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Transition type solutions for some classes of quasilinear elliptic Allen-Cahn equations

by

Renan Jackson Soares Isneri

Campina Grande - PB
November 2023

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Advised by

Prof. Dr. Claudianor Oliveira Alves

Thesis presented to the Associate Graduate Program in Mathematics UFPB/UFCG as partial fulfillment of the requirements for the degree of Doctor of Mathematics.

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ABSTRACT

The goal of this thesis is to develop and study the structure rich of the set of transition type solutions of some classes of elliptic PDEs of the form

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \ \text{ in } \ \mathbb{R}^2, \tag{PDE}$$

where Δ_Φ is a quasilinear operator in divergence form involving the N-function Φ that does not increase more rapidly than exponential functions, $A(x,y)$ is periodic in all its arguments and V is a double-well potential with minima at $t = \pm\alpha$. An important prototype of V is given by $V(t) = \Phi(|t^2 - \alpha^2|)$, which was inspired by the classical double-well Ginzburg-Landau potential. One of our motivations for looking for such solutions derives from a classic Allen-Cahn model of phase transitions that can be seen as a very special case of (PDE). In our investigations, such solutions are obtained by variational approaches using minimization methods to look for minima of an action functional on a reasonable class of admissible functions contained in the usual Orlicz-Sobolev space $W^{1,\Phi}_{\text{loc}}(\mathbb{R}^2)$. We provide several qualitative and quantitative properties for these solutions and a number of difficulties had to be overcome in our approach. For this reason, it was necessary to develop new estimates by using for example Harnack type inequalities found in [91], $C^{1,\alpha}$ regularity by Lieberman [67] and a new uniqueness result for a class of quasilinear ODEs of the type

$$-\left(\phi(|q'|)q'\right)' + a(t)V'(q) = 0 \ \text{ in } \ \mathbb{R}, \tag{ODE}$$

where $a(t)$ belongs to $L^\infty(\mathbb{R})$ and $\phi(t) = \Phi'(t)/t$ for $t > 0$.

Among the transition type solutions, heteroclinic and saddle-type solutions stand out in this work. Moreover, in this thesis, it is also of particular interest to study the existence of basic heteroclinic solutions for the relatively simple one-dimensional equation

(ODE), that is, to determine solutions that naturally connect the stationary points $\pm\alpha$ and that lie between $-\alpha$ and α. The development of such solutions to (ODE) serves as support for the construction of more complex solutions of spatial phase-transition problems. In particular, serves to characterize the asymptotic behavior of the saddle-type solution for (PDE).

Finally, we will discuss how variants of what was just described for (PDE) hold equally well for prescribed mean curvature equation of the type

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+A(x,y)V'(u)=0 \text{ in } \mathbb{R}^2.$$

Using the cutting techniques for the differential operator involved we build auxiliary equations of the form (PDE) to show that such equation also has a rich variety of transition type solutions whenever the distance between the roots of the symmetric potential V is small and V is similar to $V(t)=(t^2-\alpha^2)^2$. Not least, we will provide sufficient conditions for the existence of basic heteroclinic solutions for the following one-dimensional model

$$-\left(\frac{q'}{\sqrt{1+(q')^2}}\right)'+a(t)V'(q)=0 \text{ in } \mathbb{R}.$$

Moreover, uniqueness results are also explored under appropriate conditions on a and V.

Keywords: Transition type solutions; Heteroclinic solutions; Saddle-type solutions; Quasilinear Allen-Cahn equations; Prescribed mean curvature equation; Orlicz-Sobolev spaces; Minimization methods.

RESUMO

O objetivo desta tese é desenvolver e estudar a rica estrutura do conjunto de soluções do tipo transição de algumas classes de EDPs elípticas da forma

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \quad \text{em} \quad \mathbb{R}^2, \qquad (EDP)$$

em que Δ_Φ é um operador quaselinear na forma de divergência envolvendo a N-função Φ que não cresce mais rapidamente do que funções exponenciais, $A(x,y)$ é periódico em todos os seus argumentos e V é um potencial de poço duplo com mínimos em $t = \pm\alpha$. Um importante protótipo de V é dado por $V(t) = \Phi(|t^2 - \alpha^2|)$, que foi inspirado no clássico potencial de poço duplo de Ginzburg-Landau. Uma das nossas motivações para procurar tais soluções deriva de um modelo clássico de Allen-Cahn de transições de fase que pode ser visto como um caso muito especial de (EDP). Em nossas investigações, tais soluções são obtidas por abordagens variacionais usando métodos de minimização para procurar mínimos de um funcional ação em uma classe razoável de funções admissíveis contida no espaço usual de Orlicz-Sobolev $W^{1,\Phi}_{\text{loc}}(\mathbb{R}^2)$. Fornecemos diversas propriedades qualitativas e quantitativas para essas soluções e uma série de dificuldades tiveram que ser superadas na nossa abordagem. Por esta razão, foi necessário desenvolver novas estimativas usando por exemplo desigualdades do tipo Harnack encontradas em [91], $C^{1,\alpha}$ regularidade por Lieberman [67] e um novo resultado de unicidade para uma classe de EDOs quaselineares do tipo

$$-\left(\phi(|q'|)q'\right)' + a(t)V'(q) = 0 \quad \text{em} \quad \mathbb{R}, \qquad (EDO)$$

em que $a(t)$ pertence a $L^\infty(\mathbb{R})$ e $\phi(t) = \Phi'(t)/t$ para $t > 0$.

Dentre as soluções do tipo transição, destacam-se neste trabalho as soluções heteroclínicas e do tipo sela. Além disso, nesta tese, é também de particular interesse

estudar a existência de soluções heteroclínicas básicas para a equação unidimensional relativamente simples (EDO), ou seja, determinar soluções que conectam naturalmente os pontos estacionários $\pm\alpha$ e que ficam entre $-\alpha$ e α. O desenvolvimento de tais soluções para (EDO) serve como suporte para a construção de soluções mais complexas de problemas espaciais de transição de fase. Em particular, serve para caracterizar o comportamento assintótico da solução do tipo sela para (EDP).

Por fim, discutiremos como variantes do que acabamos de descrever para (EDP) se mantêm igualmente bem para a equação de curvatura média prescrita do tipo

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + A(x,y)V'(u) = 0 \text{ em } \mathbb{R}^2.$$

Usando as técnicas de trucamento para o operador diferencial envolvido construímos equações auxiliares da forma (EDP) para mostrar que tal equação também possui uma rica variedade de soluções do tipo transição sempre que a distância entre as raízes do potencial simétrico V for pequena e V é semelhante a $V(t) = (t^2 - \alpha^2)^2$. Não menos importante, forneceremos condições suficientes para a existência de soluções heteroclínicas básicas para o seguinte modelo unidimensional

$$-\left(\frac{q'}{\sqrt{1+(q')^2}}\right)' + a(t)V'(q) = 0 \text{ em } \mathbb{R}.$$

Além disso, resultados de unicidade também são explorados sob condições apropriadas em a e V.

Palavras-chave: Soluções do tipo transição; Soluções heteroclínicas; Soluções do tipo sela; Equações quasilineares de Allen-Cahn; Equação de curvatura média prescrita; Espaços de Orlicz-Sobolev; Métodos de minimização.

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Sincerely, Renan J. S. Isneri

"I can do all things through Christ which strengtheneth me."

Philippians 4:13

DEDICATION

This thesis is dedicated to my family for their support and love. In particular, my mother, Rilvânia Soares da Silva, my father, Jardel Jackson Gomes Isneri, and my wife, Beatriz Santos Isneri. They are my constant guiding light.

CONTENTS

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INTRODUCTION

The problem of existence and classification of bounded solutions of stationary Allen-Cahn type equations

$$-\Delta u + A(x)V'(u) = 0 \ \text{ in } \ \mathbb{R}^n \tag{1}$$

has been widely studied in the last years, providing a rich amount of differently shaped families of solutions, such as periodic, heteroclinic, saddle and multibump solution. The Allen-Cahn equation was introduced in 1979 by S. Allen and J. Cahn in [11] as a model for phase transitions in binary alloys. The standard model of V is the classical double well Ginzburg-Landau potential

$$V(t) = \frac{1}{4}(t^2 - 1)^2, \quad t \in \mathbb{R}.$$

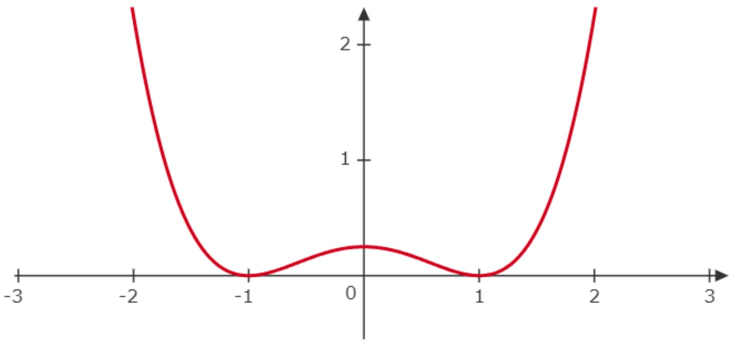

Figure 1: The double well potential $V(t) = \frac{1}{4}(t^2 - 1)^2$.

The function u is a phase parameter describing pointwise the state of the material and the global minima of V represents the pure phases of the system. Different values of u depict mixed configurations and by transition solutions we mean entire solutions of (1) which are asymptotic in different directions to the pure phases of the system. In the equation (1) the presence of the (positive) oscillatory factor $A(x)$ models an inhomogeneous behavior of the system.

When A is a positive constant function (for example, $A(x) = 1$), a long standing problem is to characterize the set of the solutions $u \in C^2(\mathbb{R}^n)$ of (1) satisfying $|u(x)| \leq 1$ and $\partial_{x_1} u(x) > 0$. This problem was pointed out by De Giorgi in [34], where he conjectured that, when $n \leq 8$ and $V(t) = (t^2 - 1)^2$, the whole set of these solutions reduces, up to translations, to the unique solution $q_+ \in C^2(\mathbb{R})$ of the one dimensional problem

$$-q''(t) + V'(q(t)) = 0 \ \text{ in } \ \mathbb{R}, \quad q(0) = 0 \quad \text{and} \quad \lim_{t \to \pm\infty} q(t) = \pm 1.$$

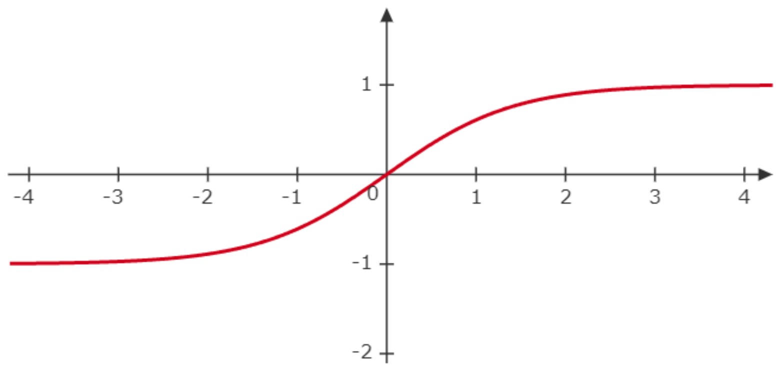

Figure 2: The graph of q_+.

