(\ell_{p_m}(E)\right)\cdots\right)\right),$$

namely, a vector matrix $(x_{i_1\ldots i_m})_{i_1,\ldots,i_m=1}^{\infty} \in \ell_{\mathbf{p}}(E)$ if, and only if,

$$\left\|(x_{i_1\ldots i_m})_{i_1,\ldots,i_m=1}^{\infty}\right\|_{\ell_{\mathbf{p}}(E)} := \left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_m=1}^{\infty}\|x_{i_1\ldots i_m}\|_E^{p_m}\right)^{\frac{p_{m-1}}{p_m}}\cdots\right)^{\frac{p_2}{p_1}}\right)^{\frac{1}{p_1}} < +\infty.$$

When $E = \mathbb{K}$, we simply write $\ell_{\mathbf{p}}$. Taking into account all that we have done in previous chapters, the following definition seems natural:

Definition 5.1. *Let $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. A multilinear operator $T : E_1 \times \cdots \times E_m \to F$ is multiple $(\mathbf{q}; \mathbf{p})$-summing if there exist a constant $C > 0$ such that*

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\left\|T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)})\right\|_F^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,p_k}$$

for all $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^{w}(E_k)$. We represent the class of all multiple $(\mathbf{q}; \mathbf{p})$-summing operators by $\Pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}^{m}(E_1, \ldots, E_m; F)$.

Of course, when $q_1 = \cdots = q_m = q$, then

$$\Pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}^{m}(E_1, \ldots, E_m; F) = \Pi_{\mathrm{mult}(q;\mathbf{p})}^{m}(E_1, \ldots, E_m; F).$$

As it happens with absolutely and multiple summing operators, the following result characterizes the multiple $(\mathbf{q};\mathbf{p})$-summing operators.

Proposition 5.2. *Let $T : E_1 \times \cdots \times E_m \to F$ be a continuous multilinear operator and $\mathbf{p},\mathbf{q} \in [1,+\infty)^m$. The following are equivalent:*

(1) T is multiple $(\mathbf{q};\mathbf{p})$-summing;

(2) $\left(T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)})\right)_{j_1,\ldots,j_m=1}^{\infty} \in \ell_{\mathbf{q}}(F)$ whenever $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^{w}(E_k)$.

(3) There exist a constant $C>0$ such that

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left\|T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)})\right\|_F^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \le C\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{n}\right\|_{w,p_k}$$

for all positive integer n and all $(x_j^{(k)})_{j=1}^{n} \in \ell_{p_k}^{w}(E_k)$.

Proof. By definition, it follows that *(1) ⇒ (2)*. Let us prove now that *(2) ⇒ (1)*. Supposing *(2)*, we can define the m-linear operator

$$\begin{array}{rcccl}\widehat{T} & : & \ell_{p_1}^{w}(E_1)\times\cdots\times\ell_{p_m}^{w}(E_m) & \to & \ell_{\mathbf{q}}(F) \\ & & \left((x_j^{(1)})_{j=1}^{\infty},\ldots,(x_j^{(m)})_{j=1}^{\infty}\right) & \mapsto & \left(T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)})\right)_{j_1,\ldots,j_m=1}^{\infty}.\end{array} \tag{5.1}$$

Observe that $\widehat{T}$ is a continuous m-linear operator. In fact, let $((x_{j,s}^{(k)})_{j=1}^{\infty})_{s=1}^{\infty} \subset \ell_{p_k}^{w}(E_k)$, $k=1,\ldots,m$, such that

$$(x_{j,s}^{(k)})_{j=1}^{\infty} \to (x_j^{(k)})_{j=1}^{\infty} \text{ in } \ell_{p_k}^{w}(E_k) \tag{5.2}$$

and

$$\widehat{T}\left((x_{j_1,s}^{(1)})_{j_1=1}^{\infty},\ldots,(x_{j_m,s}^{(m)})_{j_m=1}^{\infty}\right) \to (y_{j_1,\ldots,j_m})_{j_1,\ldots,j_m=1}^{\infty} \text{ in } \ell_{\mathbf{q}}(F). \tag{5.3}$$

From (5.2) we have that for every $k\in\{1,\ldots,m\}$, given $\epsilon>0$, there exist $N\in\mathbb{N}$ which verify

$$s\ge N \Rightarrow \sup_{\varphi\in B_{E_k^*}}\left(\sum_{j=1}^{\infty}\left|\varphi(x_{j,s}^{(k)}-x_j^{(k)})\right|^{p_k}\right)^{\frac{1}{p_k}} < \epsilon.$$

So

$$s\ge N \Rightarrow \sum_{j=1}^{\infty}\left|\varphi(x_{j,s}^{(k)}-x_j^{(k)})\right|^{p_k} < \epsilon^{p_k} \text{ for all } \varphi\in B_{E_k^*} \text{ and all } k\in\{1,\ldots,m\}$$

and thus $|\varphi(x_{j,s}^{(k)}-x_j^{(k)})|<\epsilon$ for all $\varphi\in B_{E_k^*}$ and all $\{j,k\}\in\mathbb{N}\times\{1,\ldots,m\}$. Then, from the Hahn–Banach Theorem we conclude that

$$s\ge N \Rightarrow \left\|x_{j,s}^{(k)}-x_j^{(k)}\right\|_{E_k} = \sup_{\varphi\in B_{E_k^*}}\left|\varphi(x_{j,s}^{(k)}-x_j^{(k)})\right| \le \epsilon \text{ for all } \{j,k\}\in\mathbb{N}\times\{1,\ldots,m\},$$

i.e., $x_{j,s}^{(k)} \to x_j^{(k)}$ in E_k for all $j\in\mathbb{N}$ and all $k\in\{1,\ldots,m\}$. Since T is a continuous multilinear operator, it follows that $T(x_{j_1,s}^{(1)},\ldots,x_{j_m,s}^{(m)}) \to T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)})$ in F for all fixed

$j_1, ..., j_k \in \mathbb{N}$. From (5.3), given $\epsilon > 0$, there exist $M \in \mathbb{N}$ such that

$$s \geq M \Rightarrow \left\| \widehat{T}\left((x_{j_1,s}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m,s}^{(m)})_{j_m=1}^{\infty} \right) - (y_{j_1,...,j_m})_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} < \epsilon,$$

from which we can obtain that, for $s \geq M$, $\left\| T(x_{j_1,s}^{(1)}, ..., x_{j_m,s}^{(m)}) - y_{j_1,...,j_m} \right\|_F < \epsilon$ for all fixed $j_1, ..., j_k \in \mathbb{N}$. We deduce from the uniqueness of the limit that $T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) = y_{j_1,...,j_m}$ for every $j_1, ..., j_k \in \mathbb{N}$. Hence

$$\widehat{T}\left((x_{j_1,s}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m,s}^{(m)})_{j_m=1}^{\infty} \right) = \left(T(x_{j_1,s}^{(1)}, ..., x_{j_m,s}^{(m)}) \right)_{j_1,...,j_m=1}^{\infty} = (y_{j_1,...,j_m})_{j_1,...,j_m=1}^{\infty}$$

and then, from the Closed Graph Theorem, we obtain that $\widehat{T}$ is a continuous m-linear operator. Therefore, there is $C > 0$ such that

$$\begin{aligned} \left\| \left(T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} &= \left\| \widehat{T}\left((x_{j_1}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m}^{(m)})_{j_m=1}^{\infty} \right) \right\|_{\ell_{\mathbf{q}}(F)} \\ &\leq C \left\| (x_j^{(1)})_{j=1}^{\infty} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{\infty} \right\|_{w,p_m}. \end{aligned}$$

(1) ⇒ (3). Fix $n \in \mathbb{N}$ and let $(x_j^{(1)})_{j=1}^{n} \in E_1, ..., (x_j^{(m)})_{j=1}^{n} \in E_m$. Then $(x_j^{(k)})_{j=1}^{\infty} = (x_1^{(k)}, x_2^{(k)}, ..., x_n^{(k)}, 0, 0, ...) \in \ell_{p_k}^{w}(E_k)$ for every $k \in \{1, ..., m\}$. Thus, using *(1)*, we get

$$\begin{aligned} &\left\| \left(T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)} \right) \right)_{j_1,...,j_m=1}^{n} \right\|_{\ell_{\mathbf{q}}(F} = \left\| \left(T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \\ &\leq C \left\| (x_j^{(1)})_{j=1}^{\infty} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{\infty} \right\|_{w,p_m} = C \left\| (x_j^{(1)})_{j=1}^{n} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{n} \right\|_{w,p_m}. \end{aligned}$$

(3) ⇒ (1). Consider $(x_j^{(1)})_{j=1}^{\infty} \in \ell_{p_1}^{w}(E_1), ..., (x_j^{(m)})_{j=1}^{\infty} \in \ell_{p_m}^{w}(E_m)$. Therefore

$$\begin{aligned} &\left\| \left(T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F} = \sup_n \left\| \left(T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{n} \right\|_{\ell_{\mathbf{q}}(F)} \\ &\leq C \sup_n \left\| (x_j^{(1)})_{j=1}^{n} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{n} \right\|_{w,p_m} = C \left\| (x_j^{(1)})_{j=1}^{\infty} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{\infty} \right\|_{w,p_m}. \end{aligned}$$

This concludes the proof. □

It is not to difficult to prove that $\Pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}^{m}(E_1, ..., E_m; F)$ is a subspace of $\mathcal{L}(E_1, ..., E_m; F)$ and the infimum of the constants satisfying the above definition (Definition 5.1), i.e.,

$$\inf \left\{ C \geq 0 \; ; \; \begin{array}{c} \left\| \left(T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \leq C \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,p_k}, \\ \text{for all } (x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^{w}(E_k), k = 1, ..., m \end{array} \right\}$$

defines a norm in $\Pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}^{m}(E_1, \ldots, E_m; F)$, which will be denoted by $\pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T)$.

