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Hardy–Littlewood/Bohnenblust–Hille multilinear inequalities and Peano curves on topological vector spaces

por

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João Pessoa – PB
Dezembro de 2014

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Tese apresentada ao Corpo Docente do Programa em Associação de Pós Graduação em Matemática UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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"Give me a fulcrum, and I shall move the world."

Archimedes of Syracuse (c. 287 BC – c. 212 BC)

Resumo

Este trabalho é dividido em dois temas. O primeiro diz respeito às Desigualdades multilineares de Bohnenblust–Hille e Hardy–Littlewood. Obtemos generalizações ótimas e definitivas para ambas desigualdades. Mais ainda, a abordagem apresentada fornece demonstrações mais simples e diretas do que as conhecidas anteriormente, além de sermos capazes de mostrar que os expoentes envolvidos são ótimos em várias situações. A técnica utilizada combina ferramentas probabilísticas e interpolativas; esta última é ainda usada para melhorar as estimativas das versões vetoriais da desigualdade de Bohnenblust–Hille. O segundo tema possui como ponto de partida a existência de espaços de Peano, ou seja, os espaços de Hausdorff que são imagem contínua do intervalo unitário. Sob o ponto de vista da lineabilidade, analisamos o conjunto das sobrejeções contínuas de um espaço euclidiano arbitrário em um espaço topológico que, de certa forma, é coberto por espaços de Peano, e concluímos que *grandes* álgebras são encontradas nas famílias estudadas. Fornecemos vários resultados ótimos e definitivos em espaços euclideanos, e, mais ainda, um resultado de lineabilidade ótimo naqueles espaços vetoriais topológicos especiais.

Palavras-chave: Bohnenblust–Hille, Hardy–Littlewood, lineabilidade, curvas de Peano, espaços vetoriais topológicos.

Abstract

This work is divided in two subjects. The first concerns about the Bohnenblust–Hille and Hardy–Littlewood multilinear inequalities. We obtain optimal and definitive generalizations for both inequalities. Moreover, the approach presented provides much simpler and straightforward proofs than the previous one known, and we are able to show that in most cases the exponents involved are optimal. The technique used is a combination of probabilistic tools and of an interpolative approach; this former technique is also employed in this thesis to improve the constants for vector-valued Bohnenblust–Hille type inequalities. The second subject has as starting point the existence of Peano spaces, that is, Haurdorff spaces that are continuous image of the unit interval. From the point of view of lineability we analyze the set of continuous surjections from an arbitrary euclidean spaces on topological spaces that are, in some natural sense, covered by Peano spaces, and we conclude that *large* algebras are found within the families studied. We provide several optimal and definitive result on euclidean spaces, and, moreover, an optimal lineability result on those special topological vector spaces.

Keywords: Bohnenblust–Hille, Hardy–Littlewood, lineability, Peano curves, topological vector spaces.

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Introduction

Part I: Hardy–Littlewood and Bohnenblust–Hille inequalities

The subject of this part of the work was born with a problem stated in 1913 by Harald Bohr in [35] concerning convergence of scalar valued Dirichlet series, which is a series with the form

$$\sum_{n=1}^{\infty} a_n \frac{1}{n^s},$$

where the coefficients a_n are complex and s is a complex variable. H. F. Bohnenblust and E. Hille in the notorious paper

> [34] *On the absolute convergence of Dirichlet series, Annals of Mathematics, vol. 32, 600-622, 1931,*

solved the problem proposed by H. Bohr and for the proof they established in [34, Theorem I] their famous multilinear inequality, which is of independent high interest:

Multilinear Bohnenblust–Hille's inequality (1931)**.** *For each positive integer $m \geq 1$, there exists a constant $C_m \geq 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left\| A(e_{i_1}, \ldots, e_{i_m}) \right\|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \left\| A \right\|, \tag{1}$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{C}$. Moreover, the exponent $\frac{2m}{m+1}$ is optimal.

The case $m = 2$ is the well-known Littlewood's 4/3 inequality [86, Theorem 1]. These inequalities, and the growth of the constants involved in it, have important applications in various fields of analysis and mathematical physics (see, for instance, [29,32,41,51,53,56,57,62,89,102]).

Hardy and Littlewood (see [76] and [77, p.224]) provided an ℓ_p-version Littlewood's 4/3 inequality and then Praciano-Pereira studied in [99, Theorems A and B] the effect of replacing c_0 by ℓ_p in the Bohnenblust-Hille inequality, obtaining an general forms of it for multilinear form on ℓ_p spaces, nowadays (*unfortunately*) known as Hardy–Littlewood's multilinear inequality (we set a convenient notation: $X_p := \ell_p$, $1 \leq p < +\infty$ and $X_\infty := c_0$; for $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty]^m$, $\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$):

Multilinear Hardy–Littlewood's inequality. *Let* $\mathbf{p} \in [1, +\infty]^m$ *with* $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$. *Then there exists a constant* $C_{\mathbf{p}} > 0$ *such that, for every continuous* m*-linear form* $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{C}$,

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} |A(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C_{\mathbf{p}} \|A\|.$$

This part of the thesis is devoted to generalize the previous notorious inequalities. In Chapter 1 we verse about a part of the paper

> [5] *Sharp generalizations of the multilinear Bohnenblust-Hille inequality, Journal of Functional Analysis, vol. 266, no. 6, 3726–3740, 2014.*

which is an joint work with F. Bayart, D. Pellegrino and J. Seoane. We will prove that the multilinear Bohnenblust–Hille inequality is, *de facto*, a particular case of a quite general family of optimal inequalities (which will be a particular case of vector valued results present in the Chapter 2). More precisely we obtain

Theorem. *Let* $m \geq 1$, *let* $q_1, \ldots, q_m \in [1, 2]$. *The following assertions are equivalent:*

(1) There is a constant $C_{q_1 \ldots q_m} \geq 1$ *such that*

$$\left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2=1}^{\infty} \left(\cdots \left(\sum_{i_{m-1}=1}^{\infty} \left(\sum_{i_m=1}^{\infty} \|A(e_{i_1}, \ldots, e_{i_m})\|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_{m-2}}{q_{m-1}}} \cdots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{q_1 \ldots q_m} \|A\|$$

for all continuous m*-linear forms* $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2}$.

Here, as usual, $\mathbb{K}$ denotes the field of real or complex scalars. The Bohnenblust–Hille inequality is just the particular case $q_1 = \cdots = q_m = \frac{2m}{m+1}$. The *ingredients* to attain this are nothing else than Minkowski's inequality, a Hölder's interpolative inequality and a mixed (ℓ_1, ℓ_2)-estimate consequence of Khintchine's inequality.

In Chapter 2 we discourse about the results of the paper

> [6] *Optimal Hardy-Littlewood type inequalities for polynomials and multilinear operators, Israel Journal of Mathematics, in press.*

which also is joint work with F. Bayart, D. Pellegrino and J. Seoane. It concerns about definitive generalizations of the Littlewood's 4/3 inequality. As soon as J. E. Littlewood proved his famous inequality in [86, Theorem 1], it was rapidly extended to more general frameworks. For instance,

- The multilinear Bohnenblust–Hille inequality ([34, Theorem I], 1931)
- The multilinear Hardy–Littlewood inequality ([76], and [99, Theorems A and B], 1981)

- (Defant and Sevilla-Peris, [57, Theorem 1], 2009) If $1 \le s \le q \le 2$, there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : c_0 \times \cdots \times c_0 \to \ell_s$, then

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\frac{2m}{m+2\left(\frac{1}{s}-\frac{1}{q}\right)}}\right)^{\frac{m+2\left(\frac{1}{s}-\frac{1}{q}\right)}{2m}} \le C\|A\|.$$

Furthermore, the previous results were generalized by the author, F. Bayart, D. Pellegrino and J. Seoane and also by Dimant and Sevilla-Peris:

- ([5, Corollary 1.3], 2013) Let $1 \le s \le q \le 2$ and $\mathbf{p} \in [1,+\infty]^m$ such that

$$\frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right| \ge 0.$$

  Then there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\frac{2m}{m+2\left(\frac{1}{s}-\frac{1}{q}-\left|\frac{1}{\mathbf{p}}\right|\right)}}\right)^{\frac{m+2\left(\frac{1}{s}-\frac{1}{q}-\left|\frac{1}{\mathbf{p}}\right|\right)}{2m}} \le C\|A\|$$

  and the exponent is optimal.

- (Dimant and Sevilla-Peris, [59, Proposition 4.4], 2013) Let $\mathbf{p} \in [1,+\infty]^m$ and $s, q \in [1,+\infty]$ be such that $s \le q$. Then there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A\left(e_{i_1},\ldots,e_{i_m}\right)\|_{\ell_q}^{\rho}\right)^{\frac{1}{\rho}} \le C\|A\|,$$

  where ρ is given by

  (i) If $s \le q \le 2$, and
  - (a) if $0 \le \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s} - \frac{1}{q}$, then $\frac{1}{\rho} = \frac{1}{2} + \frac{1}{m}\left(\frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|\right)$.
  - (b) if $\frac{1}{s} - \frac{1}{q} \le \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2} + \frac{1}{s} - \frac{1}{q}$, then $\frac{1}{\rho} = \frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|$.

  (ii) If $s \le 2 \le q$, and
  - (a) if $0 \le \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s} - \frac{1}{2}$, then $\frac{1}{\rho} = \frac{1}{2} + \frac{1}{m}\left(\frac{1}{s} - \frac{1}{2} - \left|\frac{1}{\mathbf{p}}\right|\right)$.
  - (b) if $\frac{1}{s} - \frac{1}{2} \le \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s}$, then $\frac{1}{\rho} = \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|$.

  (iii) If $2 \le s \le q$ and $0 \le \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s}$, then $\frac{1}{\rho} = \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|$.

  Moreover, the exponents in the cases (ia),(iib) and (iii) are optimal. Also, the exponent in (ib) is optimal for $\frac{1}{s} - \frac{1}{q} \le \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2}$.

In Chapter 2 we investigate, among other results, in depth the remaining cases of the Dimant and Sevilla-Peris result and obtain the following

Theorem. *Let* $\mathbf{p} \in [1,+\infty]^m$ *and let* $\rho > 0$*. Assume moreover that either* $q \geq 2$ *or* $q < 2$ *and* $\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2}$*. Let*

$$\frac{1}{\lambda} := \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q,2\}} - \left|\frac{1}{\mathbf{p}}\right| > 0.$$

Then there exists $C > 0$ *such that, for every continuous* m*-linear operator* $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$*, we have*

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\rho}\right)^{\frac{1}{\rho}} \leq C\|A\|$$

if and only if

$$\frac{m}{\rho} \leq \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda, s, 2\}}.$$

The following table summarizes the optimal value of $\frac{1}{\rho}$ following the respective values of $s, q, p_1, ..., p_m$:

$1 \leq s \leq q \leq 2,\ \lambda < 2$	$\frac{1}{2} + \frac{1}{ms} - \frac{1}{mq} - \frac{1}{m} \times \left\|\frac{1}{\mathbf{p}}\right\|$
$1 \leq s \leq q \leq 2,\ \lambda \geq 2,\ \left\|\frac{1}{\mathbf{p}}\right\| < \frac{1}{2}$	$\frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \left\|\frac{1}{\mathbf{p}}\right\|$
$1 \leq s \leq 2 \leq q,\ \lambda < 2$	$\frac{m-1}{2m} + \frac{1}{ms} - \frac{1}{m} \times \left\|\frac{1}{\mathbf{p}}\right\|$
$1 \leq s \leq 2 \leq q,\ \lambda \geq 2$	$\frac{1}{s} - \left\|\frac{1}{\mathbf{p}}\right\|$
$2 \leq s \leq q$	$\frac{1}{s} - \left\|\frac{1}{\mathbf{p}}\right\|$

The technique we used is a combination of probabilistic tools and of an interpolative approach; this former technique is also employed in this paper to improve the constants for vector-valued Bohnenblust–Hille type inequalities.

Part II: Peano curves on topological vector spaces

Throughout history there have always been mathematical objects that have contradicted the intuition of the working mathematician. To cite some one these objects, let us recall the famous Weierstrass Monster. It came as a general shock when, in 1872 and during a presentation before the Berlin Academy, K. Weierstrass provided a classical example of a function that was continuous everywhere but differentiable nowhere. The particular example was defined as

$$f(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is an odd integer and $ab > 1 + 3\pi/2$ (see Figure 1). Although the first published example is certainly due to Weierstrass, already in 1830 the Czech mathematician B. Bolzano exhibited a continuous nowhere differentiable function (see [105] for a thorough study of these citations).

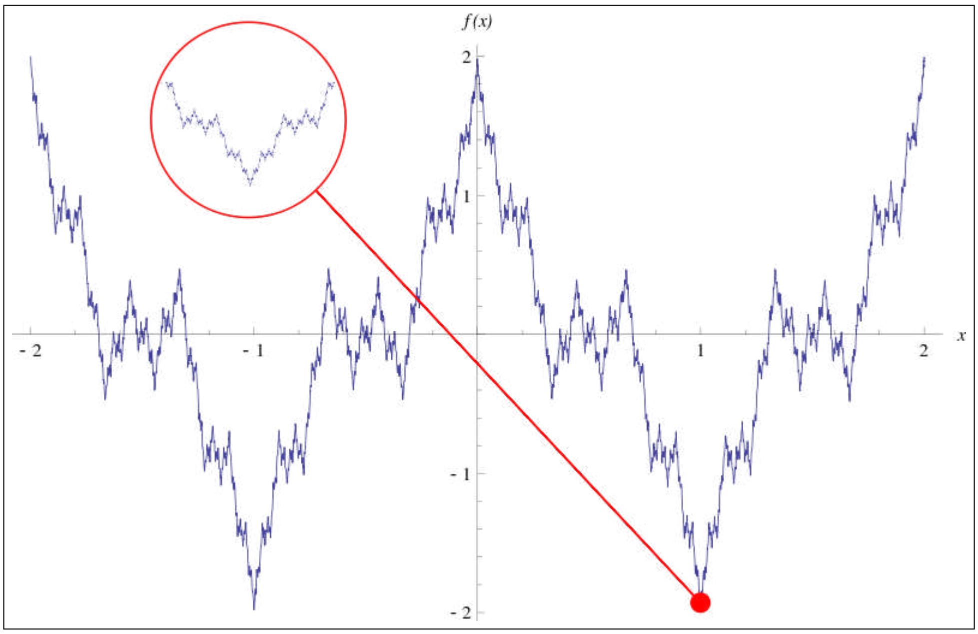

Figure 1: Weiestrass' Monster

One may think that, once such an object is found, not many more like it can possibly exist. History has proven this last statement wrong. For the last decade there has been a generalized trend in mathematics toward the search for large algebraic structures of special objects (and sometimes called *pathological* in the literature [68, 104]). One of the first results illustrating this was due to B. Levine and D. Milman [84].

Theorem (Levine and Milman, 1940). *The subset of* $\mathcal{C}[0,1]$ *of all continuous functions on* $[0,1]$ *of bounded variation does not contains a closed infinite linear space.*

Later, the following famous result on the set of continuous nowhere differentiable functions was proved by V. I. Gurariy [74]:

Theorem (Gurariy, 1966). *The set of continuous nowhere differentiable functions on* $[0,1]$ *contains an infinite linear space.*

Somehow, what we are seeing is that what one could expect to be an isolated phenomenon can actually even have a nice algebraic structure (in the form of infinite dimensional subspaces).

Let us provide a more formal and complete definition for the concepts motivated by this that, by now, are widely known (see, e.g., [10–12, 17, 27, 46, 48, 64]).

Definition (Lineability and spaceability, [10, 104])**.** *Let X be a topological vector space and M a subset of X. Let μ be a cardinal number.*

(1) *M is said to be* μ-lineable *if $M \cup \{0\}$ contains a vector space of dimension μ. At times, we shall be referring to the set M as simply* lineable *if the existing subspace is infinite dimensional.*

(2) *When the above linear space can be chosen to be dense (infinite dimensional and closed, resp.) in X we shall say that M is* μ-dense-lineable (spaceable, resp.)*.*

Moreover, L. Bernal introduced in [25] the notion of *maximal lineable* (and that of *maximal dense-lineable*) in X, meaning that, when keeping the above notation, the dimension of the existing linear space equals $\dim(X)$. Besides asking for linear spaces one could also study other structures, such as algebrability and some related ones, which were presented in [11, 12, 15, 104].

Definition. *Given an algebra $\mathcal{A}$, a subset $\mathcal{B} \subset \mathcal{A}$, and a cardinal number κ, we say that $\mathcal{B}$ is:*

(1) algebrable *if there is a subalgebra $\mathcal{C}$ of $\mathcal{A}$ so that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any system of generators of $\mathcal{C}$ is infinite.*

(2) κ-algebrable *if there exists a κ-generated subalgebra $\mathcal{C}$ of $\mathcal{A}$ with $\mathcal{C} \subset \mathcal{B} \cup \{0\}$.*

(3) strongly κ-algebrable *if there exists a κ-generated free algebra $\mathcal{C}$ contained in $\mathcal{B} \cup \{0\}$.*

Lately the study of the linear structure of certain subsets of surjective functions in $\mathbb{R}^{\mathbb{R}}$ (such as everywhere surjective functions, perfectly everywhere surjective functions, or Jones functions) has attracted the attention of several authors working on Real Analysis and Set Theory (see, e.g. [10, 12, 27, 70, 71]). The previously mentioned functions are, indeed, very "exotic": for instance an everywhere surjective function f in $\mathbb{R}^{\mathbb{R}}$ verifies that $f(I) = \mathbb{R}$ for every interval $I \subset \mathbb{R}$ and the other classes (perfectly everywhere surjective functions and Jones functions) are particular cases of everywhere surjective functions and, thus, with even "worse" behavior. It has been shown [69] that there exists a $2^{\mathfrak{c}}$-dimensional vector space every non-zero element of which is a Jones function and, thus, everywhere surjective (here, $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$). Of course, this previous result is optimal in terms of dimension since $\dim(\mathbb{R}^{\mathbb{R}}) = 2^{\mathfrak{c}}$. However, all the previous classes are nowhere continuous, thus, it is natural to ask about the set of continuous surjections.

In Chapter 3 we discourse about all the possible frameworks concerning lineability of the continuous surjections on euclidean spaces (thus a more general framework than that of $\mathbb{R}^{\mathbb{R}}$). For instance, we present the results of the paper

[4] *Maximal lineability of the set of continuous surjections, Bulletin of the Belgian Mathematical Society Simon Stevin, vol. 21, 83–87, 2014.*

which solves (optimal and positively) the lineability problem. More precisely we prove that (for a topological space X, $\mathcal{CS}(\mathbb{R}^m, X)$ denotes the set of continuous surjection from $\mathbb{R}^m$ on X)

Theorem. *For every pair $m, n \in \mathbb{N}$, the set $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$ is maximal lineable.*

Still in Chapter 3, we go further and verse about a part of the paper

> [7] *Peano curves on topological vector spaces, Linear Algebra and its Applications, vol. 460, 81–96, 2014,*

which is a joint work with L. Bernal, D. Pellegrino and J. Seoane concerning Peano curves on topological vector spaces. We bring comments on the positive answer of Bernal and Ordóñez (see [26, Theorem 3.2]) on the spaceability problem (for a topological space X, $\mathcal{CS}_\infty(\mathbb{R}^m, X)$ gathers the continous maps $f : \mathbb{R}^m \to X$ such that each point $a \in X$ is assumed on an *unbounded* subset of $\mathbb{R}^m$):

Theorem (Bernal and Ordóñez, 2014)**.** *For each pair $m, n \in \mathbb{N}$, the set $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n)$ is maximal dense-lineable and spaceable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$. In particular, it is maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$.*

We close Chapter 3 by proving the following complement of the previous results

Theorem. *For every $m \in \mathbb{N}$, the set $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{C}^n)$ is maximal strongly algebrable in $\mathcal{C}(\mathbb{R}^m, \mathbb{C}^n)$.*

This solves the algebrability remaining problem on euclidean spaces. In order to attain this we make use of some results and machinery from Complex Analysis, and we also provide some new results from Complex Analysis which are of independent interest (see, e.g., Lemma 3.15).

Chapter 4 presents the remaining part of the paper [7]: it moves on to the next natural step on trying to generalize the previous result to infinite dimensional spaces. In order to this, we make use of the notorious

Hahn–Mazurkiewicz's theorem. *A non-empty Hausdorff topological space is a continuous image of the unit interval if and only if it is a compact, connected, locally connected metrizable topological space.*

Hausdorff spaces that are the continuous image of the unit interval are called *Peano spaces.* The Hahn-Mazurkiewicz's theorem allows us to investigate topological vector spaces that are continuous image of the real line, from which we introduce the notion of σ-Peano space (see Definition 4.1) and use it to provide an optimal general lineability result

Theorem. *Let $\mathcal{X}$ be a σ-Peano topological vector space. Then $\mathcal{CS}_\infty(\mathbb{R}^m, \mathcal{X})$ is maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathcal{X})$.*

In addition, we will show how, by just starting with separable normed spaces, one can obtain σ-Peano spaces. We analyze Peano spaces in the framework of sequence spaces and also study Peano space in real and complex function spaces.

Preliminaries and Notation

To seek conciseness and to avoid boring technicalities, we introduce here several useful notations throughout this text.

- Our main interest are the Banach spaces over $\mathbb{K}$, which shall stand for the complex $\mathbb{C}$ or real $\mathbb{R}$ fields.

- *multi-index notation*: for a positive integer m and a non-void subset $D \subset \mathbb{N}$ we denote the set of multi-indices $\mathbf{i} = (i_1, \ldots, i_m)$, with each $i_k \in D$, by

$$\mathcal{M}(m, D) := \{\mathbf{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m;\ i_k \in D,\ k = 1, \ldots, m\} = D^m$$

  and it is also convenient set $\mathcal{M}(m, N) := \mathcal{M}(m, \{1, 2, \ldots, N\})$, and also $\mathcal{P}_k(m)$ stands for the set of subsets $S \subseteq \{1, \ldots, m\}$ with card(S)= k, $\widehat{S} := \{1, \ldots, m\} \setminus S$ and $\mathbf{i}_S := (i_j)_{j \in S}$.

- *multiple exponent notation*: for a positive integer m, $\mathbf{p}$ stands for a multiple exponent $(p_1, \ldots, p_m) \in [1, \infty]^m$ and

$$\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

- We set $X_\infty := c_0$ and $X_p := \ell_p$, for $1 \leq p < \infty$.

- ℓ_p^N denotes the scalar space $\mathbb{K}^N$ with the p-norm, for $p \in [1, \infty]$.

- For m-linear vector valued operator $A : X_p \times \cdots \times X_p \to Y$ and a multi-index $\mathbf{i} := (i_1, \ldots, i_m)$, we set $Ae_{\mathbf{i}} := A(e_{i_1}, \ldots, e_{i_m})$.

- p^* will denote the conjugate exponent of $p \in [1, \infty]$.

- The symbol $\sum_{\widehat{i_k}}$ will always means that we are fixing the k-th index and that we are summing over all the remaining indices.

- $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$.

- For any topological spaces X, Y, $\mathcal{C}(X, Y)$ denotes the set of continuous functions from X to Y. The following notation it will be also useful on our purposes:

$$\mathcal{CS}_\infty(\mathbb{R}^m, X) := \left\{ f \in \mathcal{C}(\mathbb{R}^m, X) : f^{-1}(\{a\}) \text{ is unbounded for every } a \in X \right\}.$$

Part I

Hardy–Littlewood and Bohnenblust–Hille inequalities

Chapter 1

Sharp generalizations of the multilinear Bohnenblust–Hille inequality

This chapter is devoted to part of the paper

[5] *Sharp generalizations of the multilinear Bohnenblust-Hille inequality, Journal of Functional Analysis, vol. 266, no. 6, 3726–3740, 2014.*

which is an joint work with F. Bayart, D. Pellegrino and J. Seoane. Here we verse about scalar valued generalizations of the multilinear Bohnenblust–Hille inequality and prove that this is, *de facto*, a particular case of a quite general family of optimal inequalities. In the next chapter, among other results, we deal with the vector valued versions of the remaining part of [5] concerning generalizations of the Hardy–Littlewood inequality.

1.1 Motivation and main results

The starting point of this chapter is the classical multilinear Bohnenblust–Hille inequality, which has the following precise form

Theorem 1.1 (Multilinear Bohnenblust-Hille's inequality, 1931, [34])**.** *For each positive integer $m \geq 1$, there exists a constant $C_m \geq 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \left\| A(e_{i_1},\ldots,e_{i_m}) \right\|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \left\| A \right\|, \tag{1.1}$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$. Moreover, the exponent $\frac{2m}{m+1}$ is optimal.

The case $m = 2$ is the well-known Littlewood's 4/3 inequality [86, Theorem 1]. These inequalities, and the growth of the constants involved in it, have important applications in various fields of analysis and mathematical physics (see, for instance, [29,32,41,51,53,56,57,62,89,102]).

We will prove that the Bohnenblust–Hille inequality is a very particular case of a large family of sharp inequalities. More precisely, we prove the following general result:

Theorem 1.2. *Let $m \geq 1$, let $q_1, \ldots, q_m \in [1,2]$. The following assertions are equivalent:*

(1) There is a constant $C_{q_1 \ldots q_m} \geq 1$ such that

$$\left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2=1}^{\infty} \left(\cdots \left(\sum_{i_{m-1}=1}^{\infty} \left(\sum_{i_m=1}^{\infty} \| A(e_{i_1}, \ldots, e_{i_m}) \|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_{m-2}}{q_{m-1}}} \cdots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{q_1 \ldots q_m} \|A\|$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2}$.

The Bohnenblust–Hille inequality is just the particular case

$$q_1 = \cdots = q_m = \frac{2m}{m+1}.$$

This is a particular case of a more general version presented in [5, Theorem 1.2] concerning ℓ_q-valued multilinear operator on ℓ_p spaces. We deal with this in the next chapter, when we introduce generalizations of the Hardy–Littlewood inequality.

This kind of inequalities was already considered in [28] when $m = 2$, as generalizations of Littlewood's 4/3 inequality. The strategy for the proof of (2) $\Rightarrow$ (1) in Theorem 1.2 will be very simple, maybe simpler than all previous known proofs of the Bohnenblust-Hille inequality. The starting point is the generalized Littlewood mixed (ℓ_1, ℓ_2)-norm inequality, that is that Theorem 1.2 is true when $(q_1, \ldots, q_m) = (1, 2, \ldots, 2)$. This property is "well-known" and it is a consequence of Khintchine's inequality. Using this and nothing else than Minkowski's inequality and an Hölder's interpolative inequality, we will infer the general case.