The conjecture has been firstly proved in the planar case by Ghoussoub and Gui in [57] even for more general double well potential V. In the case $n = 3$ it has been proved by Ambrosio and Cabrè in [19] and, assuming

$$u(x) \to \pm 1 \ \text{ as } \ x_1 \to \pm\infty,$$

the same rigidity result has been obtained in dimension $n \leq 8$ by Savin in [88], paper to which we refer also for an extensive bibliography on the argument. Del Pino, Kowalczyk and Wei showed in [37,38] that the 1-D symmetry of these solutions is generally lost when $n \geq 9$. We refer also to [21,22,40], where a weaker version of the De Giorgi conjecture, known as Gibbons conjecture, has been obtained for all the dimensions n and in more general settings. These results show that when A is a positive constant and u is a bounded solution of (1) satisfying

$$u(x) \to \pm 1 \ \text{ as } \ x_1 \to \pm\infty \ \text{ uniformly with respect to } \ (x_2, \dots, x_n) \in \mathbb{R}^{n-1} \qquad (2)$$

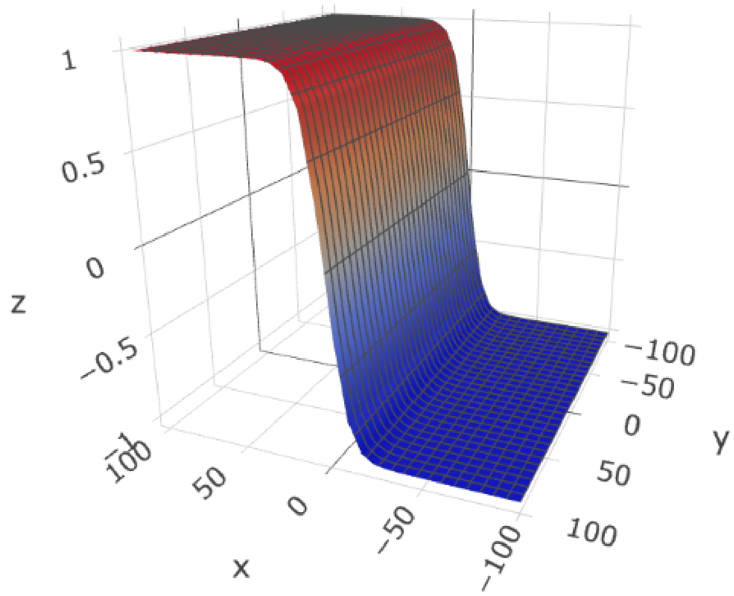

Figure 3: Graph of u in $\mathbb{R}^3$

then $u(x) = q_+(x_1)$ for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

Solutions of (1)-(2) are said to be heteroclinic from -1 to 1. The study of existence and qualitative properties of heteroclinic solutions and various generalizations has been widely studied and received special attention in recent years, because this type of solution appears in many mathematical models associated with problems that appear in Physics, Biology, Mechanics and Chemistry. Generally the heteroclinic solutions appear as physical processes involving variable transitions from an unstable equilibrium to a stable one, frontal propagation in equations of reaction-diffusion and phase-transition. For a quite comprehensive account, the interested reader may start by reading the papers [24, 69] and their references. For example, a simple description of heteroclinic solutions can be found in the mathematical equation of the pendulum

$$q''(t) + b\sin(q(t)) = 0 \quad \text{in} \quad \mathbb{R},$$

where $b > 0$ depends on the acceleration due to gravity and the length of the rod. In this case, the phase plane analysis shows that there is a heteroclinic solution q from $-\pi$ to π, that is,

$$\lim_{t\to-\infty} q(t) = -\pi, \quad \lim_{t\to+\infty} q(t) = \pi \quad \text{and} \quad \lim_{t\to\pm\infty} q'(t) = 0.$$

Physically, this solution corresponds to motions that are asymptotic to the unstable vertical equilibrium. For a deeper discussion see [24, 70].

This kind of heteroclinic type transition solutions persist when A is not constant in (1). The heteroclinic type problem was first studied by variational methods for more general elliptic equations of the form

$$-\Delta u = g(x, y, u) \quad \text{in} \quad \mathbb{R} \times \Omega \tag{3}$$

by Rabinowitz in [79], when Ω is a bounded regular domain on $\mathbb{R}^n$. Assuming that the nonlinearity g to be even and periodic in the variable x, Rabinowitz showed the existence of solutions for (3) satisfying Dirichlet or Neumann boundary condition on $\partial\Omega$ and being asymptotic as $x \to \pm\infty$ to different minimal solutions $u_\pm$, periodic in the variable x. This result was generalized by Alves in [13] for different conditions on g, including the case in which g is only asymptotically periodic in the variable x. A related variational approach was used to study the heteroclinic type problem for equation (1) in the case in which A is periodic in all variables in [5, 82, 83], showing the existence of (minimal) solutions $u(x)$ that are periodic in the variable $(x_2, \dots x_n)$ and such that u is asymptotic to different minima of the potential V as $x_1 \to \pm\infty$. Starting from the existence of this "basic" heteroclinic solutions, these papers show how the presence of a truly oscillatory factor $A(x, y)$ gives generically the existence of complex classes of other heteroclinic type transition solutions in contrast with the above described rigidity results characterizing the autonomous case (see also [9, 27, 84]).

When referred to the semilinear equation

$$-q''(t) + a(t)V'(q(t)) = 0 \quad \text{in} \quad \mathbb{R}, \tag{4}$$

the problem of the existence of heteroclinic solutions is a classical topic in the theory of ordinary differential equations. In recent years there has been a large number of works that study the existence of heteroclinic solution for (4) by considering different classes of functions $a(t)$. For example:

Class 1: [24] $a(t)$ is a positive constant.

Class 2: [24] $a(t)$ is a continuous function such that

$$0 < \inf_{t \in \mathbb{R}} a(t) \quad \text{and} \quad a(t+1) = a(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

Class 3: [24, 54] $a(t)$ is a continuous function such that there are $a_1, a_2 > 0$ verifying

$$a_1 \le a(t) \le a_2, \ \forall t \in \mathbb{R} \quad \text{and} \quad a(t) \to a_2 \text{ as } |t| \to +\infty,$$

where $a(t) < a_2$ in some set of nonzero measure.

Class 4: [12, 54] $a(t)$ is asymptotically periodic at infinity, that is, there is a continuous periodic function $a_p : \mathbb{R} \to \mathbb{R}$ satisfying

$$|a(t) - a_p(t)| \to 0 \quad \text{as} \quad |t| \to +\infty \quad \text{and} \quad 0 < \inf_{t\in\mathbb{R}} a(t) \leq a(t) < a_p(t), \ \forall t \in \mathbb{R}.$$

Class 5: [12] $a \in L^\infty(\mathbb{R})$ and

$$0 < a(0) = \inf_{t\in\mathbb{R}} a(t) < a_\infty = \liminf_{|t|\to+\infty} a(t).$$

Class 6: [52, 53] $a \in L^\infty(\mathbb{R})$ and there are $l, L \in (0, +\infty)$ such that

$$l \leq a(t) \leq L \text{ almost everywhere and } a(t) \to L \text{ as } |t| \to +\infty$$

where L/l is suitably bounded from above.

Class 7: [52] $a(t)$ is even and there are $l, L \in (0, +\infty)$ such that

$$l \leq a(t) \leq L \text{ almost everywhere in } \mathbb{R}.$$

Class 8: [52, 53] $a \in L^\infty(\mathbb{R}, [0, +\infty)$ and there are $l > 0$, $S < T$, such that

$$a(t) = l \text{ for } t \notin [S, T].$$

Class 9: [55] There is $t_0 \in \mathbb{R}$ such that $a(t)$ is increasing in $(-\infty, t_0]$ and $a(t)$ is decreasing in $[t_0, +\infty)$. Moreover,

$$\lim_{|t|\to+\infty} a(t) = l > 0 \quad \text{and} \quad \lim_{|t|\to+\infty} |t|(l - a(t)) = 0.$$

Class 10: [90] There are $l, \underline{l} > 0$ such that

$$a(t) \to l \quad \text{as} \quad |t| \to +\infty \quad \text{and} \quad \underline{l} \leq a(t) \leq L \equiv l + 4\nu\sqrt{l\underline{l}} \Big/ \int_{-1}^{1} \sqrt{V(s)}ds \text{ for all } t \in \mathbb{R},$$

where

$$\nu = \min\left\{ \int_{-1}^{\epsilon_-} \sqrt{V(s)}ds, \int_{\epsilon_+}^{1} \sqrt{V(s)}ds \right\},$$

with $\epsilon_- = \min\{s : s > -1,\ V'(s) = 0\}$ and $\epsilon_+ = \max\{s : s < 1\ V'(s) = 0\}$.

Another kind of transition solutions for (1) was introduced by Dang, Fife and Peletier in [33]. In the planar case $n = 2$, when V is an even double well potential and A is a positive constant, they showed by a sub-supersolution method that (1) has a unique bounded solution $u \in C^2(\mathbb{R}^2)$ with the same sign as x_1x_2, odd in both the variables x_1 and x_2 and symmetric with respect to the diagonals $x_2 = \pm x_1$. Along any directions not parallel to the coordinate axes the *saddle* solution u is asymptotic to the minima of the potential V representing a phase transition with cross interface. Note that, even if it is related to minimal transition heteroclinic solutions, being asymptotic to q_+ as $x_2 \to +\infty$, it no longer has minimal character (see [63, 89]). Many extensions for Allen-Cahn models have been considered. In the planar case we refer to [3] for a variational study of saddle type solutions with dihedral symmetries of order k (see also [61] for a global variational approach to the saddle problem) and to [39, 59] for a general study regarding k-end solutions. Further generalizations of the study of saddle-type solutions have been made in higher dimensions. For example, in [6] and [7], Alessio and Montecchiari established the existence of saddle-type solutions on $\mathbb{R}^3$. In [2], Alama, Bronsard and Gui studied a vectorial version of saddle-type solution, where systems of autonomous Allen-Cahn equations have been considered on the plane (see [60] and [8] for related studies on $\mathbb{R}^3$). A generalization of the variational framework considered in [2] can be found in [10]. For other interesting papers in higher dimension, we mention [28, 29, 77] for the equations case and to [2] for the case of systems of autonomous Allen-Cahn equations.