Proposition 5.3. *Let* $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. *If* $T \in \Pi^m_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(E_1, \ldots, E_m; F)$, *then*

$$\|T\|_{\mathcal{L}(E_1,\ldots,E_m;F)} \leq \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T).$$

Proof. Consider $x_j \in B_{E_j}$, $j = 1, ..., m$, and define $(x_i^{(j)})_{i=1}^{\infty} := (x_j, 0, ...)$. It is clear that $(x_i^{(j)})_{i=1}^{\infty} \in \ell^w_{p_j}(E_j)$ for every $j = 1, ..., m$. Therefore, for $T \in \Pi^m_{(\mathbf{q};\mathbf{p})}(E_1, ..., E_m; F)$,

$$\begin{aligned}
&\|T(x_1, ..., x_m)\|_F \\
&= \left(\sum_{j_1=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \left\| T(x^{(1)}_{j_1}, ..., x^{(m)}_{j_m}) \right\|_F^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \\
&\leq \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T) \prod_{j=1}^{m} \left\| (x_i^{(j)})_{i=1}^{\infty} \right\|_{w,p_j} \\
&= \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T) \prod_{j=1}^{m} \sup_{\varphi \in B_{E_j^*}} \left(\sum_{i=1}^{\infty} \left| \varphi(x_i^{(j)}) \right|^{p_j} \right)^{\frac{1}{p_j}} \\
&= \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T) \prod_{j=1}^{m} \sup_{\varphi \in B_{E_j^*}} |\varphi(x_j)| \\
&= \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T) \prod_{j=1}^{m} \|x_j\|_{E_j} = \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T),
\end{aligned}$$

as required. □

Given $T \in \Pi^m_{(\mathbf{q};\mathbf{p})}(E_1, \ldots, E_m; F)$, we have defined in (5.1) the continuous m-linear operator $\widehat{T}$. Let us prove now that

$$\|\widehat{T}\| = \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T). \tag{5.4}$$

In fact, first note that

$$\begin{aligned}
\left\| \left(T(x^{(1)}_{j_1}, ..., x^{(m)}_{j_m}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} &= \left\| \widehat{T} \left((x_j^{(1)})_{j=1}^{\infty}, ..., (x_j^{(m)})_{j=1}^{\infty} \right) \right\|_{\ell_{\mathbf{q}}(F)} \\
&\leq \|\widehat{T}\| \textstyle\prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,p_k},
\end{aligned}$$

that is, $\pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T) \leq \|\widehat{T}\|$. On the other hand, we have

$$\begin{aligned}
\|\widehat{T}\| &= \sup_{(x_j^{(k)})_{j=1}^{\infty} \in B_{\ell^w_{p_k}(E_k)}} \left\| \widehat{T} \left((x_j^{(1)})_{j=1}^{\infty}, ..., (x_j^{(m)})_{j=1}^{\infty} \right) \right\| \\
&= \sup_{(x_j^{(k)})_{j=1}^{\infty} \in B_{\ell^w_{p_k}(E_k)}} \left\| \left(T(x^{(1)}_{j_1}, ..., x^{(m)}_{j_m}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \\
&\leq \sup_{(x_j^{(k)})_{j=1}^{\infty} \in B_{\ell^w_{p_k}(E_k)}} \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T) \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,p_k} \\
&= \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T),
\end{aligned}$$

which proves (5.4).

We can naturally define the continuous operator

$$\begin{array}{rcccl}\widehat{\theta} & : & \Pi^m_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(E_1,\ldots,E_m;F) & \to & \mathcal{L}\left(\ell^w_{p_1}(E_1),\ldots,\ell^w_{p_m}(E_m);\ell_{\mathbf{q}}(F)\right) \\ & & T & \mapsto & \widehat{T},\end{array}$$

which, due to equation (5.4), is an isometry. These facts allow us to prove the following:

Theorem 5.4. *Let* $\mathbf{p},\mathbf{q}\in[1,+\infty)^m$. *Then* $(\Pi^m_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(E_1,...,E_m;F),\pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(\,\cdot\,))$ *is a Banach space.*

Proof. Let $(T_j)_{j=1}^\infty$ be a Cauchy sequence in $\Pi^m_{(\mathbf{q};\mathbf{p})}(E_1,...,E_m;F)$. Since $\|\cdot\|\leq\pi_{(\mathbf{q};\mathbf{p})}(\cdot)$ (Proposition 5.3), it follows that $(T_j)_{j=1}^\infty$ is also a Cauchy sequence in $\mathcal{L}(E_1,...,E_m;F)$. Thus, consider $T\in\mathcal{L}(E_1,...,E_m;F)$ such that $T_j\to T$ in $\mathcal{L}(E_1,...,E_m;F)$. Let us prove that $T\in\Pi^m_{(\mathbf{q};\mathbf{p})}(E_1,...,E_m;F)$. In fact, let $(x_j^{(k)})_{j=1}^\infty\subset\ell^w_{p_k}(E_k)$, $k=1,...,m$. It is enough to prove that $(T(x_{j_1}^{(1)},...,x_{j_m}^{(m)}))_{j_1,...,j_m=1}^\infty\in\ell_{\mathbf{q}}(F)$. Since $\widehat{\theta}$ is an isometry, $(\widehat{T}_j)_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{L}(\ell^w_{p_1}(E_1),...,\ell^w_{p_m}(E_m);\ell_{\mathbf{q}}(F))$, which is a Banach space because $\ell_{\mathbf{q}}(F)$ is a Banach space. Thus, there exist $S\in\mathcal{L}(\ell^w_{p_1}(E_1),...,\ell^w_{p_m}(E_m);\ell_{\mathbf{q}}(F))$ such that $\widehat{T}_j\to S$ in $\mathcal{L}(\ell^w_{p_1}(E_1),...,\ell^w_{p_m}(E_m);\ell_{\mathbf{q}}(F))$. Therefore, if we consider $P_{k_1,...,k_m}:\ell_{\mathbf{q}}(F)\to F$ the continuous linear operator given by

$$(y_{j_1\cdots j_m})_{j_1,...,j_m=1}^\infty\mapsto y_{k_1\cdots k_m},$$

and $\epsilon>0$ a positive real number, there exist a positive integer N such that

$$\begin{aligned}
&\left\|P_{k_1,...,k_m}\left(S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)-T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F\\
&\leq\left\|P_{k_1,...,k_m}\left(\widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)\right.\\
&\qquad\left.-\ P_{k_1,...,k_m}\left(S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)\right\|_F\\
&\qquad+\left\|P_{k_1,...,k_m}\left(\widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)-T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F\\
&=\left\|P_{k_1,...,k_m}\left(\widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right.\right.\\
&\qquad\left.\left.-\ S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)\right\|_F\\
&\qquad+\left\|P_{k_1,...,k_m}\left((T_j(x_{j_1}^{(1)},...,x_{j_m}^{(m)}))_{j_1,...,j_m=1}^\infty\right)-T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F\\
&\leq\|P_{k_1,...,k_m}\|\left\|\widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right.\\
&\qquad\left.-\ S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right\|_{\ell_{\mathbf{q}}(F)}\\
&\qquad+\left\|T_j(x_{k_1}^{(1)},...,x_{k_m}^{(m)})-T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F\\
&\leq C\|P_{k_1,...,k_m}\|\|\widehat{T}_j-S\|+C\|T_j-T\|\\
&<\epsilon
\end{aligned}$$

for every $j\geq N$. Then $P_{k_1,...,k_m}(S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty))=T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})$ for all

$k_1, ..., k_m \in \mathbb{N}$, and consequently

$$S((x_{j_1}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m}^{(m)})_{j_m=1}^{\infty}) = (T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}))_{j_1,...,j_m=1}^{\infty}. \tag{5.5}$$

This proves that $(T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}))_{j_1,...,j_m=1}^{\infty} \in \ell_{\mathbf{q}}(F)$, as required.

By definition we have $\widehat{T}((x_{j_1}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m}^{(m)})_{j_m=1}^{\infty}) = (T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}))_{j_1,...,j_m=1}^{\infty}$. Replacing the above expression in (5.5) we conclude that $\widehat{T} = S$. Thus, given $\epsilon > 0$, it follows from (5.4) that, for sufficiently large j, $\pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(T_j - T) = \|\widehat{T_j - T}\| = \|\widehat{T}_j - \widehat{T}\| = \|\widehat{T}_j - S\| < \epsilon$, that is, $T_j \to T$ in $\Pi_{(\mathbf{q};\mathbf{p})}^m(E_1, ..., E_m; F)$, and this proves that $(\Pi_{(\mathbf{q};\mathbf{p})}^m(E_1, ..., E_m; F), \pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(\,\cdot\,))$ is a Banach space. □

Using that $\ell_q \setminus \ell_p \neq \emptyset$ if $1 \leq p < q \leq \infty$, let us prove the following result.

Proposition 5.5. *If $q_j < p_j$ for some $j \in \{1, \ldots, m\}$, then*

$$\Pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}^m(E_1, \ldots, E_m; F) = \{0\}.$$

Proof. Since $q_j < p_j$, we know that there is a sequence $(\alpha_i)_{i=1}^{\infty} \in \ell_{p_j} \setminus \ell_{q_j}$. Let $x_j \in E_j \setminus \{0\}$. Then for all $\varphi \in E_j'$ we have

$$\sum_{i=1}^{\infty} |\varphi(\alpha_i x_j)|^{p_j} \leq \sum_{i=1}^{\infty} \|\varphi\|^{p_j} |\alpha_i|^{p_j} \|x_j\|^{p_j} = \|\varphi\|^{p_j} \|x_j\|^{p_j} \sum_{i=1}^{\infty} |\alpha_i|^{p_j} < \infty,$$

i.e., $(\alpha_i x_j)_{i=1}^{\infty} \in \ell_{p_j}^w(E_j)$. By means of contradiction, assume that there exists

$$T \in \Pi_{(\mathbf{q};\mathbf{p})}^m(E_1, ..., E_m; F) \setminus \{0\}.$$

Then, we can take $x_k \in E_k \setminus \{0\}$, $k \in \{1, ..., m\} \setminus \{j\}$, such that $T(x_1, ..., x_m) \neq 0$. For each $k \in \{1, ..., m\} \setminus \{j\}$ let us consider $(x_i^{(k)})_{i=1}^{\infty} = (x_k, 0, ...)$. Since $(x_i^{(k)})_{i=1}^{\infty} \in \ell_{p_k}^w(E_k)$ for every $k \in \{1, ..., m\} \setminus \{j\}$ and $(\alpha_i x_j)_{i=1}^{\infty} \in \ell_{p_j}^w(E_j)$, Proposition 5.2 ensures that

$$\begin{aligned}
&\left\| \left(T(x_{i_1}^{(1)}, ..., x_{i_{j-1}}^{(j-1)}, \alpha_{i_j} x_j, x_{i_{j+1}}^{(j+1)}, ..., x_{i_m}^{(m)}) \right)_{i_1,...,i_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \\
&\leq C \left(\prod_{\substack{k=1 \\ k \neq j}}^{m} \left\| (x_i^{(k)})_{i=1}^{\infty} \right\|_{w,p_k} \right) \|(\alpha_i x_j)_{i=1}^{\infty}\|_{w,p_j}.
\end{aligned}$$

However,

$$\begin{aligned}
&\left\| \left(T(x_{i_1}^{(1)}, ..., x_{i_{j-1}}^{(j-1)}, \alpha_{i_j} x_j, x_{i_{j+1}}^{(j+1)}, ..., x_{i_m}^{(m)}) \right)_{i_1,...,i_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \\
&= \left(\sum_{i_j=1}^{\infty} \left\| T\left(x_1, ..., x_{j-1}, \alpha_{i_j} x_j, x_{j+1}, ..., x_m\right) \right\|^{q_j} \right)^{\frac{1}{q_j}} \\
&= \|T(x_1, ..., x_m)\| \left(\sum_{i=1}^{\infty} |\alpha_i|^{q_j} \right)^{\frac{1}{q_j}},
\end{aligned}$$

from where we can conclude

$$\|T(x_1,...,x_m)\| \left(\sum_{i=1}^{\infty} |\alpha_i|^{q_j}\right)^{\frac{1}{q_j}} \leq C \left(\prod_{\substack{k=1\\k\neq j}}^{m} \left\|(x_i^{(k)})_{i=1}^{\infty}\right\|_{w,p_k}\right) \|(\alpha_i x_j)_{i=1}^{\infty}\|_{w,p_j}.$$

Therefore, $\sum_{i=1}^{\infty} \left|\alpha_{i_j}\right|^{q_j} < \infty$, which is a contradiction since $(\alpha_i)_{i=1}^{\infty} \in \ell_{p_j} \setminus \ell_{q_j}$. □

Using the generalized Bohnenblust–Hille inequality (Theorem 1.1) together with the fact that $\mathcal{L}(c_0, E)$ and $\ell_1^w(E)$ are isometrically isomorphic (see [72, Proposition 2.2]), it is possible to prove the following result (recall the notation of the constants $B_{\mathbb{K},m,(q_1,\dots,q_m)}^{\text{mult}}$ in Theorem 1.1). The proof is similar to the proof of Theorem 4.3 and we omit it.