1.2 Interpolation on mixed norm sequence spaces

It will be useful deal with strongly summable sequences (for more details see [58, p.32]): for a Banach space X and $p \in [1, \infty]$, we shall denote by $\ell_p(X)$ the space of strongly p-summable sequences (alternatively, strong ℓ_p sequences), that is, a vector sequence $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ belongs to $\ell_p(X)$ if the corresponding scalar sequence $(\|x_n\|_X)_{n \in \mathbb{N}}$ is in ℓ_p. Naturally $\ell_p(X)$ equipped with the p-norm

$$\|(x_n)_n\|_p := \|(\|x_n\|_X)_n\|_p$$

it is a Banach space. For a multiple exponent $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$, the mixed norm sequence space

$$\ell_{\mathbf{p}}(X) := \ell_{p_1}\left(\ell_{p_2}\left(\ldots\left(\ell_{p_m}(X)\right)\ldots\right)\right)$$

(it is a Banach space when endowed with the norm $\|\cdot\|_{\mathbf{p}}$ and) is formed by all multi-index vector valued matrices $(x_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^m}$ with finite $\mathbf{p}$-norm (recall the notation for multi-indexes $\mathbf{i}$ presented in Section "Preliminaries and Notation"). Namely, when $\mathbf{p} \in [1,\infty)^m$, a vector valued matrix $(x_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^m} \in \ell_{\mathbf{p}}(X)$ if, and only if,

$$\|(x_{\mathbf{i}})_{\mathbf{i}}\|_{\mathbf{p}} := \left(\sum_{i_1=1}^{\infty}\left(\sum_{i_2=1}^{\infty}\left(\cdots\left(\sum_{i_{m-1}=1}^{\infty}\left(\sum_{i_m=1}^{\infty}\|x_{\mathbf{i}}\|_X^{p_m}\right)^{\frac{p_{m-1}}{p_m}}\right)^{\frac{p_{m-2}}{p_{m-1}}}\cdots\right)^{\frac{p_2}{p_3}}\right)^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} < \infty.$$

When $X = \mathbb{K}$, we just write $\ell_{\mathbf{p}}$ instead of $\ell_{\mathbf{p}}(\mathbb{K})$.

The next interpolation result on these mixed norm sequences spaces has a central role on our techniques. We verse on this subject in Appendix A and present a proof of it therein (see Corollary A.5).

Proposition 1.3 (Hölder's interpolative inequality for mixed $\ell_{\mathbf{p}}$ spaces)**.** *Let m, n, N be positive integers, $\mathbf{r}, \mathbf{p}(1), \ldots, \mathbf{p}(N) \in [1,\infty]^m$ and $\theta_1, \ldots, \theta_N \in [0,1]$ be such that $\theta_1 + \cdots + \theta_N = 1$ and*

$$\frac{1}{r_j} = \sum_{k=1}^{N}\frac{\theta_k}{p_j(k)} = \frac{\theta_1}{p_j(1)} + \cdots + \frac{\theta_N}{p_j(N)}, \qquad \text{for } j = 1, \ldots, m. \tag{1.2}$$

Then, for all scalar matrices $\mathbf{a} := (\mathrm{a}_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$, we have

$$\|\mathbf{a}\|_{\mathbf{r}} \le \|\mathbf{a}\|_{\mathbf{p}(1)}^{\theta_1} \cdots \|\mathbf{a}\|_{\mathbf{p}(N)}^{\theta_N}.$$

In particular, if each $\mathbf{p}(k) \in [1,\infty)$, the previous inequality means that

$$\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|\mathrm{a}_{\mathbf{i}}|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}$$
$$\le \prod_{k=1}^{N}\left[\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|\mathrm{a}_{\mathbf{i}}|^{p_m(k)}\right)^{\frac{p_{m-1}(k)}{p_m(k)}}\cdots\right)^{\frac{p_1(k)}{p_2(k)}}\right)^{\frac{1}{p_1(k)}}\right]^{\theta_k}.$$

Remark 1.4. *The previous condition* (1.2)*, means precisely that the point $\left(\frac{1}{r_1}, \ldots, \frac{1}{r_m}\right)$ belongs to the convex hull of the points $\left(\frac{1}{p_1(k)}, \ldots, \frac{1}{p_m(k)}\right)$, $k = 1, \ldots, N$. In this situation, we will simple say that the multiple exponent $\mathbf{r} := (r_1, \ldots, r_m)$* "is obtained by interpolation" *(or* "comes by interpolation"*) of the multiple exponents $\mathbf{p}(k) := (p_1(k), \ldots, p_m(k))$, $k = 1, \ldots, N$.*

1.3 Applications of Minkowski's inequality

Minkowski's inequality is a very well-known result that helps to prove that L_p spaces are Banach spaces: it is the triangle inequality for L_p spaces. We need a somewhat well known result, which is a corollary of one of the many versions of Minkowski's inequality, whose proof can be found, for instance, in [72, Corollary 5.4.2].

Lemma 1.5 (Minkowski's inequality)**.** *For any $0 < p \leq q < \infty$ and for any scalar matrix $(a_{ij})_{i,j\in\mathbb{N}}$,*

$$\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}|a_{ij}|^{p}\right)^{q/p}\right)^{1/q} \leq \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}|a_{ij}|^{q}\right)^{p/q}\right)^{1/p}.$$

Combining the Hölder interpolative inequality for mixed $\ell_{\mathbf{p}}$ spaces (Proposition 1.3) with the previous Minkowski inequality (Proposition 1.5) we have a very useful inequality (see [20, Remark 2.2]):

Corollary 1.6 (Blei's general inequality)**.** *Let m,n be positive integers, $1 \leq k \leq m$ and $1 \leq s \leq q$. Then for all scalar matrix $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$,*

$$\left(\sum_{\mathbf{i}\in\mathcal{M}(m,n)}|a_{\mathbf{i}}|^{\rho}\right)^{\frac{1}{\rho}} \leq \prod_{S\in\mathcal{P}_k(m)}\left(\sum_{\mathbf{i}_S}\left(\sum_{\mathbf{i}_{\widehat{S}}}|a_{\mathbf{i}}|^{q}\right)^{\frac{s}{q}}\right)^{\frac{1}{s}\cdot\frac{1}{\binom{m}{k}}},$$

where

$$\rho := \frac{msq}{kq+(m-k)s}$$

and $\mathcal{P}_k(m)$ stands for the set of subsets $S \subseteq \{1,\dots,m\}$ with card(S)$=k$, $\widehat{S} := \{1,\dots,m\}\setminus S$ and $\mathbf{i}_S := (i_j)_{j\in S}$.

The next proposition is simple multi-index variant of the previous Minkowski's inequality. It shows that an (ℓ_p,ℓ_q)-mixed norm inequality implies many other of this kind. Recall that the symbol $\sum_{\widehat{i_k}}$ shall mean that we are fixing the k-th index summing over all the remaining indices.

Proposition 1.7. *Let $m,N \geq 1$ be positive integers, $1 \leq p < q < \infty$, for each $k = 1,\dots,m$, let $\mathbf{p}(k) := (q,\dots,q,p,q,\dots,q) \in [1,+\infty)^m$ (p at k-th position) and also let $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}$ be a scalar valued matrix. Then, for each $k = 1,\dots,m$,*

$$\left\|(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}\right\|_{\mathbf{p}(k)} := \left(\sum_{i_1,\dots,i_{k-1}=1}^{N}\left(\sum_{i_k=1}^{N}\left(\sum_{i_{k+1},\dots,i_m=1}^{N}|a_{\mathbf{i}}|^{q}\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leq \left(\sum_{i_k=1}^{N}\left(\sum_{\widehat{i_k}=1}^{N}|a_{\mathbf{i}}|^{q}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}.$$

Proof. The result follows by induction on k combined with Minkowski's inequality. Let us assume that the property is true up to the $(k-1)$-th position. The integers $i_1,\dots,i_{k-1}$ being fixed, we set

$$\alpha_{i_{k-1},i_k} := \left(\sum_{i_{k+1},\dots,i_m=1}^{N}|a_{\mathbf{i}}|^{q}\right)^{p/q}$$

Applying Minkowski's inequality with exponents 1 and q/p,

$$\sum_{i_{k-1}=1}^{N}\left(\sum_{i_k=1}^{N}\alpha_{i_{k-1},i_k}\right)^{q/p}\leq\left(\sum_{i_k=1}^{N}\left(\sum_{i_{k-1}=1}^{N}\alpha_{i_{k-1},i_k}^{q/p}\right)^{p/q}\right)^{q/p}.$$

Hence,

$$\left(\sum_{i_1,\ldots,i_{k-1}=1}^{N}\left(\sum_{i_k=1}^{N}\left(\sum_{i_{k+1},\ldots,i_m=1}^{N}|a_{\mathbf{i}}|^q\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}\leq\left(\sum_{i_1,\ldots,i_{k-2}=1}^{N}\left(\sum_{i_k=1}^{N}\left(\sum_{i_{k-1},i_{k+1},\ldots,i_m=1}^{N}|a_{\mathbf{i}}|^q\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}$$

Defining the scalar matrix $(b_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}$ with entries $b_{i_1,\ldots,i_{k-1},i_k,\ldots,i_m}:=a_{i_1,\ldots,i_k,i_{k-1},\ldots,i_m}$, the previous inequality means that

$$\left\|(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}\right\|_{\mathbf{p}(k)}\leq\left\|(b_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}\right\|_{\mathbf{p}(k-1)},$$

thus by the induction hypothesis, we conclude the result:

$$\left\|(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}\right\|_{\mathbf{p}(k)}\leq\left\|(b_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}\right\|_{\mathbf{p}(k-1)}\leq\left(\sum_{i_{k-1}=1}^{N}\left(\sum_{\widehat{i_{k-1}}=1}^{N}|b_{\mathbf{i}}|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}=\left(\sum_{i_k=1}^{N}\left(\sum_{\widehat{i_k}=1}^{N}|a_{\mathbf{i}}|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}.$$

□

1.4 Applications of Khinchine's inequality

The Khinchine inequality in its modern presentation has its roots in [107]. An accessible proof (for Rademacher independet functions) is in [58, Theorem 1.10].

Lemma 1.8 (Khinchine's inequality)**.** *For any $p\in[1,\infty)$, there exists a(n) (optimal) constant $\mathrm{A}_{\mathbb{K},p}$ with the following properties:*

(1) (Real case) For any sequence $(a_n)_n$ of real numbers, we have

$$\left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\leq\mathrm{A}_{\mathbb{R},p}\left(\int_0^1\left|\sum_{n=1}^{\infty}a_nr_n(t)\right|^p dt\right)^{1/p},$$

with $(r_n)_{n\in\mathbb{N}}$ denoting the sequence of Rademacher functions $r_n:[0,1]\to\{\pm1\}$. Moreover, the optimal constants $\mathrm{A}_{\mathbb{R},p}$ are

$$\mathrm{A}_{\mathbb{R},p}=\begin{cases}2^{\frac{1}{p}-\frac{1}{2}}\le \mathrm{A}_{\mathbb{R},1}=\sqrt{2}, & \text{if } 1\le p\le p_0\approx 1.847;\\ 2^{-\frac{1}{2}}\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{1+p}{2}\right)}\right)^{\frac{1}{p}}, & \text{if } p_0<p<2;\\ 1, & \text{if } 2\le p<\infty.\end{cases}$$

(2) (Complex case) For any sequence $(a_n)_n$ of complex numbers with finite support, we have

$$\left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\le \mathrm{A}_{\mathbb{C},p}\left(\int_{\mathbb{T}^\infty}\left|\sum_{n=1}^{\infty}a_nz_n\right|^p dz\right)^{1/p},$$

with $\mathbb{T}^\infty:=\left\{z=(z_n)_{n\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}:|z_n|=1 \text{ for all } n\in\mathbb{N}\right\}$ (the infinite polycircle) and dz denoting the standard Lebesgue probability measure on $\mathbb{T}^\infty$. Moreover, the optimal constants $\mathrm{A}_{\mathbb{C},p}$ are

$$\mathrm{A}_{\mathbb{C},p}=\begin{cases}\Gamma\left(\frac{2+p}{2}\right)^{-\frac{1}{p}}, & \text{if } 1\le p<2;\\ 1, & \text{if } 2\le p<\infty.\end{cases}$$

The (apparently) *strange* value $p_0\approx 1.8474$ is, to be precise, the unique number $p_0\in(1,2)$ satisfying $\Gamma\left(\frac{p_0+1}{2}\right)=\frac{\sqrt{\pi}}{2}$. The notation $\mathrm{A}_{\mathbb{K},p}$ for the optimal constants will be kept throughout this. For complex scalars it more useful to use Steinhaus variables (also called Khinchine's inequality with Steinhaus variables) instead of the Rademacher functions, since it gives better constants. The best constants $\mathrm{A}_{\mathbb{R},p}$ and $\mathrm{A}_{\mathbb{C},p}$ were obtained by Haagerup and König, respectively (see [75] and [83]). The particular values of these when $p=1$ will be very useful for the estimates we will present:

$$\mathrm{A}_{\mathbb{R},1}=\sqrt{2}\qquad\text{and}\qquad \mathrm{A}_{\mathbb{C},1}=\left[\Gamma\left(3/2\right)\right]^{-1}=\frac{2}{\sqrt{\pi}}.$$

Using Fubini's theorem and Minkowski's inequality (see, for instance, [54, Lemma 2.2] for the real case and [93, Theorem 2.2] for the complex case), these inequalities have a multilinear version.

Lemma 1.9 (Multilinear Khinchine's inequality)**.** *Let $N,m\ge 1$ be positive integers.*

(1) (Real case) For any real valued matrix $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}$,

$$\left(\sum_{\mathbf{i}\in\mathcal{M}(m,N)}|a_{\mathbf{i}}|^2\right)^{1/2}\le \mathrm{A}_{\mathbb{R},p}^m\left(\int_{[0,1]^m}\left|\sum_{\mathbf{i}\in\mathcal{M}(m,N)}r_{i_1}(t_1)\cdots r_{i_m}(t_m)a_{\mathbf{i}}\right|^p dt_1\cdots dt_m\right)^{1/p}.$$

(2) (Complex case) For any complex valued matrix $(a_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,N)}$,

$$\left(\sum_{\mathbf{i}\in\mathcal{M}(m,N)}|a_{\mathbf{i}}|^2\right)^{1/2}\le \mathrm{A}_{\mathbb{C},p}^m\left(\int_{(\mathbb{T}^\infty)^m}\left|\sum_{\mathbf{i}\in\mathcal{M}(m,N)}a_{\mathbf{i}}z_{i_1}^{(1)}\ldots z_{i_m}^{(m)}\right|^p dz^{(1)}\ldots dz^{(m)}\right)^{1/p}.$$

With this we are able to prove the following "well-known" (ℓ_1,ℓ_2)-norm inequality.

Proposition 1.10. *Let $m \geq 1$ be a positive integer. Then, for any $k \in \{1, \ldots, m\}$,*

$$\sum_{i_k=1}^{\infty} \left(\sum_{\widehat{i_k}=1}^{\infty} |A(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \leq (\mathrm{A}_{\mathbb{K},1})^{m-1} \|A\|,$$

holds for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

Proof. By symmetry, we just need to prove the result for $k = 1$. Also we will just prove the result for the real case (the argument is the same for the complex case). The multilinear version of Khinchine's inequality (Lemma 1.9) delivers

$$\begin{aligned}
&\left(\sum_{\widehat{i_1}} |A(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \\
&\quad \leq \mathrm{A}_{\mathbb{R},1}^{m-1} \int_{[0,1]^{m-1}} \left| \sum_{\widehat{i_1}} r_{i_2}(t_2) \cdots r_{i_m}(t_m) A(e_{i_1}, \ldots, e_{i_m}) \right| dt_2 \cdots dt_m \\
&\quad = \mathrm{A}_{\mathbb{R},1}^{m-1} \int_{[0,1]^{m-1}} \left| A\left(e_{i_1}, \sum_{i_2} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m} r_{i_m}(t_m) e_{i_m} \right) \right| dt_2 \cdots dt_m \\
&\quad \leq \mathrm{A}_{\mathbb{R},1}^{m-1} \cdot \sup_{t_2, \ldots, t_m \in [0,1]} \left| A\left(e_{i_1}, \sum_{i_2} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m} r_{i_m}(t_m) e_{i_m} \right) \right|. \qquad (*)
\end{aligned}$$

Fixed $t_2, \ldots, t_m \in [0,1]$, $A_{t_2,\ldots,t_m} := A\left(\cdot, \sum_{i_2} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m} r_{i_m}(t_m) e_{i_m}\right)$ defines a real linear form on c_0. Thus applying the Bohnenblust–Hille inequality for $m = 1$ (which in fact it is a well known equality that holds with $C_1 = 1$) we get

$$\sum_{i_1} \left| A\left(e_{i_1}, \sum_{i_2} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m} r_{i_m}(t_m) e_{i_m} \right) \right| \leq \|A\| \prod_{k=2}^{m} \left\| \sum_n r_n(t_k) e_n \right\|.$$

So using this combined with the well-know fact

$$\sup_{t \in [0,1]} \left\| \sum_n r_n(t) e_n \right\| \leq \sup_{|\alpha_n| \leq 1} \left\| \sum_n \alpha_n e_n \right\| = 1,$$

we get that

$$\sum_{i_1} \left| A\left(e_{i_1}, \sum_{i_2} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m} r_{i_m}(t_m) e_{i_m} \right) \right| \leq \|A\|.$$

Therefore, using this estimate combined with inequality $(*)$ we conclude the result

$$\sum_{i_1} \left(\sum_{\widehat{i_1}} |A(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \leq \mathrm{A}_{\mathbb{R},1}^{m-1} \|A\|.$$

□

For a bounded m-linear form $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$ and a multi-index $\mathbf{i} := (i_1, \ldots, i_m)$, let $Ae_{\mathbf{i}} := A(e_{i_1}, \ldots, e_{i_m})$. The following estimates will lead us to obtain the general Bohnenblust–Hille inequality (Theorem 1.2).

Proposition 1.11 (Mixed (ℓ_1, ℓ_2)-estimates)**.** *Let $\mathbf{p}(k) := (2, \ldots, 2, 1, 2, \ldots, 2) \in [1, +\infty)^m$, for $k = 1, \ldots, m$ (1 at k-th position). Then,*

$$\left\| (Ae_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^m} \right\|_{\mathbf{p}(k)} := \left(\sum_{i_1,\ldots,i_{k-1}} \left(\sum_{i_k} \left(\sum_{i_{k+1},\ldots,i_m=1} |Ae_{\mathbf{i}}|^2 \right)^{\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \leq (\mathrm{A}_{\mathbb{K},1})^{m-1} \|A\|,$$

for $k = 1, \ldots, m$ and for any bounded m-linear form $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$.

Proof. The results follows by applying the multi-index Minkowski's inequality (Proposition 1.7), the (ℓ_1, ℓ_2)-estimates (Proposition 1.10), respectively:

$$\begin{aligned} \left\| (Ae_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^m} \right\|_{\mathbf{p}(k)} &:= \left(\sum_{i_1,\ldots,i_{k-1}} \left(\sum_{i_k} \left(\sum_{i_{k+1},\ldots,i_m=1} |Ae_{\mathbf{i}}|^2 \right)^{\frac{1}{2}} \right)^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{i_k=1} \left(\sum_{\widehat{i_k}} |A(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \\ &\leq (\mathrm{A}_{\mathbb{K},1})^{m-1} \|A\|. \end{aligned}$$

□

1.5 First part of the proof of Theorem 1.2

Now we will prove that (2) implies (1) in Theorem 1.2. By the monotonicity of the norm on the sequences spaces, that is, $\|\cdot\|_s \leq \|\cdot\|_r$ as soon as $r \leq s$, it is sufficient to deal with the boundary situation:

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2},$$

which means that the exponent $\mathbf{q} := (q_1, \ldots, q_m) \in [1, 2]^m$ comes from the interpolation of the multiple exponents $\mathbf{p}(k) := (2, \ldots, 2, 1, 2, \ldots, 2) \in [1, 2]^m$, $k = 1, \ldots, m$ where 1 is at the k-th position (recall Remark 1.4), as guaranteed by the following straightforward characterization of a convex hull:

Lemma 1.12. *Let $m \geq 2$ be a positive integer, $0 < a < b$ be positive real numbers and also let $P_k := (a, \ldots, a, b, a \ldots, a)$, for $k \in \{1, \ldots, m\}$, where b is at the k-th position. Then the convex hull of $P_1, \ldots, P_m$ is the set of points $(x_1, \ldots, x_m)$ such that $x_1 + \cdots + x_m = (m-1)a + b$ and $x_k \in [a, b]$ for any $k \geq 1$.*

Given a bounded m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{K}$, the mixed (ℓ_1, ℓ_2)-estimates (Proposition 1.11) assures that

$$\left\|(Ae_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^m}\right\|_{\mathbf{p}(k)} \leq (\mathrm{A}_{\mathbb{K},1})^{m-1} \|A\|,$$

for $k = 1, \ldots, m$. Since $\mathbf{q}$ becames from the interpolation of the exponents $\mathbf{p}(k)$, we may apply the Hölder's interpolative inequality for mixed $\ell_{\mathbf{p}}$ spaces (Proposition 1.3) for the matrix $(Ae_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^m}$, and combine with the previous (ℓ_1, ℓ_2)-estimates to conclude the result:

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_m=1}^{\infty}\|A(e_{i_1},\ldots,e_{i_m})\|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$

$$=: \left\|(Ae_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^m}\right\|_{\mathbf{q}} \leq \prod_{k=1}^{N}\left\|(Ae_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^m}\right\|_{\mathbf{p}(k)}^{\theta_k} \leq \left[(\mathrm{A}_{\mathbb{K},1})^{m-1}\|A\|\right]^{\theta_1+\cdots+\theta_N} = (\mathrm{A}_{\mathbb{K},1})^{m-1}\|A\|.$$

Remark 1.13. *If we take care of the constant in the general Bohnenblust-Hille inequality (Theorem 1.2), our method shows that it is valid with constant $C_m \leq (\sqrt{2})^{m-1}$ in the real case and $C_m \leq (2/\sqrt{\pi})^{m-1}$ in the complex. This constant comes from the best known constant in the mixed (ℓ_1, ℓ_2)-Littlewood inequality (Proposition 1.11). However, Bayart, Pellegrino and Seoane provided subpolynomial constants for the multilinear case (see* [20, Corollary 3.2 and 3.3] *and* [94]*) and, among other results, that the complex polynomial constants are actually subpolynomial and solve a secular and outstanding problem concerning the Bohr radius (see* [53]*).*

1.6 On the optimality of the exponents

1.6.1 A Kahane-Salem-Zygmund inequality

A way to prove that the exponent $\frac{2m}{m+1}$ is optimal in the Bohnenblust-Hille inequality is to use the Kahane-Salem-Zygmund inequality, which allows to control the infinite norm of random polynomials. We need a variant of this inequality for multilinear forms on ℓ_p^n, which represents $\mathbb{K}^n$ with the p-norm.

Lemma 1.14 (Kahane-Salem-Zygmund's multilinear inequality)**.** *Let $m, N \geq 1$, $p_1, \ldots, p_m \in [1, +\infty]^m$ and let, for $p \geq 1$,*

$$\alpha(p) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{p}, & \textit{if } p \geq 2; \\ 0, & \textit{otherwise.} \end{cases}$$

Then there exists a m-linear form $A : \ell_{p_1}^N \times \cdots \times \ell_{p_m}^N \to \mathbb{K}$ which may be written as

$$A(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1,\ldots,i_m=1}^{N} \pm z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}$$

such that

$$\|A\| \leq C_m N^{\frac{1}{2}+\alpha(p_1)+\cdots+\alpha(p_m)}.$$

We present the proof this result in Appendix B.

1.6.2 Optimality of Theorem 1.2

Let $q_1, \ldots, q_m \in [1,2]$ satisfying (1) of Theorem 1.2. Let $A : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{K}$ be the m-linear map given by the previous Lemma 1.14. Then

$$\|A\| \leq C_{q_1 \ldots q_m} N^{\frac{m+1}{2}}$$

whereas, for any $i_1, \ldots, i_m$,

$$|A(e_{i_1}, \ldots, e_{i_m})| = \left| \sum_{j_1,\ldots,j_m=1}^{N} \pm \delta_{i_1 j_1} \cdots \delta_{i_m j_m} \right| = 1.$$

It is then easy to show by induction that

$$\left(\sum_{i_1=1}^{N} \left(\cdots \left(\sum_{i_m=1}^{N} |A(e_{i_1}, \ldots, e_{i_m})|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} = N^{\frac{1}{q_1}+\cdots+\frac{1}{q_m}}.$$

So,

$$N^{\frac{1}{q_1}+\cdots+\frac{1}{q_m}} \leq C_{q_1 \ldots q_m} N^{\frac{m+1}{2}}$$

holds for all positive integers N and, therefore, we conclude that for the condition (1) of Theorem 1.2 to be true, it is necessary that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2}.$$

Chapter 2

Optimal Hardy–Littlewood type inequalities for multilinear operators

In this chapter is we present the results of the paper

> [6] *Optimal Hardy-Littlewood type inequalities for polynomials and multilinear operators, Israel Journal of Mathematics, in press.*

a joint work with F. Bayart, D. Pellegrino and J. Seoane which concerning general and definitive forms of the Hardy–Littlewood inequality, which provides much simpler and straightforward proofs when restricted to the original particular cases, and we are able to show that in most cases the exponents involved are optimal. The technique we used is a combination of probabilistic tools and of an interpolative approach; this former technique is also employed in this paper to improve the constants for vector-valued Bohnenblust–Hille type inequalities.

2.1 Motivation and main results

This part of the thesis has as the starting point a result of 1930 due to J. E. Littlewood [86, Theorem 1] which is the following result concerning bilinear forms on $c_0 \times c_0$, now called Littlewood's 4/3 inequality:

Theorem 2.1 (Littlewood, 1930 [86]). *For any bounded bilinear form $A : c_0 \times c_0 \to \mathbb{C}$,*

$$\left(\sum_{i,j=1}^{+\infty} |A(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2}\|A\|$$

and, moreover, the exponent $4/3$ is optimal.

As soon as Littlewood's 4/3 inequality appeared, it was rapidly extended to more general frameworks. For instance, the first step was to generalize to multilinear forms, which is due to Bohnenbluts and Hille [34, Theorem I],

Multilinear Bohnenblust–Hille's inequality (1931). *For each positive integer $m \geq 1$, there exists a constant $C_m \geq 1$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} \|A(e_{i_1},\ldots,e_{i_m})\|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C \|A\|, \tag{2.1}$$

for all continuous m-linear forms $A : c_0 \times \cdots \times c_0 \to \mathbb{C}$. Moreover, the exponent $\frac{2m}{m+1}$ is optimal.

Hardy-Littlewood (see [76] and [77, p.224]) provided an ℓ_p-version Littlewood's 4/3 inequality and then Praciano-Pereira studied in [99, Theorems A and B] the effect of replacing c_0 by ℓ_p in the Bohnenblust-Hille inequality, obtaining a general form of it for multilinear form on ℓ_p spaces, nowadays (*unfortunately*) known as Hardy–Littlewood's multilinear inequality (recall that $X_p := \ell_p$, $1 \leq p < +\infty$ and $X_\infty := c_0$; for $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty]^m$, $\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$):

Multilinear Hardy–Littlewood's inequality (1981). *Let $\mathbf{p} \in [1, +\infty]^m$ with $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$. Then there exists a constant $C_{\mathbf{p}} > 0$ such that, for every continuous m-linear form $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{C}$,*

$$\left(\sum_{i_1,\ldots,i_m=1}^{\infty} |A(e_{i_1},\ldots,e_{i_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C_{\mathbf{p}} \|A\|. \tag{2.2}$$

A. Defant and P. Sevilla-Peris in [57, Theorem 1] provided a vector valued form of the Bohnenblust-Hille inequality,

Theorem 2.2 (Defant and Sevilla-Peris, 2009). *If $1 \leq s \leq q \leq 2$, there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : c_0 \times \cdots \times c_0 \to \ell_s$, then*

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\frac{2m}{m+2\left(\frac{1}{s}-\frac{1}{q}\right)}} \right)^{\frac{m+2\left(\frac{1}{s}-\frac{1}{q}\right)}{2m}} \leq C\|A\|.$$

Very recently the previous results were generalized by the author, F. Bayart, D. Pellegrino and J. Seoane in [5, Corollary 1.3] and also by Dimant and Sevilla-Peris [59, Proposition 4.4]:

Theorem 2.3 (–, Bayart, Pellegrino and Seoane, 2013). *Let $1 \leq s \leq q \leq 2$ and $\mathbf{p} \in [1, +\infty]^m$ such that*

$$\frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right| \geq 0. \tag{2.3}$$

Then there exists a constant $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\frac{2m}{m+2\left(\frac{1}{s}-\frac{1}{q}-\left|\frac{1}{\mathbf{p}}\right|\right)}} \right)^{\frac{m+2\left(\frac{1}{s}-\frac{1}{q}-\left|\frac{1}{\mathbf{p}}\right|\right)}{2m}} \leq C\|A\|$$

and the exponent is optimal.