The analogous for saddle type solutions for (1) in the planar case, when $A \in C(\mathbb{R}^2)$ is positive, even, periodic and symmetric with respect to the plane diagonal $x_2 = x_1$ has been introduced in [4] where a variational procedure was introduced to find as in the autonomous case a solution u of (1) on $\mathbb{R}^2$ which is odd with respect to both its variables, symmetric with respect to the diagonal, strictly positive on the first quadrant and is asymptotic to the minima of V along any directions not parallel to the coordinate axes. Moreover in [4] it is shown that, as $y \to +\infty$ (uniformly with respect to $x \in \mathbb{R}$), the solution u is asymptotic to the set of the x-odd minimal heteroclinic type solutions of (1) which are periodic in the variable y.

Throughout this thesis, we tackled the problem of existence of heteroclinic and saddle type solutions via variational methods for the analogous of Allen-Cahn model in the quasilinear setting of the form

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \text{ in } \mathbb{R}^2, \tag{5}$$

where $\Delta_\Phi u = \operatorname{div}(\phi(|\nabla u|)\nabla u)$ and $\Phi : \mathbb{R} \to [0,+\infty)$ is an N-function of the type

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds \tag{6}$$

for a $\phi \in C^1([0,+\infty),[0,+\infty))$ such that:

(ϕ_1) $\phi(t) > 0$ and $(\phi(t)t)' > 0$ for any $t > 0$.

(ϕ_2) There are $l, m \in \mathbb{R}$ with $1 < l \le m$ such that

$$l - 1 \le \frac{(\phi(t)t)'}{\phi(t)} \le m - 1 \text{ for all } t > 0.$$

(ϕ_3) There exist constants $c_1, c_2, \eta > 0$ and $s > 1$ satisfying

$$c_1 t^{s-1} \le \phi(t)t \le c_2 t^{s-1} \text{ for } t \in (0,\eta).$$

(ϕ_4) ϕ is non-decreasing on $(0,+\infty)$.

We would like to point out that in the study of quasilinear elliptic problems driven by the Φ-Laplacian operator, the conditions (ϕ_1)-(ϕ_2) are well-known and guarantee that Φ and its complementary function $\tilde{\Phi}$ are N-functions that check the so called Δ_2-condition (see for instance Appendix A). Those conditions ensure that Φ behaves in such a way that the Orlicz-Sobolev space associated to Φ is reflexive and separable.

In recent years, facing the need of a mathematical description of physical problems, there has been a growing number of works involving the Φ-Laplacian operator Δ_Φ and its theory is by now rather developed. As a first example we may consider the case

$$\Phi(t) = |t|^p, \quad t \in \mathbb{R}, \quad p \in (1,+\infty),$$

which is related to the celebrated p-Laplacian operator that often appears in physical models, for example in Newtonian and non-Newtonian fluids (see [35, 36] and references therein). Motivated by concrete examples of equations arising from fluid mechanics and plasticity theory, Seregin and Fuchs in [46, 47] (see also [45]) were led to the minimization of integrals where appears the logarithmic model

$$\Phi(t) = |t|^p \ln(1+|t|), \quad t \in \mathbb{R}, \quad p \in [1,+\infty),$$

which is an N-function of the type (4). Other model of N-function of the form (4) that often arises in a lot of fields of physics and related sciences, such as biophysics and chemical reaction design, is

$$\Phi(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q, \quad t \in \mathbb{R}, \quad 1 < p < q < +\infty.$$

The differential operator associated with this N-function is known as the (p, q)-Laplacian operator and the prototype for these models can be written in the form

$$u_t = -\Delta_\Phi + f(x, u).$$

In this configuration, the function u generally describes a concentration, Δ_Φ corresponds to the diffusion and $f(x, u)$ is the reaction term that corresponds to source and loss processes. For a quite comprehensive account, the interested reader might start by referring to [20, 41]. Finally, it is worth mentioning that the N-function given by

$$\Phi(t) = (1 + t^2)^\gamma - 1, \quad t \in \mathbb{R}, \quad \gamma > 1,$$

appears in the works [49, 50], where the authors report that the studies of quasilinear equations involving the associated operator Δ_Φ are motivated by nonlinear elasticity models. For other examples of N-functions of the type (4) and more applications we refer the reader to [45, 48] and the bibliography therein.

This thesis is a collection of the published and submitted papers listed below:

(P1) *Existence of saddle-type solutions for a class of quasilinear problems in* $\mathbb{R}^2$, Topol. Methods Nonlinear Anal., 61(2), 2023, 825-868. (with Claudianor Alves and Piero Montecchiari).

(P2) *Existence of heteroclinic and saddle type solutions for a class of quasilinear problems in whole* $\mathbb{R}^2$, Commun. Contemp. Math., 2022. (with Claudianor Alves and Piero Montecchiari).

(P3) *Existence of heteroclinic solutions for the prescribed curvature equation*, J. Differential Equations, 362, 2023, 484-513. (with Claudianor Alves).

(P4) *Heteroclinic solutions for some classes of prescribed mean curvature equations in whole* $\mathbb{R}^2$, preprint. (with Claudianor Alves).

(P5) *Saddle solutions for Allen-Cahn type equations involving the prescribed mean curvature operator*, in preparation.

We would like to emphasize that each paper is presented as a chapter and the exposition of the chapters varies slightly from the presentations of the papers in order to complement the studies performed there.

Another paper that complements this thesis is the following:

(P6) *Uniqueness of heteroclinic solutions in a class of autonomous quasilinear ODE problems*, preprint. (with Claudianor Alves and Piero Montecchiari).

Next, we describe the organization of this thesis and we present a brief overview of the topics studied in the chapters.

In Chapter 1, we present the joint paper with professors Claudianor Alves and Piero Montecchiari [16]. The main goal of this chapter is to prove the existence of saddle-type solutions for the following class of quasilinear equations

$$-\Delta_\Phi u + V'(u) = 0 \;\; \text{in} \;\; \mathbb{R}^2, \tag{7}$$

where the potential V satisfies the following conditions:

(V_1) $V(t) \geq 0$ for all $t \in \mathbb{R}$ and $V(t) = 0 \Leftrightarrow t = -\alpha, \alpha$ for $\alpha > 0$.

(V_2) $V(-t) = V(t)$ for any $t \in \mathbb{R}$.

(V_3) There are $\delta_\alpha \in (0, \alpha)$ and $w_1, w_2 > 0$ such that

$$w_1\Phi(|t-\alpha|) \leq V(t) \leq w_2\Phi(|t-\alpha|) \;\; \forall t \in (\alpha - \delta_\alpha, \alpha + \delta_\alpha).$$

(V_4) There are $\omega_1, \omega_2, \omega_3, \omega_4, \tau > 0$ such that

$$-\omega_3\phi(\omega_4|\alpha - t|)(\alpha - t)t \leq V'(t) \leq -\omega_1\phi(\omega_2|\alpha - t|)(\alpha - t)t \;\; \forall t \in [0, \alpha + \tau].$$

(V_5) There is $\delta_0 > 0$ such that V' is increasing on $(\alpha - \delta_0, \alpha)$.

(V_6) There are $\gamma, \epsilon > 0$ such that

$$\tilde{\Phi}(V'(t)) \leq \gamma\Phi(|\alpha - t|), \;\; \forall t \in (\alpha - \epsilon, \alpha).$$

It is worth mentioning that an important example of a potential V that checks the conditions (V_1)-(V_6) is given by

$$V(t) = \Phi(|t^2 - \alpha^2|), \quad t \in \mathbb{R},$$

where Φ is an N-function of the form (6) verifying (ϕ_1)-(ϕ_2), which was inspired by the classical double well Ginzburg-Landau potential $V(t) = (t^2 - 1)^2$.

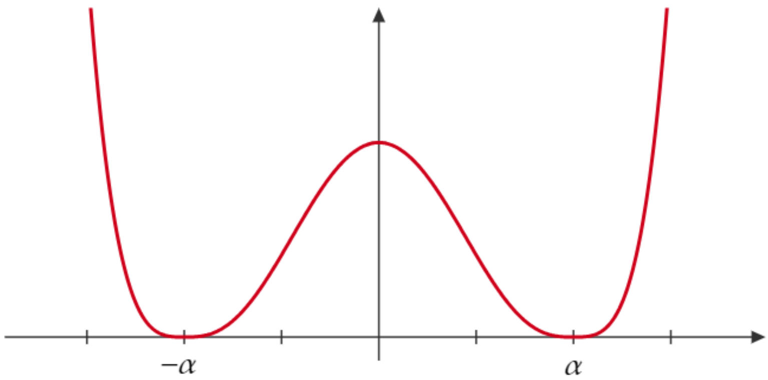

Figure 4: The potential $V(t) = \Phi(|t^2 - \alpha^2|)$.

Our main result involving saddle-type solutions is the following:

Theorem 0.1 *Assume (ϕ_1)-(ϕ_4) and (V_1)-(V_6). Then, For each $j \geq 2$ there exists $v_j \in C^{1,\gamma}_{loc}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$ such that v_j is a weak solution of (7) satisfying*

(a) $0 < \tilde{v}_j(\rho, \theta) < \alpha$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *and* $\rho > 0$,

(b) $\tilde{v}_j(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_j(\rho, \frac{\pi}{2} - \theta)$ *for all* $(\rho, \theta) \in [0, +\infty) \times \mathbb{R}$,

(c) $\tilde{v}_j(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_j(\rho, \theta)$ *for all* $(\rho, \theta) \in [0, +\infty) \times \mathbb{R}$,

(d) $\tilde{v}_j(\rho, \theta) \to \alpha$ *as* $\rho \to +\infty$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$,

where $\tilde{v}_j(\rho, \theta) = v_j(\rho\cos(\theta), \rho\sin(\theta))$.

The item (d) of Theorem 0.1 is a characterization of the asymptotic behavior of v_j, which guarantees that for $k = 0, ..., 2j-1$ there results

$$\tilde{v}_j(\rho, \theta) \to (-\alpha)^{k+1} \;\text{ as }\; \rho \to +\infty \;\text{ whenever }\; \theta \in \left(\frac{\pi}{2} + k\frac{\pi}{j}, \frac{\pi}{2} + (k+1)\frac{\pi}{j}\right).$$

Therefore, the saddle-type solution can be seen as a phase transition with cross interface.