Proposition 5.6 (Generalized Bohnenblust–Hille *re-written*)**.** *If* $\mathbf{q} = (q_1,\dots,q_m) \in [1,2]^m$ *are such that* $|1/\mathbf{q}| \leq (m+1)/2$, *then*

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\left|T(x_{j_1}^{(1)},\dots,x_{j_m}^{(m)})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq B_{\mathbb{K},m,\mathbf{q}}^{\text{mult}} \|T\| \prod_{k=1}^{m} \left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,1},$$

for all m-linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$ and all sequences $(x_j^{(k)})_{j=1}^{\infty} \in \ell_1^w(E_k)$, $k = 1,\dots,m$. In other words, if $\mathbf{q} \in [1,2]^m$ are such that $|1/\mathbf{q}| \leq (m+1)/2$ we have the following coincidence result:

$$\Pi_{\text{mult}(\mathbf{q};1,\dots,1)}^{m}(E_1,\dots,E_m;\mathbb{K}) = \mathcal{L}(E_1,\dots,E_m;\mathbb{K}).$$

With the same idea of the proof of Proposition 5.6 (but now using $\mathcal{L}(c_0,E) = \ell_1^w(E)$ and $\mathcal{L}(\ell_p,E) = \ell_{p^*}^w(E)$), we can re-write the Theorems 1.1 and 1.2 (recall the notation for the constants on each result):

Proposition 5.7. *Let $m \geq 1$, $\mathbf{p} := (p_1,\dots,p_m) \in [1,\infty]^m$.*

(1) (Generalized Hardy–Littlewood inequality for $0 \leq |1/\mathbf{p}| \leq 1/2$ *re-written*) *Let $0 \leq |1/\mathbf{p}| \leq 1/2$ and $\mathbf{q} := (q_1,\dots,q_m) \in \left[(1-|1/\mathbf{p}|)^{-1}, 2\right]^m$ such that $|1/\mathbf{q}| \leq (m+1)/2 - |1/\mathbf{p}|$. Then, for all continuous m-linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,*

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_m=1}^{\infty}\left|T\left(x_{i_1}^{(1)},\dots,x_{i_m}^{(m)}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\text{mult}} \|T\| \prod_{k=1}^{m} \left\|(x_i^{(k)})_{i=1}^{\infty}\right\|_{w,p_k^*},$$

regardless of the sequences $(x_i^{(k)})_{i=1}^{\infty} \in \ell_{p_k^*}^w(E_k)$, $k = 1,\dots,m$. In other words, if $\mathbf{p} = (p_1,\dots,p_m) \in [1,\infty]^m$ and $\mathbf{q} = (q_1,\dots,q_m) \in \left[(1-|1/\mathbf{p}|)^{-1}, 2\right]^m$ are such that $0 \leq |1/\mathbf{p}| \leq 1/2$ and $|1/\mathbf{q}| \leq (m+1)/2 - |1/\mathbf{p}|$, then

$$\Pi_{\text{mult}(\mathbf{q};p_1^*,\dots,p_m^*)}^{m}(E_1,\dots,E_m;\mathbb{K}) = \mathcal{L}(E_1,\dots,E_m;\mathbb{K}).$$

(2) (Hardy–Littlewood inequality for $1/2 \leq |1/\mathbf{p}| < 1$ *re-written*) *If* $1/2 \leq |1/\mathbf{p}| < 1$, *then, for all continuous* m*-linear forms* $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \left| T\left(x_{i_1}^{(1)}, \ldots, x_{i_m}^{(m)} \right) \right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D_{\mathbb{K},m,\mathbf{p}}^{\mathrm{mult}} \|T\| \prod_{k=1}^{m} \left\| (x_i^{(k)})_{i=1}^{\infty} \right\|_{w,p_k^*}$$

regardless of the sequences $(x_i^{(k)})_{i=1}^{\infty} \in \ell_{p_k^*}^{w}(E_k)$, $k = 1, \ldots, m$. *In other words, if* $1/2 \leq |1/\mathbf{p}| < 1$, *then*

$$\Pi^{m}_{\mathrm{mult}\left((1-|1/\mathbf{p}|)^{-1}; p_1^*, \ldots, p_m^*\right)} (E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}).$$

The following proposition illustrates how, within this framework, coincidence results for m-linear forms can be extended to $m+1$-linear forms.

Proposition 5.8. *Let* $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. *If*

$$\Pi^{m}_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}),$$

then

$$\Pi^{m+1}_{\mathrm{mult}(\mathbf{q},2;\mathbf{p},1)}(E_1, \ldots, E_m, E_{m+1}; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m, E_{m+1}; \mathbb{K}).$$

Proof. Let us first prove that, for all continuous $(m+1)$-linear forms $T : E_1 \times \cdots \times E_m \times c_0 \to \mathbb{K}$, there exist a constant $C > 0$ such that

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_{m+1}=1}^{n} \left| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}} \right) \right|^2 \right)^{\frac{q_m}{2}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C A_{\mathbb{K},q_m}^{-1} \|T\| \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{n} \right\|_{w,p_k}, \tag{5.6}$$

where $A_{\mathbb{K},q_m}$ is the constant of the Khintchine inequality (1). In fact, from Khintchine's inequality, we have

$$\begin{aligned} & A_{\mathbb{K},q_m} \left(\sum_{j_{m+1}=1}^{n} \left| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}} \right) \right|^2 \right)^{\frac{1}{2}} \\ & \leq \left(\int_0^1 \left| \sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}} \right) \right|^{q_m} dt \right)^{\frac{1}{q_m}} \\ & = \left(\int_0^1 \left| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, \sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) e_{j_{m+1}} \right) \right|^{q_m} dt \right)^{\frac{1}{q_m}}. \end{aligned}$$

Thus,

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_{m+1}=1}^{n} \left| T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}}) \right|^2 \right)^{\frac{q_m}{2}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}}$$

$$\leq A_{\mathbb{K},q_m}^{-1}\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\int_0^1\left|T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)},\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}})\right|^{q_m}dt\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$= A_{\mathbb{K},q_m}^{-1}\left(\sum_{j_1=1}^{n}\left(\cdots\left(\int_0^1\sum_{j_m=1}^{n}\left|T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)},\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}})\right|^{q_m}dt\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$\leq A_{\mathbb{K},q_m}^{-1}\sup_{t\in[0,1]}\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)},\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$\leq A_{\mathbb{K},q_m}^{-1}\sup_{t\in[0,1]}\pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}\Big(T(\cdot,\ldots,\cdot,\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}})\Big)\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{n}\right\|_{w,p_k}$$

Since $\|\cdot\|\leq\pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(\,\cdot\,)$ (see Proposition 5.3) and since, by hypothesis

$$\mathcal{L}(E_1,\ldots,E_m;\mathbb{K})=\Pi^m_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(E_1,\ldots,E_m;\mathbb{K}),$$

the Open Mapping Theorem ensures that the norms $\pi_{\mathrm{mult}(\mathbf{q};\mathbf{p})}(\,\cdot\,)$ and $\|\cdot\|$ are equivalents. Therefore, there exists a constant $C>0$ such that

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_{m+1}=1}^{n}\left|T\left(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)},e_{j_{m+1}}\right)\right|^{2}\right)^{\frac{q_m}{2}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$\leq CA_{\mathbb{K},q_m}^{-1}\sup_{t\in[0,1]}\left\|T\left(\cdot,\ldots,\cdot,\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}}\right)\right\|\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{n}\right\|_{w,p_k}$$

$$\leq CA_{\mathbb{K},q_m}^{-1}\|T\|\sup_{t\in[0,1]}\left\|\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}}\right\|\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{n}\right\|_{w,p_k}$$

$$\leq CA_{\mathbb{K},q_m}^{-1}\|T\|\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{n}\right\|_{w,p_k}.$$

Let $T\in\mathcal{L}(E_1,\ldots,E_m,E_{m+1};\mathbb{K})$, $(x_j^{(k)})_{j=1}^{n}\in\ell_{p_k}^{w}(E_k)$, $k=1,\ldots,m$, and $(x_j^{(m+1)})_{j=1}^{n}\in\ell_1^w(E_{m+1})$. From [72, Proposition 2.2] we have the boundedness of the linear operator $u:c_0\to E_{m+1}$ such that $e_j\mapsto u(e_j)=x_j^{(m+1)}$ and $\|u\|=\|(x_j^{(m+1)})_{j=1}^{n}\|_{1,w}$. Then, $S:E_1\times\cdots\times E_m\times c_0\to\mathbb{K}$ defined by $S(y_1,\ldots,y_{m+1})=T(y_1,\ldots,y_m,u(y_{m+1}))$ is a continuous $(m+1)$-linear form and $\|S\|\leq\|T\|\|u\|$. Therefore, from (5.6),

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_{m+1}=1}^{n}\left|T\left(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)},x_{j_{m+1}}^{(m+1)}\right)\right|^{2}\right)^{\frac{q_m}{2}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$=\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{jm+1=1}^{n}\left|S\left(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)},e_{j_{m+1}}\right)\right|^{2}\right)^{\frac{q_m}{2}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$
\begin{aligned}
&\le CA_{\mathbb{K},q_m}^{-1} \|T\| \|u\| \left\|(x_j^{(1)})_{j=1}^n\right\|_{w,p_1} \cdots \left\|(x_j^{(m)})_{j=1}^n\right\|_{w,p_m} \\
&= CA_{\mathbb{K},q_m}^{-1} \|T\| \left\|(x_j^{(1)})_{j=1}^n\right\|_{w,p_1} \cdots \left\|(x_j^{(m)})_{j=1}^n\right\|_{w,p_m} \left\|(x_j^{(m+1)})_{j=1}^n\right\|_{w,1},
\end{aligned}
$$

i.e., $T \in \Pi^{m+1}_{\mathrm{mult}(\mathbf{q},2;\mathbf{p},1)}(E_1, \ldots, E_{m+1}; \mathbb{K})$. □

5.2 Partially multiple summing operators: The unifying concept

In addition to Bohnenblust–Hille and Hardy–Littlewood inequalities (see (1.1) and Theorems 1.1 and 1.2, respectively), the following results on summability of m-linear forms $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$ are well known.