This extends Defant and Sevilla-Peris result to the ℓ_p-case (and we get the same result if we choose $p_1 = \cdots = p_m = \infty$) and also implies the Hardy–Littlewood(/Praciano-Pereira)'s inequality. To show this, it suffices to choose $q = 2$, $s = 1$ and to consider only m-linear mappings which have their range in the span of the first basis vector.

The generalization of Dimant and Sevilla-Peris is the following.

Theorem 2.4 (Dimant and Sevilla-Peris, 2013)**.** *Let* $\mathbf{p} \in [1, +\infty]^m$ *and* $s, q \in [1, +\infty]$ *be such that* $s \leq q$*. Then there exists a constant* $C > 0$ *such that, for every continuous* m*-linear mapping* $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$*, we have*

$$\left(\sum_{i_1, \ldots, i_m = 1}^{+\infty} \| A(e_{i_1}, \ldots, e_{i_m}) \|_{\ell_q}^{\rho} \right)^{\frac{1}{\rho}} \leq C \|A\|,$$

where ρ *is given by*

(i) If $s \leq q \leq 2$*, and*

- *(a) if* $0 \leq \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s} - \frac{1}{q}$*, then* $\frac{1}{\rho} = \frac{1}{2} + \frac{1}{m}\left(\frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|\right)$.
- *(b) if* $\frac{1}{s} - \frac{1}{q} \leq \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2} + \frac{1}{s} - \frac{1}{q}$*, then* $\frac{1}{\rho} = \frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|$.

(ii) If $s \leq 2 \leq q$*, and*

- *(a) if* $0 \leq \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s} - \frac{1}{2}$*, then* $\frac{1}{\rho} = \frac{1}{2} + \frac{1}{m}\left(\frac{1}{s} - \frac{1}{2} - \left|\frac{1}{\mathbf{p}}\right|\right)$.
- *(b) if* $\frac{1}{s} - \frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s}$*, then* $\frac{1}{\rho} = \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|$.

(iii) If $2 \leq s \leq q$ *and* $0 \leq \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{s}$*, then* $\frac{1}{\rho} = \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|$.

Moreover, the exponents in the cases (ia),(iib) and (iii) are optimal. Also, the exponent in (ib) is optimal for $\frac{1}{s} - \frac{1}{q} \leq \left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2}$.

Our main intention in this chapter, is to improve the previous theorems in three directions.

(I) We study in depth the remaining cases of the Dimant and Sevilla-Peris result. Surprisingly, we show that in case (iia), the exponent given above is optimal whereas it is not optimal in case (ib) when $\left|\frac{1}{\mathbf{p}}\right| > \frac{1}{2}$. We give a better exponent in that case and show a necessary condition on it. These two bounds coincide when $s = 1$. We can summarize this into the two following statements.

Theorem 2.5. *Let* $\mathbf{p} \in [1, +\infty]^m$ *and let* $\rho > 0$*. Assume moreover that either* $q \geq 2$ *or* $q < 2$ *and* $\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2}$*. Let*

$$\frac{1}{\lambda} := \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{q, 2\}} - \left|\frac{1}{\mathbf{p}}\right| > 0.$$

Then there exists $C>0$ such that, for every continuous m-linear operator $A : X_{p_1}\times\cdots\times X_{p_m}\to X_s$, we have

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty}\|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\rho}\right)^{\frac{1}{\rho}}\le C\|A\|$$

if and only if

$$\frac{m}{\rho}\le\frac{1}{\lambda}+\frac{m-1}{\max\{\lambda,s,2\}}.$$

The following table summarizes the optimal value of $\frac{1}{\rho}$ following the respective values of $s,q,p_1,...,p_m$:

$1\le s\le q\le 2,\ \lambda<2$	$\frac{1}{2}+\frac{1}{ms}-\frac{1}{mq}-\frac{1}{m}\times\left\|\frac{1}{\mathbf{p}}\right\|$
$1\le s\le q\le 2,\ \lambda\ge 2,\ \left\|\frac{1}{\mathbf{p}}\right\|<\frac{1}{2}$	$\frac{1}{2}+\frac{1}{s}-\frac{1}{q}-\left\|\frac{1}{\mathbf{p}}\right\|$
$1\le s\le 2\le q,\ \lambda<2$	$\frac{m-1}{2m}+\frac{1}{ms}-\frac{1}{m}\times\left\|\frac{1}{\mathbf{p}}\right\|$
$1\le s\le 2\le q,\ \lambda\ge 2$	$\frac{1}{s}-\left\|\frac{1}{\mathbf{p}}\right\|$
$2\le s\le q$	$\frac{1}{s}-\left\|\frac{1}{\mathbf{p}}\right\|$

We note that (1.1) and (2.2) are recovered by Theorem 2.5 just by choosing $s=1$ and $q=2$.

When $q<2$ and $\left|\frac{1}{\mathbf{p}}\right|>\frac{1}{2}$ (observe that this automatically implies $\lambda\ge 2$), the situation is more difficult and we get the following statement.

Theorem 2.6. *Let $\mathbf{p}\in[1,+\infty]^m$, $\left|\frac{1}{\mathbf{p}}\right|>\frac{1}{2}$, $1\le s\le q\le 2$ and let $\rho>0$. Let us consider the following property.*

> *There exists $C>0$ such that, for every continuous m-linear operator $A : X_{p_1}\times\cdots\times X_{p_m}\to X_s$, we have*
>
> $$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty}\|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{\rho}\right)^{\frac{1}{\rho}}\le C\|A\|.$$

(A) The property is satisfied as soon as

$$\frac{1}{\rho}\le\frac{\left(\frac{1}{s}-\frac{1}{q}\right)\left(\frac{1}{s}-\left|\frac{1}{\mathbf{p}}\right|\right)}{\frac{1}{2}-\frac{1}{s}}.$$

(B) If the property is satisfied, then

$$\frac{1}{\rho} \leq 2\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)\left(\frac{1}{s}-\frac{1}{q}\right).$$

In particular, if $s=1$, then the property is satisfied if and only if

$$\frac{1}{\rho} \leq 2\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)\left(1-\frac{1}{q}\right).$$

(II) We give a simpler proof of the sufficient part of the Dimant and Sevilla-Peris theorem. It turns out that it is easier to prove a more general result.

Theorem 2.7. *Let $\mathbf{p} \in [1,+\infty]^m$ and $1 \leq s \leq q \leq \infty$ be such that*

$$\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{2}+\frac{1}{s}-\frac{1}{\min\{q,2\}}.$$

Also let

$$\frac{1}{\lambda} := \frac{1}{2}+\frac{1}{s}-\frac{1}{\min\{q,2\}}-\left|\frac{1}{\mathbf{p}}\right|.$$

If $\lambda > 0$ and $t_1, \ldots, t_m \in [\lambda, \max\{\lambda, s, 2\}]$ are such that

$$\frac{1}{t_1}+\cdots+\frac{1}{t_m} \leq \frac{1}{\lambda}+\frac{m-1}{\max\{\lambda, s, 2\}}, \tag{2.4}$$

then there exists $C>0$ satisfying, for every continuous m-linear map $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$,

$$\left(\sum_{i_1=1}^{+\infty}\left(\cdots\left(\sum_{i_m=1}^{+\infty}\left\|A\left(e_{i_1}, \ldots, e_{i_m}\right)\right\|_{\ell_q}^{t_m}\right)^{\frac{t_{m-1}}{t_m}} \cdots\right)^{\frac{t_1}{t_2}}\right)^{\frac{1}{t_1}} \leq C\|A\|. \tag{2.5}$$

Moreover, the exponents are optimal except eventually if $q \leq 2$ and $\left|\frac{1}{\mathbf{p}}\right| > \frac{1}{2}$.

Remark 2.8. *The optimality in the above theorem shall be understood in a strong sense: when $\lambda < 2$, we prove that if $t_1, \ldots, t_m \in [1,+\infty)$ are so that (2.5) holds then (2.4) is valid. When $\lambda \geq 2$, note that $\lambda = \max\{\lambda, s, 2\}$ and we prove that if $t = t_1 = \cdots = t_m$ are in $[1,+\infty)$ and (2.5) is valid, then we have (2.4) and, as a direct consequence, $t \geq \lambda$.*

(III) We prove similar results for m-linear mappings with arbitrary codomains which assume their cotype. For a Banach space X, let $q_X = \inf\{q \geq 2;\ X \text{ has cotype } q\}$.

The proof that (B) implies (A) in the theorem below appears in [59, Proposition 4.3].

Theorem 2.9. *Let* $\mathbf{p} \in [2, +\infty]^m$*, let* X *be an infinite dimensional Banach space with cotype* q_X*,* $\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{q_X}$*, and let* $\rho > 0$*. The following assertions are equivalent:*

(A) Every bounded m*-linear operator* $A : X_{p_1} \times \cdots \times X_{p_m} \to X$ *is such that*

$$\sum_{i_1,...,i_m=1}^{+\infty} \left\|A(e_{i_1}, ..., e_{i_m})\right\|^{\rho} < +\infty.$$

(B) $\frac{1}{\rho} \leq \frac{1}{q_X} - \left|\frac{1}{\mathbf{p}}\right|$.

Finally, in the last section of the paper we obtain better estimates for the constants of vector-valued Bohnenblust–Hille inequalities.

We conclude this introduction by noting that our theorems can be naturally stated in the context of homogeneous polynomials. Given an m-homogeneous polynomial $P : X \to Y$, we denote its coefficients $(c_\alpha(P))$. In [57, Lemma 5], it is shown that an inequality

$$\left(\sum_{\alpha} \|c_\alpha(P)\|^{\rho}\right)^{\frac{1}{\rho}} \leq C\|P\|$$

holds for every m-homogeneous polynomial $P : X \to Y$ if and only if a similar inequality

$$\left(\sum_{i_1,...,i_m} \|A(e_{i_1}, \ldots, e_{i_m})\|^{\rho}\right)^{\frac{1}{\rho}} \leq C'\|T\|$$

is satisfied for every m-linear mapping $A : X \times \cdots \times X \to Y$, where X is a Banach sequence space.

2.2 Proof of Theorem 2.7 (sufficiency)

Let $1 \leq q \leq +\infty$. We recall that a Banach space X has *cotype* q if there is a constant $\kappa > 0$ such that, no matter how we select finitely many vectors $x_1, ..., x_n \in X$,

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{\frac{1}{q}} \leq \kappa \left(\int_I \left\|\sum_{k=1}^{n} r_k(t)x_k\right\|^2 dt\right)^{\frac{1}{2}}$$

where $I = [0, 1]$ and r_k denotes the k-th Rademacher function. To cover the case $q = +\infty$, the left hand side should be replaced by $\max_{1\leq k\leq n} \|x_k\|$. The smallest of all these constants is denoted by $C_q(X)$ and named the cotype q constant of X.

An operator between Banach spaces $v : X \to Y$ is (r, s)-summing (with $s \leq r \leq +\infty$) if there exists $C > 0$ such that, for all $n \geq 1$ and for all vectors $x_1, \ldots, x_n \in X$,

$$\left(\sum_{k=1}^{n} \|vx_k\|^r\right)^{\frac{1}{r}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{n} |x^*(x_k)|^s\right)^{\frac{1}{s}}.$$

The smallest constant in this inequality is denoted by $\pi_{r,s}(v)$.

We need a cotype q version of [5, Proposition 4.1], whose proof can be found in [59, Proposition 3.1]:

Proposition 2.10. *Let X be a Banach space, let Y be a cotype q space, let $r \in [1,q]$ and let $\mathbf{p} \in [1,+\infty]^m$ with*

$$\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{r} - \frac{1}{q}.$$

Define

$$\frac{1}{\lambda} := \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|.$$

Then, for every continuous m-linear map $A : X_{p_1} \times \cdots \times X_{p_m} \to X$ and every $(r,1)$-summing operator $v : X \to Y$, we have

$$\left(\sum_{i_k}\left(\sum_{\widehat{i_k}} \|vA(e_{i_1},\cdots,e_{i_m})\|_Y^q\right)^{\lambda/q}\right)^{1/\lambda} \leq \left(\sqrt{2}C_q(Y)\right)^{m-1}\pi_{r,1}(v)\|A\| \tag{2.6}$$

for all $k = 1,\ldots,m$.

Recall that the symbol $\sum_{\widehat{i_k}}$ means that we are fixing the k-th index and that we are summing over all the remaining indices. We shall deduce from this lemma the following theorem, which extends results of [5] and [59]:

Theorem 2.11. *Let $\mathbf{p} \in [1,+\infty]^m$, X be a Banach space, Y be a cotype q space and $1 \leq r \leq q$, with $\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{r}$. Define*

$$\frac{1}{\lambda} := \frac{1}{r} - \left|\frac{1}{\mathbf{p}}\right|.$$

If $t_1,\ldots,t_m \in [\lambda, \max\{\lambda,q\}]$ are such that

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} \leq \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda,q\}},$$

then, for every continuous m-linear map $A : X_{p_1} \times \cdots \times X_{p_m} \to X$ and every $(r,1)$-summing operator $v : X \to Y$, we have

$$\left(\sum_{i_1=1}^{+\infty}\left(\cdots\left(\sum_{i_m=1}^{+\infty}\|vA\left(e_{i_1},\ldots,e_{i_m}\right)\|_Y^{t_m}\right)^{\frac{t_{m-1}}{t_m}}\cdots\right)^{\frac{t_1}{t_2}}\right)^{\frac{1}{t_1}} \leq \left(\sqrt{2}C_{\max\{\lambda,q\}}(Y)\right)^{m-1}\pi_{r,1}(v)\left\|A\right\|. \tag{2.7}$$

Proof. If $\lambda < q$, from Lemma 2.10, we have (2.7) for

$$(t_1,...,t_m) = (\lambda,q,...,q).$$

Since $\lambda < q$, the mixed $(\ell_\lambda,\ell_q)-$ norm inequality (the multi-index Minkowski inequality, Proposition 1.7), we also have (2.7) for the exponents

$$(t_1, ..., t_m) = (q, ..., q, \lambda, q, ..., q)$$

with λ in the k-th position, for all $k = 1, ..., m$. Now, using the Hölder's interpolative inequality for mixed $\ell_{\mathbf{p}}$ spaces (Proposition 1.3), we get (2.7) for all $(t_1, \ldots, t_m) \in [\lambda, q]^m$ such that

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} = \frac{1}{\lambda} + \frac{m-1}{q} = \frac{1}{\lambda} + \frac{m-1}{\max\{\lambda, q\}}.$$

If $\lambda \geq q$, for any $\varepsilon > 0$, let $q_\varepsilon = \lambda + \varepsilon$. So $\lambda < q_\varepsilon$ and this automatically implies that

$$\left|\frac{1}{\mathbf{p}}\right| < \frac{1}{r} - \frac{1}{q_\varepsilon}.$$

Since Y has cotype $q_\varepsilon > q$, we may apply Lemma 2.10 to get

$$\left(\sum_{i_1=1}^{N}\left(\sum_{i_2,\ldots,i_m=1}^{N} \|vA\left(e_{i_1}, \ldots, e_{i_m}\right)\|^{\lambda+\varepsilon}\right)^{\frac{\lambda}{\lambda+\varepsilon}}\right)^{\frac{1}{\lambda}} \leq \left(\sqrt{2}C_{\lambda+\varepsilon}(Y)\right)^{m-1} \pi_{r,1}(v)\|A\|$$

for all positive integer N. Making $\varepsilon \to 0$, we get

$$\left(\sum_{i_1,\ldots,i_m=1}^{N} \|vA\left(e_{i_1}, \ldots, e_{i_m}\right)\|^{\lambda}\right)^{\frac{1}{\lambda}} \leq \left(\sqrt{2}C_{\lambda}(Y)\right)^{m-1} \pi_{r,1}(v)\|A\|,$$

for all N and the proof is done. □

Remark 2.12. *If we take $t_1 = \cdots = t_m$, then, upon polarization, we recover exactly* [59, Theorem 1.2] *with a much simpler proof due to the fact that the inequality is simpler to prove for the extremal values of $(t_1, \ldots, t_m)$.*

We are now ready for the proof of the sufficient part of Theorem 2.7. We split the proof into three cases, and we combine Theorem 2.11 with the inequalities due (independently) to G. Bennet and B. Carl ([22, 23, 47]):

Bennett-Carl's inequalities. *For $1 \leq s \leq q \leq +\infty$, the inclusion map $\ell_s \hookrightarrow \ell_q$ is $(r, 1)$-summing, where the optimal r is given by*

$$\frac{1}{r} := \frac{1}{2} + \frac{1}{s} - \frac{1}{\min\{2, q\}}.$$

(i) $s \leq q \leq 2$: The Bennet-Carl-inequalities ensure that the inclusion map $\ell_s \hookrightarrow \ell_q$ is $(r, 1)$-summing with $\frac{1}{r} = \frac{1}{2} + \frac{1}{s} - \frac{1}{q}$, so the results follow from Theorem 2.11, with $t_1, \ldots, t_m$ satisfying

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} = \frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right| + \frac{m-1}{\max\{\lambda, 2\}}.$$

(ii) $s \leq 2 \leq q$: Also by using Bennet-Carl inequalities, $\ell_s \hookrightarrow \ell_2$ is $(s,1)$-summing, thus we get (2.5) applying Theorem 2.11, with $t_1, \ldots, t_m$ satisfying

$$\frac{1}{t_1} + \cdots + \frac{1}{t_m} = \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right| + \frac{m-1}{\max\{\lambda, 2\}}.$$

(iii) $2 \leq s \leq q$: Since $\ell_s \hookrightarrow \ell_s$ is $(s,1)$-summing, the result follows from Theorem 2.11, with $t_1 = \cdots = t_m = \lambda$ and $\lambda \geq s$, since $r = s$ and

$$\frac{1}{\lambda} := \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{s}.$$

Remark 2.13. *Let us set*

$$c_{qs} := \begin{cases} q\,, & \text{if } s \leq q \leq 2\,, \\ 2\,, & \text{if } s \leq 2 \leq q\,, \\ s\,, & \text{if } 2 \leq s \leq q. \end{cases}$$

With the above notations, a careful look at the proof shows that the constant C which appears in Theorem 2.7 is dominated by

$$\left(\sqrt{2}C_{\max\{\lambda,s,2\}}(\ell_{c_{qs}})\right)^{m-1} \pi_{r,1}(\ell_s \hookrightarrow \ell_{c_{qs}}).$$

2.3 Proof of Theorem 2.7 (optimality)

In this section we show that the exponents in Theorem 2.7 are optimal except when $q \leq 2$ and $\left|\frac{1}{\mathbf{p}}\right| > \frac{1}{2}$. More precisely, if $(t_1, \ldots, t_m) \in [1, +\infty)^m$ are such that there exists $C > 1$ satisfying, for any continuous multilinear map $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$,

$$\left(\sum_{i_1=1}^{+\infty}\left(\cdots\left(\sum_{i_m=1}^{+\infty}\|A(e_{i_1},\ldots,e_{i_m})\|_{\ell_q}^{t_m}\right)^{\frac{t_{m-1}}{t_m}}\cdots\right)^{\frac{t_1}{t_2}}\right)^{\frac{1}{t_1}} \leq C\|A\|, \tag{2.8}$$

then we prove that (2.4) holds. When $\lambda \geq 2$, we will always assume that $t_1 = \cdots = t_m = t$, since $\lambda = \max\{\lambda, s, 2\}$ and our inequality holds true when all the exponents are equal. We split the proof into several cases. Most of the cases are a consequence of a random construction. The main tool is the following lemma, which is a vector-valued version of the Kahane-Salem-Sygmund multilinear inequality presented in Lemma 1.14:

Lemma 2.14 (Vector valued Kahane-Salem-Zygmund's multilinear inequality)**.** *Let $d, N \geq 1$, $p_1, \ldots, p_{d+1} \in [1, +\infty]^{d+1}$ and let, for $p \geq 1$,*

$$\alpha(p) = \begin{cases} \frac{1}{2} - \frac{1}{p} & \text{if } p \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a d-linear mapping $A : \ell_{p_1}^N \times \cdots \times \ell_{p_d}^N \to \ell_{p_{d+1}}^N$ *which may be written*

$$A\big(x^{(1)}, \ldots, x^{(d)}\big) = \sum_{i_1,\ldots,i_{d+1}=1}^{N} \pm x_{i_1}^{(1)} \cdots x_{i_d}^{(d)} e_{i_{d+1}}$$

such that

$$\|A\| \leq C_d N^{\frac{1}{2}+\alpha(p_1)+\cdots+\alpha(p_d)+\alpha(p_{d+1}^*)}.$$

Proof. There is an isometric correspondence between d-linear maps $\ell_{p_1}^N \times \cdots \times \ell_{p_d}^N \to \ell_s^N$ and $(d+1)$-linear maps $\ell_{p_1}^N \times \cdots \times \ell_{p_d}^N \times \ell_{s^*}^N \to \mathbb{K}$. The correspondence is given by

$$\left(z \mapsto \sum_{i_1,\ldots,i_{d+1}} a_{i_1,\ldots,i_{d+1}} z_{i_1}^{(1)} \cdots z_{i_d}^{(d)} e_{i_{d+1}}\right) \mapsto \left(z \mapsto \sum_{i_1,\ldots,i_{d+1}} a_{i_1,\ldots,i_{d+1}} z_{i_1}^{(1)} \cdots z_{i_{d+1}}^{(d+1)}\right).$$

Therefore, the result follows by combining this isometry with Lemma 1.14. □

2.3.1 Case 1: $1 \leq s \leq q \leq 2$ and $\lambda < 2$

This case follows by using Lemma 2.14 with $d = m$ and $(p_1, \ldots, p_{m+1}) = (p_1, \ldots, p_m, s)$. Let $t_1, \ldots, t_m \in [\lambda, 2]$ satisfying (1) of Theorem 2.7. Let $A : \ell_{p_1}^N \times \cdots \times \ell_{p_m}^N \to \ell_s^N$ be given by Lemma 2.14. Then

$$\|A\| \leq C_m N^{\frac{1}{2}+\frac{m+1}{2}-\frac{1}{p_1}-\cdots-\frac{1}{p_m}-\frac{1}{s^*}} = C_m N^{\frac{m}{2}-\left|\frac{1}{\mathbf{p}}\right|+\frac{1}{s}}$$

whereas, for any $i_1, \ldots, i_m$,

$$\|A(e_{i_1}, \ldots, e_{i_m})\|_q = N^{1/q}.$$

Then,

$$\left(\sum_{i_m=1}^{N} \|A(e_{i_1}, \ldots, e_{i_m})\|_q^{q_m}\right)^{\frac{q_{m-1}}{q_m}} = N^{\frac{q_{m-1}}{q_m}+\frac{q_{m-1}}{q}}.$$

It is then easy to show by induction that

$$\left(\sum_{j_1=1}^{N}\left(\cdots\left(\sum_{j_m=1}^{N} \|A\left(e_{j_1}, \ldots, e_{j_m}\right)\|_q^{q_m}\right)^{\frac{q_{m-1}}{q_m}} \cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} = N^{\frac{1}{q}+\frac{1}{q_1}+\cdots+\frac{1}{q_m}}.$$

For (1) of Theorem 2.7 to be true, it is necessary that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m}{2} - \left|\frac{1}{\mathbf{p}}\right| + \frac{1}{s} - \frac{1}{q}.$$

2.3.2 Case 2: $1 \leq s \leq q \leq 2$, $\lambda \geq 2$ and $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$

This case has already been solved in [59, Proposition 4.4(ib)] using a Fourier matrix. We shall give an alternative probabilistic proof. Let $p \in [2, +\infty]$ be such that $\frac{1}{p} = \left|\frac{1}{\mathbf{p}}\right|$. By Lemma

2.14, there exists a linear map $T:\ell_p^n\to\ell_s^n$ which may be written $T(x)=\sum_{i,j}\varepsilon_{i,j}x_ie_j$ with $\varepsilon_{i,j}=\pm1$ and such that

$$\|T\|\leq Cn^{\frac12+\frac12-\frac1p+\frac12-\frac1{s^*}}=Cn^{\frac12+\frac1s-\left|\frac1{\mathbf{p}}\right|}.$$

Let $A:\ell_{p_1}^n\times\cdots\times\ell_{p_m}^n\to\ell_s^n$ defined by

$$A\big(x^{(1)},\dots,x^{(m)}\big):=\sum_{i,j}\varepsilon_{i,j}x_i^{(1)}\cdots x_i^{(m)}e_j.$$

By Hölder's inequality, it is plain that $\|A\|\leq\|T\|\leq Cn^{\frac12+\frac1s-\left|\frac1{\mathbf{p}}\right|}$. On the other hand, since $A(e_{i_1},\dots,e_{i_m})\neq0$ if and only if $i_1=\ldots=i_m$, and

$$\|A(e_i,\dots,e_i)\|_{\ell_q}=n^{1/q},$$

we have

$$\left(\sum_{\mathbf{i}\in\mathcal{M}(m,n)}\|A\left(e_{i_1},\dots,e_{i_m}\right)\|_{\ell_q}^t\right)^{\frac1t}=n^{\frac1q+\frac1t}.$$

This clearly implies

$$\frac1t\leq\frac12+\frac1s-\frac1q-\left|\frac1{\mathbf{p}}\right|.$$

2.3.3 Case 3: $1\leq s\leq2\leq q$ and $\lambda<2$

Let $p\in[0,+\infty]$ be defined by

$$\frac1p=\frac1{p_m}+\frac1{s^*}.$$

Since $\lambda<2$, it is easy to check that $p\geq2$ and that $p_i\geq2$ for any $i=1,\dots,m$. We then apply Lemma 2.14 with $d=m-1$ and $(q_1,\dots,q_m)=(p_1,\dots,p_{m-1},p^*)$. We get an $(m-1)$-linear form $T:\ell_{p_1}^n\times\cdots\times\ell_{p_{m-1}}^n\to\ell_{p^*}^n$ which can be written

$$T\big(x^{(1)},\dots,x^{(m-1)}\big)=\sum_{i_1,\dots,i_m}\varepsilon_{i_1,\dots,i_m}x_{i_1}^{(1)}\cdots x_{i_{m-1}}^{(m-1)}e_{i_m}$$

and such that

$$\|T\|\leq Cn^{\frac12+\frac m2-\left|\frac1{\mathbf{p}}\right|-\frac1{s^*}}=Cn^{\frac{m-1}2-\left|\frac1{\mathbf{p}}\right|+\frac1s}.$$

We then define $A:\ell_{p_1}^n\times\cdots\times\ell_{p_m}^n\to\ell_s^n$ by

$$A\big(x^{(1)},\dots,x^{(m)}\big)=\sum_{i_1,\dots,i_m}\varepsilon_{i_1,\dots,i_m}x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}e_{i_m}.$$

Then, for any $x^{(1)},\ldots,x^{(m)} \in B_{\ell_{p_1}^n}\times\cdots\times B_{\ell_{p_m}^n}$,

$$\begin{aligned}
\|A(x^{(1)},\ldots,x^{(m)})\| &= \sup_{y\in B_{\ell_{s^*}^n}}\left|\sum_{i_1,\ldots,i_m}\varepsilon_{i_1,\ldots,i_m}x_{i_1}^{(1)}\cdots x_{i_m}^{(m)}y_{i_m}\right| \\
&\leq \sup_{z\in B_{\ell_p^n}}\left|\sum_{i_1,\ldots,i_m}\varepsilon_{i_1,\ldots,i_m}x_{i_1}^{(1)}\cdots x_{i_{m-1}}^{(m-1)}z_{i_m}\right| \\
&\leq \|T\|.
\end{aligned}$$

Moreover, given any $\mathbf{i}\in\mathcal{M}(m,n)$, $\|A(e_{i_1},\ldots,e_{i_m})\|_q = \|e_{i_m}\|_q = 1$, so that

$$\left(\sum_{i_1=1}^{+\infty}\left(\cdots\left(\sum_{i_m=1}^{+\infty}\|A\left(e_{i_1},\ldots,e_{i_m}\right)\|_{\ell_q}^{t_m}\right)^{\frac{t_{m-1}}{t_m}}\cdots\right)^{\frac{t_1}{t_2}}\right)^{\frac{1}{t_1}} = n^{\frac{1}{t_1}+\cdots+\frac{1}{t_m}}.$$

Hence, provided (2.8) is satisfied, $(t_1,\ldots,t_m)$ has to satisfy

$$\frac{1}{t_1}+\cdots+\frac{1}{t_m}\leq\frac{m-1}{2}+\frac{1}{s}-\left|\frac{1}{\mathbf{p}}\right|.$$

2.3.4 Case 4 and Case 5: $1\leq s\leq 2\leq q$ and $\lambda\geq 2$, $2\leq s\leq q$

These cases have a deterministic proof, as noted in [59, Proposition 4.4 (iib), (iii)], considering $A:\ell_{p_1}^n\times\cdots\times\ell_{p_m}^n\to\ell_s^n$ given by

$$A\big(x^{(1)},\ldots,x^{(m)}\big):=\sum_{i=1}^{n}x_i^{(1)}\cdots x_i^{(m)}e_i.$$

2.3.5 The proof of Theorem 2.5

From Theorem 1.3, by choosing $t_1=\ldots=t_m$ we conclude that provided

$$\left|\frac{1}{\mathbf{p}}\right|<\frac{1}{2}+\frac{1}{s}-\frac{1}{\min\{q,2\}},$$

the best exponent ρ in Theorem 2.5 satisfies

$$\frac{m}{\rho}=\frac{1}{\lambda}+\frac{m-1}{\max\{\lambda,s,2\}}.$$

To conclude the proof, it remains to prove that, whenever

$$\left|\frac{1}{\mathbf{p}}\right|\geq\frac{1}{2}+\frac{1}{s}-\frac{1}{\min\{q,2\}},$$

we cannot find an exponent $\rho>0$ such that (2.5) is satisfied for all m-linear operators $A:X_{p_1}\times\cdots\times X_{p_m}\to X_s$. In fact, everything has already been done before: if $q\leq 2$, then we have just to follow the lines of Case 2 and if $q \geq 2$, then we may consider the m-linear mapping of Cases 4 and 5.