In the proof of Theorem 0.1 it is crucial to prove the existence and uniqueness of the minimal odd heteroclinic solution for

$$-(\phi(|q'|)q')' + V'(q) = 0 \quad \text{in} \quad \mathbb{R}. \tag{8}$$

After that, we use the heteroclinic solutions as support to characterize the asymptotic behavior of the saddle-type solution for (7). The main tool used is the variational method on Orlicz-Sobolev spaces, more precisely, minimization technique on a set of admissible function. The idea is looking for minima of the action functional

$$F(q) = \int_{\mathbb{R}} (\Phi(|q'|) + V(q))\, dt$$

on the class

$$E_\Phi = \{q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}) : \ q \text{ is odd a.e. in } \mathbb{R}\}.$$

Denoting by K_Φ the set of minima of F on E_Φ, we have the following result:

Theorem 0.2 *Assume* (ϕ_1)-(ϕ_3) *and* (V_1)-(V_6)*. Then, there exists a unique* $q \in K_\Phi$ *such that it is a weak solution of* (8) *being heteroclinic from* $-\alpha$ *to* α*, that is,*

$$q(t) \to -\alpha \quad as \quad t \to -\infty \quad and \quad q(t) \to \alpha \quad as \quad t \to +\infty.$$

Moreover, $q \in C^{1,\gamma}_{loc}(\mathbb{R})$ *for some* $\gamma \in (0,1)$ *and satisfies the following properties:*

(a) $q(t) = -q(-t)$ *for any* $t \in \mathbb{R}$*,*

(b) $0 < q(t) < \alpha$ *for all* $t > 0$*,*

(c) q *is increasing on* $\mathbb{R}$*,*

(d) $q'(t) \to 0$ *as* $t \to \pm\infty$*,*

(e) q' *is non-increasing on* $[0, +\infty)$*,*

(f) $q'(t) > 0$ *for any* $t \in \mathbb{R}$*.*

We would like to point out that the Theorem 0.1 complements the study made in [3], because in that paper the authors considered the Laplacian operator, while in our study we considered a large class of equations involving quasilinear operators. However, it is important to mention that some estimate found in [3] can not be used here, as for example some maximum principles, C^2 regularity for the Laplacian operator as well as existence and uniqueness of solution for second order ordinary differential equations. Here, it was necessary to develop new estimates by using for example a Harnack-type inequality, $C^{1,\alpha}$ regularity by Lieberman [67] and a new uniqueness result for a class of ordinary differential equations driven by a quasilinear operator.

Now, some result involving heteroclinic solutions and its generalizations will be discussed briefly. For one dimensional problems, we would like to cite the papers by Rabinowitz [80, 81] and Gavioli and Sanchez [56] and their references, where the reader can find interesting results about the existence of heteroclinic solutions for related problems. Further generalizations of the study of heteroclinic-type solutions have been made in higher dimensions, see for example Rabinowitz [79], Alves [13], Rabinowitz and Stredulinsky [82]. Related to elliptic system we cite the paper by Byeon, Montecchiari and Rabinowitz [27]. In the literature we also find some papers that study the existence of heteroclinic solution for classes of quasilinear problems, see for example Feliz [71–73] and for a vectorial version, we recommend the paper by Ruan [86]. Finally, for a recent account about heteroclinic solutions involving the fractional Laplacian operator, we refer the reader to [30, 31] where the authors showed the existence and uniqueness of the following problem

$$(-\partial^2_{zz})^s q + q^3 - q = 0 \ \text{ in } \ \mathbb{R}, \quad q(0) = 0, \quad \lim_{t\to\pm\infty} q(t) = \pm 1, \quad q' > 0.$$

The existence of heteroclinic solutions to higher dimensional problems has been explored by Alves, Ambrosio and Torres Ledesma [14].

In Chapter 2 we present the paper [17], which is a joint work with professors Claudianor Alves and Piero Montecchiari. In this chapter, we study the existence of related weak heteroclinic and saddle-type solutions of the non-autonomous version of the equation (7), which is given by

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{9}$$

where $A : \mathbb{R}^2 \to \mathbb{R}$ satisfies:

(A_1) A is a continuous function and $A(x,y) > 0$ for each $(x,y) \in \mathbb{R}^2$,

(A_2) $A(x,y) = A(-x,y) = A(x,-y)$ for all $(x,y) \in \mathbb{R}^2$,

(A_3) $A(x,y) = A(x+1,y) = A(x,y+1)$ for any $(x,y) \in \mathbb{R}^2$,

(A_4) $A(x,y) = A(y,x)$ for all $(x,y) \in \mathbb{R}^2$.

An interesting model for A is given by

$$A(x,y) = \cos(2\pi x)\cos(2\pi y) + c \text{ with } c > 1.$$

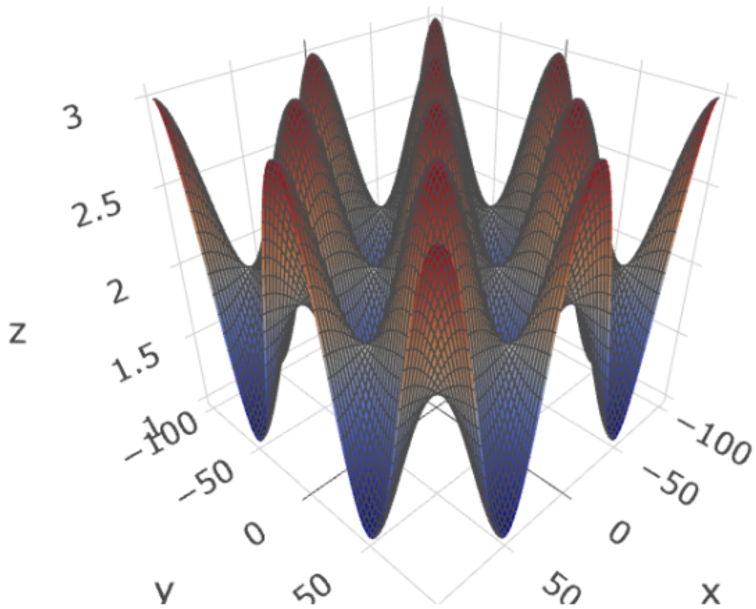

Figure 5: Graph of $A(x,y) = \cos(2\pi x)\cos(2\pi y) + 2$.

In this chapter, we use variational methods related to the ones introduced in [4] and [16] to establish the existence of (minimal) heteroclinic type solutions form $-\alpha$ to α of (9), that is, weak solutions $v \in C^{1,\beta}_{\text{loc}}(\mathbb{R}^2)$ which are 1-periodic in the variable y such that

$$v(x,y) \to -\alpha \text{ as } x \to -\infty \text{ and } v(x,y) \to \alpha \text{ as } x \to +\infty \text{ uniformly in } y \in \mathbb{R}.$$

Moreover, we borrow some ideas developed in [4] and [79] to look for minima of the action functional

$$I(u) = \int_{\mathbb{R}}\int_0^1 (\Phi(|\nabla u|) + A(x,y)V(u))\, dydx,$$

on the class

$$E_\Phi(\alpha) = \left\{ u \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}\times[0,1]) : \ 0 \le u(x,y) \le \alpha \text{ for } x > 0 \text{ and } u \text{ is odd in } x \right\},$$

where $W^{1,\Phi}_{\text{loc}}(\mathbb{R}\times[0,1])$ denotes the usual Orlicz-Sobolev space. Denoting by $K_\Phi(\alpha)$ the set of minima of I on $E_\Phi(\alpha)$, we show that $K_\Phi(\alpha)$ is not empty and constituted by (minimal) heteroclinic type solutions of (9). The minimality properties of these heteroclinic type solutions allows us, as a second step, to build up a variational framework inspired to the one introduced in [4] to detect the existence of saddle type solution of (9), characterizing their the asymptotic behavior. More precisely, we have the following results:

Theorem 0.3 *Assume* (ϕ_1)*-*(ϕ_3)*,* $V \in C^1(\mathbb{R}, \mathbb{R})$*,* (V_1)*-*(V_3) *and* (A_1)*-*(A_3)*. Then, there exists* $v \in C^{1,\beta}_{loc}(\mathbb{R}^2)$ *for some* $\beta \in (0,1)$ *such that* v *is a weak solution of* (9) *that verifies the following:*

(a) $v(x,y) = -v(-x,y)$ *for all* $(x,y) \in \mathbb{R}^2$*,*

(b) $v(x,y) = v(x,y+1)$ *for any* $(x,y) \in \mathbb{R}^2$*,*

(c) $0 < v(x,y) < \alpha$ *for each* $x > 0$ *and* $y \in \mathbb{R}$*.*

Moreover, v *is a heteroclinic solution from* $-\alpha$ *to* α*.*

Theorem 0.4 *Assume* (ϕ_1)*-*(ϕ_4)*,* $V \in C^1(\mathbb{R}, \mathbb{R})$*,* (V_1)*-*(V_4) *and* (A_1)*-*(A_4)*. Then, there is* $v \in C^{1,\beta}_{loc}(\mathbb{R}^2)$ *for some* $\beta \in (0,1)$ *such that* v *is a weak solution of* (9) *that verifies the following:*

(a) $0 < v(x,y) < \alpha$ *on the fist quadrant in* $\mathbb{R}^2$*,*

(b) $v(x,y) = -v(-x,y) = -v(x,-y)$ *for all* $(x,y) \in \mathbb{R}^2$*,*

(c) $v(x,y) = v(y,x)$ *for any* $(x,y) \in \mathbb{R}^2$*,*

(d) There is $u_0 \in K_\Phi(\alpha)$ *such that* $\|v - \tau_j u_0\|_{L^\infty(\mathbb{R}\times[j,j+1])} \to 0$ *as* $j \to +\infty$*,*

where $\tau_j u_0(x,y) = u_0(x, y-j)$ *for all* $(x,y) \in \mathbb{R}^2$*.*

The item (d) of Theorem 0.4 characterizes the asymptotic behavior of v. It guarantees that along directions parallel to the coordinate axes the saddle type solution is asymptotic to the minimal heteroclinic set $K_\Phi(\alpha)$. This implies that along any direction not parallel to the coordinate axes v is asymptotic at infinity to $\pm\alpha$ and therefore the saddle type solution can be seen as a phase transition solution with cross interface.

We would like to point out that Theorems 0.3 and 0.4 improve the results of Chapter 1 not only in the fact that the function $A(x,y)$ is allowed to be not constant but also because, unlike the Chapter 1, the assumptions (V_5) and (V_6) are not needed. Moreover, we note that even though the variational approach is inspired by the one used in [4], many tools used in the classical Laplacian context, such as for example some maximum principles, C^2 regularity, existence and local uniqueness theorems, are no more available in the present framework. To show that (9) admits transition type solutions it was necessary to develop new estimates based on the Harnack type inequalities found in [91] and on results about $C^{1,\alpha}$ regularity for quasilinear problems as obtained by Lieberman in [67].