- Aron and Globevnik ([20], 1989): For every continuous m-linear form $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$,
$$\sum_{i=1}^{\infty} |T(e_i, \ldots, e_i)| \le \|T\|, \tag{5.7}$$
and the exponent 1 is optimal.

- Zalduendo ([137], 1993): Let $|1/\mathbf{p}| < 1$. For every continuous m-linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$,
$$\left(\sum_{i=1}^{\infty} |T(e_i, ..., e_i)|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \le \|T\|, \tag{5.8}$$
and the exponent $1/(1 - |1/\mathbf{p}|)$ is optimal.

Our aims in this section is to present a unified version of the Bohnenblust–Hille and the Hardy–Littlewood inequalities with partial sums (i.e., it was shown what happens when some of the indices of the sums $i_1, \ldots, i_m$ are repeated) which also encompasses Zalduendo's and Aron–Globevnik's inequalities and to present a new class of summing multilinear operators, recovering the class of absolutely and multiple summing operators. To achieve this purpose, we will first establish some notations and results.

Let us establish the following notation: for Banach spaces $E_1, \ldots, E_m$ and an element $x_j \in E_j$, for some $j \in \{1, \ldots, m\}$, the symbol $x_j \cdot e_j$ represents the vector $x_j \cdot e_j \in E_1 \times \cdots \times E_m$ such that its j-th coordinate is $x_j \in E_j$, and 0 otherwise. The next result (for a detailed proof see [2]) will be an important tool to obtain the forthcoming Lemma 5.10, wich is crucial to the proof of the Hardy–Littlewood inequalities with partial sums. In the following we keep the notation of the constant $B^{\mathrm{mult}}_{\mathbb{K},k,(q_1,\ldots,q_k)}$ from Theorem 1.1.

Theorem 5.9 (Generalized Bohnenblust–Hille inequality with partial sums)**.** *Let m, k be positive integers with $1 \le k \le m$, and $\mathbf{q} := (q_1, \ldots, q_k) \in [1, 2]^k$ such that $1/q_1 + \cdots + 1/q_k \le (k + 1)/2$. Let also $\mathcal{I} = \{I_1, \ldots, I_k\}$ be a family of non-void disjoints subsets of $\{1, \ldots, m\}$ such that $\cup_{i=1}^k I_i = \{1, \ldots, m\}$, that is, $\mathcal{I}$ is a partition of $\{1, \ldots, m\}$. Then,* *for all bounded m-linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$,*

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_k=1}^{\infty}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} e_{i_n}\cdot e_j\right)\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq B^{\mathrm{mult}}_{\mathbb{K},k,(q_1,\ldots,q_k)}\|T\|.$$

Lemma 5.10. *Let $m, N \geq 1$ and let $1 \leq k \leq m$ such that $\{1,...,m\}$ is the disjoint union of non-void proper subsets $I_1,...,I_k$. Assume $\mathbf{p} := (p_1,...,p_m) \in [1,\infty]^m$ is such that $1/p_1 + \cdots + 1/p_m \leq 1/2$ and let $\lambda = 1/(1-|1/\mathbf{p}|)$. Then for every continuous m-linear form $T : \ell_{p_1}^N \times \cdots \times \ell_{p_m}^N \to \mathbb{K}$ we have, for each $r \in \{1,...,k\}$,*

$$\left(\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} e_{i_n}\cdot e_j\right)\right|^{2}\right)^{\frac{\lambda}{2}}\right)^{\frac{1}{\lambda}} \leq B^{\mathrm{mult}}_{\mathbb{K},k,(1,2,\ldots,2)}\|T\|.$$

Proof. Let $C = B^{\mathrm{mult}}_{\mathbb{K},k,(1,2,\ldots,2)}$. Let us suppose that $1 \leq s \leq m$ and that

$$\left(\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} e_{i_n}\cdot e_j\right)\right|^{2}\right)^{\frac{\lambda_{s-1}}{2}}\right)^{\frac{1}{\lambda_{s-1}}} \leq C\|T\|$$

is true for all continuous m–linear forms $T : \ell_{p_1}^N \times \cdots \times \ell_{p_{s-1}}^N \times \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{K}$ and for all $r \in \{1,...,k\}$, where $\lambda_i = 1/(1-(1/p_1 + \cdots + 1/p_i))$, $i = 0,...,m$. Let us prove that

$$\left(\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} e_{i_n}\cdot e_j\right)\right|^{2}\right)^{\frac{\lambda_s}{2}}\right)^{\frac{1}{\lambda_s}} \leq C\|T\|$$

for all continuous m–linear forms $T : \ell_{p_1}^N \times \cdots \times \ell_{p_s}^N \times \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{K}$ and for all $r \in \{1,...,k\}$. The initial case ($p_1 = \cdots = p_m = \infty$) is a consequence of the Theorem 5.9. In fact, we just need to observe that $\lambda_0 = 1$ and that $(k-1)/2 + 1/\lambda_0 = (k+1)/2$. Consider $T \in \mathcal{L}(\ell_{p_1}^N,...,\ell_{p_s}^N,\ell_\infty^N,...,\ell_\infty^N;\mathbb{K})$ and for each $x \in B_{\ell_{p_s}^N}$ define

$$\begin{array}{rcll} T^{(x)} & : & \ell_{p_1}^N \times \cdots \times \ell_{p_{s-1}}^N \times \ell_\infty^N \times \cdots \times \ell_\infty^N & \to \mathbb{K} \\ & & (z^{(1)},...,z^{(m)}) & \mapsto T(z^{(1)},...,z^{(s-1)},xz^{(s)},z^{(s+1)},...,z^{(m)}), \end{array}$$

with $xz^{(s)} = (x_i z_i^{(s)})_{i=1}^N$. Observe that $\|T\| \geq \sup\{\|T^{(x)}\| : x \in B_{\ell_{p_s}^N}\}$. Consider $k_s \in \{1,...,k\}$ such that $s \in I_{k_s}$. By applying the induction hypothesis to $T^{(x)}$, we get

$$\left(\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} e_{i_n}\cdot e_j\right)\right|^{2}\left|x_{i_{k_s}}\right|^2\right)^{\frac{\lambda_{s-1}}{2}}\right)^{\frac{1}{\lambda_{s-1}}}$$

$$= \left(\sum_{i_r=1}^{N} \left(\sum_{\widehat{i_r}=1}^{N} \left| T \left(\sum_{\substack{n=1 \\ n \neq k_s}}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j + \sum_{j \in I_{k_s} \setminus \{s\}} e_{i_{k_s}} \cdot e_j + x e_{i_{k_s}} \cdot e_s \right) \right|^2 \right)^{\frac{\lambda_{s-1}}{2}} \right)^{\frac{1}{\lambda_{s-1}}}$$

$$= \left(\sum_{i_r=1}^{N} \left(\sum_{\widehat{i_r}=1}^{N} \left| T^{(x)} \left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_{s-1}}{2}} \right)^{\frac{1}{\lambda_{s-1}}} \leq C \|T^{(x)}\| \leq C\|T\| \tag{5.9}$$

for all $r = 1, ..., k$.

We will analyze two cases:

- $r = k_s$.

Since $(p_i/\lambda_{i-1})^* = \lambda_i/\lambda_{i-1}$ for all $i = 1, ..., m$, we conclude that

$$\begin{aligned}
&\left(\sum_{i_{k_s}=1}^{N} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_s}{2}} \right)^{\frac{1}{\lambda_s}} \\
&= \left(\sum_{i_{k_s}=1}^{N} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_{s-1}}{2} \left(\frac{p_s}{\lambda_{s-1}} \right)^*} \right)^{\frac{1}{\lambda_{s-1}} \frac{1}{\left(\frac{p_s}{\lambda_{s-1}} \right)^*}} \\
&= \left\| \left(\left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_{s-1}}{2}} \right)_{i_{k_s}=1}^{N} \right\|_{\left(\frac{p_s}{\lambda_{s-1}} \right)^*}^{\frac{1}{\lambda_{s-1}}} \\
&= \left(\sup_{y \in B_{\ell^N_{\frac{p_s}{\lambda_{s-1}}}}} \sum_{i_{k_s}=1}^{N} |y_{i_{k_s}}| \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_{s-1}}{2}} \right)^{\frac{1}{\lambda_{s-1}}} \\
&= \left(\sup_{x \in B_{\ell^N_{p_s}}} \sum_{i_{k_s}=1}^{N} |x_{i_{k_s}}|^{\lambda_{s-1}} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_{s-1}}{2}} \right)^{\frac{1}{\lambda_{s-1}}} \\
&= \sup_{x \in B_{\ell^N_{p_s}}} \left(\sum_{i_{k_s}=1}^{N} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T \left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^2 |x_{i_{k_s}}|^2 \right)^{\frac{\lambda_{s-1}}{2}} \right)^{\frac{1}{\lambda_{s-1}}} \\
&\leq C\|T\|
\end{aligned}$$

where the last inequality holds by (5.9).

- $r \neq k_s$.

Let us first suppose that $1/p_1 + \cdots + 1/p_s < 1/2$. It is important to note that in this case $\lambda_{s-1} < \lambda_s < 2$ for all $s \in \{1, ..., m\}$. Denoting, for $r = 1, ..., k$,

$$S_r = \left(\sum_{\widehat{i_r}=1}^{N} \left| T\left(\sum_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{1}{2}}$$

we get

$$\begin{aligned}
&\sum_{i_r=1}^{N} \left(\sum_{\widehat{i_r}=1}^{N} \left| T\left(\sum_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_s}{2}} = \sum_{i_r=1}^{N} S_r^{\lambda_s} = \sum_{i_r=1}^{N} S_r^{\lambda_s - 2} S_r^2 \\
&= \sum_{i_r=1}^{N} \sum_{\widehat{i_r}=1}^{N} \frac{\left| T\left(\sum\limits_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2}{S_r^{2-\lambda_s}} = \sum_{i_{k_s}=1}^{N} \sum_{\widehat{i_{k_s}}=1}^{N} \frac{\left| T\left(\sum\limits_{n=1}^{k} \sum\limits_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2}{S_r^{2-\lambda_s}} \\
&= \sum_{i_{k_s}=1}^{N} \sum_{\widehat{i_{k_s}}=1}^{N} \frac{\left| T\left(\sum\limits_{n=1}^{k} \sum\limits_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^{\frac{2(2-\lambda_s)}{2-\lambda_{s-1}}}}{S_r^{2-\lambda_s}} \left| T\left(\sum_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^{\frac{2(\lambda_s - \lambda_{s-1})}{2-\lambda_{s-1}}}.
\end{aligned}$$