2.4 The case $1 \leq s \leq q \leq 2$, $\lambda \geq 2$ and $\left|\frac{1}{\mathbf{p}}\right| > \frac{1}{2}$

2.4.1 A reformulation of the Hardy-Littlewood type inequalities

We shall improve in this section the bound given by Theorem 2.5. We shall proceed by interpolation. To do this, we need a reformulation of the result of this theorem, as Villanueva and Perez-Garcia reformulated the Bohnenblust-Hille inequality in [98]. The forthcoming result is a variant of [38, Proposition 2.2]; its proof will be omitted.

Theorem 2.15. *Let $1 \leq p_1, ..., p_m \leq +\infty$, $1 \leq s \leq q \leq \infty$ and let $\rho > 0$. The following assertions are equivalent.*

(A) There exists $C > 0$ such that, for every continuous m-linear mapping $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$, we have

$$\left(\sum_{i_1,\ldots,i_m} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^{\rho} \right)^{1/\rho} \leq C\|A\|.$$

(B) There exists $C > 0$ such that, for any $n \geq 1$, for any Banach spaces $Y_1, \ldots, Y_m$, for any continuous m-linear mapping $S : Y_1 \times \cdots \times Y_m \to X_s$, the induced operator

$$\begin{array}{rcl} T : \ell^n_{p_1^*,w}(Y_1) \times \cdots \times \ell^n_{p_m^*,w}(Y_m) & \to & \ell_\rho^{n^m}(X_q) \\ \big(x^{(1)}, \ldots, x^{(m)}\big) & \mapsto & \big(S(x^{(1)}_{i_1}, \ldots, x^{(m)}_{i_m})\big)_{\mathbf{i} \in \mathcal{M}(m,n)} \end{array}$$

satisfies $\|T\| \leq C\|S\|$.

We recall that, for any $p \in [1, +\infty]$ and any Banach space Y,

$$\ell^n_{p,w}(Y) = \left\{ (x_j)_{j=1}^n \subset Y;\ \|(x_j)\|_{w,p} := \sup_{\varphi \in B_{Y^*}} \left(\sum_{j=1}^n |\varphi(x_j)|^p \right)^{1/p} < +\infty \right\}$$

with the appropriate modifications for $p = \infty$.

2.4.2 Proof of the sufficient condition

We now prove our better upper bound in the case $1 \leq s \leq q \leq 2$, $\left|\frac{1}{\mathbf{p}}\right| > \frac{1}{2}$ (namely we prove the first part of Theorem 2.6). Let $n \geq 1$, let $Y_1, \ldots, Y_m$ be Banach spaces and let $S : Y_1 \times \cdots \times Y_m \to X_s$ be bounded. Let $n \geq 1$ and let T be the operator induced by S on $\mathcal{Y} = \ell^n_{p_1^*,w}(Y_1) \times \cdots \times \ell_{p_m^*,w}(Y_m)$, defined by

$$T\big(x^{(1)}, \ldots, x^{(m)}\big) = \big(S(x^{(1)}_{i_1}, \ldots, x^{(m)}_{i_m})\big).$$

Then T is bounded as an operator from $\mathcal{Y}$ into $\ell_\infty^{n^m}(X_s)$. T is also bounded as an operator from $\mathcal{Y}$ into $\ell_\rho^{n^m}(X_s)$ with $\frac{1}{\rho} = \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|$ (this is Theorem 2.5 for $1 \leq s \leq 2$ and $q \geq 2$). We can interpolate between these two extreme situations. Hence, let $q \in [s,2]$ and let $\theta \in [0,1]$ be such that

$$\frac{1}{q} = \frac{1-\theta}{s} + \frac{\theta}{2} \iff \theta = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{s} - \frac{1}{2}}.$$

By [24, Theorem 4.4.1], T is bounded as an operator from $\mathcal{Y}$ into $\ell_t^{n^m}(X_q)$ where

$$\frac{1}{t} = \frac{1-\theta}{\infty} + \frac{\theta}{\rho} = \frac{\left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|\right)}{\frac{1}{s} - \frac{1}{2}}.$$

Remark 2.16. *It is easy to check that, for $1 \leq s \leq q \leq 2$ and $\left|\frac{1}{\mathbf{p}}\right| \geq \frac{1}{2}$, then the bound $\frac{\left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|\right)}{\frac{1}{s} - \frac{1}{2}}$ is always better (namely larger) than the bound $\frac{1}{2} + \frac{1}{s} - \frac{1}{q} - \left|\frac{1}{\mathbf{p}}\right|$ obtained in Theorem 2.5.*

2.4.3 The necessary condition

We now prove the second part of Theorem 2.6. It also uses a probabilistic device for linear maps when the two spaces do not need to have the same dimension. The forthcoming lemma can be found in [23, Proposition 3.2].

Lemma 2.17. *Let $n, d \geq 1$, $1 \leq p, s \leq 2$. There exists $T : \ell_p^d \to \ell_s^n$, $T(x) = \sum_{i,j} \pm x_j e_i$ such that*

$$\|T\| \leq C_{p,s} \max\left(d^{1/s}, n^{1-\frac{1}{p}} d^{\frac{1}{s}-\frac{1}{2}}\right).$$

Coming back to the proof of Theorem 2.6, we first observe that we may always assume that $\left|\frac{1}{\mathbf{p}}\right| < 1$. Otherwise, we can consider the m-linear map $A : X_{p_1} \times \cdots \times X_{p_m} \to X_s$ defined by

$$A\big(x^{(1)}, \ldots, x^{(m)}\big) = \sum_{i \geq 1} x_i^{(1)} \ldots x_i^{(m)} e_0$$

and observe that it is bounded whereas it has infinitely many coefficients equal to 1. We then define $p \in [1,2]$ by $\frac{1}{p} = \left|\frac{1}{\mathbf{p}}\right|$ and we consider $T : \ell_p^d \to \ell_s^n$, $T(x) = \sum_{i,j} \varepsilon_{i,j} x_j e_i$ the map given by Lemma 2.17. We then define

$$\begin{array}{rcl} A : \ell_{p_1}^d \times \cdots \times \ell_{p_m}^d & \to & \ell_s^n \\ \big(x^{(1)}, \ldots, x^{(m)}\big) & \mapsto & \displaystyle\sum_{i,j} \varepsilon_{i,j} x_j^{(1)} \cdots x_j^{(m)} e_i \end{array}$$

and we observe that, by Hölder's inequality, $\|A\| \leq \|T\|$. Furthermore,

$$\left(\sum_{i_1, \ldots, i_m} \|A(e_{i_1}, \ldots, e_{i_m})\|_{\ell_q}^t\right)^{1/t} = n^{1/t} d^{1/q}.$$

Taking $d^{1/2} = n^{1-\frac{1}{p}}$ (this is the optimal relation between d and n), we get that if

$$\left(\sum_{i_1,\dots,i_m} \|A(e_{i_1},\dots,e_{i_m})\|_{\ell_q}^t\right)^{1/t} \leq C\|A\|,$$

then it is necessary that

$$\frac{1}{t} \leq 2\left(1-\frac{1}{p}\right)\left(\frac{1}{s}-\frac{1}{q}\right).$$

Remark 2.18. *This last condition is optimal when $s = 1$ or when $\left|\frac{1}{\mathbf{p}}\right| = \frac{1}{2}$ (with, in fact, the same proof as in Case 2 above). When $1 < s < 2$, another necessary condition is*

$$\frac{1}{t} \leq \frac{1}{s} - \left|\frac{1}{\mathbf{p}}\right|$$

(see Case 4 or Case 5 above).

2.5 Optimal estimates under cotype assumptions

For a Banach space X, let $q_X := \inf\{q \geq 2;\ X \text{ has cotype } q\}$. For scalar-valued multilinear operators it is easy to observe that summability in multiple indexes behaves in a quite different way than summability in just one index. For instance, for any bounded bilinear form $A : c_0 \times c_0 \to \mathbb{C}$,

$$\left(\sum_{i,j=1}^{+\infty} |A(e_i,e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq \sqrt{2}\|A\|$$

and the exponent $4/3$ is optimal. But, if we sum diagonally ($i = j$) the exponent $4/3$ can be reduced to 1 since

$$\sum_{i=1}^{+\infty} |A(e_i,e_i)| \leq \|A\|$$

for any bounded bilinear form $A : c_0 \times c_0 \to \mathbb{C}$. Now we prove Theorem 2.9 which shows that when replacing the scalar field by infinite-dimensional spaces the situation is quite different.

Proof. $(A) \Rightarrow (B)$. From a deep result of Maurey and Pisier ([88] and [58, Section 14]), ℓ_{q_X} is finitely representable in X, which means that, for any $n \geq 1$, one may find unit vectors $z_1,\dots,z_n \in X$ such that, for any $a_1,\dots,a_n \in \mathbb{C}$,

$$\sum_{i=1}^{n} \|a_i z_i\|_X \leq 2\left(\sum_{i=1}^{n} |a_i|^{q_X}\right)^{1/q_X}.$$

We then consider the m-linear map $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to X$ defined by

$$A\left(x^{(1)},\cdots,x^{(m)}\right) := \sum_{i=1}^{n} x_i^{(1)} \cdots x_i^{(m)} z_i.$$

Then, for any $(x^{(1)}, \dots, x^{(m)})$ belonging to $B_{\ell_{p_1}^n} \times \cdots \times B_{\ell_{p_m}^n}$,

$$\left\| A\left(x^{(1)}, \cdots, x^{(m)}\right) \right\| \leq 2 \left(\sum_{i=1}^{n} |x_i^{(1)}|^{q_X} \cdots |x_i^{(m)}|^{q_X} \right)^{1/q_X}$$
$$\leq 2n^{\frac{1}{q_X} - \left|\frac{1}{\mathbf{p}}\right|}$$

where the last inequality follows from Hölder's inequality applied to the exponents

$$\frac{p_1}{q_X}, \dots, \frac{p_m}{q_X}, \left(1 - q_X \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}.$$

On the other hand,

$$\left(\sum_{i=1}^{n} \| A\left(e_i, \dots, e_i\right) \|^{\rho} \right)^{1/\rho} = n^{\frac{1}{\rho}}$$

and we obtain (3).

$(B) \Rightarrow (A)$. This implication is proved in [59, Proposition 4.3]. □

If X does not have cotype q_X, the condition remains necessary. But now we just have the following sufficient condition:

$$\frac{m}{\rho} < \frac{1}{q_X} - \left|\frac{1}{\mathbf{p}}\right|.$$

Of course, it would be nice to determine what happens in this case. A look at [58, page 304] shows that the situation does not look simple.

As a consequence of the previous result we conclude that under certain circumstances the concepts of absolutely summing multilinear operator and multiple summing multilinear operator (see [39, 87, 97]) are precisely the same.

Corollary 2.19. *Let $p \in [2, +\infty]$, let X be an infinite dimensional Banach space with cotype $q_X < \frac{p}{m}$ and let $\rho > 0$. The following assertions are equivalent:*

(A) Every bounded m-linear operator $A : X_p \times \cdots \times X_p \to X$ is absolutely $(\rho; p^*)$-summing.

(B) Every bounded m-linear operator $A : X_p \times \cdots \times X_p \to X$ is multiple $(\rho; p^*)$-summing.

(C) $\frac{1}{\rho} \leq \frac{1}{q_X} - \frac{m}{p}$.

We stress the equivalence between (A) and (B) is not true, in general. For instance, every bounded bilinear operator $A : \ell_2 \times \ell_2 \to \ell_2$ is absolutely $(1; 1)$-summing but this is no longer true for multiple summability.

2.6 Constants of vector-valued Bohnenblust–Hille inequalities

A particular case of our main result is the following vector-valued Bohnenblust–Hille inequality (see [57, Lemma 3] and also [103, Section 2.2]):

Theorem 2.20. *Let X be a Banach space, Y a cotype q space and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r \leq q$. Then, for all m-linear operators $T : c_0 \times \cdots \times c_0 \to X$,*

$$\left(\sum_{i_1,\ldots,i_m=1}^{+\infty} \|vT(e_{i_1},\ldots,e_{i_m})\|_Y^{\frac{qrm}{q+(m-1)r}} \right)^{\frac{q+(m-1)r}{qrm}} \leq C_{Y,m}\pi_{r,1}(v)\|T\|$$

with $C_{Y,m} = \left(\sqrt{2}C_q(Y)\right)^{m-1}$.

In this section, in Theorem 2.21, we improve the above estimate for $C_{Y,m}$. The proof of Theorem 2.21 follows almost word by word the proof of [20, Proposition 3.1] using [54, Lemma 2.2] and Kahane's inequality instead of the Khinchine inequality. We present the proof for the sake of completeness. We need the following inequality due to Kahane:

Kahane's Inequality. *Let $0 < p, q < +\infty$. Then there is a constant $K_{p,q} > 0$ for which*

$$\left(\int_I \left\| \sum_{k=1}^n r_k(t)x_k \, dt \right\|^q \right)^{\frac{1}{q}} \leq K_{p,q} \left(\int_I \left\| \sum_{k=1}^n r_k(t)x_k \, dt \right\|^p \right)^{\frac{1}{p}},$$

regardless of the choice of a Banach space X and of finitely many vectors $x_1, \ldots, x_n \in X$.

Theorem 2.21. *For all m and all $1 \leq k < m$,*

$$C_{Y,m} \leq \left(C_q(Y) K_{\frac{qrk}{q+(k-1)r},2} \right)^{m-k} C_{Y,k}.$$

Proof. Let $\rho := \frac{qrm}{q+(m-1)r}$ and to simplify notation let us write

$$vTe_{\mathbf{i}} = vT(e_{i_1},\ldots,e_{i_m}).$$

Let us make use of the general Blei inequality (Proposition 1.6) with $m \geq 2$, $1 \leq k \leq m-1$ and $s = \frac{qrk}{q+(k-1)r}$. So we have

$$\left(\sum_{\mathbf{i}} \|vTe_{\mathbf{i}}\|_Y^{\rho} \right)^{\frac{1}{\rho}} \leq \prod_{S \in P_k(m)} \left(\sum_{\mathbf{i}_S} \left(\sum_{\mathbf{i}_{\hat{S}}} \|vT(e_{\mathbf{i}_S}, e_{\mathbf{i}_{\hat{S}}})\|_Y^q \right)^{\frac{s}{q}} \right)^{\frac{1}{s\binom{m}{k}}}, \tag{2.9}$$

where $P_k(m)$ denotes the set of all subsets of $\{1, \ldots, m\}$ with cardinality k. For sake of clarity, we shall assume that $S = \{1, \ldots, k\}$. By the multilinear cotype inequality (see [54, Lemma 2.2]) and the Kahane inequality, we have

$$\begin{aligned}
&\left(\sum_{\mathbf{i}_{\hat{S}}}\|vT\left(e_{\mathbf{i}_S},e_{\mathbf{i}_{\hat{S}}}\right)\|_Y^q\right)^{\frac{s}{q}} \\
&\qquad\leq \left(C_q(Y)K_{s,2}\right)^{s(m-k)}\int_{I^{m-k}}\left\|\sum_{\mathbf{i}_{\hat{S}}}r_{\mathbf{i}_{\hat{S}}}(t_{\hat{S}})vT\left(e_{\mathbf{i}_S},e_{\mathbf{i}_{\hat{S}}}\right)\right\|_Y^s dt_{\hat{S}} \\
&\qquad= \left(C_q(Y)K_{s,2}\right)^{s(m-k)}\int_{I^{m-k}}\left\|vT\left(e_{\mathbf{i}_S},\sum_{\mathbf{i}_{\hat{S}}}r_{\mathbf{i}_{\hat{S}}}(t_{\hat{S}})e_{\mathbf{i}_{\hat{S}}}\right)\right\|_Y^s dt_{\hat{S}},
\end{aligned}$$

which the term inside the $\|\cdot\|_Y$ in the last inequality is precisely

$$v\left(T\left(e_{i_1},\dots,e_{i_k},\sum_{i_{k+1}}r_{k+1}(t_{k+1})e_{k+1},\dots,\sum_{i_m}r_m(t_m)e_m\right)\right)$$

But for a fixed choice of $(t_{k+1},\dots,t_m)\in I^{m-k}=[0,1]^{m-k}$, we know, by Theorem 2.20, that

$$\sum_{i_1,\dots,i_k}\left\|v\left(T\left(e_{i_1},\dots,e_{i_k},\sum_{i_{k+1}}r_{k+1}(t_{k+1})e_{k+1},\dots,\sum_{i_m}r_m(t_m)e_m\right)\right)\right\|_Y^s \leq \left(C_{Y,k}\pi_{r,1}(v)\|T\|\right)^s.$$

Thus,

$$\begin{aligned}
&\sum_{\mathbf{i}_S}\left(\sum_{\mathbf{i}_{\hat{S}}}\|vT\left(e_{\mathbf{i}_S},e_{\mathbf{i}_{\hat{S}}}\right)\|_Y^q\right)^{\frac{s}{q}} \\
&\quad\leq \left(C_q(Y)K_{s,2}\right)^{s(m-k)}\times \qquad (2.10)\\
&\qquad\times\sum_{i_1,\dots,i_k}\left\|v\left(T\left(e_{i_1},\dots,e_{i_k},\sum_{i_{k+1}}r_{k+1}(t_{k+1})e_{k+1},\dots,\sum_{i_m}r_m(t_m)e_m\right)\right)\right\|_Y^s \\
&\quad\leq \left(\left(C_q(Y)K_{s,2}\right)^{m-k}\pi_{r,1}(v)C_{Y,k}\|T\|\right)^s, \qquad (2.11)
\end{aligned}$$

namely

$$\left(\sum_{\mathbf{i}_S}\left(\sum_{\mathbf{i}_{\hat{S}}}\|vT\left(e_{\mathbf{i}_S},e_{\mathbf{i}_{\hat{S}}}\right)\|_Y^q\right)^{\frac{s}{q}}\right)^{\frac{1}{s}} \leq \left(C_q(Y)K_{s,2}\right)^{m-k}\pi_{r,1}(v)C_{Y,k}\|T\|.$$

From (2.9) we conclude that

$$\left(\sum_{\mathbf{i}}\|vTe_{\mathbf{i}}\|_Y^{\rho}\right)^{\frac{1}{\rho}} \leq \left(C_q(Y)K_{s,2}\right)^{m-k}C_{Y,k}\pi_{r,1}(v)\|T\|.$$

$\square$

When m is even, the case $k = \frac{m}{2}$ recovers the constants from [103].

Corollary 2.22. *For all m,*

$$C_{Y,m} \leq C_q(Y)^{m-1} \prod_{k=1}^{m-1} K_{\frac{qrk}{q+(k-1)r},2}.$$

2.7 Other exponents

From now on $1 \leq r \leq q$ and $(q_1, \ldots, q_m) \in [r,q]^m$ so that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{q+(m-1)r}{qr} = \frac{1}{r} + \frac{m-1}{q}$$

are called vector-valued Bohnenblust–Hille exponents. From Theorem 2.11 we have:

Theorem 2.23 (Multiple exponent vector-valued Bohnenblust–Hille inequality)**.** *Let X be a Banach space and Y a cotype q space with $1 \leq r \leq q$. If $(q_1, \ldots, q_m) \in [r,q]^m$ are vector-valued Bohnenblust–Hille exponents, then there exists $C_{Y,q_1,\ldots,q_m} \geq 1$ such that, for all m-linear operators $T : c_0 \times \cdots \times c_0 \to X$ and every $(r,1)$-summing operator $v : X \to Y$, we have*

$$\left(\sum_{i_1=1}^{+\infty} \cdots \left(\sum_{i_m=1}^{+\infty} \|vTe_{\mathbf{i}}\|_Y^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{1}{q_1}} \leq C_{Y,q_1,\ldots,q_m} \pi_{r,1}(v) \|T\|, \tag{2.12}$$

with $C_{Y,q_1,\ldots,q_m} = \left(\sqrt{2} C_q(Y)\right)^{m-1}$.

Our final result gives better estimates for the constants $C_{Y,q_1,\ldots,q_m}$:

Theorem 2.24. *If $(q_1, \ldots, q_m)$ is a vector-valued Bohnenblust–Hille exponent, then*

$$C_{Y,q_1\ldots,q_m} \leq \prod_{k=1}^{m} \left(\left(C_q(Y) K_{\frac{kqr}{q+(k-1)r},2} \right)^{m-k} C_{Y,k} \right)^{\theta_k}$$

with

$$\theta_m = m \left(\frac{1}{r} - \frac{1}{q} \right)^{-1} \left(\frac{1}{q_{\sigma(m)}} - \frac{1}{q} \right) \tag{2.13}$$

and

$$\theta_k = k \left(\frac{1}{r} - \frac{1}{q} \right)^{-1} \left(\frac{1}{q_{\sigma(k)}} - \frac{1}{q_{\sigma(k+1)}} \right), \quad \textit{for } k = 1, \ldots, m-1. \tag{2.14}$$

where σ is a permutation of the indexes $\{1, \ldots, m\}$ such that $q_{\sigma(1)} \leq \cdots \leq q_{\sigma(m)}$.

Proof. First let us suppose that $q_i \leq q_j$ whenever $i \leq j$. For each $k = 1, \ldots, m$, define

$$s_k = \frac{kqr}{q+(k-1)r}.$$

From the proof of Theorem 2.21 we have (2.12) for each exponent $\left(s_k, \overset{k \text{ times}}{\ldots}, s_k, q, \ldots, q\right)$. More precisely, from (2.11) we have

$$\left(\sum_{i_1,\ldots,i_k} \left(\sum_{i_{k+1},\ldots,i_m} \|vTe_{\mathbf{i}}\|_Y^q \right)^{\frac{s_k}{q}} \right)^{\frac{1}{s_k}} \leq \left(C_q(Y) K_{s_k,2}\right)^{m-k} C_{Y,k} \pi_{r,1}(v) \|T\|.$$

Consequently,

$$C_{Y,s_k,\overset{k \text{ times}}{\ldots},s_k,q\ldots,q} \leq \left(C_q(Y) K_{s_k,2}\right)^{m-k} C_{Y,k},$$

for each $k = 1, \ldots, m$. Since every vector-valued Bohnenblust–Hille exponent $(q_1, \ldots, q_m)$ with $q_1 \leq \cdots \leq q_m$ could be obtained by interpolation of $\alpha_1, \ldots, \alpha_m$ with $\alpha_k := \left(s_k, \overset{k \text{ times}}{\ldots}, s_k, q, \ldots, q\right)$, and $\theta_1, \ldots, \theta_m$ as in (2.13) and (2.14), we conclude that

$$C_{Y,q_1,\ldots,q_m} \leq \prod_{k=1}^{m} \left(C_{Y,s_k,\overset{k \text{ times}}{\ldots},s_k,q,\ldots,q} \right)^{\theta_k} \leq \prod_{k=1}^{m} \left(\left(C_q(Y) K_{s_k,2}\right)^{m-k} C_{Y,k} \right)^{\theta_k}.$$

When the indexes $i \in \{1, \ldots, m\}$ does not match with de order of the q_i's, one just need to use the result to the exponents $q_{\sigma(1)} \leq \cdots \leq q_{\sigma(m)}$ and apply the Minkowski inequality successively (Proposition 1.5). $\square$

A particular case of Kahane's inequality is Khintchine's inequality (Proposition 1.8), whose optimal constants $\mathrm{A}_{\mathbb{K},p}$ we will use here also. Taking $X = Y = \mathbb{K}$ and $r = 1$ we obtain estimates for the constants of the scalar-valued Bohnenblust–Hille inequality with multiple exponents:

Corollary 2.25. *If $(q_1, \ldots, q_m) \in [1,2]^m$ are such that $q_{\sigma(1)} \leq \cdots \leq q_{\sigma(m)}$ and*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2},$$

then

$$\left(\sum_{i_1=1}^{+\infty} \cdots \left(\sum_{i_m=1}^{+\infty} \left| T\left(e_{i_1}, \ldots, e_{i_m}\right) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{1}{q_1}}$$
$$\leq C_{\mathbb{K},m}^{2m\left(\frac{1}{q_{\sigma(m)}} - \frac{1}{2}\right)} \left(\prod_{k=1}^{m-1} \left(\mathrm{A}_{\mathbb{K},\frac{2k}{k+1}}^{m-k} C_{\mathbb{K},k} \right)^{2k\left(\frac{1}{q_{\sigma(k)}} - \frac{1}{q_{\sigma(k+1)}}\right)} \right) \|T\|,$$

for all m-linear operators $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$. In particular, for complex scalars, the left hand *side of the above inequality can be replaced by*

$$\left(\prod_{j=1}^{m}\Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{2m\left(\frac{1}{q_{\sigma(m)}}-\frac{1}{2}\right)}\times$$

$$\times\left(\prod_{k=1}^{m-1}\left(\Gamma\left(\frac{3k+1}{2k+2}\right)^{\left(\frac{-k-1}{2k}\right)(m-k)}\prod_{j=1}^{k}\Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{2k\left(\frac{1}{q_{\sigma(k)}}-\frac{1}{q_{\sigma(k+1)}}\right)}\right)\|T\|.$$

Part II

Peano curves on topological vector spaces

Chapter 3

Peano curves on Euclidean spaces

In this chapter is we present the results of the papers

> [4] *Maximal lineability of the set of continuous surjections, Bulletin of the Belgian Mathematical Society Simon Stevin, vol. 21, 83–87, 2014.*

and some portion of

> [7] *Peano curves on topological vector spaces, Linear Algebra and its Applications, vol. 460, 81–96, 2014.*

Tthe last one is a joint work with L. Bernal, D. Pellegrino and J. Seoane. We deal with Peano curves on Euclidean spaces. The topological and algebraic structure of the set of these functions (as well as extensions to spaces with higher dimensions) is analyzed from the modern point of view of lineability, from which large dense vector spaces and algebras are found within the families studied.