It is nowadays a well-known fact that the prescribed mean curvature operator

$$div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \tag{10}$$

has been extensively studied in the recent years, due to the close connection with capillarity theory [43]. After the pioneering works of Young [92], Laplace [66], and Gauss [51] in the early 18th century about the mean curvature of a capillary surface, much has already been produced in the literature and it is difficult and exhaustive to measure here the vastness of physical applications involving the (10) operator, however for the interested reader in this subject, we could cite here some problems that appear in optimal transport [25] and in minimal surfaces [58]. Moreover, (10) also appears in some problems involving reaction-diffusion processes which occur frequently in a wide variety of physical and biological settings. For example, in [65], Kurganov and Rosenau observed that when the saturation of the diffusion is incorporated into these processes, it may cause a deep impact on the the morphology of the transitions connecting the equilibrium states, as now not only do discontinuous equilibria become permissible, but traveling waves can arise in their place. A specific class of such processes is modeled by the following equation

$$u_t = div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - aV'(u), \tag{11}$$

where the reaction function V is the classical double well Ginzburg-Landau potential and a is a constant. The impact of saturated diffusion on reaction-diffusion processes was investigated by them in the straight line and in the plane.

As indicated in the previous paragraph, [65] provided a significant physical motivation for the study of equations of the form (11) having as main objective the existence and classification of transition-type solutions, that is, entire solutions of (11) which are asymptotic in different directions to the equilibrium states of the system. In this sense, Bonheure, Obersnel and Omari in [23] investigated the existence of a heteroclinic solution of the one-dimensional equation

$$-\left(\frac{q'}{\sqrt{1+(q')^2}}\right)' + a(t)V'(q) = 0 \ \text{ in } \ \mathbb{R}, \tag{12}$$

looking for minima of an action functional on a convex subset of $BV_{\rm loc}(\mathbb{R})$ made of all functions satisfying an asymptotic condition at infinity, where the authors considered as usual V a double-well potential with minima at $t = \pm 1$ with the function a asymptotic to a positive periodic function, that is, $a \in L^\infty(\mathbb{R})$ with $0 < \operatorname{ess}\inf_{t\in\mathbb{R}} a(t)$ and there is $a^* \in L^\infty(\mathbb{R})$ τ-periodic, for some $\tau > 0$, such that $a(t) \le a^*(t)$ almost everywhere on $\mathbb{R}$ satisfying

$$\operatorname{ess}\lim_{|t|\to+\infty} (a^*(t) - a(t)) = 0.$$

In Chapter 3 we present the work [14], which is a joint work with professor Claudianor Alves. This chapter is concerned with the existence and qualitative properties of heteroclinic solutions of the prescribed curvature equation (12). The basic idea is to truncate the mean curvature operator to build up a variational framework inspired to the one introduced in [74] on Orlicz-Sobolev space $W^{1,\Phi}_{\rm loc}(\mathbb{R})$, that is, to obtain an auxiliary equation of the form

$$-\left(\phi(|q'|)q'\right)' + a(t)V'(q) = 0 \quad \text{in} \quad \mathbb{R}, \tag{13}$$

where $\phi : (0, +\infty) \to (0, +\infty)$ is a C^1 function verifying (ϕ_1)-(ϕ_3), in order to establish the existence of a heteroclinic solution for (12) in the case where the function a belongs to the following class of functions

Class 11: $a \in L^\infty(\mathbb{R})$ is an even non-negative function satisfying

$$0 < a_0 := \inf_{t\ge M} a(t) \quad \text{for some} \quad M > 0.$$

Throughout this chapter, we say that a function q is a *heteroclinic solution from $-\alpha$ to α of* (12) ((13)) if $q \in C^{1,\beta}_{\rm loc}(\mathbb{R})$ for some $\beta \in (0, 1)$ and satisfies the equation (12) ((13)) for all $t \in \mathbb{R}$, and Moreover,

$$\lim_{t\to-\infty} q(t) = -\alpha, \quad \lim_{t\to+\infty} q(t) = \alpha \quad \text{and} \quad \lim_{t\to\pm\infty} q'(t) = 0.$$

Our main result in this chapter is the following:

Theorem 0.5 *Assume that a belongs to Class 11, $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2) and that*

(V_7) (i) $V''(\pm\alpha) > 0$.

(ii) There are $\tilde{\alpha} > 0$ and $C = C(\tilde{\alpha}) > 0$ such that $\sup_{|t|\in[0,\alpha]} |V'(t)| \le C$ for all $\alpha \in (0, \tilde{\alpha})$.

Then, for each $L > 0$ there exists $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$ equation (12) possesses a heteroclinic solution q_α from $-\alpha$ to α satisfying:

(a) $q_\alpha(t) = -q_\alpha(-t)$ *for all* $t \in \mathbb{R}$,

(b) $0 < q_\alpha(t) < \alpha$ *for all* $t > 0$,

(c) $|q'_\alpha(t)| < \sqrt{L}$ *for any* $t \in \mathbb{R}$.

The assumption (V_7)-(ii) is a uniform condition on the potentials V that depends on $\alpha > 0$ and a class of such potentials for which (V_1)-(V_2) and (V_7) are all satisfied is

$$V(t) = (t^2 - \alpha^2)^2, \quad \alpha > 0,$$

which includes the classical double well Ginzburg-Landau potential when $\alpha = 1$. The reader is invited to see that the theorem above is true for potentials of the Ginzburg-Landau type, for example when α is small.

Our second main result is the following:

Theorem 0.6 *Assume* (ϕ_1)-(ϕ_2), $V \in C^1(\mathbb{R}, \mathbb{R})$, (V_1)-(V_3) *and that* a *belongs to Class 11. Then equation* (13) *has a heteroclinic solution from* $-\alpha$ *to* α *satisfying*

(a) $q(t) = -q(-t)$ *for any* $t \in \mathbb{R}$,

(b) $0 \leq q(t) \leq \alpha$ *for all* $t > 0$.

Moreover, taking into account the assumptions (ϕ_3) *and*

(V_8) *There are* $d_1, d_2 > 0$ *and* $\lambda > 0$ *such that*

$$|V'(t)| \leq d_1\phi(d_2|t - \alpha|)|t - \alpha| \text{ for all } t \in [\alpha - \lambda, \alpha + \lambda],$$

then the inequalities in (b) are strict.

Moreover, the classical case $\Phi(t) = \dfrac{t^2}{2}$ corresponds to the equation (4), and in this case the Theorem 0.6 can be written of the following way

Theorem 0.7 *Assume* $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2), $(V_5) - (i)$ *and that* a *belongs to Class 11. Then equation* (4) *has a heteroclinic solution from* $-\alpha$ *to* α *in* $C^2(\mathbb{R})$ *such that*

(a) $q(t) = -q(-t)$ *for any* $t \in \mathbb{R}$,

(b) $0 < q(t) < \alpha$ *for all* $t > 0$.

Theorem 0.6 holds for all $\alpha > 0$ and complements the study done in Section 1.1 of Chapter 1, because $a = 1$ there. Furthermore, Theorem 0.7 also complements some papers on the study of heteroclinic solutions, because here we are considering a new class of functions a that allows to be null in a symmetric interval compact in $\mathbb{R}$, and Theorem 0.5 complements the study made in [23], because in that article the authors considered the case $\inf_{\mathbb{R}} a(t) > 0$ and applied variational methods in the space $BV_{loc}(\mathbb{R})$, while here we use variational methods in the Orlicz-Sobolev spaces by adapting for our case some ideas found in [74], $\inf_{\mathbb{R}} a(t) = 0$ and we prove some results involving the uniqueness of heteroclinic solution for (12) when $a(t)$ is constant.

In Chapter 4, we present another joint paper with Professor Claudianor [15]. The main goal of this chapter is to use variational methods to show the existence of heteroclinic solutions for prescribed mean curvature equation of the type

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + A(\epsilon x, y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2 \tag{14}$$

taking into account different geometric conditions on function $A : \mathbb{R}^2 \to \mathbb{R}$ with $\epsilon > 0$. Throughout Chapter 4, we mean by heteroclinic solution a function u that is a weak solution of (14) and has the following asymptotic property at infinity

$$u(x,y) \to \alpha \text{ as } x \to -\infty \text{ and } u(x,y) \to \beta \text{ as } x \to +\infty \text{ uniformly in } y \in \mathbb{R},$$

where α and β are global minima of $V : \mathbb{R} \to \mathbb{R}$ that satisfies the following assumptions:

$(\tilde{V}_1)$ $V \in C^1(\mathbb{R}, \mathbb{R})$.

$(\tilde{V}_2)$ $\alpha < \beta$ and $V(\alpha) = V(\beta) = 0$.

$(\tilde{V}_3)$ $V(t) \geq 0$ for any $t \in \mathbb{R}$ and $V(t) > 0$ for all $t \in (\alpha, \beta)$.

$(\tilde{V}_4)$ There are $\lambda > 0$ and $C(\lambda) > 0$ such that $\sup_{t \in (\alpha,\beta)} |V'(t)| \leq C(\lambda)$ when $\max\{|\alpha|, |\beta|\} \in (0, \lambda)$.

We would like to point out that the condition $(\tilde{V}_4)$ is uniform with respect to the roots α and β of potentials V and a class of such potentials of the Ginzburg-Landau type for which $(\tilde{V}_1)$-$(\tilde{V}_4)$ are satisfied is

$$V(t) = (t-\alpha)^2(t-\beta)^2. \tag{15}$$

Moreover, when $\alpha = -\beta$, another class of potentials V of the Sine-Gordon type can be given by

$$V(t) = \beta + \beta \cos\left(\frac{t\pi}{\beta}\right). \tag{16}$$

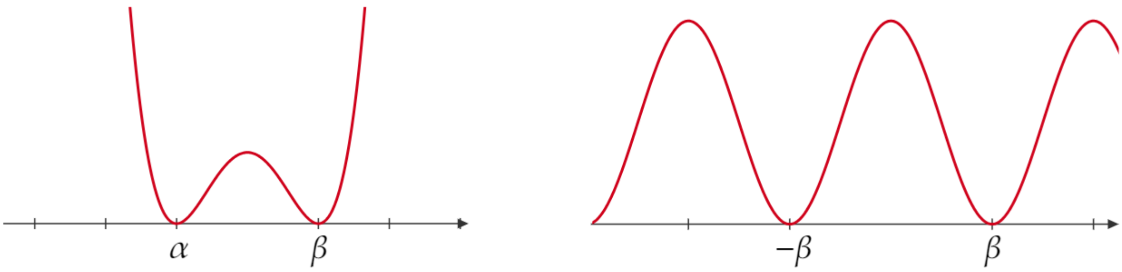

Figure 6: The potentials $V(t) = (t-\alpha)^2(t-\beta)^2$ and $V(t) = \beta + \beta \cos\left(\frac{t\pi}{\beta}\right)$ respectively.