Therefore, using Hölder's inequality twice we obtain

$$\begin{aligned}
&\sum_{i_r=1}^{N} \left(\sum_{\widehat{i_r}=1}^{N} \left| T\left(\sum_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_s}{2}} \\
&\leq \sum_{i_{k_s}=1}^{N} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \frac{\left| T\left(\sum\limits_{n=1}^{k} \sum\limits_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2}{S_r^{2-\lambda_{s-1}}} \right)^{\frac{2-\lambda_s}{2-\lambda_{s-1}}} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T\left(\sum_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_s - \lambda_{s-1}}{2-\lambda_{s-1}}} \\
&\leq \left(\sum_{i_{k_s}=1}^{N} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \frac{\left| T\left(\sum\limits_{n=1}^{k} \sum\limits_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2}{S_r^{2-\lambda_{s-1}}} \right)^{\frac{\lambda_s}{\lambda_{s-1}}} \right)^{\frac{\lambda_{s-1}}{\lambda_s} \frac{2-\lambda_s}{2-\lambda_{s-1}}} \\
&\quad \times \left(\sum_{i_{k_s}=1}^{N} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T\left(\sum_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_s}{2}} \right)^{\frac{1}{\lambda_s} \frac{2(\lambda_s - \lambda_{s-1})}{2-\lambda_{s-1}}}. \qquad (5.10)
\end{aligned}$$

We know from the case $r = k_s$ that

$$\left(\sum_{i_{k_s}=1}^{N} \left(\sum_{\widehat{i_{k_s}}=1}^{N} \left| T\left(\sum_{n=1}^{k} \sum_{j\in I_n} e_{i_n} \cdot e_j \right) \right|^2 \right)^{\frac{\lambda_s}{2}} \right)^{\frac{1}{\lambda_s} \frac{2(\lambda_s - \lambda_{s-1})}{2-\lambda_{s-1}}} \leq (C\|T\|)^{\frac{2(\lambda_s - \lambda_{s-1})}{2-\lambda_{s-1}}}. \qquad (5.11)$$

Now we investigate the first factor in (5.10). From Hölder's inequality and (5.9) it follows that

$$\left(\sum_{i_{k_s}=1}^{N}\left(\sum_{\widehat{i_{k_s}}=1}^{N}\frac{\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2}{S_r^{2-\lambda_{s-1}}}\right)^{\frac{\lambda_s}{\lambda_{s-1}}}\right)^{\frac{\lambda_{s-1}}{\lambda_s}} = \left\|\left(\left(\sum_{\widehat{i_{k_s}}}\frac{\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2}{S_r^{2-\lambda_{s-1}}}\right)\right)_{i_{k_s}=1}^{N}\right\|_{\left(\frac{p_s}{\lambda_{s-1}}\right)^*}$$

$$= \sup_{y\in B_{\ell^N_{\frac{p_s}{\lambda_{s-1}}}}}\sum_{i_{k_s}=1}^{N}|y_{i_{k_s}}|\sum_{\widehat{i_{k_s}}=1}^{N}\frac{\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2}{S_r^{2-\lambda_{s-1}}} = \sup_{x\in B_{\ell^N_{p_s}}}\sum_{i_{k_s}=1}^{N}\sum_{\widehat{i_{k_s}}=1}^{N}\frac{\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2}{S_r^{2-\lambda_{s-1}}}|x_{i_{k_s}}|^{\lambda_{s-1}}$$

$$= \sup_{x\in B_{\ell^N_{p_s}}}\sum_{i_r=1}^{N}\sum_{\widehat{i_r}=1}^{N}\frac{\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^{2-\lambda_{s-1}}}{S_r^{2-\lambda_{s-1}}}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^{\lambda_{s-1}}\left|x_{i_{k_s}}\right|^{\lambda_{s-1}}$$

$$\leq \sup_{x\in B_{\ell^N_{p_s}}}\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\frac{\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2}{S_r^{2}}\right)^{\frac{2-\lambda_{s-1}}{2}}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2\left|x_{i_{k_s}}\right|^2\right)^{\frac{\lambda_{s-1}}{2}}$$

$$= \sup_{x\in B_{\ell^N_{p_s}}}\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2|x_{i_{k_s}}|^2\right)^{\frac{\lambda_{s-1}}{2}} \leq (C\|T\|)^{\lambda_{s-1}}. \tag{5.12}$$

Replacing (5.11) and (5.12) in (5.10) we finally conclude that

$$\begin{aligned}\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2\right)^{\frac{\lambda_s}{2}} &\leq (C\|T\|)^{\lambda_{s-1}\frac{2-\lambda_s}{2-\lambda_{s-1}}}\,(C\|T\|)^{\frac{2(\lambda_s-\lambda_{s-1})}{2-\lambda_{s-1}}}\\ &= (C\|T\|)^{\lambda_s}.\end{aligned}$$

It remains to consider when $1/p_1+\cdots+1/p_s = 1/2$. In this case it follows that $\lambda_s = 2$ and we have a more simple situation since

$$\begin{aligned}&\left(\sum_{i_r=1}^{N}\left(\sum_{\widehat{i_r}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2\right)^{\frac{\lambda_s}{2}}\right)^{\frac{1}{\lambda_s}}\\ &= \left(\sum_{i_{k_s}=1}^{N}\left(\sum_{\widehat{i_{k_s}}=1}^{N}\left|T\left(\sum_{n=1}^{k}\sum_{j\in I_n}e_{i_n}\cdot e_j\right)\right|^2\right)^{\frac{\lambda_s}{2}}\right)^{\frac{1}{\lambda_s}}\\ &\leq C\|T\|,\end{aligned}$$

where the inequality is due to the case $r = k_s$. This concludes the proof. □

Now we will show a generalization of the Bohnenblust–Hille and Hardy–Littlewood multilinear inequalities, which ensures that these results are in fact, corollaries of a unique yet general result.

Theorem 5.11 (Hardy–Littlewood with partial sums[1]). *Let m,k be positive integers with $1 \leq k \leq m$, and $\mathcal{I} = \{I_1, \ldots, I_k\}$ a partition of $\{1, \ldots, m\}$. Also, let us set $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$ with $0 \leq |1/\mathbf{p}| < 1$.*

(1) If $0 \leq |1/\mathbf{p}| \leq 1/2$ and $\mathbf{q} := (q_1, \ldots, q_k) \in \left[(1-|1/\mathbf{p}|)^{-1}, 2\right]^k$ are such that $|1/\mathbf{q}| \leq (k+1)/2 - |1/\mathbf{p}|$ then, for every continuous m–linear forms $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$, there exists a constant $C^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p},\mathbf{q}} \geq 1$ such that

$$\left(\sum_{i_1=1}^{\infty} \left(\cdots \left(\sum_{i_k=1}^{\infty} \left| T\left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^{q_k} \right)^{\frac{q_{k-1}}{q_k}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p},\mathbf{q}} \|T\|,$$

with $C^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p},\mathbf{q}} \leq B^{\text{mult}}_{\mathbb{K},k,(1,2,\ldots,2)}$.

(2) If $1/2 \leq |1/\mathbf{p}| < 1$ then, there exists a constant $D^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p}} \geq 1$ such that, for all continuous m–linear forms $T : \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{K}$,

$$\left(\sum_{i_1,\ldots,i_k=1}^{\infty} \left| T\left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p}} \|T\|$$

with $D^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p}} \leq D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$. Moreover, the exponent is optimal.

Proof. (1) Since $\lambda = (1-|1/\mathbf{p}|)^{-1} \leq 2$, using the Minkowski inequality as in [6], it is possible to prove that we have, for all fixed $j \in \{1, ..., k\}$, similar inequalities to the inequality of the previous lemma with the exponents $\mathbf{q}(j) := (2, ..., 2, \lambda, 2 ..., 2) \in [\lambda, 2]^k$ with λ in the j–th position. The multiple exponent $(q_1, ..., q_k) \in [\lambda, 2]^k$ can be obtained by interpolating the multiple exponents $\mathbf{q}(1), ..., \mathbf{q}(k)$ in the sense of [6] with $\theta_1 = \cdots = \theta_k = 1/k$. Therefore $1/q_1 + \cdots + 1/q_k = 1/\lambda + (k-1)/2 = (k+1)/2 - |1/\mathbf{p}|$.

(2) The case $k = m$ is exactly the Theorem 1.2. Let $1 \leq k < m$ and observe that

$$\begin{aligned} \left(\sum_{i_1,\ldots,i_k=1}^{\infty} \left| T\left(\sum_{n=1}^{k} \sum_{j \in I_n} e_{i_n} \cdot e_j \right) \right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} &\leq \left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left| T\left(e_{i_1}, \ldots, e_{i_m}\right) \right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}} \right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \\ &\leq D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \|T\|. \end{aligned}$$

It remains to prove the optimality of the exponent. The argument is a variant of [137, Proposition 1]. Let $(\beta_n)_n$ be a strictly increasing sequence converging to $(|1/\mathbf{p}| - 1)$. For each positive integer n, let us define the bounded m-linear form $\Phi_n : \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{K}$ by

$$\Phi_n\left(e_{i_1}, \ldots, e_{i_m}\right) := \begin{cases} j^{\beta_n}, & \text{if } i_1 = \cdots = i_m = j; \\ 0, & \text{otherwise.} \end{cases}$$

[1]The main result of [3] improves Theorem 5.11. We emphasize that the proof of that result is based on a tensorial perspective and, since this approach does not fit on the context of this thesis we will not present it. It is important to mention that in [3] it is shown that the exponent in *(1)* is also optimal and that $C^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p},\mathbf{q}} \leq C^{\text{mult}}_{\mathbb{K},k,\mathbf{r},\mathbf{q}}$ and $D^{\text{mult}}_{\mathbb{K},k,m,\mathcal{I},\mathbf{p}} \leq D^{\text{mult}}_{\mathbb{K},k,\mathbf{r}}$ with $\mathbf{r} := (r_1, ..., r_k)$, $1/r_i = \sum_{j \in I_i} 1/p_j$, $1 \leq i \leq k$.