3.1 Motivation and main results

Lately, many authors have been interested in the study of the set of surjections in $\mathbb{K}^{\mathbb{K}}$. From this study, many different classes of functions have been either recovered from the old literature or introduced. Some of these classes are, for instance, we say that $f \in \mathbb{R}^{\mathbb{R}}$ is

(i) everywhere surjective functions (ES, see [10]): ($f \in ES(\mathbb{R})$) if $f(I) = \mathbb{R}$ for every nontrivial interval $I \subset \mathbb{R}$;

(ii) strongly everywhere surjective functions (SES, see [71]): ($f \in SES(\mathbb{R})$) if f takes all values $\mathfrak{c}$ times on any interval (where $\mathfrak{c}$ stands for the cardinality of $\mathbb{R}$), *i.e.*,

$$\forall\, I \subset \mathbb{R} \text{ (non-trivial) },\ \forall\, a \in \mathbb{R},\ \text{card } \left(f^{-1}(a) \cap I\right) = \mathfrak{c}.$$

(iii) perfectly everywhere surjective functions (PES, see [71]): ($f \in PES(\mathbb{R})$) if $f(P) = \mathbb{R}$, for every perfect set P;

(iv) Jones functions (J, see [69, 70, 81]): ($f \in J(\mathbb{R})$) if its graph intersects every closed subset of $\mathbb{R}^2$ with uncountable projection on the x-axis.

If S and CS stand, respectively, for the set of surjections and continuous surjections on $\mathbb{R}$, the above functions (when defined on $\mathbb{R}$) enjoy the following strict inclusions:

$$\begin{tikzcd} J \arrow[r] \arrow[drrr] & PES \arrow[r] \arrow[drr] & SES \arrow[r] \arrow[dr] & ES \arrow[d] \\ CS \arrow[rrr] & & & S \end{tikzcd}$$

Authors have studied the previous classes of functions in depth, to the point of even finding large algebraic structures (infinite dimensional linear spaces or infinitely generated algebras) inside the previous sets of functions. However, "most" of these functions, although surjective, also are *nowhere continuous* on their domains (as expected). Thus, a natural question rises when one tries to consider *continuous surjections*.

Inspired by Cantor's counterintuitive result stating that the unit interval $[0, 1]$ has the same cardinality as the infinite number of points in any finite-dimensional manifold (such as the unit square), Peano constructed the (no doubt!) most famous space filling curve, also known as the *Peano curve* [95, 101] (see Figure 3.1).

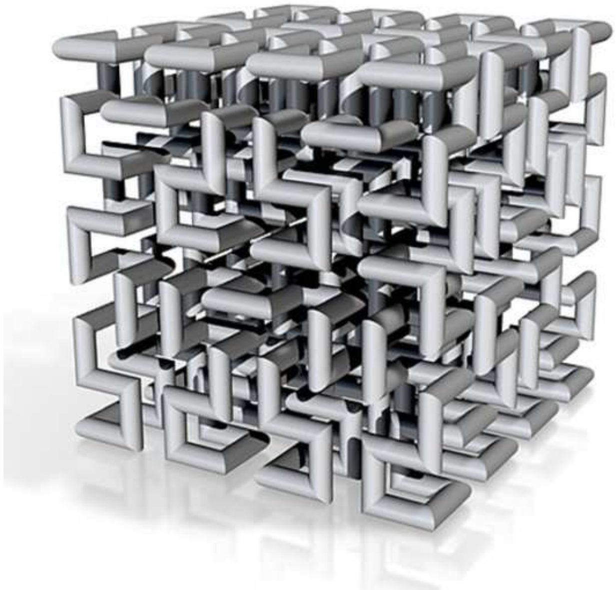

Figure 3.1: Sketch of an iteration of a tridimensional space-filling curve.

Later on, the Hahn-Mazurkiewicz theorem (see, e.g., [106, Theorem 31.5] or [80]) helped in characterizing the Hausdorff spaces that are the continuous image of $[0, 1]$, which are called *Peano spaces*, namely:

Theorem 3.1 (Hahn-Mazurkiewicz). *A non-empty Hausdorff topological space is a continuous* *image of the unit interval if and only if it is a compact, connected, locally connected, and second-countable space.*

Hausdorff spaces that are the continuous image of the unit interval will be called Peano spaces, so that, a *Peano space* is a Hausdorff, compact, connected, locally connected second-countable topological space. Equivalently, by well-known metrization theorems, a Peano space is a compact, connected, locally connected metrizable topological space. From Peano's example one can easily construct a continuous function from the real line $\mathbb{R}$ onto the plane $\mathbb{R}^2$ (see, e.g., [4]). If X and Y are topological spaces, by $\mathcal{C}(X,Y)$ and $\mathcal{CS}(X,Y)$ we will denote, respectively, the set of all continuous mappings $X \to Y$ and the subfamily of all continuous *surjective* mappings.

This chapter focuses on studying the algebraic structure of the set of continuous surjections between Euclidean spaces. Before carrying on, let us recall some concepts that, by now, are widely known (see, e.g., [10–12, 17, 27, 46, 48, 64]).

Definition 3.2 (Lineability and spaceability, [10, 104]). *Let X be a topological vector space and M a subset of X. Let μ be a cardinal number.*

(1) *M is said to be* μ-lineable *if $M \cup \{0\}$ contains a vector space of dimension μ. At times, we shall be referring to the set M as simply* lineable *if the existing subspace is infinite dimensional.*

(2) *When the above linear space can be chosen to be dense (infinite dimensional and closed, resp.) in X we shall say that M is* μ-dense-lineable (spaceable, resp.)*.*

Moreover, L. Bernal introduced in [25] the notion of *maximal lineable* (and that of *maximal dense-lineable*) in X, meaning that, when keeping the above notation, the dimension of the existing linear space equals $\dim(X)$. Besides asking for linear spaces one could also study other structures, such as algebras and some related ones, which were presented in [11, 12, 15, 104].

Definition 3.3. *Given an algebra $\mathcal{A}$, a subset $\mathcal{B} \subset \mathcal{A}$, and a cardinal number κ, we say that $\mathcal{B}$ is:*

(1) algebrable *if there is a subalgebra $\mathcal{C}$ of $\mathcal{A}$ so that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any system of generators of $\mathcal{C}$ is infinite.*

(2) κ-algebrable *if there exists a κ-generated subalgebra $\mathcal{C}$ of $\mathcal{A}$ with $\mathcal{C} \subset \mathcal{B} \cup \{0\}$.*

(3) strongly κ-algebrable *if there exists a κ-generated free algebra $\mathcal{C}$ contained in $\mathcal{B} \cup \{0\}$.*

If $\mathcal{A}$ is commutative, the last sentence means that there is a set $C \subset \mathcal{A}$ with $\mathrm{card}(C) = \kappa$ such that, for every finite set $\{x_1, \ldots, x_N\} \subset C$ and every nonzero polynomial P of N variables without constant term, one has $P(x_1, \ldots, x_N) \in \mathcal{B} \setminus \{0\}$. Being strongly algebrable implies being algebrable (the converse is not true, see [15]). When, in part (3) of Definition 3.3, one can take as κ the supremum of the cardinalities of all algebraically free systems in $\mathcal{A}$, then $\mathcal{B}$ will be called *maximal strongly algebrable* in $\mathcal{A}$.

This chapter is arranged in four sections. Section 3.2 is devoted to the study the lineability of the set $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$ of continuous surjections from $\mathbb{R}^m$ to $\mathbb{R}^n$, for any pair (m, n) of positive integers. In Section 3.3 we comment how Bernal and Ordóñez solved the spaceability of $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$ as a particular case of a more general problem, which is presented in the ingenious paper [26]. In Section 3.4 we deal with the algebrability of the set of continuous surjections from $\mathbb{R}^m$ to $\mathbb{C}^n$, where we improve the results from [4, 26] by showing that the subset of continuous surjections from $\mathbb{R}^m$ to $\mathbb{C}^n$ such that each value $a \in \mathbb{C}^n$ is assumed *on an unbounded set* of $\mathbb{R}^m$ is, actually, (maximal) strongly algebrable (Theorem 3.17). In order to achieve this we shall need to make use of some results and machinery from Complex Analysis, such as the order of growth of an entire function. While doing this, we also provide some new results from Complex Analysis which are of independent interest (see, e.g., Lemma 3.15).

3.2 Lineability of the set $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$

Our starting point is to build a curve from the real that fills the plane, that is, to prove that $\mathcal{CS}(\mathbb{R}, \mathbb{R}^2) \neq \varnothing$. In the next chapter we will indicate a similar construction on special topological spaces, but for the lack of references on the subject, we write here a prove of this simple situation. Recall the I denote the unit compact interval $[0, 1]$.

Lemma 3.4. *There exists a continuous surjection from the real line $\mathbb{R}$ to the plane $\mathbb{R}^2$,* i.e., *the family $\mathcal{CS}(\mathbb{R}, \mathbb{R}^2)$ is non-void.*

Proof. Let us fix a Peano curve $\phi : I \to I^2$. We may assume that the curve ϕ starts and ends at origin. For each positive integer n, I and $I_n = [n, n+1]$, as I^2 and $[-n, n]^2$, are homeomorphic. Thus we may assume that the continuous function ϕ maps I onto $[-1, 1]^2$, with the origin (of $\mathbb{R}^2$) being the start and end point. Thus the map $h_n : I_n \to [-n, n]^2$ defined by $h_n(t) \stackrel{\text{def}}{=} n \cdot \phi(t-n)$ is a CS map (a continuous surjection) that starts and ends at origin. Pasting the maps h_n ($h_0 = \phi$), one after another, we get a CS $H : [0, +\infty) \to \mathbb{R}^2$. We may take an extension $\tilde{H} : \mathbb{R} \to\!\!\to \mathbb{R}^2$ of H mapping all $(-\infty, 0]$ into the origin and pasting with H (or just by using Tiestz's extension theorem). Explicitly, $\tilde{H}$ may be defined by $\tilde{H}(t) = 0$, if $t \in (-\infty, 0]$, and $\tilde{H}(t) = h_n(t)$, if $t \in [n, n+1]$, with $n \geq 0$. $\square$

The following result guarantees that, *de facto*, there exists a continuous surjection beetween any Euclidean spaces. It uses the fact that $\mathcal{CS}(\mathbb{R}, \mathbb{R}^2) \neq \varnothing$.

Proposition 3.5. *For any pair (m, n) of positive integers, there exists a continuous surjection from $\mathbb{R}^m$ to $\mathbb{R}^n$,* i.e., *$\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n) \neq \varnothing$.*

Proof. Let us take $f \in \mathcal{CS}(\mathbb{R}, \mathbb{R}^2)$. If $f_i := \pi_i \circ f,\ i = 1, 2$ denotes the i-coordinates functions of f ($f = (f_1, f_2)$), then the map $id_{\mathbb{R}} \times f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $(id_{\mathbb{R}} \times f)(t, s) := (t, f_1(s), f_2(s))$ is a continuous surjection. Thus, $(id_{\mathbb{R}} \times f) \circ f$ belongs to $\mathcal{CS}(\mathbb{R}, \mathbb{R}^3)$. Proceeding by induction, we can assure the existence of a function g belonging to $\mathcal{CS}(\mathbb{R}, \mathbb{R}^n)$ for every $n \in \mathbb{N}$. Hence, defining $F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ by $F := g \circ \pi_1$, i.e.,

$$F(x) = F(x_1, \ldots, x_m) = g(x_1)\ , \quad \text{for all } x = (x_1, \ldots, x_m) \in \mathbb{R}^m$$

($\pi_1 : \mathbb{R}^m \longrightarrow \mathbb{R}$ denotes the canonical projection over the first coordinate), we conclude that $F \in \mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$. $\square$

Attempting maximal lineability of $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$ (that is, $\mathfrak{c}$-lineability, with $\mathfrak{c}$ standing for the cardinality of $\mathbb{R}$) we make use of the following remark (inspired in a result from [10]), which indicates a method to obtain our main result.

Remark 3.6. *Given a continuous surjection $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, suppose we have $\mathcal{X} \subset \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ a subset of $\mathfrak{c}$-many linearly independent functions such that every nonzero element of span($\mathcal{X}$) is a continuous surjection. Then, we have that*

$$\mathcal{Y} := \{F \circ f\}_{F \in \mathcal{X}} \subset \mathcal{C}(\mathbb{R}^m; \mathbb{R}^n)$$

has cardinality $\mathfrak{c}$, is linearly independent and is formed just by continuous surjections. Moreover,

$$\mathit{span}(\mathcal{Y}) \subset \mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n) \cup \{0\},$$

obtaining the $\mathfrak{c}$-lineability of $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$.

In order to continue we shall need two lemmas and some notation. First, let us consider (for $r > 0$) the homeomorphism $\phi_r : \mathbb{R} \to \mathbb{R}$ given by

$$\phi_r(t) := e^{rt} - e^{-rt}.$$

Lemma 3.7. *The subset $\mathfrak{A} := \{\phi_r\}_{r \in \mathbb{R}^+}$ of $\mathbb{R}^{\mathbb{R}}$ is linearly independent, has cardinality $\mathfrak{c}$, and every nonzero element of span($\mathfrak{A}$) is continuous and surjective.*

Proof. First let us prove that every nonzero element $\phi = \sum_{i=1}^{k} \alpha_i \cdot \phi_{r_i} \in \operatorname{span}(\mathfrak{A})$ is surjective. We may suppose that $r_1 > r_2 > \cdots > r_k$ and $\alpha_1 \neq 0$. Writing

$$\phi(t) = e^{r_1 t} \cdot \left(\alpha_1 + \sum_{i=2}^{k} \alpha_i \cdot e^{(r_i - r_1)t} \right) - \sum_{i=1}^{k} \alpha_i \cdot e^{-r_i t},$$

we conclude that $\lim_{t \to +\infty} \phi(t) = \operatorname{sign}(\alpha_1) \cdot \infty$ and $\lim_{t \to -\infty} \phi(t) = -\operatorname{sign}(\alpha_1) \cdot \infty$. Thus, the continuity of ϕ assures its surjection. Now let us see that $\mathfrak{A}$ is linearly independent: suppose that $\psi = \sum_{i=1}^{n} \lambda_i \cdot \phi_{s_i} = 0$. If there is some $\lambda_j \neq 0$, we may suppose that $s_1 > \cdots > s_n$ and $\lambda_1 \neq 0$. Repeating the argument above, we obtain

$$\lim_{t \to +\infty} \psi(t) = \operatorname{sign}(\lambda_1) \cdot \infty \ \text{ and } \ \lim_{t \to -\infty} \psi(t) = -\operatorname{sign}(\lambda_1) \cdot \infty,$$

which contradicts $\psi = 0$. This proves that $\mathfrak{A}$ is linearly independent. The other assertions are easy to prove. $\square$

For each $r = (r_1, \dots, r_n) \in (\mathbb{R}^+)^n$, let φ_r be the homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $\varphi_r = (\phi_{r_1}, \dots, \phi_{r_n})$, *i.e.*,

$$\varphi_r(x) := (\phi_{r_1}(x_1), \dots, \phi_{r_n}(x_n)), \quad \text{for all } x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Working on each coordinate, and using the previous lemma, we have the following.

Lemma 3.8. *The set $\mathfrak{B} = \{\varphi_r\}_{r \in (\mathbb{R}^+)^n}$ of $\mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$ is linearly independent, has cardinality $\mathfrak{c}$, and every nonzero element of span$(\mathfrak{B})$ is continuous and surjective.*

The following is a well know result which will guarantees that the lineability result is optimal, *i.e.*, the result is the best possible in terms of dimension. We sketch the proof.

Lemma 3.9. $\dim \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n) = \mathfrak{c}$.

Proof. Since each point of $\mathbb{R}^m$ may be see as a constant function in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$, we get a injection $\mathbb{R}^m \hookrightarrow \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$, and thus

$$\text{card } \mathbb{R}^m = \mathfrak{c} \leq \text{card } \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n).$$

On the other hand, $\mathbb{R}^m$ is separable ($\mathbb{Q}^m$ is dense in $\mathbb{R}^m$) and we may enumerate $\mathbb{Q}^m = \{r_n\}_{n \in \mathbb{N}}$. Defining a map that assigns each continuous function $f \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$ to the sequence of its rational points images $(f(r_n))_{n \in \mathbb{N}}$, we get a map $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n) \to (\mathbb{R}^n)^{\mathbb{N}}$ that is injective as consequence of the separability of $\mathbb{R}^m$. Since card $(\mathbb{R}^n)^{\mathbb{N}} = \text{card } \mathbb{R}^{(n \cdot \mathbb{N})} = \text{card } \mathbb{R}^{\mathbb{N}} = \mathfrak{c}$, we get

$$\text{card } \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n) \leq \mathfrak{c},$$

and therefore conclude that card $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n) = \mathfrak{c}$. ☐

Now it is time to state and prove the main lineability result.

Theorem 3.10 (Albuquerque, 2014)**.** *For every pair $m, n \in \mathbb{N}$, the set $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$ is maximal lineable.*

Proof. Let $f \in \mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$. Using the notation of the previous lemma and the ideas of the Remark 3.6, we now prove that the set $\mathfrak{C} = \{F \circ f\}_{F \in \mathfrak{B}}$ is so that span$(\mathfrak{C})$ is the space we are looking for.

The surjectivity of f assures that $G \circ f = 0$ implies $G = 0$, for every function G from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus, if $G_i \in \mathfrak{B},\ i = 1, \dots, k$ and

$$0 = \sum_{i=1}^{k} \alpha_i \cdot G_i \circ f = \left(\sum_{i=1}^{k} \alpha_i G_i \right) \circ f,$$

then $\sum_{i=1}^{k} \alpha_i G_i = 0$; so since $\mathfrak{B}$ is linearly independent, we conclude that $\alpha_i = 0,\ i = 1, \dots, k$ and thus, $\mathfrak{C}$ is linearly independent. Clearly, it has cardinality $\mathfrak{c}$. Furthermore, any nonzero function

$$\sum_{i=1}^{l} \lambda_i \cdot F_i \circ f = \left(\sum_{i=1}^{l} \lambda_i F_i \right) \circ f$$

of span($\mathfrak{C}$) is continuous and surjective, since it is the composition of continuous surjective functions (recall that, from Lemma 3.8, $\sum_{i=1}^{l} \lambda_i F_i$ is a continuous surjective function). Therefore, span($\mathfrak{C}$) only contains, except for the zero function, continuous surjective functions. $\square$

Remark 3.11. *The case of injective functions deserves some comments. In $\mathbb{R}^{\mathbb{R}}$ the set of surjective functions is $2^{\mathfrak{c}}$-lineable, while the set of injective functions is only 1-lineable and, consequently, also the set of bijections in $\mathbb{R}^{\mathbb{R}}$. In fact, given two linearly independent injective functions $f, g : \mathbb{R} \to \mathbb{R}$, take $x \neq y$ in $\mathbb{R}$ and $\alpha = \frac{f(x)-f(y)}{g(y)-g(x)} \in \mathbb{R}$. Then the function $h := f + \alpha g \in span(f, g)$ satisfies $h(x) = h(y)$ and, therefore, is not injective. This argument can be easily adapted to functions from $\mathbb{R}^n$ to $\mathbb{R}$.*

3.3 Spaceability of $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n)$

In [26, Theorem 3.2] Bernal and Ordóñez provide the following general spaceability criteria. Let us recall some concepts and notation: $\mathcal{P}(\Omega)$ denotes, as usual, the family of subsets of a set Ω; $\sigma(f)$ will denote the support of a function $f : \Omega \to \mathbb{K}$, that is, the set $\sigma(f) = \{x \in \Omega;\ f(x) \neq 0\}$; an F-space is a complete metrizable topological vector space; an F-norm on a vector space X is a functional $\|\cdot\| : X \to [0, \infty)$ satisfying, for all $x, y \in X$ and $\lambda \in \mathbb{K}$, the following properties $\|x + y\| \leq \|x\| + \|y\|$, $\|\lambda x\| \leq \|x\|$ if $|\lambda| \leq 1$, $\|x\| = 0$ only if $x = 0$, and $\|\lambda x\| \to 0$ if $\lambda \to 0$.

Theorem 3.12 (Bernal,Ordóñez, 2014)**.** *Let Ω be a nonempty set and Z a topogical vector space on $\mathbb{K}$. Assume that X is an F-space on $\mathbb{K}$ consisting of Z-valued functions on Ω and that $\|\cdot\|$ is an F-norm defining the topology of X. Suppose, in addition, that S is a nonempty subset of X and $\mathcal{S} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is a set function with $A \subset \mathcal{S}(A)$ for all $A \in \mathcal{P}(\Omega)$ satisfying the following properties:*

(i) If $(g_n)_n \subset X$ satisfies $g_n \to g$ in X, then there is a subsequence $(n_k) \subset \mathbb{N}$ such that, for every $x \in \Omega$, $g_{n_k}(x) \to g(x)$;

(ii) There is a constant $C \in (0, +\infty)$ such that $\|f + g\| \geq C\|f\|$ for all $f, g \in X$ with $\sigma(f) \cap \sigma(g) = \emptyset$;

(iii) $\alpha f \in S$ for all $\alpha \in \mathbb{K}$ and all $f \in S$;

(iv) If $f, g \in X$ are such that $f + g \in \mathcal{S}$ and $\mathcal{S}(\sigma(f)) \cap \sigma(g) = \emptyset$, then $f \in S$;

(v) There is a sequence of functions $(f_n)_n \subset X \setminus S$, such that $\mathcal{S}(\sigma(f_m)) \cap \sigma(f_n) = \emptyset$, for all $m \neq n$.

Then $X \setminus S$ is spaceable in X.

For each pair $m, n \in \mathbb{N}$, setting

$$\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n) := \left\{ f \in \mathcal{C}\left(\mathbb{R}^m, \mathbb{R}^n\right) : f^{-1}(\{a\}) \text{ is unbounded for every } a \in \mathbb{R}^n \right\}.$$

(obviously) we have that $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n)$ is a "smaller" part of $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$. Endowing $\mathcal{C}\left(\mathbb{R}^m, \mathbb{R}^n\right)$ with the compact-open topology and using the previous spaceability criteria, Bernal and Ordóñez solved problem of spaceability of the set of continuous surjections between Euclidean spaces:

Theorem 3.13 (Bernal and Ordóñez, 2014)**.** *For each pair $m, n \in \mathbb{N}$, the set $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n)$ is maximal dense-lineable and spaceable in $\mathcal{C}\left(\mathbb{R}^m, \mathbb{R}^n\right)$. In particular, it is maximal lineable in $\mathcal{C}\left(\mathbb{R}^m, \mathbb{R}^n\right)$.*

3.4 Algebrability of $\mathcal{CS}_\infty\left(\mathbb{R}^m, \mathbb{C}^n\right)$

Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Recall that, for any topological space X, we set

$$\mathcal{CS}_\infty(\mathbb{R}^m, X) := \left\{ f \in \mathcal{C}\left(\mathbb{R}^m, X\right) : f^{-1}(\{a\}) \text{ is unbounded for every } a \in X \right\}.$$

Once solved the lineability and spaceability (Theorems 3.10 and 3.13 respectively), a natural question would be to ask about the algebrability of the set $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n)$. Clearly, algebrability cannot be obtained in the real context, since for any $f \in \mathbb{R}^{\mathbb{R}}$, f^2 is always non-negative. However, in the complex frame it is actually possible to obtain algebrability. Before that, let us recall some results related to the growth of an entire function (see, e.g., [30, p.9]).

Remark 3.14. (*Order of an entire function and consequences*). By $\mathcal{H}\left(\mathbb{C}\right)$ we denote the space of all entire functions from $\mathbb{C}$ to $\mathbb{C}$. For $r > 0$ and $f \in \mathcal{H}\left(\mathbb{C}\right)$, we set $M\left(f, r\right) := \max_{|z|=r} |f(z)|$. The function $M(f, \cdot)$ increases strictly towards $+\infty$ as soon as f is non-constant.

(a) The (growth) *order* $\rho(f)$ of an entire function $f \in \mathcal{H}\left(\mathbb{C}\right)$ is defined as the infimum of all positive real numbers α with the following property: $M\left(f, r\right) < e^{r^\alpha}$ for all $r > r(\alpha) > 0$. Note that $\rho(f) \in [0, +\infty]$. Trivially, the order of a constant map is 0. If f is non-constant, we have
$$\rho(f) = \limsup_{r \to +\infty} \frac{\log \log M\left(f, r\right)}{\log r}.$$

(b) If $f(z) = \sum_{n=1}^\infty a_n z^n$ is the MacLaurin series expansion of f then
$$\rho(f) = \limsup_{n \to +\infty} \frac{n \log n}{\log\left(1/|a_n|\right)}.$$
In particular, given $\alpha > 0$, $f_\alpha(z) := \displaystyle\sum_{n=1}^\infty \frac{z^n}{n^{n/\alpha}}$ satisfies $\rho(f_\alpha) = \alpha$.