These kinds of potentials arise in various branches of Mathematical Physics, for example in models of phase transitions in binary metallic alloys and propagation of dislocations in crystals, respectively, where the prototype of these models can be represented by stationary Allen-Cahn type equations (1). Generally the introduction of a factor $A(x)$ can be used to study inhomogeneous materials. For a deeper discussion of these applications, we refer the interested reader to [11, 44].

In what follows, associated with function A we assume the assumptions:

$(\tilde{A}_1)$ A is continuous and there is $A_0 > 0$ such that $A(x,y) \geq A_0$ for all $(x,y) \in \mathbb{R}^2$.

$(\tilde{A}_2)$ $A(x,y) = A(x,-y)$ for all $(x,y) \in \mathbb{R}^2$.

$(\tilde{A}_3)$ $A(x,y) = A(x,y+1)$ for any $(x,y) \in \mathbb{R}^2$.

Now let us mention the classes of A that we will considered in this thesis.

Class A: A satisfies $(\tilde{A}_1)$-$(\tilde{A}_3)$ and is 1-periodic on the variable x.

Class B: A satisfies $(\tilde{A}_1)$-$(\tilde{A}_3)$ and there exists a continuous function $A_p : \mathbb{R}^2 \to \mathbb{R}$, which is 1-periodic on x, satisfying $A(x,y) < A_p(x,y)$ for all $(x,y) \in \mathbb{R}^2$ and

$$|A(x,y) - A_p(x,y)| \to 0 \text{ as } |(x,y)| \to +\infty.$$

Class C: A satisfies $(\tilde{A}_1)$-$(\tilde{A}_3)$ and

$$\inf_{\mathbb{R}^2} A(x,y) \leq \sup_{y\in[0,1]} A(0,y) < \liminf_{|(x,y)|\to+\infty} A(x,y) = A_\infty < +\infty.$$

Class D: A satisfies $(\tilde{A}_2)$-$(\tilde{A}_3)$, is a continuous non-negative function, even in x, $A \in L^\infty(\mathbb{R}^2)$ and there exists $K > 0$ such that

$$\inf_{|x|\geq K,\ y\in[0,1]} A(x,y) > 0.$$

We would like to highlight that some of these conditions are well known in the context of the Laplacian operator. For example, a condition like Class A was studied by Rabinowitz [79] to show the existence of heteroclinic solution for a class of second order partial differential equations in which he includes the equation of the form

$$-\Delta u + A(x,y)V'(u) = 0 \ \text{ in } \ \Omega, \tag{17}$$

where the set Ω is a cylindrical domain in $\mathbb{R}^n$ given by $\Omega = \mathbb{R}\times D$ with D being a bounded open set in $\mathbb{R}^{n-1}$ such that $\partial D \in C^1$. In the literature we also find interesting works that study the equation (17) in the case that $A(x,y)$ is periodic in all variables when $\Omega = \mathbb{R}^2$, see for example Rabinowitz and Stredulinsky [82] and Alessio, Gui and Montecchiari [4]. Related to the Classes B and C we cite a paper by Alves [13], where the author established the existence of classical solutions of (17) on a cylindrical domain that are heteroclinic in the variable x. Finally, the Class D was introduced in [14].

The main results of this chapter can be now stated in the following form.

Theorem 0.8 *Assume $(\tilde{V}_1)$-$(\tilde{V}_4)$, $\epsilon = 1$ and that A belongs to Class A or B. Given $L > 0$ there exists $\delta > 0$ such that if $\max\{|\alpha|,|\beta|\} \in (0,\delta)$ then equation (14) possesses a heteroclinic solution $u_{\alpha,\beta}$ from α to β in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, satisfying*

(a) $u_{\alpha,\beta}$ is 1-periodic on y.

(b) $\alpha \le u_{\alpha,\beta}(x,y) \le \beta$ for any $(x,y) \in \mathbb{R}^2$.

(c) $\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}$.

Moreover, if $V \in C^2(\mathbb{R},\mathbb{R})$ then the inequalities in (b) are strict.

Theorem 0.9 *Assume $(\tilde{V}_1)$-$(\tilde{V}_4)$ and that A belongs to Class C. There is $\epsilon_0 > 0$ such that for each $\epsilon \in (0,\epsilon_0)$ and $L > 0$ there exists $\delta > 0$ such that if $\max\{|\alpha|,|\beta|\} \in (0,\delta)$ then equation (14) possesses a heteroclinic solution $u_{\alpha,\beta}$ from α to β in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, verifying*

(a) $u_{\alpha,\beta}$ is 1-periodic on y.

(b) $\alpha \le u_{\alpha,\beta}(x,y) \le \beta$ for any $(x,y) \in \mathbb{R}^2$.

(c) $\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}$.

Moreover, if $V \in C^2(\mathbb{R},\mathbb{R})$ occurs then the inequalities in (b) are strict.

Demanding a little more of the potential V we may relax the conditions on the function A to ensure the existence of a heteroclinic solution for (14), as the following result says.

Theorem 0.10 *Assume $V \in C^2(\mathbb{R},\mathbb{R})$, $(\tilde{V}_2)$-$(\tilde{V}_4)$ with $\alpha = -\beta$, $\epsilon = 1$ and that A belongs to Class D. Moreover, assume (V_2) and*

$(\tilde{V}_5)$ $V''(-\beta), V''(\beta) > 0$.

Then, for each $L > 0$ there exists $\delta > 0$ such that if $\beta \in (0,\delta)$ then equation (14) possesses a heteroclinic solution u_β from $-\beta$ to β in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, verifying

- *(a) $u_\beta(x,y) = -u_\beta(-x,y)$ for any $(x,y) \in \mathbb{R}^2$.*
- *(b) $u_\beta(x,y) = u_\beta(x,y+1)$ for all $(x,y) \in \mathbb{R}^2$.*
- *(c) $0 < u_\beta(x,y) < \beta$ for $x > 0$.*
- *(d) $\|\nabla u_\beta\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}$.*

The reader is invited to see that the above theorems are true for the Ginzburg-Landau (15) and Sine-Gordon (16) potentials when roots α and β have a small distance between them.

Motivated by the ideas of Chapter 3, in the proof of the theorems above, we truncate the differential operator involved in (17) of such way may that the new operator can be seen as a quasilinear operator in divergence form. For this reason, as a first step in the present chapter, we study quasilinear equations of the form

$$-\Delta_\Phi u + A(\epsilon x, y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{18}$$

where Φ is an N-function of the form (6) with $\phi : (0,+\infty) \to [0,+\infty)$ being a C^1 function verifying the conditions (ϕ_1)-(ϕ_3). The solutions of (18) are found as minima of the action functional

$$I(w) = \sum_{j\in\mathbb{Z}} \left(\int_0^1 \int_j^{j+1} (\Phi(|\nabla w|) + A(\epsilon x, y)V(w))\, dxdy \right)$$

on the class of admissible functions

$$\Gamma_\Phi(\alpha,\beta) = \left\{ w \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}\times(0,1)) : \tau_k w \to \alpha\ (\beta) \text{ in } L^\Phi((0,1)\times(0,1)) \text{ as } k \to -\infty\ (+\infty) \right\}.$$

Our results involving the quasilinear equation (18) are stated below:

Theorem 0.11 *Assume (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$, $\epsilon = 1$ and that A belongs to Class A or B. Then equation (18) has a heteroclinic solution from α to β in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$ such that*

- *(a) $u(x,y) = u(x,y+1)$ for any $(x,y) \in \mathbb{R}^2$.*

(b) $\alpha \leq u(x,y) \leq \beta$ *for all* $(x,y) \in \mathbb{R}^2$.

Moreover, taking into account the assumptions (ϕ_3) *and*

$(\tilde{V}_6)$ *There are* $d_1, d_2, d_3, d_4 > 0$ *and* $\lambda > 0$ *such that*

$$|V'(t)| \leq d_1 \phi(d_2|t-\beta|)|t-\beta| \text{ for all } t \in [\beta-\lambda, \beta+\lambda]$$

and

$$|V'(t)| \leq d_3 \phi(d_4|t-\alpha|)|t-\alpha| \text{ for all } t \in [\alpha-\lambda, \alpha+\lambda],$$

then the inequalities in (b) are strict.

Theorem 0.12 *Assume* (ϕ_1)*-*(ϕ_2)*,* $(\tilde{V}_1)$*-*$(\tilde{V}_3)$ *and that* A *belongs to Class C. Then, there is a constant* $\epsilon_0 > 0$ *such that for each* $\epsilon \in (0, \epsilon_0)$ *equation (18) has a heteroclinic solution from* α *to* β *in* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that*

(a) $u(x,y) = u(x, y+1)$ *for any* $(x,y) \in \mathbb{R}^2$.

(b) $\alpha \leq u(x,y) \leq \beta$ *for all* $(x,y) \in \mathbb{R}^2$.

Moreover, assuming (ϕ_3) *and* $(\tilde{V}_6)$ *we have that the inequalities in (b) are strict.*

Theorem 0.13 *Assume* (ϕ_1)*-*(ϕ_2)*,* $(\tilde{V}_1)$*-*$(\tilde{V}_3)$ *and* (V_2) *with* $\alpha = -\beta$*,* $\epsilon = 1$ *and that* A *belongs to Class D. Also consider the following assumption*

$(\tilde{V}_7)$ *There are* $\mu > 0$ *and* $\theta \in (0, \beta)$ *such that*

$$\mu \Phi(|t-\beta|) \leq V(t), \quad \forall t \in (\beta-\theta, \beta+\theta).$$

Then equation (18) possesses a heteroclinic solution u *from* $-\beta$ *to* β *in* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that*

(a) $u(x,y) = -u(-x,y)$ *for any* $(x,y) \in \mathbb{R}^2$.

(b) $u(x,y) = u(x,y+1)$ *for all* $(x,y) \in \mathbb{R}^2$.

(c) $0 \leq u(x,y) \leq \beta$ *for any* $x > 0$ *and* $y \in \mathbb{R}$.

Moreover, if (ϕ_3) *and* $(\tilde{V}_6)$ *occur then the inequalities in (c) are strict.*

Here it is worth mentioning that an example of potential V that satisfies the conditions $(\tilde{V}_1)$-$(\tilde{V}_7)$ is given by

$$V(t) = \Phi(|(t-\alpha)(t-\beta)|), \tag{19}$$

where Φ is an N-function of the type (6) verifying (ϕ_1)-(ϕ_2). Moreover, the classical case $\Phi(t) = \dfrac{t^2}{2}$ corresponds to the Laplacian operator, and in this case, as we are considering a new class of functions A, we can rewrite Theorem 0.13 as follows

Theorem 0.14 *Assume $\alpha = -\beta$, $V \in C^2(\mathbb{R}, \mathbb{R})$, $(\tilde{V}_2)$-$(\tilde{V}_3)$, $(\tilde{V}_5)$, (V_2) and that A belongs to Class D. Then equation (17) with $\Omega = \mathbb{R}^2$ possesses a heteroclinic (classical) solution u from $-\beta$ to β such that*

(a) $u(x, y) = -u(-x, y)$ *for any* $(x, y) \in \mathbb{R}^2$.