If $s < \lambda$ we may take $n \in \mathbb{N}$ large enough such that $-1/s < \beta_n < (|1/\mathbf{p}| - 1) < 0$, which results in $1 > -s\beta_n > 0$. Let us define $\Phi := \Phi_n$. Then,

$$\sum_{i_1,\ldots,i_k=1}^{\infty} \left|\Phi\left(e_{i_1}^{n_1},\ldots,e_{i_k}^{n_k}\right)\right|^s = \sum_{j=1}^{\infty} |\Phi(e_j,\ldots,e_j)|^s = \sum_{j=1}^{\infty} j^{s\beta_n} = \sum_{j=1}^{\infty} \frac{1}{j^{-s\beta_n}}$$

diverges and, therefore, the exponent is optimal. □

This theorem motivated us to give the following unifying notion of absolutely summing multilinear operators (the essence of the notion of partially multiple summing operators (below) was first sketched in [114, Definition 2.2.1] but it has not been explored since):

Definition 5.12. *Let $E_1,\ldots,E_m,F$ be Banach spaces, m,k be positive integers with $1 \leq k \leq m$, and $(\mathbf{p},\mathbf{q}) := (p_1,\ldots,p_m,q_1,\ldots,q_k) \in [1,\infty)^{m+k}$. Let also $\mathcal{I} = \{I_1,\ldots,I_k\}$ a family of non-void disjoints subsets of $\{1,\ldots,m\}$ such that $\cup_{i=1}^k I_i = \{1,\ldots,m\}$. A multilinear operator $T : E_1\times\cdots\times E_m \to F$ is $\mathcal{I}$-partially multiple $(\mathbf{q};\mathbf{p})$-summing if there exists a constant $C > 0$ such that*

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_k=1}^{\infty}\left\|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} x_{i_n}^{(j)}\cdot e_j\right)\right\|_F^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\prod_{j=1}^{m}\left\|(x_i^{(j)})_{i=1}^{\infty}\right\|_{w,p_j}$$

for all $(x_i^{(j)})_{i=1}^{\infty} \in \ell_{p_j}^w(E_j)$, $j = 1,\ldots,m$. We represent the class of all $\mathcal{I}$-partially multiple $(\mathbf{q};\mathbf{p})$-summing operators by $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1,\ldots,E_m;F)$. The infimum taken over all possible constants $C > 0$ satisfying the previous inequality defines a norm in $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1,\ldots,E_m;F)$, which is denoted by $\pi_{(\mathbf{q};\mathbf{p})}$.

As usual, $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1,\ldots,E_m;F)$ is a subspace of $\mathcal{L}(E_1,\ldots,E_m;F)$. Moreover, note that when

- $k = 1$, we recover the class of absolutely $(q;\mathbf{p})$-summing operators, with $q := q_1$;
- $k = m$ and $q_1 = \cdots = q_m =: q$, we recover the class of multiple $(q;\mathbf{p})$-summing operators;
- $k = m$, we recover the class of multiple $(\mathbf{q};\mathbf{p})$-summing operators, as we have defined in the section 5.1.

Example 5.13. As in Proposition 5.6, it is possible to prove the following result (now using the generalized Bohnenblust–Hille inequality with partial sums (Theorem 5.9): if m,k are positive integers with $1 \leq k \leq m$, $\mathcal{I} = \{I_1,\ldots,I_k\}$ is a partition of $\{1,\ldots,m\}$, and $\mathbf{q} = (q_1,\ldots,q_k) \in [1,2]^k$ is such that $|1/\mathbf{q}| \leq (k+1)/2$, then

$$\Pi_{\left(\mathbf{q};1,\overset{m \text{ times}}{\cdots},1\right)}^{k,m,\mathcal{I}}(E_1,\ldots,E_m;\mathbb{K}) = \mathcal{L}(E_1,\ldots,E_m;\mathbb{K}).$$

More generally, with the same idea of Proposition 5.7, we can re-written the Hardy–Littlewood inequalities with partial sums (Theorem 5.11): Let m,k be positive integers with $1 \leq k \leq m$, and $\mathcal{I} = \{I_1,\ldots,I_k\}$ a partition of $\{1,\ldots,m\}$. Also, let us set $\mathbf{p} := (p_1,\ldots,p_m) \in [1,\infty]^m$ with $0 \leq |1/\mathbf{p}| < 1$.

(1) If $0 \leq |1/\mathbf{p}| \leq 1/2$ and $\mathbf{q} := (q_1, \ldots, q_k) \in \left[(1-|1/\mathbf{p}|)^{-1}, 2\right]^k$ are such that $|1/\mathbf{q}| \leq (k+1)/2 - |1/\mathbf{p}|$, then

$$\Pi^{k,m,\mathcal{I}}_{\left(\mathbf{q};p_1^*,\ldots,p_m^*\right)}(E_1,\ldots,E_m;\mathbb{K}) = \mathcal{L}\left(E_1,\ldots,E_m;\mathbb{K}\right);$$

(2) If $1/2 \leq |1/\mathbf{p}| < 1$, we have

$$\Pi^{k,m,\mathcal{I}}_{\left((1-|1/\mathbf{p}|)^{-1};p_1^*,\ldots,p_m^*\right)}(E_1,\ldots,E_m;\mathbb{K}) = \mathcal{L}\left(E_1,\ldots,E_m;\mathbb{K}\right).$$

The basis of this theory can be developed in the same lines as those from the previous section, as we will be presenting in what follows. From now on, m, k are positive integers with $1 \leq k \leq m$, $(\mathbf{p},\mathbf{q}) := (p_1,\ldots,p_m,q_1,\ldots,q_k) \in [1,\infty)^{m+k}$ and $\mathcal{I} = \{I_1,\ldots,I_k\}$ is a partition of $\{1,\ldots,m\}$.

Proposition 5.14. *Let $T : E_1 \times \cdots \times E_m \to F$ be a continuous multilinear operator. The following assertions are equivalent:*

(1) T is $\mathcal{I}$-partially multiple $(\mathbf{q};\mathbf{p})$-summing;

(2) $\left(T\left(\sum\limits_{n=1}^{k}\sum\limits_{j\in I_n} x_{i_n}^{(j)}\cdot e_j\right)\right)_{i_1,\ldots,i_k=1}^{\infty} \in \ell_{\mathbf{q}}(F)$ *whenever* $(x_i^{(j)})_{i=1}^{\infty} \in \ell_{p_j}^{w}(E_j)$, *for* $j = 1,\ldots,m$.

(3) There exist a constant $C > 0$ such that

$$\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_k=1}^{n}\left\|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} x_{i_n}^{(j)}\cdot e_j\right)\right\|_F^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\prod_{k=1}^{m}\left\|(x_i^{(j)})_{i=1}^{n}\right\|_{w,p_j}$$

for all positive integer n and all $(x_i^{(j)})_{i=1}^{n} \in \ell_{p_j}^{w}(E_j)$, $j = 1,...,m$.

Proposition 5.15. *If $T \in \Pi^{k,m,\mathcal{I}}_{(\mathbf{q};\mathbf{p})}(E_1,\ldots,E_m;F)$, then $\|T\|_{\mathcal{L}(E_1,\ldots,E_m;F)} \leq \pi_{(\mathbf{q};\mathbf{p})}(T)$.*

Given $T \in \Pi^{k,m,\mathcal{I}}_{(\mathbf{q};\mathbf{p})}(E_1,\ldots,E_m;F)$ we may define the m-linear operator

$$\begin{array}{rcll} \widehat{T} & : & \ell_{p_1}^{w}(E_1)\times\cdots\times\ell_{p_m}^{w}(E_m) & \to \ \ell_{\mathbf{q}}(F) \\ & & \left((x_i^{(1)})_{i=1}^{\infty},\ldots,(x_i^{(m)})_{i=1}^{\infty}\right) & \mapsto \ \left(T\left(\sum_{n=1}^{k}\sum_{j\in I_n} x_{i_n}^{(j)}\cdot e_j\right)\right)_{i_1,\ldots,i_k=1}^{\infty}. \end{array} \tag{5.13}$$

By using both, the Closed Graph and the Hahn–Banach Theorems, it is possible to prove that $\widehat{T}$ is a continuous m-linear operator. Furthermore, we can prove that $\|\widehat{T}\| = \pi_{(\mathbf{q};\mathbf{p})}(T)$, therefore, naturally we define the isometric operator

$$\begin{array}{rccc} \widehat{\theta} & : & \Pi^{k,m,\mathcal{I}}_{(\mathbf{q};\mathbf{p})}(E_1,\ldots,E_m;F) & \to & \mathcal{L}\left(\ell_{p_1}^{w}(E_1),\ldots,\ell_{p_m}^{w}(E_m);\ell_{\mathbf{q}}(F)\right) \\ & & T & \mapsto & \widehat{T}. \end{array}$$

These facts lead us to the following results:

Theorem 5.16. $\left(\Pi^{k,m,\mathcal{I}}_{(\mathbf{q};\mathbf{p})}(E_1,\ldots,E_m;F),\pi_{(\mathbf{q};\mathbf{p})}(\cdot)\right)$ *is a Banach space.*

Proposition 5.17. *If there exists* $n\in\{1,\ldots,k\}$ *such that* $1/q_n>\sum_{j\in I_n}1/p_j$, *then*

$$\Pi^{k,m,\mathcal{I}}_{(\mathbf{q};\mathbf{p})}(E_1,\ldots,E_m;F)=\{0\}.$$

Part III

Strange functions

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Chapter 6

Lineability in function spaces

A real-valued function on $\mathbb{R}$ satisfying the property that it takes on each real value in any nonempty open set is called *everywhere surjective*. In [21] the authors proved that the set of everywhere surjective functions $f : \mathbb{R} \to \mathbb{R}$ is $2^{\mathfrak{c}}$-lineable, which is the best possible result in terms of dimension. In other words, the last set is maximal lineable in the space of all real functions. Other results establishing the degree of lineability of more stringent classes of functions can be found in [38] and the references contained in it.

As usual, the symbol $\mathcal{C}(I)$ stands for the vector space of all real continuous functions defined on an interval $I \subset \mathbb{R}$. In the special case $I = \mathbb{R}$, the space $\mathcal{C}(\mathbb{R})$ will be endowed with the topology of the convergence in compacta. It is well known that $\mathcal{C}(\mathbb{R})$ under this topology is an F-space, that is, a complete metrizable topological vector space. Turning to the setting of more regular functions, in [83] the following results are proved: the set of *differentiable* functions on $\mathbb{R}$ whose derivatives are discontinuous almost everywhere is $\mathfrak{c}$-lineable; given a non-void compact interval $I \subset \mathbb{R}$, the family of differentiable functions whose derivatives are discontinuous almost everywhere on I is dense-lineable in the space $\mathcal{C}(I)$, endowed with the supremum norm; and the class of differentiable functions on $\mathbb{R}$ that are monotone on no interval is $\mathfrak{c}$-lineable.

Finally, recall that every bounded variation function on an interval $I \subset \mathbb{R}$ (that is, a function satisfying $\sup\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : \{t_1 < t_2 < \cdots < t_n\} \subset I,\, n \in \mathbb{N}\} < \infty$) is *differentiable almost everywhere.* A continuous bounded variation function $f : I \to \mathbb{R}$ is called strongly singular whenever $f'(x) = 0$ for almost every $x \in I$ and, in addition, f is nonconstant on any subinterval of I. Balcerzak *et al.* [25] showed that the set of strongly singular functions on $[0,1]$ is densely strongly $\mathfrak{c}$-algebrable in $\mathcal{C}([0,1])$.

A number of results related to the above ones will be shown in the next two sections.

6.1 Measurable functions

Our aim in this section is to study the lineability of the family of Lebesgue measurable functions $f : \mathbb{R} \to \mathbb{R}$ that are everywhere surjective, denoted $\mathcal{MES}$. This result is, in some sense, unexpected since (as we can see in [82, 83]) the class of everywhere surjective functions contains a $2^{\mathfrak{c}}$-lineable set of non-measurable ones (called *Jones functions*).