(c) For every $f \in \mathcal{H}\left(\mathbb{C}\right)$, every $N \in \mathbb{N}$ and every $\alpha \in \mathbb{C} \setminus \{0\}$,
$$\rho\left(\alpha f^N\right) = \rho\left(f\right).$$

(d) For every $f, g \in \mathcal{H}(\mathbb{C})$,

$$\rho(f \cdot g) \leq \max\{\rho(f), \rho(g)\}$$

and

$$\rho(f + g) \leq \max\{\rho(f), \rho(g)\}.$$

Moreover, if f and g have different orders, then

$$\rho(f + g) = \max\{\rho(f), \rho(g)\} = \rho(f \cdot g),$$

where it is assumed $f \not\equiv 0 \not\equiv g$ for the second equality.

(e) (Corollary to Hadamard's theorem): Every non-constant entire function f with $\infty > \rho(f) \notin \mathbb{N}$ is surjective (see, e.g., [2, Corollary, p.211] or [73, Thm 9.3.10]).

As a consequence of the previous properties, we obtain the following result (of independent interest) concerning the order of a polynomial of several variables evaluated on entire functions with different orders. First, we need to establish some notation: for a non-constant polynomial in M complex variables $P \in \mathbb{C}[z_1, \dots, z_M]$, let $\mathcal{I}_P \subset \{1, \dots, M\}$ be the set of indexes k such that the variable z_k explicitly appears in some monomial (with non-zero coefficient) of P; that is, $\mathcal{I}_P = \{n \in \{1, \dots, M\} : \frac{\partial P}{\partial z_n} \not\equiv 0\}$.

Lemma 3.15. *Let $f_1, \dots, f_M \in \mathcal{H}(\mathbb{C})$ such that $\rho(f_i) \neq \rho(f_j)$ whenever $i \neq j$. Then*

$$\rho\left(P\left(f_1, \dots, f_M\right)\right) = \max_{k \in \mathcal{I}_P} \rho\left(f_k\right),$$

for all non-constant polynomials $P \in \mathbb{C}[z_1, \dots, z_M]$. Moreover, $(f_k)_{k=1}^M$ is algebraically independent and generates a free algebra.

Proof. The second part of the statement follows straightforwardly from the first one. In order to prove the first part, it is clear that we may assume, without loss of generality, that $M > 1$ and the entire functions $f_1, \dots, f_M$ satisfy $\rho(f_1) < \rho(f_2) < \cdots < \rho(f_M)$. Given a non-constant polynomial $P \in \mathbb{C}[z_1, \dots, z_M]$, properties (c) and (d) of Remark 3.14 assure that $\rho\left(P\left(f_1, \dots, f_M\right)\right) \leq \max_{k \in \mathcal{I}_P} \rho\left(f_k\right)$. Therefore, we just need to prove that

$$\rho\left(P\left(f_1, \dots, f_M\right)\right) \geq \max_{k \in \mathcal{I}_P} \rho\left(f_k\right).$$

Let be $N = \max_{k \in \mathcal{I}_P}$ (so that $\max_{k \in \mathcal{I}_P} \rho\left(f_k\right) = \rho_N > 0$). We can write

$$P(f_1, \dots, f_M) = \sum_{i=0}^{m} P_i(f_1, \dots, f_{N-1}) \cdot f_N^i, \tag{1}$$

with some $m > 0$ and $P_m \in \mathbb{C}[z_1, \dots, z_{N-1}] \setminus \{0\}$. Let $\varepsilon > 0$ such that

$$\rho(f_{N-1}) < \rho(f_N) - 2\varepsilon < \rho(f_N) =: \rho_N.$$

Now, parts (c) and (d) of Remark 3.14 allow us to estimate the order of each one of terms of the sum in (1):

$$\rho\left(P_i(f_1,\ldots,f_{N-1})\right) \leq \rho(f_{N-1}) < \rho_N \quad \text{for all } i=0,\ldots,m \ \textit{and}$$

$$\rho\left(P_m(f_1,\ldots,f_{N-1})\cdot f_N\right) = \rho_N.$$

By the definition of order, there exist a sequence of positive real numbers, $(r_n)_n$, going to $+\infty$ and complex numbers z_n, of modulus r_n, such that, for n large enough, the following inequalities hold:

$$|P_m(f_1,\ldots,f_{N-1})(z_n)|\cdot|f_N(z_n)| > e^{r_n^{\rho_N-\varepsilon}} \quad \text{and}$$

$$|P_i(f_1,\ldots,f_{N-1})(z_n)| < e^{r_n^{\rho_N-2\varepsilon}} \quad \text{for all } i=0,\ldots,m.$$

In particular,

$$|f_N(z_n)| > e^{r_n^{\rho_N-\varepsilon}-r_n^{\rho_N-2\varepsilon}} \quad \text{for } n \text{ large.}$$

Thus,

$$\begin{aligned}
&|P(f_1,\ldots,f_M)(z_n)| \geq \\
&\geq |P_m(f_1,\ldots,f_{N-1})(z_n)|\cdot|f_N(z_n)|^m - \sum_{i=0}^{m-1}|P_i(f_1,\ldots,f_{N-1})(z_n)|\cdot|f_N(z_n)|^i \\
&> e^{r_n^{\rho_N-\varepsilon}}\cdot|f_N(z_n)|^{m-1} - e^{r_n^{\rho_N-2\varepsilon}}\cdot\sum_{i=0}^{m-1}|f_N(z_n)|^i \\
&= e^{r_n^{\rho_N-\varepsilon}}\cdot|f_N(z_n)|^{m-1}\cdot\left[1-e^{r_n^{\rho_N-2\varepsilon}-r_n^{\rho_N-\varepsilon}}\cdot\sum_{i=0}^{m-1}|f_N(z_n)|^{i-(m-1)}\right].
\end{aligned}$$

Note that the expression inside the brackets in the last formula tends to 1 as $n\to\infty$: indeed, $e^{r_n^{\rho_N-2\varepsilon}-r_n^{\rho_N-\varepsilon}} \to 0$ and $|f_N(z_n)|^{-1} < e^{r_n^{\rho_N-2\varepsilon}-r_n^{\rho_N-\varepsilon}} \to 0$. Thus, it is greater than some constant $C\in(0,1)$ for n large enough. Furthermore, we also have

$$e^{r_n^{\rho_N-\varepsilon}}\cdot|f_N(z_n)|^{m-1} > e^{r_n^{\rho_N-\varepsilon}}\cdot e^{(m-1)r_n^{\rho_N-\varepsilon}-(m-1)r_n^{\rho_N-2\varepsilon}}$$

$$= e^{m\,r_n^{\rho_N-\varepsilon}-(m-1)r_n^{\rho_N-2\varepsilon}} > e^{(m/2)\,r_n^{\rho_N-\varepsilon}}$$

for n large enough. Consequently, one has for n large that

$$M\left(P(f_1,\ldots,f_M),r_n\right) \geq C\cdot e^{(m/2)r_n^{\rho_N-\varepsilon}},$$

which implies

$$\begin{aligned}
\rho\left(P(f_1,\dots,f_M)\right) &= \limsup_{r\to+\infty} \frac{\log\log M\left(P(f_1,\dots,f_M),r\right)}{\log r} \\
&\geq \limsup_{n\to\infty} \frac{\log\log M\left(P(f_1,\dots,f_M),r_n\right)}{\log r_n} \\
&\geq \lim_{n\to\infty} \frac{\log\log(C\cdot e^{(m/2)r_n^{\rho_N-\varepsilon}})}{\log r_n} \\
&= \rho_N - \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, the above inequalities prove

$$\rho\left(P(f_1,\dots,f_M)\right) \geq \rho_N = \max_{k\in\mathcal{I}(P)} \rho\left(f_k\right),$$

as required. □

Remark 3.16. Lemma 3.15 very probably plays the same role in the complex case as the *exponential-like functions method* plays in the real case. The last method was developed in the paper [13] by Balcerzak *et al.* and it has numerous applications, see for instance [14].

From this lemma we can prove that $\mathcal{CS}_\infty\left(\mathbb{R}^m,\mathbb{C}^n\right)$ is maximal strongly algebrable, which means that the set is strongly $\mathfrak{c}$-algebrable.

Theorem 3.17. *For every $m\in\mathbb{N}$, the set $\mathcal{CS}_\infty\left(\mathbb{R}^m,\mathbb{C}^n\right)$ is maximal strongly algebrable in $\mathcal{C}\left(\mathbb{R}^m,\mathbb{C}^n\right)$.*

Proof. It suffices to consider the case $n=m=1$. In fact, the case $m>1$ follows from the $m=1$ by considering the projection map from $\mathbb{R}^m$ to $\mathbb{R}$. The case $n>1$ is obtained from $n=1$ by working on each coordinate.

For each $s>0$, select an entire function $\varphi_s:\mathbb{C}\to\mathbb{C}$ of order $s>0$. Let $A:=(0,+\infty)\setminus\mathbb{N}$. Lemma 3.15 assures that the set $\{\varphi_s\}_{s\in A}$ is a system of cardinality $\mathfrak{c}$ generating a free algebra $\mathcal{A}$.

Next, notice that any element $\varphi\in\mathcal{A}\setminus\{0\}$ may be written as a non-constant polynomial P without constant term evaluated on some φ_{s_1}, φ_{s_2}, …, φ_{s_N}:

$$\varphi = P(\varphi_{s_1},\varphi_{s_2},\dots,\varphi_{s_N}) = \sum_{|\alpha|\leq m} c_\alpha\cdot\varphi_{s_1}^{\alpha_1}\cdot\varphi_{s_2}^{\alpha_2}\cdots\varphi_{s_N}^{\alpha_N}.$$

By Lemma 3.15, there exists $j\in\{1,\dots,N\}$ such that $\rho(\varphi)=\rho(\varphi_{s_j})=s_j\notin\mathbb{N}_0$. Thus Remark 3.14 (e) guarantees that φ is surjective. Finally, take any $F\in\mathcal{CS}_\infty\left(\mathbb{R},\mathbb{C}\right)$ and consider the algebra

$$\mathcal{B} := \{\varphi\circ F\}_{\varphi\in\mathcal{A}}.$$

Then it is plain that $\mathcal{B}$ is freely $\mathfrak{c}$-generated and that $\mathcal{B}\setminus\{0\}\subset\mathcal{CS}_\infty\left(\mathbb{R},\mathbb{C}\right)$, as required. □

Chapter 4

Peano curves on topological vector spaces

In this chapter is we present the remaining results of the paper

> [7] *Peano curves on topological vector spaces, Linear Algebra and its Applications, vol. 460, 81–96, 2014.*

(which is a joint work with L. Bernal, D. Pellegrino and J. Seoane) concerning Peano curves on topological vector spaces. The Hahn-Mazurkiewicz's theorem allows us to investigate topological vector spaces that are continuous image of the real line, from which we provide an optimal lineability result.

4.1 Motivation and main results

This chapter moves on to the study of generalizations of the previous results to topological vector spaces that are, in some natural sense, covered by Peano spaces. We introduce the notion of σ-Peano space (see Definition 4.1) and use it to show (among other results) that given any topological vector space $\mathcal{X}$ that is also a σ-Peano space, then the set

$$\mathcal{CS}_\infty(\mathbb{R}^m, \mathcal{X}) := \left\{ f \in \mathcal{C}\left(\mathbb{R}^m, \mathcal{X}\right) : f^{-1}(\{a\}) \text{ is unbounded for every } a \in \mathcal{X} \right\}$$

is $\mathfrak{c}$-lineable (hence maximal lineable in $\mathcal{C}\left(\mathbb{R}^m, \mathcal{X}\right)$), where $\mathfrak{c}$ stands for the continuum (see Theorem 4.6). In addition, we will show how, by just starting with separable normed spaces, one can obtain σ-Peano spaces. We analyze Peano spaces in the framework of sequence spaces and also study Peano space in real and complex function spaces.

4.2 σ-Peano spaces

As mentioned in the introduction, the theorem of Hahn and Mazurkiewicz provides a topological characterization of Hausdorff topological spaces that are continuous image of the unit interval $I := [0,1]$: these are precisely the Peano spaces. In this section we investigate topological spaces that are continuous image of the *real line* and for this task the following definition seems natural.

Definition 4.1. *A topological space X is a σ-*Peano space *if there exists an increasing sequence of subsets*

$$K_1 \subset K_2 \subset \cdots \subset K_m \subset \cdots \subset X,$$

such that each one of them is a Peano space (endowed with the topology inherited from X) and its union amounts to the whole space, that is, $\bigcup_{n\in\mathbb{N}} K_n = X$.

From now on, CS will stand for an abbreviation of "continuous surjective".

Proposition 4.2. *Let X be a Hausdorff topological space. The following assertions are equivalent:*

(a) *X is a σ-Peano space.*

(b) $\mathcal{CS}_\infty(\mathbb{R}, X) \neq \varnothing$.

(c) $\mathcal{CS}(\mathbb{R}, X) \neq \varnothing$.

Proof. (a) $\Rightarrow$ (b): Let $K_1 \subset K_2 \subset \cdots$ be an increasing sequence of Peano spaces in X such that its union is the whole X. Fix a point $x_0 \in X$. Without loss of generality, we may suppose that $x_0 \in K_n$, for all $n \geq 1$. Since Peano spaces are arcwise connected [106, Theorem 31.2], for each $n \geq 1$ there is a Peano map $f_n : [n, n+1] \to K_n$, that starts and ends at x_0, i.e., $f_n(n) = x_0 = f_n(n+1)$. Joining all these Peano maps with the constant path $t \in (-\infty, 1] \mapsto x_0 \in K_1$, one obtains a CS map $F : \mathbb{R} \to X$.
(c) $\Rightarrow$ (a): Let f be a map in $\mathcal{CS}(\mathbb{R}, X)$. Therefore,

$$X = f(\mathbb{R}) = f\left(\bigcup_{n\in\mathbb{N}} [-n, n]\right) = \bigcup_{n\in\mathbb{N}} f([-n, n]).$$

Since (b) $\Rightarrow$ (c) is obvious, the proof is done. □

Example 4.3. (*Spaces that are σ-Peano*).

(a) Trivially, Euclidean spaces $\mathbb{R}^n$ and Peano spaces are σ-Peano. For $1 < p \leq \infty$, the Hilbert cube

$$\mathrm{C}_p := \prod_{n\in\mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right] \subset \ell_p,$$

considered as a topological subspace of ℓ_p, is a compact metric space, so it is a Peano space. For each natural k, let $k\,\mathrm{C}_p$ be the Hilbert cube after applying an "k-homogeneous dilation" to it. Therefore, the union of Hilbert cubes $\bigcup_{k\in\mathbb{N}} k\,\mathrm{C}_p$ is a σ-Peano topological vector space, when endowed with the topology inherited from ℓ_p.

(b) Let $\mathcal{X}$ be a separable topological vector space and $\mathcal{X}'$ be its topological dual endowed with the weak*-topology. If $\mathcal{X}'$ is covered by an increasing sequence of (weak*-)compact subsets, then it is σ-Peano. Indeed, when the topological dual is endowed with the weak*-topology, its weak*-compact subsets are metrizable (see, for instance, [100, Theorem 3.16]). Therefore, it will be a σ-Peano space. Clearly, this holds on the topological dual (endowed with weak*-topology) of separable normed spaces.

Recall that an F-*space* is a topological vector space with complete translation-invariant metric which provides its topology.

Example 4.4. (*Spaces that are not σ-Peano*).

(a) *Every σ-Peano space is separable. Indeed, continuity preserves separability. In particular, ℓ_∞ is not σ-Peano.*

(b) *No infinite dimensional* F*-space is σ-compact (i.e., a countable union of compact spaces), and, therefore, is not σ-Peano. This is a consequence of the Baire category theorem combined with the fact that on infinite dimensional topological vector spaces, compact sets have empty interior. In particular, no infinite dimensional Banach space is σ-Peano.*

Remark 4.5. *If X is a σ-Peano space, then card $\mathcal{C}(\mathbb{R}, X) \leq \mathfrak{c}$. The argument is similar to Lemma 3.9: indeed, this is consequence of card $X \leq \mathfrak{c}$ (as an image of the real line),* in tandem *with the fact that the separability of $\mathbb{R}$ implies that each map of $\mathcal{C}(\mathbb{R}, X)$ is uniquely determined the sequence of its rational images, which defines an injective map $\mathcal{C}(\mathbb{R}, X) \hookrightarrow X^{\mathbb{N}}$ and, therefore, card $\mathcal{C}(\mathbb{R}, X) \leq$ card $X^{\mathbb{N}} \leq \mathfrak{c}$.*

Now we state and prove the main result of this section, which provides maximal lineability of *Peano curves* on arbitrary topological vector spaces that are also σ-Peano spaces. As in [26], we work with some *particular* Peano maps, namely, with those continuous surjections assuming each value on an unbounded set.

It is convenient to recall a well-known fact from set theory: a family $\{A_\lambda\}_{\lambda\in\Lambda}$ of infinite subsets of $\mathbb{N}$ is called *almost disjoint* if $A_\lambda \cap A_{\lambda'}$ is finite whenever $\lambda \neq \lambda'$. The usual procedure to generate such a family is the following (see, e.g., [3]): denote by $\{q_n\}_{n\in\mathbb{N}}$ an enumeration of the rational numbers. For every irrational α, we choose a subsequence $\{q_{n_k}\}_{k\in\mathbb{N}}$ of $\{q_n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to+\infty} q_{n_k} = \alpha$ and define $A_\alpha := \{n_k\}_{k\in\mathbb{N}}$. By construction, we obtain that $\{A_\alpha\}_{\alpha\in\mathbb{R}\setminus\mathbb{Q}}$ is an almost disjoint uncountable family of subsets of $\mathbb{N}$.

Theorem 4.6. *Let $\mathcal{X}$ be a σ-Peano topological vector space. Then $\mathcal{CS}_\infty(\mathbb{R}^m, \mathcal{X})$ is maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathcal{X})$.*

Proof. It is sufficient to prove the result for $m = 1$. Take $g : \mathbb{N}_0 \to \mathbb{N} \times \mathbb{N}$ a bijection, and set

$$I_{k,n} := \left[g^{-1}(k,n), g^{-1}(k,n) + 1\right],$$

for all $k, n \in \mathbb{N}$, thus $\{I_{k,n}\}_{k,n\in\mathbb{N}}$ is a family of compact intervals of $[0, +\infty)$ such that $\bigcup_{k,n\in\mathbb{N}} I_{k,n} = [0, +\infty)$, the intervals $I_{k,n}$ having pairwise disjoint interiors, and $\bigcup_{k\in\mathbb{N}} I_{k,n}$ is unbounded for every

n. Proceeding as in the construction presented in Proposition 4.2, for each n, we can build a CS map $f_n : \mathbb{R} \to \mathcal{X}$ with the following properties:

- $f_n \left(\bigcup_{k\in\mathbb{N}} I_{k,n}\right) = \mathcal{X}$;
- for each $k \in \mathbb{N}$, on the interval $I_{k,n}$, f_n starts and ends at the origin $0_{\mathcal{X}} \in \mathcal{X}$ and covers the k-th Peano subset of $\mathcal{X}$;
- $f_n \equiv 0$ on $\bigcup_{k\in\mathbb{N}} I_{k,m}$, for all $m \neq n$.

Notice that each $f_n \in \mathcal{CS}_\infty(\mathbb{R}, \mathcal{X})$, since $\bigcup_{k\in\mathbb{N}} I_{k,n}$ is unbounded.

Now let $\{J_\lambda\}_{\lambda\in\Lambda}$ be an almost disjoint family with cardinality $\mathfrak{c}$ consisting of infinite subsets of $\mathbb{N}$. Define, for each $\lambda \in \Lambda$,

$$F_\lambda := \sum_{n\in J_\lambda} f_n : \mathbb{R} \to \mathcal{X}.$$

The pairwise disjointness of the interior of the intervals $I_{k,n}$ (together with the above properties of f_n) guarantees that F_λ is well-defined, as well as continuous. We assert that the set

$$\{F_\lambda\}_{\lambda\in\Lambda}$$

provides the desired maximal lineability. The crucial point is the following argument: let $F_{\lambda_1}, \ldots, F_{\lambda_N}$ be distinct and $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, with $\alpha_N \neq 0$. Since $J_{\lambda_N} \setminus \left(\cup_{i=1}^{N-1} J_{\lambda_i}\right)$ is infinite, we may fix $n_0 \in J_{\lambda_N} \setminus \left(\cup_{i=1}^{N-1} J_{\lambda_i}\right)$. Notice that

$$F_{\lambda_1} = \cdots = F_{\lambda_{N-1}} \equiv 0 \quad \text{and} \quad F_{\lambda_N} = f_{n_0} \quad \text{on} \quad \bigcup_{k\in\mathbb{N}} I_{k,n_0}.$$

Consequently,

$$\sum_{k=1}^{N} \alpha_k \cdot F_{\lambda_k} = \alpha_N \cdot f_{n_0} \quad \text{on} \quad \bigcup_{k\in\mathbb{N}} I_{k,n_0}.$$

Then $F := \sum_{k=1}^{N} \alpha_k \cdot F_{\lambda_k}$ is an element of $\mathcal{CS}_\infty(\mathbb{R}, \mathcal{X})$, because the image of $\mathbb{R}$ under F contains $\alpha_N \cdot f_{n_0}(\bigcup_{k\in\mathbb{N}} I_{k,n_0}) = \alpha_N \mathcal{X} = \mathcal{X}$ and each vector of $\mathcal{X}$ is the image by f_{n_0} of an unbounded set. Hence, one may easily prove that the set $\{F_\lambda\}_{\lambda\in\Lambda}$ has $\mathfrak{c}$-many linearly independent elements, and each non-zero element of its linear span also belongs to $\mathcal{CS}_\infty(\mathbb{R}, \mathcal{X})$. The maximal lineability follows from Remark 4.5. □

Observe that this result recovers Theorem 3.10 and the second part of Theorem 3.13. Moreover, together with Example 4.3, item (b), provides,

Corollary 4.7. *Let $\mathcal{N}$ be a separable normed space and $\mathcal{N}'$ be its topological dual endowed with the weak*-topology. Then $\mathcal{CS}_\infty(\mathbb{R}^m, \mathcal{N}')$ is maximal lineable.*

Notice that this result holds in a more general framework: if $\mathcal{X}$ is a separable topological vector space and its topological dual $\mathcal{X}'$ (endowed with the weak*-topology) is covered by an increasing sequence of (weak*-)compact subsets, then $\mathcal{CS}_\infty(\mathbb{R}^m, \mathcal{X}')$ is maximal lineable.

4.3 Peano curves on sequence spaces

Throughout this section we shall deal with the space of real sequences $\mathbb{R}^{\mathbb{N}}$ and some of its variants. Recall that $\mathbb{R}^{\mathbb{N}}$ is an F-space under the metric

$$\mathrm{d}\left((x_n)_n, (y_n)_n\right) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|},$$

and also this metric provides the product topology on $\mathbb{R}^{\mathbb{N}}$ (see [92, p.175]). From Example 4.4, it is clear that $\mathbb{R}^{\mathbb{N}}$ is not a σ-Peano space.

Looking for infinite dimensional "smaller" subspaces of $\mathbb{R}^{\mathbb{N}}$ that could be σ-Peano, we easily find the following example.

Example 4.8. *The space c_{00} of eventually null sequences (with its natural topology induced by the sup norm) is a σ-Peano space. Indeed, $I_n := [-n, n]^n \times \{0\}^{\mathbb{N}} \subset c_{00}$ defines a increasing sequence of Peano spaces in c_{00}, whose union results in the entire space.*

Therefore, Theorem 4.6 immediately gives the following:

Proposition 4.9. *The set $\mathcal{CS}_\infty(\mathbb{R}^m, c_{00})$ is maximal lineable.*

It possible to provide a more "constructive" proof of the previous result, by just making some adjustments to an argument provided in [4]. The proof is presented below and will be used later in order to obtain algebrability results.

2nd proof of Proposition 4.9. Let $\mathbb{R}^+ := (0, +\infty)$ and $\ell_\infty^+ := (\mathbb{R}^+)^{\mathbb{N}} \cap \ell_\infty$. For $\mathbf{r} = (r_n)_{n\in\mathbb{N}} \in \ell_\infty^+$, let us define $\Phi_{\mathbf{r}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$\Phi_{\mathbf{r}}\left((t_n)_{n\in\mathbb{N}}\right) := (\phi_{r_n}(t_n))_{n\in\mathbb{N}},$$

where $\phi_r(t) := e^{rt} - e^{-rt}$ for each $r \in \mathbb{R}^+$. Observe that each ϕ_r is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and, consequently, $\Phi_{\mathbf{r}}$ is a bijection. Notice that the restriction $\Phi_{\mathbf{r}} : \ell_\infty \to \ell_\infty$ is well defined and surjective because, for $(t_n)_n \in \ell_\infty$, one has

$$|\phi_{r_n}(t_n)| = \left|e^{r_n t_n} - e^{-r_n t_n}\right| \le e^{r_n|t_n|} \le e^C \quad (n = 1, \dots),$$

for some positive constant C. Moreover, the map $\Phi_{\mathbf{r}} : \ell_\infty \to \ell_\infty$ is continuous (when ℓ_∞ is endowed with its natural topology). Indeed, fix $s \in \mathbb{R}$ and let $t \in [s-1, s+1]$. By the mean value theorem, there exists $\zeta = \zeta(t, s, n) \in [s-1, s+1]$ such that

$$|\phi_{r_n}(t) - \phi_{r_n}(s)| = \left|\phi'_{r_n}(\zeta)\right| \cdot |t - s|.$$

But since $\mathbf{r} \in \ell_\infty^+$,

$$|\phi'_{r_n}(u)| = r_n\left(e^{r_n u} - e^{-r_n u}\right) \le r_n e^{r_n|u|} \le \|\mathbf{r}\|_\infty e^{\|\mathbf{r}\|_\infty |u|},$$

for all real numbers u. Thus,

$$|\phi_{r_n}(t) - \phi_{r_n}(s)| \leq \|\mathbf{r}\|_\infty e^{\|\mathbf{r}\|_\infty (1+|s|)} \cdot |t-s| \,.$$

If now we fix $\mathbf{s} = (s_n)_{n\in\mathbb{N}} \in \ell_\infty$ and consider $\mathbf{t} = (t_n)_{n\in\mathbb{N}} \in \ell_\infty$ with $\|\mathbf{t}-\mathbf{s}\|_\infty \leq 1$ then we get

$$|\phi_{\mathbf{r}}(\mathbf{t}) - \phi_{\mathbf{r}}(\mathbf{s})| \leq \|\mathbf{r}\|_\infty e^{\|\mathbf{r}\|_\infty (1+\|s\|_\infty)} \cdot \|\mathbf{t}-\mathbf{s}\|_\infty,$$

which yields the desired continuity. Since $\phi_r(0) = 0$ for all $r \in \mathbb{R}^+$, we may restrict again to $\Phi_{\mathbf{r}} : c_{00} \to c_{00}$, being a continuous mapping as well. Then, for a fixed map $F \in \mathcal{CS}_\infty(\mathbb{R}, c_{00})$, it is plain that the set

$$\{\Phi_{\mathbf{r}} \circ F\}_{\mathbf{r}\in\ell_\infty^+}$$

only contains functions in $\mathcal{CS}_\infty(\mathbb{R}, c_{00})$. Working on each coordinate and using the properties of the maps ϕ_r as in [4], this family provides the desired maximal lineability. □

The following result extends Theorem 3.17 to the framework of sequence spaces.