(b) $u(x, y) = u(x, y + 1)$ *for all* $(x, y) \in \mathbb{R}^2$.

(c) $0 < u(x, y) < \beta$ *for any* $x > 0$ *and* $y \in \mathbb{R}$.

We now point out some interactions of our results with other works already known in the literature. For example, Theorems 0.8, 0.9 and 0.10 complement the study carried out in [14] and [23], because in those articles the authors considered the one-dimensional equation (12), while we treat (14) and investigated the existence of a heteroclinic solution for (14) for other classes of functions A. Moreover, Theorems 0.11 and 0.12 complement the results obtained in [13], because in that paper the author considered the Laplacian operator while here we considered a large class of quasilinear operators.

Finally, in Chapter 5 we introduce the study of the paper [62], which combines the arguments developed in previous chapters to study the existence and qualitative properties of saddle solutions for some classes of mean curvature equations prescribed as follows

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + A(x, y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{20}$$

The main theorems of this chapter are listed below.

Theorem 0.15 *Assume $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2), (V_7), and (A_1)-(A_4). Given $L > 0$ there exists $\delta > 0$ such that if $\alpha \in (0, \delta)$ then the prescribed mean curvature equation (20) possesses a weak solution $v_{\alpha,L}$ in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0, 1)$, satisfying the following properties:*

(a) $0 < v_{\alpha,L}(x, y) < \alpha$ *on the fist quadrant in* $\mathbb{R}^2$,

(b) $v_{\alpha,L}(x, y) = -v_{\alpha,L}(-x, y) = -v_{\alpha,L}(x, -y)$ *for all* $(x, y) \in \mathbb{R}^2$,

(c) $v_{\alpha,L}(x, y) = v_{\alpha,L}(y, x)$ *for any* $(x, y) \in \mathbb{R}^2$,

(d) $v_{\alpha,L}(x, y) \to \alpha$ *as* $x \to \pm\infty$ *and* $y \to \pm\infty$,

(e) $v_{\alpha,L}(x, y) \to -\alpha$ *as* $x \to \mp\infty$ *and* $y \to \pm\infty$,

(f) $\|\nabla v_{\alpha,L}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$.

When $A(x,y)$ is a positive constant, we obtain an infinite number of geometrically distinct saddle-type solutions to the equation (20). This fact is reported in the following result.

Theorem 0.16 *Assume $V \in C^2(\mathbb{R},\mathbb{R})$, (V_1)-(V_2), (V_7), and that $A(x,y)$ is a positive constant. Then, given $L>0$ there is $\delta>0$ such that if $\alpha \in (0,\delta)$ then for each $j \geq 2$ the prescribed mean curvature equation (20) possesses a weak solution $v_{\alpha,L,j}$ in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, satisfying*

(a) $0 < \tilde{v}_{\alpha,L,j}(\rho,\theta) < \alpha$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *and* $\rho > 0$,

(b) $\tilde{v}_{\alpha,L,j}(\rho, \frac{\pi}{2}+\theta) = -\tilde{v}_{\alpha,L,j}(\rho, \frac{\pi}{2}-\theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$,

(c) $\tilde{v}_{\alpha,L,j}(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_{\alpha,L,j}(\rho,\theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$,

(d) $\tilde{v}_{\alpha,L,j}(\rho,\theta) \to (-\alpha)^{k+1}$ *as* $\rho \to +\infty$ *whenever* $\theta \in \left(\frac{\pi}{2} + k\frac{\pi}{j}, \frac{\pi}{2} + (k+1)\frac{\pi}{j}\right)$ *for* $k = 0, \ldots, 2j-1$,

(e) $\|\nabla v_{\alpha,L,j}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$,

where $\tilde{v}_{\alpha,L,j}(\rho,\theta) = v_{\alpha,L,j}(\rho\cos(\theta), \rho\sin(\theta))$.

The solutions $v_{\alpha,L,j}$ described in the theorem above are characterized by the fact that, along different directions parallel to the end lines, they are uniformly asymptotic to $\pm\alpha$ and such solutions may appropriately be termed "pizza solutions". Moreover, to prove Theorems 0.15 and 0.16 it was necessary to extend the results of Chapters 1 and 2 on saddle solutions for a larger class of N-functions and the interested reader can immediately consult Section 5.1.

In Appendix A, we write some results involving Orlicz and Orlicz-Sobolev spaces for unfamiliar readers with the topic. Such results are crucial for a good understanding of this work.

This thesis ends with Appendix B, where we detail some properties about a class of double well potentials, which were frequently mentioned throughout the text.

To end this introduction, we would like to point out that other interesting results of this thesis were not listed here, however, the interested reader will be able to find such results throughout the chapters.

INTRODUÇÃO

O problema de existência e classificação das soluções limitadas das equações estacionárias do tipo Allen-Cahn

$$-\Delta u + A(x)V'(u) = 0 \quad \text{em} \quad \mathbb{R}^n \tag{21}$$

tem sido amplamente estudado nos últimos anos, fornecendo uma rica quantidade de famílias de soluções com formatos diferentes, tais como solução periódica, heteroclínica, sela e multibump. A equação de Allen-Cahn foi introduzida em 1979 por S. Allen e J. Cahn em [11] como um modelo para transições de fase em ligas binárias. O modelo padrão de V é o clássico potencial de poço duplo de Ginzburg-Landau

$$V(t) = \frac{1}{4}(t^2-1)^2, \quad t \in \mathbb{R}.$$

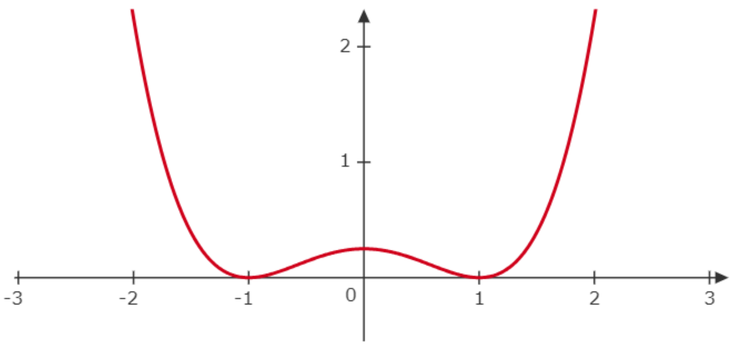

Figure 7: O potencial de poço duplo $V(t) = \frac{1}{4}(t^2-1)^2$.

A função u é um parâmetro de fase que descreve pontualmente o estado do material e os mínimos globais de V representam as fases puras do sistema. Diferentes valores de u retratam configurações mistas e por soluções de transição entendemos soluções inteiras de (21) que são assintóticas em diferentes direções para as fases puras do sistema. Na equação (21) a presença do fator oscilatório (positivo) $A(x)$ modela um comportamento não homogêneo do sistema.

Quando A é uma função constante positiva (por exemplo, $A(x) = 1$), um problema de longa data é caracterizar o conjunto das soluções $u \in C^2(\mathbb{R}^n)$ de (21) satisfazendo $|u(x)| \leq 1$ e $\partial_{x_1} u(x) > 0$. Este problema foi apontado por De Giorgi em [34], onde ele conjecturou que, quando $n \leq 8$ e $V(t) = (t^2 - 1)^2$, todo o conjunto dessas soluções se reduz, sob translações, à única solução $q_+ \in C^2(\mathbb{R})$ do problema unidimensional

$$-q''(t) + V'(q(t)) = 0 \quad \text{em} \quad \mathbb{R}, \quad q(0) = 0 \quad \text{e} \quad \lim_{t \to \pm\infty} q(t) = \pm 1.$$

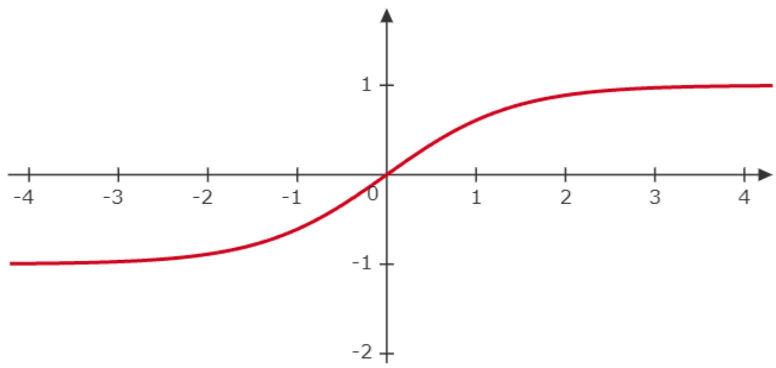

Figure 8: O gráfico de q_+.

A conjectura foi primeiramente provada no caso planar por Ghoussoub e Gui em [57] mesmo para um potencial de poço duplo mais geral V. No caso $n = 3$ foi provado por Ambrosio e Cabrè em [19] e, assumindo

$$u(x) \to \pm 1 \quad \text{quando} \quad x_1 \to \pm\infty,$$

o mesmo resultado de rigidez foi obtido na dimensão $n \leq 8$ por Savin em [88], artigo ao qual nos referimos também para uma extensa bibliografia sobre o argumento. Del Pino, Kowalczyk e Wei mostraram em [37, 38] que a simetria 1-D dessas soluções é geralmente perdida quando $n \geq 9$. Nos referimos também a [21, 22, 40], onde uma versão mais fraca da conjectura de De Giorgi, conhecida como conjectura de Gibbons, foi obtida para todas as dimensões n e em configurações mais gerais. Esses resultados mostram que quando A é uma constante positiva e u é uma solução limitada de (21) satisfazendo

$$u(x) \to \pm 1 \quad \text{quando} \quad x_1 \to \pm\infty \quad \text{uniformemente em relação a} \quad (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \tag{22}$$

então $u(x) = q_+(x_1)$ para todo $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