Theorem 6.1. *The set $\mathcal{MES}$ is $\mathfrak{c}$-lineable.*

Proof. Firstly, we consider the everywhere surjective function furnished in [83, Example 2.2]. For the sake of convenience, we reproduce here its construction. Let $(I_n)_{n\in\mathbb{N}}$ be the collection of all open intervals with rational endpoints. The interval I_1 contains a Cantor type set, call it C_1. Now, $I_2 \setminus C_1$ also contains a Cantor type set, call it C_2. Next, $I_3 \setminus (C_1 \cup C_2)$ contains, as well, a Cantor type set, C_3. Inductively, we construct a family of pairwise disjoint Cantor type sets, $(C_n)_{n\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$, $I_n \setminus (\bigcup_{k=1}^{n-1} C_k) \supset C_n$. Now, for every $n \in \mathbb{N}$, take any bijection $\phi_n : C_n \to \mathbb{R}$, and define $f\colon \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} \phi_n(x) & \text{if } x \in C_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is clearly everywhere surjective. Indeed, let I be any interval in $\mathbb{R}$. There exists $k \in \mathbb{N}$ such that $I_k \subset I$. Thus $f(I) \supset f(I_k) \supset f(C_k) = \phi_k(C_k) = \mathbb{R}$. But the novelty of the last function is that f is, in addition, zero almost everywhere, and in particular, it is (Lebesgue) *measurable*. That is, $f \in \mathcal{MES}$.

Now, taking advantage of the approach of [21, Proposition 4.2], we are going to construct a vector space that shall be useful later on. Let $\Lambda := \text{span}\,\{\varphi_\alpha : \alpha > 0\}$, where $\varphi_\alpha(x) := e^{\alpha x} - e^{-\alpha x}$. Then M is a $\mathfrak{c}$-dimensional vector space because the functions φ_α $(\alpha > 0)$ are linearly independent. Indeed, assume that there are scalars $c_1, \ldots, c_p$ (not all 0) as well as positive reals $\alpha_1, \ldots, \alpha_p$ such that $c_1\varphi_{\alpha_1}(x) + \cdots + c_p\varphi_{\alpha_p}(x) = 0$ for all $x \in \mathbb{R}$. Without loss of generality, we may assume that $p \geq 2$, $c_p \neq 0$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_p$. Then $\lim\limits_{x\to+\infty} (c_1\varphi_{\alpha_1}(x) + \cdots + c_p\varphi_{\alpha_p}(x)) = +\infty$ or $-\infty$, which is clearly a contradiction. Therefore $c_1 = \cdots = c_p = 0$ and we are done. Note that each nonzero member $g = \sum_{i=1}^p c_i\,\varphi_{\alpha_i}$ (with the c_i's and the α_i's as before) of Λ is (continuous and) surjective because $\lim_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to-\infty} g(x) = -\infty$ if $c_p > 0$ (with the values of the limits interchanged if $c_p < 0$).

Next, we define the vector space $M := \{g \circ f : g \in \Lambda\}$. Observe that, since the f is measurable and the functions g in Λ are continuous, the members of M are measurable. Fix any $h \in M \setminus \{0\}$. Then, again, there are finitely many scalars $c_1, \ldots, c_p$ with $c_p \neq 0$, and positive reals $\alpha_1 < \alpha_2 < \cdots < \alpha_p$ such that $g = c_1\varphi_{\alpha_1} + \cdots + c_p\varphi_{\alpha_p}$ and $h = g \circ f$. Now, fix a non-degenerate interval $J \subset \mathbb{R}$. Then $h(J) = g(f(J)) = g(\mathbb{R}) = \mathbb{R}$, which shows that h is everywhere surjective. Hence $M \setminus \{0\} \subset \mathcal{MES}$.

Finally, by using the linear independence of the functions φ_α and the fact that f is surjective, it is easy to see that the functions $\varphi_\alpha \circ f$ $(\alpha > 0)$ are linearly independent, which entails that M has dimension $\mathfrak{c}$, as required. □

In [136, Example 2.34] it is exhibited one sequence of measurable everywhere surjective functions tending pointwise to zero. With Theorem 6.1 in hand, we now get a plethora of such sequences, and even in a much easier way than described in [136].

Corollary 6.2. *The family of sequences $\{f_n\}_{n\geq 1}$ of Lebesgue measurable functions $\mathbb{R} \to \mathbb{R}$ such that f_n converges pointwise to zero and such that $f_n(I) = \mathbb{R}$, for any positive integer n and each non-degenerate interval I, is $\mathfrak{c}$-lineable.*

Proof. Consider the family $\widetilde{M}$ consisting of all sequences $\{h_n\}_{n\geq 1}$ given by $h_n(x) = h(x)/n$ where the functions h run over the vector space M constructed in the last theorem. It is easy to see that $\widetilde{M}$ is a $\mathfrak{c}$-dimensional vector subspace of $(\mathbb{R}^\mathbb{R})^\mathbb{N}$, that each h_n is measurable, that $h_n(x) \to 0$ $(n \to \infty)$ for every $x \in \mathbb{R}$ and that every h_n is everywhere surjective if h is not the zero function. □

Remark 6.3. It would be interesting to know whether $\mathcal{MES}$ is, likewise the set of everywhere surjective functions, maximal lineable in $\mathbb{R}^{\mathbb{R}}$ (that is, $2^{\mathfrak{c}}$-lineable).

6.2 Special differentiable functions

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a *Pompeiu function* (see Figure 6.1) provided that it is differentiable and f' vanishes on a dense set in $\mathbb{R}$. The symbols $\mathcal{P}$ and $\mathcal{DP}$ stand for the vector spaces of Pompeiu functions and of the derivatives of Pompeiu functions, respectively. In this section, we analyze the lineability of the set of Pompeiu functions that are not constant on any interval. Of course, this set is not a vector space.

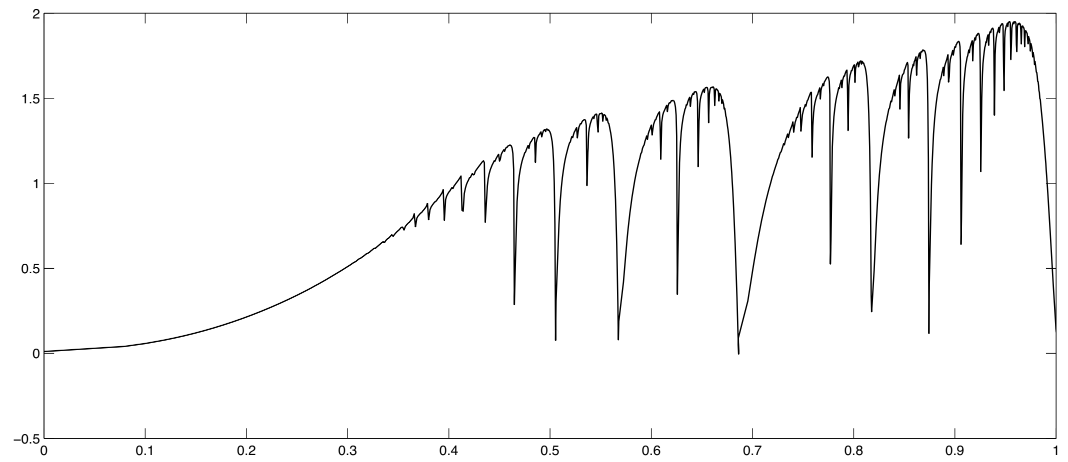

Figure 6.1: Rough sketch of the graph of Pompeiu's original example.

Firstly, the following version of the well-known Stone–Weierstrass density theorem (see e.g. [130]) for the space $\mathcal{C}(\mathbb{R})$ will be relevant to the proof of our main result. Its proof is a simple application of the original Stone–Weierstrass theorem for $\mathcal{C}(S)$ (the Banach space of continuous functions $f : S \to \mathbb{R}$, endowed with the uniform distance, where S is a compact topological space) together with the fact that convergence in $\mathcal{C}(\mathbb{R})$ means convergence on each compact subset of $\mathbb{R}$. So we omit the proof.

Lemma 6.4. *Suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{C}(\mathbb{R})$ satisfying the following properties:*

(a) *Given $x_0 \in \mathbb{R}$ there is $F \in \mathcal{A}$ with $F(x_0) \neq 0$.*

(b) *Given a pair of distinct points $x_0, x_1 \in \mathbb{R}$, there exists $F \in \mathcal{A}$ such that $F(x_0) \neq F(x_1)$.*

Then $\mathcal{A}$ is dense in $\mathcal{C}(\mathbb{R})$.

In [25, Proposition 7], Balcerzak, Bartoszewicz and Filipczak established a nice algebrability result by using the so-called *exponential-like functions,* that is, the functions $\varphi : \mathbb{R} \to \mathbb{R}$ of the form

$$\varphi(x) = \sum_{j=1}^{m} a_j e^{b_j x},$$

for some $m \in \mathbb{N}$, some $a_1, \dots, a_m \in \mathbb{R} \setminus \{0\}$ and some distinct $b_1, \dots, b_m \in \mathbb{R} \setminus \{0\}$. By $\mathcal{E}$ we denote the class of exponential-like functions. The following lemma (see [36] or [18, Chapter 7]) is a slight variant of the mentioned Proposition 7 of [25].

Lemma 6.5. *Let Ω be a nonempty set and $\mathcal{F}$ be a family of functions $f : \Omega \to \mathbb{R}$. Assume that there exists a function $f \in \mathcal{F}$ such that $f(\Omega)$ is uncountable and $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{E}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More precisely, if $H \subset (0, \infty)$ is a set with $\operatorname{card}(H) = \mathfrak{c}$ and linearly independent over the field $\mathbb{Q}$, then $\{\exp \circ (rf) : r \in H\}$ is a free system of generators of an algebra contained in $\mathcal{F} \cup \{0\}$.*

Lemma 6.6 below is an adaptation of a result that is implicitly contained in [27, Section 6]. We sketch the proof for the sake of completeness.

Lemma 6.6. *Let $\mathcal{F}$ be a family of functions in $\mathcal{C}(\mathbb{R})$. Assume that there exists a strictly monotone function $f \in \mathcal{F}$ such that $\varphi \circ f \in \mathcal{F}$ for every exponential-like function φ. Then $\mathcal{F}$ is densely strongly $\mathfrak{c}$-algebrable in $\mathcal{C}(\mathbb{R})$.*

Proof. If $\Omega = \mathbb{R}$ then $f(\Omega)$ is a non-degenerate interval, so it is an uncountable set. Then, it is sufficient to show that the algebra $\mathcal{A}$ generated by the system $\{\exp \circ (rf) : r \in H\}$ given in Lemma 6.5 is dense. For this, we invoke Lemma 6.4. Take any $\alpha \in H \subset (0, +\infty)$. Given $x_0 \in \mathbb{R}$, the function $F(x) := e^{\alpha f(x)}$ belongs to $\mathcal{A}$ and satisfies $F(x_0) \neq 0$. Moreover, for prescribed distinct points $x_0, x_1 \in \mathbb{R}$, the same function F fulfills $F(x_0) \neq F(x_1)$, because both functions f and $x \mapsto e^{\alpha x}$ are one-to-one. As a conclusion, $\mathcal{A}$ is dense in $\mathcal{C}(\mathbb{R})$. $\square$

Now we state and prove the main result of this section.