Proposition 4.10. *The set* $\mathcal{CS}_\infty(\mathbb{R}^m, c_{00}(\mathbb{C}))$ *is maximal strongly algebrable in* $\mathcal{C}(\mathbb{R}^m, c_{00}(\mathbb{C}))$.

Proof. It is sufficient to deal with the case $m = 1$. The argument combines the previous constructive proof and the ideas of Theorem 3.17: let $A := (0, +\infty) \setminus \mathbb{N}$, and let $\varphi_s : \mathbb{C} \to \mathbb{C}$ stand for an entire function of order $s > 0$ such that $\varphi_s(0) = 0$. For each $\mathbf{r} = (r_n)_n \in A^{\mathbb{N}}$, the map

$$\varPhi_{\mathbf{r}} := (\varphi_{r_n})_{n\in\mathbb{N}} : c_{00}(\mathbb{C}) \to c_{00}(\mathbb{C})$$

is well-defined, continuous and surjective. Therefore, for a fixed map

$$F \in \mathcal{CS}_\infty(\mathbb{R}, c_{00}(\mathbb{C})),$$

the set $\{\varPhi_{\mathbf{r}} \circ F\}_{\mathbf{r}\in A^{\mathbb{N}}}$ generates a free algebra, which provides the strong maximal algebrability. □

From Example 4.4, item (a), we know that ℓ_∞ is not σ-Peano. On the other hand, if we consider the product topology inherited from $\mathbb{R}^{\mathbb{N}}$, it is obvious that it becomes σ-Peano. In fact, $\ell_\infty = \bigcup_n [-n, n]^{\mathbb{N}}$. Consequently, Theorem 4.6 provides the maximal lineability of the set $\mathcal{CS}_\infty(\mathbb{R}^m, \ell_\infty)$ in $\mathcal{C}(\mathbb{R}^m, \ell_\infty)$.

Notice that, as we did earlier when we dealt with c_{00}, one could also present a constructive proof of this lineability result: for a fixed $F \in \mathcal{CS}_\infty(\mathbb{R}, \ell_\infty)$ the set

$$\{\Phi_{\mathbf{r}}|_{\ell_\infty} \circ F\}_{\mathbf{r}\in\ell_\infty^+}$$

provides the desired maximal lineability. With appropriate adaptations, a similar argument as the one employed in the proof of the algebrability of $\mathcal{CS}(\mathbb{R}^m, c_{00}(\mathbb{C}))$ will prove that the set $\mathcal{CS}(\mathbb{R}^m, \ell_\infty(\mathbb{C}))$ is maximal strongly algebrable in $\mathcal{C}(\mathbb{R}^m, \ell_\infty(\mathbb{C}))$, when ℓ_∞ is endowed with the product topology inherited from $\mathbb{R}^{\mathbb{N}}$.

4.4 Peano curves on function spaces

Let Λ be an infinite index set. Recall that the space $\mathbb{R}^\Lambda$ of real functions $f : \Lambda \to \mathbb{R}$ is a complete metric space when endowed with the metric given by

$$\mathrm{d}(f, g) := \sup_{\lambda\in\Lambda} \min \{1, |f(\lambda) - g(\lambda)|\},$$

which provides the *uniform* topology on $\mathbb{R}^\Lambda$, strictly finer that the product topology (see [92, p. 124] for more details). Note that $\mathbb{R}^\Lambda$ is not σ-compact and, thus, cannot be σ-Peano. Indeed, suppose that $\mathbb{R}^\Lambda = \bigcup_{n\in\mathbb{N}} K_n$. We may regard $\mathbb{N}$ as a subset of Λ, and consider the standard n-projection $\pi_n : \mathbb{R}^\Lambda \to \mathbb{R}$, which is continuous and, so there is $x_n \in \mathbb{R} \setminus \pi_n(K_n)$. However, the function $f : \Lambda \to \mathbb{R}$ defined by $f(n) = x_n$, for $n \in \mathbb{N}$, and $f(\lambda) = 0$, if $\lambda \notin \mathbb{N}$, does not belong to $\bigcup_n K_n = \mathbb{R}^\Lambda$.

Let Λ, Γ be infinite index sets. Clearly, if $\operatorname{card}\Lambda \geq \operatorname{card}\Gamma$, then $\mathcal{S}(\mathbb{R}^\Lambda, \mathbb{R}^\Gamma) \neq \varnothing$, i.e., the set of surjective maps from from $\mathbb{R}^\Lambda$ onto $\mathbb{R}^\Gamma$ is non-empty. In this situation, Γ may be seen as a subset of Λ. Keeping the notation of the proof of Proposition 4.9, for each $\mathbf{r} = (r_\gamma)_{\gamma\in\Gamma} \in (0, 1]^\Gamma$, define $\Phi_\mathbf{r} : \mathbb{R}^\Lambda \to \mathbb{R}^\Gamma$ by $\Phi_\mathbf{r}(f)(\gamma) := \phi_{r_\gamma}(f(\gamma))$. Since the set of coordinate maps $\{\phi_\gamma := \pi_\gamma \circ \Phi_\mathbf{r}\}_{\gamma\in\Gamma}$ is equicontinuous, $\Phi_\mathbf{r}$ is continuous. Working with the set $\left\{\Phi_\mathbf{r} : \mathbb{R}^\Lambda \to \mathbb{R}^\Gamma\right\}_{\mathbf{r}\in(0,1]^\Gamma}$ and with entire maps as in Section 2, we obtain

Proposition 4.11. *Let* $\operatorname{card}\Lambda \geq \operatorname{card}\Gamma$. *Then*

(a) $\mathcal{CS}\left(\mathbb{R}^\Lambda, \mathbb{R}^\Gamma\right)$ *is* $2^{\operatorname{card}\Gamma}$*-lineable.*

(b) $\mathcal{CS}\left(\mathbb{C}^\Lambda, \mathbb{C}^\Gamma\right)$ *is* $2^{\operatorname{card}\Gamma}$*-algebrable.*

Appendix

Appendix A

The L_p spaces with mixed norm and an interpolative Hölder's inequality

This appendix is devoted for an approach that represents fundamental role to obtain our results concerning the first part of this thesis (the Hardy–Littlewood multilinear type inequalities): Hölder's interpolative inequality. Since there exists a lack of references on this subject, we verse a little bit about the mixed norm spaces introduced in [21] (see also [67]).

A.1 L_p spaces with mixed norm

From now on, (X_i, Σ_i, μ_i), $i = 1, \ldots, m$ shall be m given σ-finite measurable spaces,

$$(\mathbf{X}, \Sigma, \mu) := \left(\prod_{i=1}^{m} X_i, \prod_{i=1}^{m} \Sigma_i, \prod_{i=1}^{m} \mu_i \right)$$

shall be the product space endowed with the product measure and $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$. The space $L_{\mathbf{p}}(\mathbf{X})$ consists in all (equivalence classes of) measurable functions $f : \mathbf{X} \to \mathbb{K}$ with the following property: for any $(x_1, \ldots, x_{m-1}) \in X_1 \times \cdots \times X_{m-1}$ the function $f(x_1, \ldots, x_{m-1}, \cdot)$ belongs to $L_{p_m}(X_m)$, that is,

$$\|f(x_1, \ldots, x_{m-1}, \cdot)\|_{p_m} < \infty,$$

and $\|f\|_{p_m}$ results in a measurable function defined in $\left(\prod_{i=1}^{m-1} X_i, \prod_{i=1}^{m-1} \Sigma_i, \prod_{i=1}^{m-1} \mu_i\right)$; successively, for $1 < k < m$ and for any $(x_1, \ldots, x_{m-k}) \in X_1 \times \cdots \times X_{m-k}$, the measurable function

$$\|f(x_1, \ldots, x_{m-k}, \cdot)\|_{p_{m-k+2}, \ldots, p_m} : X_{m-k+1} \to \mathbb{K}$$

belongs to $L_{p_{m-k+1}}(X_{m-k+1})$, *i.e.*,

$$\|f(x_1, \ldots, x_{m-k}, \cdots)\|_{p_{m-k+1}, \ldots, p_m} := \left\| \|f(x_1, \ldots, x_{m-k}, \cdots)\|_{p_{m-k+2}, \ldots, p_m} \right\|_{p_{m-k+1}} < \infty,$$

and $\|f\|_{p_{m-k+1},\ldots,p_m}$ results in a measurable function defined in $\left(\prod_{i=1}^{k} X_i, \prod_{i=1}^{k} \Sigma_i, \prod_{i=1}^{k} \mu_i\right)$; finally for $k = m$ the measurable function $\|f\|_{p_2,\ldots,p_m} : X_1 \to \mathbb{K}$ belongs to $L_{p_1}(X_1)$, which means that

$$\|f\|_{\mathbf{p}} = \|f\|_{p_1,\ldots,p_m} := \left\| \|f\|_{p_2,\ldots,p_m} \right\|_{p_{m-k+1}} < \infty.$$

For instance, when all $p_i < \infty$ a measurable function $f : \mathbf{X} \to \mathbb{K}$ it is an element of $L_{\mathbf{p}}(\mathbf{X})$ if, and only if,

$$\|f\|_{\mathbf{p}} := \left(\int_{X_1} \left(\ldots \left(\int_{X_m} |f|^{p_m} d\mu_m \right)^{\frac{p_{m-1}}{p_m}} \ldots \right)^{\frac{p_1}{p_2}} d\mu_1 \right)^{\frac{1}{p_1}} < \infty.$$

Successive applications of Minkowski's inequality will allow us to conclude that $\|\cdot\|_{\mathbf{p}}$ defines a norm on $L_{\mathbf{p}}(\mathbf{X})$. Indeed, let $f, g \in L_{\mathbf{p}}(\mathbf{X})$. Applying Minkowski's inequality with the p_m-norm of $f+g$, f and g (evaluated in a fixed vector $(x_1, \ldots, x_{m-1}) \in X_1 \times \cdots \times X_{m-1}$ that we will omit for the sake of clarity), we get

$$\|f+g\|_{p_m} \leq \|f\|_{p_m} + \|g\|_{p_m}.$$

By hypothesis, $\|f\|_{p_m} + \|g\|_{p_m}, \|f\|_{p_m}, \|g\|_{p_m}$ are measurable functions on $L_{p_{m-1}}(X_{m-1})$, thus the monotonicity and the Minkowski inequality now with the p_{m-1}-norm lead us to

$$\begin{aligned}
\|f+g\|_{p_{m-1},p_m} &:= \left\| \|f+g\|_{p_m} \right\|_{p_{m-1}}, && \text{monotonicity} \\
&\leq \left\| \|f\|_{p_m} + \|g\|_{p_m} \right\|_{p_{m-1}}, && \text{Minkowski} \\
&\leq \left\| \|f\|_{p_m} \right\|_{p_{m-1}} + \left\| \|g\|_{p_m} \right\|_{p_{m-1}} \\
&=: \|f\|_{p_{m-1},p_m} + \|g\|_{p_{m-1},p_m}.
\end{aligned}$$

Thereby, making use of Minkowski's inequality successively we conclude that

$$\|f+g\|_{\mathbf{p}} \leq \|f\|_{\mathbf{p}} + \|g\|_{\mathbf{p}},$$

and, therefore, that $L_{\mathbf{p}}(\mathbf{X})$ is a normed vector space.

A. Benedek and R. Panzone, on [21], established several properties and deep results concerning the $L_{\mathbf{p}}$ spaces, among others, the correspondent of the Monotone and Lebesgue's dominated convergence classical theorems, and, as expected, they also proved that $L_{\mathbf{p}}(\mathbf{X})$ is a Banach space (see [21, Theorem 1.b]).

We are interested on a version of Hölder's inequality for these mixed norm spaces, which implies an interpolative inequality which in turn has a crucial role in our results concerning the Hardy–Littlewood multilinear inequality. First, let us recall that given two functions $f, g \in L_{\mathbf{p}}(\mathbf{X})$, the *product* function $fg : \mathbf{X} \to \mathbb{K}$ it is defined by the *pointwise product*, that is,

$$fg(x_1, \ldots, x_m) := f(x_1, \ldots, x_m) \cdot g(x_1, \ldots, x_m), \tag{A.1}$$

for each $(x_1, \ldots, x_m) \in \mathbf{X} = X_1 \times \cdots \times X_m$. Also let us recall the classical one variable version of the Hölder inequality.

Theorem A.1 (Classical Hölder's inequality [1]). *Let (X, Σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$ (Hölder's conjugates exponents). Then, for all measurable (real or complex valued) functions $f \in L_p(X)$ and $g \in L_q(X)$, we have that $fg \in L_1(X)$. Moreover,*

$$\int_X |fg| d\mu =: \|fg\|_1 \leq \|f\|_p \cdot \|g\|_q. \tag{A.2}$$

A more general version is also very useful: let $r \in [1, \infty)$, $p_1, \ldots, p_N \in [1, \infty]$ such that $\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$. If $f_i \in L_{p_i}(X)$, $i = 1, \ldots, N$, then $f_1 \cdots f_N \in L_r(X)$ and, moreover,

$$\|f_1 \cdots f_N\|_r \leq \|f_1\|_{p_1} \cdot \|f_N\|_{p_N}. \tag{A.3}$$

By making use of induction on N, this is a straightforward consequence of the classical version. First we prove the result for $N = 2$: since $1 = r/p_1 + r/p_2$, we have

$$\int_X |f_1 f_2|^r d\mu =: \||f_1 f_2|^r\|_1 \leq \|f_1^r\|_{p_1/r} \cdot \|f_2^r\|_{p_2/r} = \|f_1\|_{p_1}^r \cdot \|f_2\|_{p_2}^r,$$

and from this we conclude that $\|f_1 f_2\|_r \leq \|f_1\|_{p_1} \cdot \|f_2\|_{p_2}$. Now let us suppose that the result is true for $N - 1$. If $p_N = \infty$, then then result follows by $|f_N| \leq \|f_N\|_\infty$ and the induction hypothesis:

$$\|f_1 \cdots f_N\|_r \leq \|f_1 \cdots f_{N-1}\|_r \cdot \|f_N\|_\infty \leq \|f_1\|_{p_1} \cdots \|f_{N-1}\|_{p_{N-1}} \cdot \|f_N\|_{p_N}.$$

If $p_N < \infty$, then note that $p := p_N/(p_N - r)$ and $q := p_N/r$ are Hölder's conjugates exponents in $(1, \infty)$, thus applying Hölder's inequality with the exponents $1/r = 1/rp + 1/rq$ and then the induction hypothesis with $\frac{1}{rp} = \left(\frac{1}{p_1} + \cdots + \frac{1}{p_{N-1}}\right)$, we conclude the result (note that $rq = p_N$)

$$\|(f_1 \cdots f_{N-1}) \cdot f_N\|_r \leq \|f_1 \cdots f_{N-1}\|_{rp} \cdot \|f_N\|_{rq} \leq \|f_1\|_{p_1} \cdots \|f_{N-1}\|_{p_{N-1}} \cdot \|f_N\|_{p_N}.$$

The corresponding result on multiple variables with mixed norm is the following.

Theorem A.2 (General mixed Hölder's inequality). *Let $\mathbf{r} \in [1, \infty)^m$ and $, \mathbf{p}(1), \ldots, \mathbf{p}(N) \in [1, \infty]^m$ be such that*

$$\frac{1}{r_j} = \frac{1}{p_j(1)} + \cdots + \frac{1}{p_j(N)}, \qquad \text{for } j = 1, \ldots, m.$$

If $f_k \in L_{\mathbf{p}(k)}(\mathbf{X})$ for $k = 1, \ldots, N$, then $f_1 f_2 \cdots f_N \in L_{\mathbf{r}}(\mathbf{X})$ and, moreover,

$$\|f_1 \cdots f_N\|_{\mathbf{r}} \leq \|f_1\|_{\mathbf{p}(1)} \cdots \|f_N\|_{\mathbf{p}(N)}. \tag{A.4}$$

[1] Leonard James Rogers (1862-1933) and Otto Hölder (1859-1937) discovered, independently, the famous inequality that (nowadays) holds Hölder's name (1889, [79]), they could have never imagined that, at that precise moment, they had just started a "revolution" in Functional Analysis.

Proof. The result follows by iterating each single-variable case and making use of the general version of Hölder's inequality (A.3). Indeed, since each measurable map f_k (evaluated in a fixed vector $(x_1, \ldots, x_{m-1}) \in X_1 \times \cdots \times X_{m-1}$ that we will omit) belongs to $L_{p_m(k)}(X_m)$ and since that $\frac{1}{r_m} = \frac{1}{p_m(1)} + \cdots + \frac{1}{p_m(N)}$, by the inequality (A.3) we get that $f_1 \cdots f_N \in L_{r_m}(X_m)$ and

$$\|f_1 \cdots f_N\|_{r_m} \leq \|f_1\|_{p_m(1)} \cdots \|f_N\|_{p_m(N)}.$$

Now since $\|f_1\|_{p_m(1)}, \ldots, \|f_N\|_{p_m(N)}$ are measurable functions in $L_{p_{m-1}(k)}(X_{m-1})$, the monotonicity of the r_{m-1}-norm and general Hölder's inequality (A.3) with $\frac{1}{r_{m-1}} = \frac{1}{p_{m-1}(1)} + \cdots + \frac{1}{p_{m-1}(N)}$ lead us to

$$\begin{aligned}
\|f_1 \cdots f_N\|_{r_{m-1},r_m} &:= \left\| \|f_1 \cdots f_N\|_{r_m} \right\|_{r_{m-1}}, && \text{monotonicity} \\
&\leq \left\| \|f_1\|_{p_m(1)} \cdots \|f_N\|_{p_m(N)} \right\|_{r_{m-1}}, && \text{Hölder} \\
&\leq \left\| \|f_1\|_{p_m(1)} \right\|_{p_{m-1}(1)} \cdots \left\| \|f_N\|_{p_m(1)} \right\|_{p_{m-1}(N)} \\
&=: \|f_1\|_{p_{m-1}(1),p_m(1)} \cdots \|f_N\|_{p_{m-1}(N),p_m(N)}.
\end{aligned}$$

Repeating this argument successively on each variable $x_{m-2}, \ldots, x_1$, we conclude the result. □

The next interpolation result it is a general version for $L_{\mathbf{p}}$ spaces of the interpolative approach we used for sequences spaces in the papers [5, 6].

Corollary A.3 (Mixed interpolative Hölder's inequality)**.** *Let* $\mathbf{r}, \mathbf{p}(1), \ldots, \mathbf{p}(N) \in [1, \infty]^m$ *and* $\theta_1, \ldots, \theta_N \in [0, 1]$ *be such that* $\theta_1 + \cdots + \theta_N = 1$ *and*

$$\frac{1}{r_j} = \sum_{k=1}^{N} \frac{\theta_k}{p_j(k)} = \frac{\theta_1}{p_j(1)} + \cdots + \frac{\theta_N}{p_j(N)}, \quad \text{for } j = 1, \ldots, m.$$

If $f \in L_{\mathbf{p}(k)}(X)$ *for* $k = 1, \ldots, N$*, then* $f \in L_{\mathbf{r}}(X)$ *and, moreover,*

$$\|f\|_{\mathbf{r}} \leq \|f\|_{\mathbf{p}(1)}^{\theta_1} \cdots \|f\|_{\mathbf{p}(N)}^{\theta_N}.$$

Proof. This follow straightforward from the previous result combined with the following fact: for a real positive number $\theta \in (0, 1]$, $\mathbf{p} \in [1, \infty]^m$ and $\mathbf{p}/\theta := (p_1/\theta, \ldots, p_m/\theta)$, it is straightforward verify that

$$\left\| |f|^\theta \right\|_{\mathbf{p}/\theta} = \|f\|_{\mathbf{p}}^\theta.$$

Since we have that, for $j = 1, \ldots, m$,

$$\frac{1}{r_j} = \frac{1}{p_j(1)/\theta_1} + \cdots + \frac{1}{p_j(N)/\theta_N},$$

the inequality (A.4) will lead us to conclude the result

$$\|f\|_{\mathbf{r}} = \left\| |f|^{\theta_1} \cdots |f|^{\theta_N} \right\|_{\mathbf{r}} \leq \left\| |f|^{\theta_1} \right\|_{\mathbf{p}(1)/\theta_1} \cdots \left\| |f|^{\theta_N} \right\|_{\mathbf{p}(N)/\theta_N} = \|f\|_{\mathbf{p}(1)}^{\theta_1} \cdots \|f\|_{\mathbf{p}(N)}^{\theta_N}.$$

$\square$

A.2 Sequences spaces with mixed norm

We shall now work with mixed normed sequence spaces, once it is the case we are interested in. We may define these spaces by considering on $\mathbb{N}$ the power set $\mathcal{P}(\mathbb{N})$ as being the sigma algebra and the counting measure $\mu_c(A)$ defined as the cardinality of each subset A of $\mathbb{N}$. In this fashion, $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$ becomes a measurable space and, for each $p \in [1, \infty]$, $\ell_p = L_p(\mathbb{N})$ gathers all sequences $(a_n) \in \mathbb{K}^{\mathbb{N}}$ with finite p-norm.

Now let us consider $\mathbf{p} \in [1, \infty]^m$. Using the definitions of mixed norm $L_{\mathbf{p}}$ spaces of the preceding section, the mixed norm sequence space

$$\ell_{\mathbf{p}} := L_{\mathbf{p}}\left(\mathbb{N}^m\right)$$

gathers all multi-index scalars valued matrices $\mathbf{a} := (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,\mathbb{N})}$ with finite $\mathbf{p}$-norm (recall the multi-index notation established: $\mathcal{M}(m, \mathbb{N}) := \mathbb{N}^m$). For instance, when $\mathbf{p} \in [1, \infty)^m$ a scalar matrix $\mathbf{a}$ belongs to $\ell_{\mathbf{p}}$ if, and only if,

$$\|\mathbf{a}\|_{\mathbf{p}} := \left(\sum_{i_1=1}^{\infty}\left(\sum_{i_2=1}^{\infty}\left(\cdots\left(\sum_{i_{m-1}=1}^{\infty}\left(\sum_{i_m=1}^{\infty}|a_{\mathbf{i}}|^{p_m}\right)^{\frac{p_{m-1}}{p_m}}\right)^{\frac{p_{m-2}}{p_{m-1}}}\cdots\right)^{\frac{p_2}{p_3}}\right)^{\frac{p_1}{p_2}}\right)^{\frac{1}{p_1}} < \infty.$$

Therewith, the mixed norm sequence space

$$\ell_{\mathbf{p}}(Z) := \ell_{p_1}\left(\ell_{p_2}\left(\ldots\left(\ell_{p_m}(Z)\right)\ldots\right)\right),$$

with Z a Banach space, introduced on Section 1.2, coincides with the class built previously since, for $Z = \mathbb{K}$,

$$\ell_{\mathbf{p}}(\mathbb{K}) = L_{\mathbf{p}}(\mathbb{N}^m) =: \ell_{\mathbf{p}}.$$

We will close this appendix with the version for mixed norm sequences spaces of the mixed Hölder inequality and its interpolation consequence (Theorem A.2 and Corollary A.3, respectively). Before that, note that the product of two multi-indexes scalar matrices $\mathbf{a} := (\mathrm{a}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$ and $\mathbf{b} := (\mathrm{b}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$ (recall that $\mathcal{M}(m,n) := \{1, 2, \ldots, n\}^m$) defined in (A.1) turns in *pointwise product*, that is,

$$\mathbf{ab} := (\mathrm{a}_{\mathbf{i}}\mathrm{b}_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}.$$

Therefore, the versions of Theorem A.2 and Corollary A.3 for mixed $\ell_{\mathbf{p}}$ spaces turns in

Theorem A.4 (Hölder's inequality for mixed $\ell_{\mathbf{p}}$ spaces)**.** *Let m, n, N be positive integers,* $\mathbf{r} \in$

$[1,\infty)^m$ *and* , $\mathbf{p}(1),\ldots,\mathbf{p}(N)\in[1,\infty]^m$ *be such that*

$$\frac{1}{r_j}=\frac{1}{p_j(1)}+\cdots+\frac{1}{p_j(N)},\qquad \text{for } j=1,\ldots,m,$$

and also let $\mathbf{a}(k):=(\mathrm{a}(k)_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$, $k=1,\ldots,N$ *be scalar matrices. Then,*

$$\|\mathbf{a}_1\cdots\mathbf{a}_N\|_{\mathbf{r}}\leq\|\mathbf{a}_1\|_{\mathbf{p}(1)}\cdots\|\mathbf{a}_N\|_{\mathbf{p}(N)}\,.$$

In particular, if each $\mathbf{p}(k)\in[1,\infty)$*, then we have*

$$\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|\mathrm{a}(1)_{\mathbf{i}}\cdots\mathrm{a}(N)_{\mathbf{i}}|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}$$
$$\leq\prod_{k=1}^{N}\left[\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|\mathrm{a}(k)_{\mathbf{i}}|^{p_m(k)}\right)^{\frac{p_{m-1}(k)}{p_m(k)}}\cdots\right)^{\frac{p_1(k)}{p_2(k)}}\right)^{\frac{1}{p_1(k)}}\right].$$

Corollary A.5 (Hölder's interpolative inequality for mixed $\ell_{\mathbf{p}}$ spaces)**.** *Let* m,n,N *be positive integers,* $\mathbf{r},\mathbf{p}(1),\ldots,\mathbf{p}(N)\in[1,\infty]^m$ *and* $\theta_1,\ldots,\theta_N\in[0,1]$ *be such that* $\theta_1+\cdots+\theta_N=1$ *and*

$$\frac{1}{r_j}=\sum_{k=1}^{N}\frac{\theta_k}{p_j(k)}=\frac{\theta_1}{p_j(1)}+\cdots+\frac{\theta_N}{p_j(N)},\qquad \text{for } j=1,\ldots,m.$$

Then, for all scalar matrix $\mathbf{a}:=(\mathrm{a}_{\mathbf{i}})_{\mathbf{i}\in\mathcal{M}(m,n)}$*, we have*

$$\|\mathbf{a}\|_{\mathbf{r}}\leq\|\mathbf{a}\|_{\mathbf{p}(1)}^{\theta_1}\cdots\|\mathbf{a}\|_{\mathbf{p}(N)}^{\theta_N}\,.$$

In particular, if each $\mathbf{p}(k)\in[1,\infty)$*, the previous inequality means that*

$$\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|\mathrm{a}_{\mathbf{i}}|^{r_m}\right)^{\frac{r_{m-1}}{r_m}}\cdots\right)^{\frac{r_1}{r_2}}\right)^{\frac{1}{r_1}}$$
$$\leq\prod_{k=1}^{N}\left[\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}|\mathrm{a}_{\mathbf{i}}|^{p_m(k)}\right)^{\frac{p_{m-1}(k)}{p_m(k)}}\cdots\right)^{\frac{p_1(k)}{p_2(k)}}\right)^{\frac{1}{p_1(k)}}\right]^{\theta_k}.$$

Under the point of view of interpolation theory it is not a complicated result but, just in march of 2014, the authors of [6] known the article [21] which bring more acessible techniques to prove the previous results. But for the sake of completeness of this article, we would also like to present the following proof of the previous result, which is based on interpolation theory presented in [24].