Theorem 6.7. *The set of functions in $\mathcal{P}$ that are nonconstant on any non-degenerated interval of $\mathbb{R}$ is densely strongly $\mathfrak{c}$-algebrable in $\mathcal{C}(\mathbb{R})$.*

Proof. From [136, Example 3.11] (see also [134, Example 13.3]) we know that there exists a derivable *strictly increasing* real-valued function $(a, b) \to (0, 1)$ (with $f((a, b)) = (0, 1)$) whose derivative vanishes on a dense set and yet does not vanish everywhere. By composition with the function $x \mapsto ((b - a)/\pi) \arctan x + (a + b)/2$, we get a strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ satisfying that $D := \{x \in \mathbb{R} : f'(x) = 0\}$ is dense in $\mathbb{R}$ but $D \neq \mathbb{R}$. Observe that, in particular, f is a Pompeiu function that is nonconstant on any interval.

According to Lemma 6.6, our only task is to prove that, given a prescribed function $\varphi \in \mathcal{E}$, the function $\varphi \circ f$ belongs to

$$\mathcal{F} := \{f \in \mathcal{P} : f \text{ is nonconstant on any interval of } \mathbb{R}\}.$$

By the chain rule, $\varphi \circ f$ is a differentiable function and $(\varphi \circ f)'(x) = \varphi'(f(x)) f'(x)$ $(x \in \mathbb{R})$. Hence $(\varphi \circ f)'$ vanishes at least on D, so this derivative vanishes on a dense set. It remains to prove that $\varphi \circ f$ is nonconstant on any open interval of $\mathbb{R}$. In order to see this, fix one such interval J. Clearly, the function φ' also belongs to $\mathcal{E}$. Then φ' is a nonzero entire function. Therefore the set $S := \{x \in \mathbb{R} : \varphi'(x) = 0\}$ is discrete in $\mathbb{R}$. In particular, it is closed in $\mathbb{R}$ and countable, so $\mathbb{R} \setminus S$ is open and dense in $\mathbb{R}$. Of course, $S \cap (0, 1)$ is discrete in $(0, 1)$. Since $f : \mathbb{R} \to (0, 1)$ is a homeomorphism, the set $f^{-1}(S)$ is discrete in $\mathbb{R}$. Hence $J \setminus f^{-1}(S)$ is a nonempty open set of J. On the other hand, since D is dense in $\mathbb{R}$, it follows that the set D^0 of all interior points of D is $\varnothing$. Indeed, if this were not true, there would exist an interval $(c,d) \subset D$. Then $f' = 0$ on (c,d), so f would be constant on (c,d), which is not possible because f is strictly increasing. Therefore $\mathbb{R} \setminus D$ is dense in $\mathbb{R}$, from which one derives that $J \setminus D$ is dense in J. Thus $(J \setminus f^{-1}(S)) \cap (J \setminus D) \neq \varnothing$. Finally, pick any point x_0 in the last set. This means that $x_0 \in J$, $f(x_0) \notin S$ (so $\varphi'(f(x_0)) \neq 0$) and $x_0 \notin D$ (so $f'(x_0) \neq 0$). Thus $(\varphi \circ f)'(x_0) = \varphi'(f(x_0))f'(x_0) \neq 0$, which implies that $\varphi \circ f$ is nonconstant on J, as required. □

Remark 6.8. 1. In view of the last theorem one might believe that the expression "$f' = 0$ on a dense set" (see the definition of $\mathcal{P}$) could be replaced by the stronger one "$f' = 0$ almost everywhere". But this is not possible because every differentiable function is an N-function –that is, it sends sets of null measure into sets of null measure– (see [134, Theorem 21.9]) and every continuous N-function on an interval whose derivative vanishes almost everywhere must be a constant (see [134, Theorem 21.10]).
2. If a real function f is a derivative then f^2 may be not a derivative (see [134, p. 86]). This leads us to conjecture that the set $\mathcal{DP}$ of Pompeiu derivatives (and of course, any subset of it) is not algebrable.
3. Nevertheless, from Theorem 3.6 (and also from Theorem 4.1) of [83] it follows that the family $\mathcal{BDP}$ of bounded Pompeiu derivatives is $\mathfrak{c}$-lineable. A quicker way to see this is by invoking the fact that $\mathcal{BDP}$ is a vector space that becomes a Banach space under the supremum norm [54, pp. 33–34]. Since it is not finite dimensional, a simple application of Baire's category theorem yields $\dim(\mathcal{BDP}) = \mathfrak{c}$. Now, on one hand, we have that, trivially, $\mathcal{BDP}$ is dense-lineable in itself. On the other hand, it is known that the set of derivatives that are positive on a dense set and negative on another is a dense G_δ set in the Banach space $\mathcal{BDP}$ [54, p. 34]. Then, as the authors of [83] suggest, it would be interesting to see whether this set is also dense-lineable.

6.3 Discontinuous functions

Let $n \geq 2$ and consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, \dots, x_n) = \begin{cases} \dfrac{x_1 \cdots x_n}{x_1^{2n} + \cdots + x_n^{2n}} & \text{if } x_1^2 + \cdots + x_n^2 \neq 0, \\ 0 & \text{if } x_1 = \cdots = x_n = 0. \end{cases} \tag{6.1}$$

Observe that f is discontinuous at the origin since arbitrarily near of $0 \in \mathbb{R}^n$ there exist points of the form $x_1 = \cdots = x_n = t$ at which f has the value $1/nt^n$. On the other hand, fixed $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$, the real-valued function of a real variable given by $\psi : x_i \mapsto f(x_1, \dots, x_n)$ is everywhere a continuous function of x_i. Indeed, this is trivial if all x_j's ($j \neq i$) are not 0, while $\psi \equiv 0$ if some $x_j = 0$. Of course, f is continuous at any point of $\mathbb{R}^n \setminus \{0\}$.

Definition 6.9. *Let $n \geq 2$ be a positive integer. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is separately continuous if, fixed $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$, the real-valued function of a real variable given by $x_i \mapsto f(x_1, \dots, x_n)$ is a continuous function of x_i. Given $x_0 \in \mathbb{R}^n$, we denote by $\mathcal{SC}(\mathbb{R}^n, x_0)$ the vector space of all separately continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that are continuous on $\mathbb{R}^n \setminus \{x_0\}$.*

Example 6.10. The function f given in (6.1) belongs to $\mathcal{SC}(\mathbb{R}^n, 0)$.

Since $\operatorname{card}(\mathcal{C}(\mathbb{R}^n \setminus \{x_0\})) = \mathfrak{c}$, it is easy to see that the cardinality (so the dimension) of $\mathcal{SC}(\mathbb{R}^n, x_0)$ is equals $\mathfrak{c}$. In Theorem 6.11 below we will show the algebrability of the family $\mathcal{DSC}(\mathbb{R}^n, x_0) := \{f \in \mathcal{SC}(\mathbb{R}^n, x_0) : f \text{ is discontinuous at } x_0\}$ in a maximal sense.

Theorem 6.11. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $x_0 \in \mathbb{R}^n$. Then the set $\mathcal{DSC}(\mathbb{R}^n, x_0)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. We can suppose without loss of generality that $x_0 = 0 = (0, 0, \ldots, 0)$. Consider the function $f \in \mathcal{DSC}(\mathbb{R}^n, 0)$ given by (6.1). For each $c > 0$, we set $\varphi_c(x) := e^{|x|^c} - e^{-|x|^c}$. It is easy to see that these functions generate a free algebra. Indeed, if $P(t_1, \ldots, t_p)$ is a nonzero polynomial in p variables with $P(0, 0, \ldots, 0) = 0$ and $c_1, \ldots, c_p$ are distinct positive real numbers, let $M := \{j \in \{1, \ldots, p\} : \text{the variable } t_j \text{ appears explicitly in the expression of } P\}$, and $c_0 := \max\{c_j : j \in M\}$. Then one derives that the function $P(\varphi_{c_1}, \ldots, \varphi_{c_p})$ has the form $D\, e^{m|x|^{c_0} + g(x)} + h(x)$, where $D \in \mathbb{R} \setminus \{0\}$, $m \in \mathbb{N}$, g is a finite sum of the form $\sum_k m_k |x|^{\alpha_k}$ with m_k integers and $\alpha_k < c_0$, and h is a finite linear combination of functions of the form $e^{q(x)}$ where, in turn, each $q(x)$ is a finite sum of the form $\sum_k n_k |x|^{\gamma_k}$, with each γ_k satisfying that either $\gamma_k < c_0$, or $\gamma_k = c_0$ and $n_k < 0$ simultaneously. Then

$$\lim_{x\to\infty} |P(\varphi_{c_1}(x), \ldots, \varphi_{c_p}(x))| = +\infty \tag{6.2}$$

and, in particular, $P(\varphi_{c_1}, \ldots, \varphi_{c_p})$ is not 0 identically. This shows that the algebra Λ generated by the φ_c's is free.

Now, define the set $\mathcal{A} := \{\varphi \circ f : \varphi \in \Lambda\}$. Plainly, $\mathcal{A}$ is an algebra of functions $\mathbb{R}^n \to \mathbb{R}$ each of them being continuous on $\mathbb{R}^n \setminus \{0\}$. But, in addition, this algebra is freely generated by the functions $\varphi_c \circ f$ $(c > 0)$. To see this, assume that $\Phi = P(\varphi_{c_1} \circ f, \ldots, \varphi_{c_p} \circ f) \in \mathcal{A}$, where $P, c_1, \ldots, c_p$ are as above. Suppose that $\Phi = 0$. Evidently, the function f is onto (note that, for example, $f(x, \ldots, x) = 1/n\, x^n$, $f(-x, x, \ldots, x) = -1/n\, x^n$ and $f(0, \ldots, 0) = 0$). Therefore $P(\varphi_{c_1}(x), \ldots, \varphi_{c_p}(x)) = 0$ for all $x \in \mathbb{R}$, so $P \equiv 0$, which is absurd because $P(\varphi_{c_1}, \ldots, \varphi_{c_p})$ becomes large as $x \to \infty$.

Hence our only task is to prove that every function $\Phi \in \mathcal{A} \setminus \{0\}$ as in the last paragraph belongs to $\mathcal{DSC}(\mathbb{R}^n, 0)$. Firstly, the continuity of each φ_c implies that $\Phi \in \mathcal{SC}(\mathbb{R}^n, 0)$. Finally, the function Φ is discontinuous at the origin. Indeed, we have for all $x \neq 0$ that

$$|\Phi(x, x, \ldots, x)| = \left| P\big(\varphi_{c_1}(\frac{1}{n\, x^n}), \ldots, \varphi_{c_p}(\frac{1}{n\, x^n})\big) \right| \longrightarrow +\infty$$

as $x \to 0$, due to (6.2). This is inconsistent with continuity at 0. The proof is finished. $\square$

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