2nd proof (using interpolative approach). We just follow the lines of [5, Proposition 2.1]. Proceeding by induction on N and using that, for any Banach space Z and $\theta \in [0,1]$,

$$\ell_{\mathbf{r}}(Z) = [\ell_{\mathbf{p}}(Z), \ell_{\mathbf{q}}(Z)]_{\theta},$$

with $\frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}$, for $i = 1, \ldots, m$ (see, for instance, [24, Theorems 5.1.1 and 5.1.2]). If

$$\frac{1}{q_i} = \frac{\theta_1}{q_i(1)} + \cdots + \frac{\theta_N}{q_i(N)},$$

with $\sum_{k=1}^{N} \theta_k = 1$ and each $\theta_k \in [0,1]$, then we also have

$$\frac{1}{q_i} = \frac{\theta_1}{q_i(1)} + \frac{1-\theta_1}{p_i},$$

setting

$$\frac{1}{p_i} = \frac{\alpha_2}{q_i(2)} + \cdots + \frac{\alpha_N}{q_i(N)}, \quad \text{and } \alpha_j = \frac{\theta_j}{1-\theta_1},$$

for $i = 1, \ldots, m$ and $j = 2, \ldots, N$. So $\alpha_j \in [0,1]$ and $\sum_{j=2}^{N} \alpha_j = 1$. Therefore, by the induction hypothesis, we conclude that

$$\|\mathbf{a}\|_{\mathbf{q}} \leq \|\mathbf{a}\|_{\mathbf{q}(1)}^{\theta_1} \cdot \|\mathbf{a}\|_{\mathbf{p}}^{1-\theta_1} \leq \|\mathbf{a}\|_{\mathbf{q}(1)}^{\theta_1} \cdot \left[\prod_{j=2}^{N} \|\mathbf{a}\|_{\mathbf{q}(j)}^{\alpha_j}\right]^{1-\theta_1} = \prod_{k=1}^{N} \|\mathbf{a}\|_{\mathbf{q}(k)}^{\theta_k}.$$

$\square$

Appendix B

Kahane–Salem–Zygmund's inequality

The essence of the Kahane–Salem–Zygmund inequalities, as we describe below, probably appeared for the first time in [82], but our approach follows the lines of Boas' paper [31]. Paraphrasing Boas, the Kahane–Salem–Zygmund inequalities use probabilistic methods to construct a homogeneous polynomial (or multilinear operator) with a relatively small supremum norm but relatively large majorant function. Both the multilinear and polynomial versions are needed for our goals.

In this appendix we present and prove Lemma 1.14. Recall that ℓ_p^n stands for the complex space $\mathbb{C}^n$ with the p-norm, $p \in [1, \infty]$ (the same argument with the same constants will provide the result for real scalars). We will need following results.

Chebyshev–Markov's Inequallity. *Let (Ω, Σ, μ) be a measure space. For every measurable numerical functions f, g on Ω, with g nonnegative and nondecreasing on the range of f, and every positive real number α,*

$$\mu\left([f \geq \alpha]\right) \leq \frac{1}{g(\alpha)} \int_\Omega g \circ f \, d\mu$$

holds. In particular,

$$\mu\left([|f| \geq \alpha]\right) \leq \frac{1}{\alpha^p} \int_\Omega |f|^p \, d\mu,$$

holds for every real $p > 0$.

Proof. (see [18, Lemma 20.1]). Let $A = \{x \in \Omega\, ;\, f(x) \geq \alpha\}$ and $B = \{x \in \Omega\, ;\, (g \circ f)(x) \geq g(\alpha)\}$. Since g is nondecreasing on the range of f, $A \subset B$ and, consequently, $\mu(A) \leq \mu(B)$. Thus, the result is concluded using monotonicity of integration with respect to μ and the following inequality:

$$\int_\Omega g \circ f \, d\mu \geq \int_{[g \circ f \geq \alpha]} g \circ f \, d\mu \geq \int_{[g \circ f \geq \alpha]} g(\alpha) \, d\mu = g(\alpha) \int_{[g \circ f \geq \alpha]} d\mu = g(\alpha)\mu(B) \geq g(\alpha)\mu(A).$$

The final assertion is obtained defining the map $g(t) \overset{\text{def}}{=} |t|^p$. $\square$

Lemma B.1 (Covering argument)**.** *Let r be a positive real number. Then the unit open ball B of ℓ_p^n can be covered by a collection of open ℓ_p^n balls of radius r, with the numbers of balls in the* *collection not exceeding* $\left(1+2r^{-1}\right)^{2n}$, *and the centers of the balls lying in the closed unit ball* $\overline{B}$ *of* ℓ_p^n.

Proof. Place arbitrarily in the open ball $B\left(1+\frac{r}{2}\right) \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n\,;\, \|z\|_p < 1+r/2\}$ a collection $\mathcal{B}$ of disjoint open ℓ_p^n balls $B_k \stackrel{\text{def}}{=} B_n^p\left(x_k, r/2\right)$ of radius $r/2$ (obviously the centers x_k lies in the closed unit ball $\overline{B}$). The (Euclidean) volume of a ℓ_p^n ball of radius $r>0$ is

$$V_{2n}^p(R) = \frac{\left(2R\cdot\Gamma\left(1+1/p\right)\right)^{2n}}{\Gamma\left(1+2n/p\right)}.$$

Comparing the volumes of $B\left(1+\frac{r}{2}\right)$ and B_k,

$$\frac{V_{2n}^p(1+r/2)}{V_{2n}^p(r/2)} = \left(\frac{1+r/2}{r/2}\right)^{2n} = \left(1+2r^{-1}\right)^{2n},$$

we conclude that the number of disjoint balls cannot exceed $\left(1+2r^{-1}\right)^{2n}$. If the collection $\mathcal{B}$ if made maximal, then every point of $\overline{B}$ must lies within $r/2$ of some point of one ball in $\mathcal{B}$ (otherwise we would get some ball of radious $r/2$ in $B\left(1+\frac{r}{2}\right)$, being not in $\mathcal{B}$). So the balls $B_k' \stackrel{\text{def}}{=} B_n^p\left(x_k, r\right),\, k=1,\ldots,\left(1+2r^{-1}\right)^{2n}$ must cover $\overline{B}$. □

H. P. Boas presented in [31, Theorem 4] the following

Multilinear Kahane-Salem-Zygmund's inequality (by Boas). *Let* $p \in [1,p]$, *and integers* $n, d \geq 1$. *Then exists a symmetric multi-linear map* $F : \left(\ell_p^n\right)^d \to \mathbb{C}$ *of the form*

$$F\left(z_1,\ldots,z_d\right) = \sum_{j_1,\ldots,j_d=1}^{n} \pm z_{j_1}^1 \cdots z_{j_d}^d,$$

such that the supremum of $\left|F\left(z_1,\ldots,z_d\right)\right|$ *when every* n*-vector* $z_k \in \ell_p^n$ *lies in the unit ball of* ℓ_p^n *is at most*

$$\sqrt{32d\log\left(6d\right)}\cdot\begin{cases} n^{\frac{1}{2}}\left(d!\right)^{\left(1-\frac{1}{p}\right)}, & \text{if } 1 \leq p < 2\,; \\ n^{\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{p}\right)d}, & \text{if } 2 \leq p \leq \infty. \end{cases}$$

When all the vectors z_k are equal, the theorem provides a special d-homogeneous polynomial. Observe that the sum that composes F has n^d monomials of degree d, and some of these are repeated. Indeed, if α_k denotes the numbers of times the integer k appears in the d-tuple $\left(j_1,\ldots,j_d\right)$, then the number of d-tuples that are permutations of the list is the multinomial coefficient $\binom{d}{\alpha}$, with $\alpha = \left(\alpha_1,\ldots,\alpha_n\right)$. Consequently, we have the following consequence of the previous result.

Corollary B.2. *Let* $p \in [1,p]$, *and integers* $n, d \geq 1$. *Then there exists a homogeneous polynomial of degree* d *in the variable* z *in* $\mathbb{C}^n$ *of the form*

$$\sum_{|\alpha|=d} \pm \binom{d}{\alpha} z^\alpha$$

such that the supremum of the modulus of the polynomial when z lies in the unit ball of ℓ_p^n is no greater than the same bound of the previous theorem.

The following is precisely Lemma 1.14.

Multilinear Kahane-Salem-Zygmund's inequality. *Let $d, n \geq 1$, $p_1, \ldots, p_d \in [1, +\infty]^d$ and, for $p \geq 1$, define*

$$\alpha(p) \stackrel{def}{=} \begin{cases} \dfrac{1}{2} - \dfrac{1}{p}, & \text{if } p \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Then there exists a d-linear map $A : \ell_{p_1}^n \times \cdots \times \ell_{p_d}^n \to \mathbb{K}$ of the form

$$A(z_1, \ldots, z_d) = \sum_{j_1, \ldots, j_d = 1}^{n} \pm z_{j_1}^1 \cdots z_{j_d}^d,$$

such that

$$\|A\| \leq C_d \cdot n^{\frac{1}{2} + \alpha(p_1) + \cdots + \alpha(p_d)}.$$

where $C_d = (d!)^{1 - \frac{1}{m(p)}} \sqrt{32 d \log(6d)}$, and $p = \max\{p_1, \ldots, p_k\}$.

Proof. Let's stablish the following notation: z denotes a d-tuple $z_1, \ldots, z_d$ of n-vectors, $\mathbf{j}$ denotes a d-tuple $j_1, \ldots, j_d$ of integers between 1 and n, sums and products run over all such d-tuples of integers, a prime on a sums or product means that it is restricted to d-tuples of integers that are arranged in non-decreasing order, and $\mathbf{j} \sim \mathbf{k}$ indicates that the d-tuples $\mathbf{j}$ and $\mathbf{k}$ are permutations of each other. The symbol $z_{\mathbf{j}}$ is shorthand for the monomial $z_{j_1}^1 \cdots z_{j_d}^d$. Thus, fixed a multi-linear map like in the statement, it can be written in the form

$$A(z) = \sum_{\mathbf{k}}{}' \left(\pm \sum_{\mathbf{j} \sim \mathbf{k}} z_{\mathbf{j}} \right),$$

and in this expression, all of the plus and minus signs are independent of each other.

The proof consists of a probabilistic estimate and a covering argument.

To begin the probabilistic argument, fix a point $z \in (\mathbb{C}^d)$ such each z_k lies in the ℓ_p^n unit ball B_p^n. For each d-tuple $\mathbf{k}$ in non-decreasing order, choose a different Rademacher function $r_{\mathbf{k}}$, and consider the random sum

$$A(t, z) = \sum_{\mathbf{k}}{}' \left(r_{\mathbf{k}}(t) \sum_{\mathbf{j} \sim \mathbf{k}} z_{\mathbf{j}} \right),$$

where t lies in the interval $I \stackrel{\text{def}}{=} [0, 1]$. The immediate goal is to make an upper estimate on the probability that this sum has large modulus. Let λ be an arbitrary positive number (after we will specify a value for λ in terms of n, d and p). Invoking the independence of the Rademacher functions, we may compute the expectation (that is, the integral with respect to t) of the exponential of the real part of $\lambda A(t, z)$, by computing the product over non-decreasing d-tuples

$\mathbf{k}$ of the expectation of $e^{\left(\lambda r_{\mathbf{k}}(t)\sum_{\mathbf{j}\sim\mathbf{k}} z_{\mathbf{j}}\right)}$. More precisely:

$$\operatorname{Re}\left[\lambda A\left(t,z\right)\right]=\sum_{\mathbf{k}}{}^{\prime}\left(\lambda r_{\mathbf{k}}(t)\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}z_{\mathbf{j}}\right)\Rightarrow e^{\operatorname{Re}\left[\lambda A(t,z)\right]}=\prod_{\mathbf{k}}{}^{\prime}e^{\left(\lambda r_{\mathbf{k}}(t)\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}z_{\mathbf{j}}\right)};$$

denoting $\alpha_{\mathbf{k}}=\lambda\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}z_{\mathbf{j}}$, using the independence and orthogonality properties of Rademacher functions, and

$$\int_I e^{\beta r_i(t)}\,dt=\int_I\left(\sum_{m=1}^{+\infty}\frac{\beta^m r_i^m(t)}{m!}\right)dt=\sum_{m=1}^{+\infty}\frac{\beta^{2m}}{(2m)!}=\cosh\left(\beta\right)$$

we get

$$\int_I e^{\operatorname{Re}\left[\lambda A(t,z)\right]}\,dt=\int_I\prod_{\mathbf{k}}{}^{\prime}e^{\alpha_{\mathbf{k}}r_{\mathbf{k}}(t)}\,dt=\prod_{\mathbf{k}}{}^{\prime}\int_I e^{\alpha_{\mathbf{k}}r_{\mathbf{k}}(t)}\,dt=\prod_{\mathbf{k}}{}^{\prime}\cosh\left(\lambda\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}z_{\mathbf{j}}\right).$$

In view of the inequality $\cosh x\leq e^{\frac{x^2}{2}}$ (that follows by $k!\cdot 2^k\leq k!\cdot(k+1)\cdots(2k)=(2k)!$ and series comparison argument), we get a upper bound for the expectation of $\operatorname{Re}\left[\lambda A\left(t,z\right)\right]$:

$$\int_I e^{\operatorname{Re}\left[\lambda A(t,z)\right]}\,dt\leq\prod_{\mathbf{k}}{}^{\prime}e^{\frac{1}{2}\left(\lambda\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}z_{\mathbf{j}}\right)^2}=e^{\frac{1}{2}\lambda^2\sum_{\mathbf{k}}^{\prime}\left(\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}\,z_{\mathbf{j}}\right)^2}.\tag{B.1}$$

Define $p\overset{\text{def}}{=}\max\{p_1,\ldots,p_k\}$, $m(t)\overset{\text{def}}{=}\min(t,2)$ and $M(t)\overset{\text{def}}{=}\min(t,2)$, for a real positive t. Hölder's inequality assures

$$\begin{aligned}\left(\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}\,z_{\mathbf{j}}\right)^2=\left|\sum_{\mathbf{j}\sim\mathbf{k}}\operatorname{Re}\,z_{\mathbf{j}}\right|^2\leq\left(\sum_{\mathbf{j}\sim\mathbf{k}}\left|\operatorname{Re}\,z_{\mathbf{j}}\right|\right)^2&\\ \leq\left(\sum_{\mathbf{j}\sim\mathbf{k}}|z_{\mathbf{j}}|\right)^2\overset{\text{Hölder}}{\leq}(d!)^{2\left(1-\frac{1}{m(p)}\right)}\cdot\left(\sum_{\mathbf{j}\sim\mathbf{k}}|z_{\mathbf{j}}|^{m(p)}\right)^{\frac{2}{m(p)}}&.\end{aligned}\tag{B.2}$$

The exponent $2/m(p)$ is equal to 1 when $p\in[2,\infty]$. When $p\in[1,2]$, it is equal to $2/p$ and belongs to $[1,2]$, so replacing it by 1 can only increase the right-hand side, since $\sum_{\mathbf{j}\sim\mathbf{k}}|z_{\mathbf{j}}|^p\leq 1$, wich is obtained from the following argument: each n-vector z_k lies in $B^n_{p_k}$, so $\sum_{j=1}^n\left|z_j^k\right|^{p_k}\leq 1$ and, using the fact that "$r\leq s\Rightarrow\|\cdot\|_{\ell_s^n}\leq\|\cdot\|_{\ell_r^n}$ *(p-norm is nondecreasing)*",

$$\begin{aligned}\sum_{\mathbf{j}\sim\mathbf{k}}|z_{\mathbf{j}}|^p=\sum_{\mathbf{j}=(j_1,\ldots,j_d)\sim\mathbf{k}}\left|z^1_{j_1}\cdots z^d_{j_d}\right|^p\leq\sum_{j_1,\ldots,j_d=1}^n\left|z^1_{j_1}\cdots z^d_{j_d}\right|^p=\prod_{k=1}^d\left(\sum_{j_k=1}^n\left|z^k_{j_k}\right|^p\right)&\\ =\left(\|z_1\|_{\ell_p^n}\cdots\|z_d\|_{\ell_p^n}\right)^p\leq\left(\|z_1\|_{\ell_{p_1}^n}\cdots\|z_d\|_{\ell_{p_d}^n}\right)^p\leq 1.&\end{aligned}\tag{B.3}$$

Inequalities (B.2) and (B.3), lead us to conclude, for both cases $1 \le p \le 2$ and $2 \le p \le \infty$, that

$$\begin{aligned}
\sum_{\mathbf{k}}{}' \left(\sum_{\mathbf{j}\sim\mathbf{k}} \operatorname{Re} z_{\mathbf{j}}\right)^2 &\le (d!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot \sum_{\mathbf{k}}{}' \left(\sum_{\mathbf{j}\sim\mathbf{k}} |z_{\mathbf{j}}|^{m(p)}\right)^{\frac{2}{m(p)}} \\
&\le (d!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot \sum_{\mathbf{k}}{}' \sum_{\mathbf{j}\sim\mathbf{k}} |z_{\mathbf{j}}|^{m(p)} \\
&= (d!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot \sum_{j_1,\ldots,j_d=1}^{n} \left|z^1_{j_1}\cdots z^d_{j_d}\right|^{m(p)} \\
&= (d!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot \left(\sum_{j_1=1}^{n} \left|z^1_{j_1}\right|^{m(p)}\right)\cdots\left(\sum_{j_d=1}^{n} \left|z^d_{j_d}\right|^{m(p)}\right) \\
&= (d!)^{2\left(1-\frac{1}{m(p)}\right)} \|z_1\|^{m(p)}_{\ell^n_{m(p)}} \cdots \|z_d\|^{m(p)}_{\ell^n_{m(p)}}.
\end{aligned}$$

Using the following facts: $m(p_k) \le m(p)$ and $\left|z^k_j\right| \le \|z_k\|_{\ell^n_{p_k}} \le 1 \Rightarrow \left|z^k_j\right|^{m(p)} \le \left|z^k_j\right|^{m(p_k)}$; $m(p_k)\cdot M(p_k)/2 = p_k$; and once more applying Hölder's inequallity; we increase further the bound:

$$\begin{aligned}
\|z_k\|^{m(p)}_{\ell^n_{m(p)}} = \sum_{j=1}^{n} \left|z^k_j\right|^{m(p)} \le \sum_{j=1}^{n} \left|z^k_j\right|^{m(p_k)} &\le n^{1-\frac{2}{M(p_k)}} \cdot \left(\sum_{j=1}^{n} \left|z^k_j\right|^{\frac{m(p_k)M(p_k)}{2}}\right)^{\frac{2}{M(p_k)}} \\
= n^{1-\frac{2}{M(p_k)}} \cdot \left(\sum_{j=1}^{n} \left|z^k_j\right|^{p_k}\right)^{\frac{2}{M(p_k)}} &= n^{1-\frac{2}{M(p_k)}} \cdot \|z_k\|^{\frac{2p_k}{M(p_k)}}_{\ell^n_{p_k}} \le n^{1-\frac{2}{M(p_k)}}. \qquad \text{(B.4)}
\end{aligned}$$

Therefore, (B.1) is bounded above by

$$\int_I e^{\lambda \operatorname{Re}[A(t,z)]}\, dt \le e^{\frac{1}{2}\lambda^2 \sum'_{\mathbf{k}}\left(\sum_{\mathbf{j}\sim\mathbf{k}} \operatorname{Re} z_{\mathbf{j}}\right)^2} \le \exp\left[\frac{1}{2}\lambda^2 (d!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}\right]. \qquad \text{(B.5)}$$

Let R be an arbitrary positive real number (it will be specified a value for R in terms of n, d and p). Using the previous upper bound and applying Chebyshev–Markov inequality with $g(t) \overset{\text{def}}{=} e^t$, we obtain a upper bound for the measure of the set $A \overset{\text{def}}{=} \{t \in I\,;\, \operatorname{Re}[A(t,z)] \ge R\} = \{t \in I\,;\, \lambda\operatorname{Re}[A(t,z)] \ge \lambda R\}$:

$$\mu(A) \le \frac{1}{e^{\lambda R}} \int_I e^{\lambda \operatorname{Re}[A(t,z)]}\, dt \le \exp\left[-R\lambda + \frac{1}{2}\lambda^2 (d!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}\right]. \qquad \text{(B.6)}$$

An similar argument (symmetric reasoning) gives the same estimate for the measure of the points $t \in I$ that $\operatorname{Re}[A(t,z)]$ is less than $-R$, wich is the same set of points that $(-\lambda)\operatorname{Re}[A(t,z)] \ge \lambda R$ (we may initiate the previous argument working with $-\lambda\operatorname{Re}[A(t,z)]$ and obtain the inequality (B.5) with $-\lambda$). Since $[|\operatorname{Re}[A(t,z)]| \ge R] = [\operatorname{Re}[A(t,z)] \ge R] \cup [\operatorname{Re}[A(t,z)] \le -R]$, we get that this set has measure at most 2 times the bound in (B.6). The same argument applies to the imaginary part of $A(t,z)$. In view of the inequality (for complex numbers) $|w| \le$

$\sqrt{2}\max\{|\mathrm{Re}\, w|, |\mathrm{Im}\, w|\}$, $|A(t,z)| \geq \sqrt{2}R$ implies $|\mathrm{Re}[A(t,z)]| \geq R$ or $|\mathrm{Im}[A(t,z)]| \geq R$, thus

$$\left[|A(t,z)| \geq \sqrt{2}R\right] \subset [|\mathrm{Re}[A(t,z)]| \geq R] \cup [|\mathrm{Im}[A(t,z)]| \geq R].$$

Therefore, the probability (or measure, since $\mu(I) = 1$) that $|A(t,z)|$ exceeds $\sqrt{2}R$ is at most

$$4 \cdot \exp\left[-R\lambda + \frac{1}{2}\lambda^2 (d!)^{2\left(1-\frac{1}{m(p)}\right)} \cdot n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}\right].$$

This probabilistic estimate holds for an arbitrary but fixed z. The second part of the proof use the covering argument of the lemma initially presented, and a simple lipschitz estimate for $A(t,z)$. Suppose that z and w are points of $(\mathbb{C}^n)^d$ such that all of the component n-vectors z_k and w_k lie in B_p^n, and $\|z_k - w_k\| \leq \varepsilon$ for every $k = 1, \ldots, n$. The multi-linearity of A implies that

$$A(t, z_1, \ldots, z_d) = A(t, z_1 - w_1, , z_2, \ldots, z_d) + A(t, w_1, z_2 - w_2, z_3 \ldots, z_d) + \cdots$$
$$\cdots + A(t, w_1, \ldots, w_{d-1}, z_d - w_d) + A(t, w_1, \ldots, w_d).$$

Consequently, the boundedness of A guarantees

$$|A(t,z) - A(t,w)| \leq \varepsilon \cdot d \cdot \sup_{\mathbf{z} \in B_{p_1}^n \times \cdots \times B_{p_d}^n} |A(t,\mathbf{z})|.$$

Now let $z \in B_{p_1}^n \times \cdots \times B_{p_d}^n$ be an arbitrary and fixed point. Taking $r = 1/2d$, the lemma assures that each open ball $B_{p_k}^n$ is covered by a collection of at most $(1+4d)^{2n}$ ball of radius $1/2d$ and with centers lying on the in the closed unit ball $\overline{B_{p_k}^n}$. Let $\mathcal{W} = \{w\}$ be the collection of points $w = (w_1, \ldots, w_d) \in \overline{B_{p_1}^n} \times \cdots \times \overline{B_{p_d}^n}$ such that each component w_k being the center of a ball from the cover of the ball $B_{p_k}^n$, $k = 1, \ldots, d$. Thus $\mathcal{W}$ do not exceed $(1+4d)^{2nd}$ points. Consequently, each component n-vectors z_k, must not exceed at most $1/2d$ of some $w_k \stackrel{\text{def}}{=} w_{n_k}$ and, therefore, $w = (w_1, \ldots, w_d) \in B_{p_1}^n \times \cdots \times B_{p_d}^n$, z, w fulfils the above lipschitz property and

$$|A(t,z)| \leq |A(t,w)| + |A(t,z) - A(t,w)| \leq \max_{w \in \mathcal{W}} |A(t,w)| + \frac{1}{2} \cdot \sup_{\mathbf{z} \in B_{p_1}^n \times \cdots \times B_{p_d}^n} |A(t,\mathbf{z})|.$$

Since z is an arbitrary point, we get

$$\sup_{\mathbf{z} \in B_{p_1}^n \times \cdots \times B_{p_d}^n} |A(t,\mathbf{z})| \leq 2 \cdot \max_{w \in \mathcal{W}} |A(t,w)|,$$

which implies

$$\left[\sup_{\mathbf{z} \in B_{p_1}^n \times \cdots \times B_{p_d}^n} |A(t,\mathbf{z})| \geq 2\sqrt{2}R\right] \subset \bigcup_{w \in \mathcal{W}} \left[|A(t,w)| \geq \sqrt{2}R\right].$$

Hence, applying the preceding probabilistic estimate to each point of the finite collection $\mathcal{W}$, we get that the measure (probability) of the set $\left[\sup_{\mathbf{z}\in B^n_{p_1}\times\cdots\times B^n_{p_d}} |A(t,\mathbf{z})| \geq 2\sqrt{2}R\right]$ is at most

$$4\left(1+4d\right)^{2nd}\cdot\exp\left[-R\lambda+\frac{1}{2}\lambda^2\left(d!\right)^{2\left(1-\frac{1}{m(p)}\right)}\cdot n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}\right].$$

Now taking the followings values for the parameters R and λ,

$$R\overset{\text{def}}{=}\left(2\left(d!\right)^{2\left(1-\frac{1}{m(p)}\right)}n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}\log\left(8\left(1+4d\right)^{2nd}\right)\right)^{\frac{1}{2}},$$

$$\lambda\overset{\text{def}}{=}\frac{R}{\left(d!\right)^{2\left(1-\frac{1}{m(p)}\right)}n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}},$$

we conclude that, with these choices, the probability that the supremum of $|A(t,\cdot)|$ over $B^n_{p_1}\times\cdots\times B^n_{p_d}$ exceeds $2\sqrt{2}R$ is at most $1/2$ (this is achieved looking for values for R and λ such that the previous probability is $1/2$, *i.e.*,

$$\exp\left(-R\lambda+\frac{\beta}{2}\lambda^2\right)=\frac{1}{2\alpha},$$

with $\alpha\overset{\text{def}}{=}4\left(1+4d\right)^{2nd}$ and $\beta\overset{\text{def}}{=}\left(d!\right)^{2\left(1-\frac{1}{m(p)}\right)}\cdot n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}$; this is equivalent to solves the following equation with respect to λ

$$\frac{\beta}{2}\lambda^2-R\lambda+\log\left(2\alpha\right)=0$$

which provides, imposing the condition $R^2-2\beta\log\left(2\alpha\right)=0$, the unique solution $\lambda=R/\beta$, and these are precisely the values presented previously). Therefore, we are sure that there exists a particular value t_p such that the supremum of $|A(t_p,\cdot)|$ over $B^n_{p_1}\times\cdots\times B^n_{p_d}$ is no more than

$$2\sqrt{2}R=\left(16\left(d!\right)^{2\left(1-\frac{1}{m(p)}\right)}n^{2\sum_{k=1}^{d}\left(\frac{1}{2}-\frac{1}{M(p_k)}\right)}\log\left(8\left(1+4d\right)^{2nd}\right)\right)^{\frac{1}{2}}.$$

The values of the Rademacher functions at this particular value t_p produce the pattern of plus and minus signs indicated in the statement of theorem. Moreover, $8\left(1+4d\right)^{2nd}<\left(6d\right)^{2nd}$ when n and d are both at least 2, so the previous upper bound is even smaller than the bound stated. $\square$

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