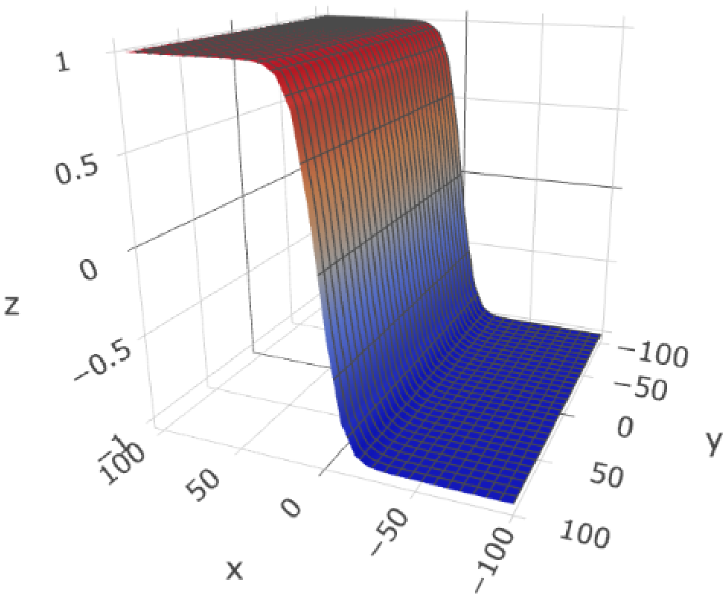

Figure 9: Gráfico de u em $\mathbb{R}^3$

Soluções de (21)-(22) são ditas heteroclínicas de -1 a 1. O estudo da existência e propriedades qualitativas das soluções heteroclínicas e várias generalizações tem sido amplamente estudado e recebido atenção especial nos últimos anos, pois esse tipo de solução aparece em muitos modelos matemáticos associados a problemas que aparecem na Física, Biologia, Mecânica e Química. Geralmente as soluções heteroclínicas aparecem como processos físicos envolvendo transições variáveis de um equilíbrio instável para um estável, propagação frontal em equações de reação-difusão e transição de fase. Para um relato bastante abrangente, o leitor interessado pode começar lendo os artigos [24, 69] e suas referências. Por exemplo, uma descrição simples de soluções heteroclínicas pode ser encontrada na equação matemática do pêndulo

$$q''(t) + b\sin(q(t)) = 0 \quad \text{em} \quad \mathbb{R},$$

em que $b > 0$ depende da aceleração da gravidade e do comprimento da barra. Neste caso, a análise do plano de fase mostra que existe uma solução heteroclínica q de $-\pi$ a π, isto é,

$$\lim_{t\to-\infty} q(t) = -\pi, \quad \lim_{t\to+\infty} q(t) = \pi \quad \text{e} \quad \lim_{t\to\pm\infty} q'(t) = 0.$$

Fisicamente, esta solução corresponde a movimentos assintóticos ao equilíbrio vertical instável. Para uma discussão mais profunda veja [24, 70].

Este tipo de soluções de transição do tipo heteroclínica persiste quando A não é constante em (21). O problema do tipo heteroclínico foi estudado pela primeira vez por métodos variacionais para equações elípticas mais gerais da forma

$$-\Delta u = g(x, y, u) \text{ em } \mathbb{R} \times \Omega \tag{23}$$

por Rabinowitz em [79], quando Ω é um domínio regular limitado em $\mathbb{R}^n$. Assumindo que a não-linearidade g seja par e periódica na variável x, Rabinowitz mostrou a existência de soluções para (23) satisfazendo a condição de contorno de Dirichlet ou Neumann em $\partial\Omega$ e sendo assintótica quando $x \to \pm\infty$ a diferentes soluções mínimas $u_{\pm}$, periódicas na variável x. Este resultado foi generalizado por Alves em [13] para diferentes condições em g, inclusive para o caso em que g é apenas assintoticamente periódico na variável x. Uma abordagem variacional relacionada foi usada para estudar o problema do tipo heteroclínico para a equação (21) no caso em que A é periódico em todas variáveis em [5, 82, 83], mostrando a existência de soluções (mínimas) $u(x)$ que são periódicas na variável $(x_2, \ldots x_n)$ e tais que u é assintótica a diferentes mínimos do potencial V quando $x_1 \to \pm\infty$. Partindo da existência dessas soluções heteroclínicas "básicas", esses artigos mostram como a presença de um fator verdadeiramente oscilatório $A(x, y)$ dá genericamente a existência de classes complexas de outras soluções de transição do tipo heteroclínica em contraste com as resultados de rigidez acima descritos caracterizando o caso autônomo (ver também [9, 27, 84]).

Quando referido à equação semilinear

$$-q''(t) + a(t)V'(q(t)) = 0 \text{ em } \mathbb{R}, \tag{24}$$

o problema de existência de soluções heteroclínicas é um tópico clássico na teoria das equações diferenciais ordinárias. Nos últimos anos tem havido um grande número de trabalhos que estudam a existência de solução heteroclínica para (24) considerando diferentes classes de funções $a(t)$. Por exemplo:

Classe 1: [24] $a(t)$ é uma constante positiva.

Classe 2: [24] $a(t)$ é uma função contínua tal que

$$0 < \inf_{t \in \mathbb{R}} a(t) \text{ e } a(t+1) = a(t) \text{ para todo } t \in \mathbb{R}.$$

Classe 3: [24, 54] $a(t)$ é uma função contínua tal que existem $a_1, a_2 > 0$ verificando

$$a_1 \leq a(t) \leq a_2, \ \forall t \in \mathbb{R} \ \text{ e } \ a(t) \to a_2 \text{ quando } |t| \to +\infty,$$

onde $a(t) < a_2$ em algum conjunto de medida diferente de zero.

Classe 4: [12, 54] $a(t)$ é assintoticamente periódica no infinito, ou seja, existe uma função periódica contínua $a_p : \mathbb{R} \to \mathbb{R}$ satisfazendo

$$|a(t) - a_p(t)| \to 0 \ \text{ quando } \ |t| \to +\infty \ \text{ e } \ 0 < \inf_{t\in\mathbb{R}} a(t) \leq a(t) < a_p(t) \ \ \forall t \in \mathbb{R}.$$

Classe 5: [12] $a \in L^\infty(\mathbb{R})$ e

$$0 < a(0) = \inf_{t\in\mathbb{R}} a(t) < a_\infty = \liminf_{|t|\to+\infty} a(t).$$

Classe 6: [52, 53] $a \in L^\infty(\mathbb{R})$ e existem $l, L \in (0, +\infty)$ tais que

$$l \leq a(t) \leq L \text{ quase em todos os lugares e } a(t) \to L \text{ quando } |t| \to +\infty$$

onde L/l é adequadamente limitado por cima.

Classe 7: [52] $a(t)$ é par e existem $l, L \in (0, +\infty)$ tais que

$$l \leq a(t) \leq L \text{ quase em todos os lugares em } \mathbb{R}.$$

Classe 8: [52, 53] $a \in L^\infty(\mathbb{R}, [0, +\infty)$ e existem $l > 0$, $S < T$, tais que

$$a(t) = l \text{ para } t \notin [S, T].$$

Classe 9: [55] Existe $t_0 \in \mathbb{R}$ tal que $a(t)$ é crescente em $(-\infty, t_0]$ e $a(t)$ é decrescente em $[t_0, +\infty)$. Além disso,

$$\lim_{|t|\to+\infty} a(t) = l > 0 \quad \text{e} \quad \lim_{|t|\to+\infty} |t|(l - a(t)) = 0.$$

Classe 10: [90] Existem $l, \underline{l} > 0$ tais que

$$a(t) \to l \ \text{ quando } \ |t| \to +\infty \ \text{ e } \ \underline{l} \leq a(t) \leq L \equiv l + 4\nu\sqrt{l\underline{l}}/\int_{-1}^{1}\sqrt{V(s)}ds \text{ para todo } t \in \mathbb{R},$$

onde

$$\nu = \min\left\{\int_{-1}^{\epsilon_-} \sqrt{V(s)}ds, \int_{\epsilon_+}^{1} \sqrt{V(s)}ds\right\},$$

com $\epsilon_- = \min\{s : s > -1,\ V'(s) = 0\}$ e $\epsilon_+ = \max\{s : s < 1\ V'(s) = 0\}$.

Outro tipo de solução de transição para (21) foi introduzido por Dang, Fife e Peletier em [33]. No caso planar $n = 2$, quando V é um potencial de poço duplo par e A é uma constante positiva, eles mostraram por um método de sub-supersolução que (21) tem uma única solução limitada $u \in C^2(\mathbb{R}^2)$ com o mesmo sinal de x_1x_2, ímpar em ambas as variáveis x_1 e x_2 e simétrica em relação às diagonais $x_2 = \pm x_1$. Ao longo de quaisquer direções não paralelas aos eixos coordenados, a solução *sela* u é assintótica aos mínimos do potencial V representando uma transição de fase com interface cruzada. Observe que, mesmo que esteja relacionado a soluções heteroclínicas de transição mínimas, sendo assintótico a q_+ quando $x_2 \to +\infty$, não possui mais caráter minimal (ver [63, 89]). Muitas extensões para modelos de Allen-Cahn foram consideradas. No caso planar, nos referimos a [3] para um estudo variacional de soluções do tipo sela com simetrias diedrais de ordem k (veja também [61] para uma abordagem variacional global para o problema de sela) e a [39, 59] para um estudo geral sobre soluções k-end. Outras generalizações do estudo de soluções do tipo sela foram feitas em dimensões maiores. Por exemplo, em [6] e [7], Alessio e Montecchiari estabeleceram a existência de soluções do tipo sela em $\mathbb{R}^3$. Em [2], Alama, Bronsard e Gui estudaram uma versão vetorial de solução do tipo sela, onde sistemas de equações autônomas de Allen-Cahn foram considerados no plano (ver [60] e [8] para estudos relacionados em $\mathbb{R}^3$). Uma generalização da estrutura variacional considerada em [2] pode ser encontrada em [10]. Para outros artigos interessantes em dimensão superior, mencionamos [28, 29, 77] para o caso das equações e [2] para o caso de sistemas de equações de Allen-Cahn autônomos.

O análogo para soluções do tipo sela para (21) no caso planar, quando $A \in C(\mathbb{R}^2)$ é positivo, par, periódico e simétrico em relação ao plano diagonal $x_2 = x_1$ foi introduzido em [4] onde um procedimento variacional foi introduzido para encontrar como no caso autônomo uma solução u de (21) em $\mathbb{R}^2$ que é ímpar com respeito a ambas as suas variáveis, simétrica em relação à diagonal, estritamente positivo no primeiro quadrante e é assintótica aos mínimos de V ao longo de quaisquer direções não paralelas aos eixos coordenados. Além disso em [4] mostra-se que, quando $y \to +\infty$ (uniformemente em relação a $x \in \mathbb{R}$), a solução u é assintótica ao conjunto dos x-ímpares soluções mínimas do tipo heteroclínica de (21) que são periódicas na variável y.

Ao longo desta tese, abordamos o problema da existência de soluções heteroclínicas e do tipo sela através dos métodos variacionais para o análogo do modelo de Allen-Cahn na configuração quaselinear da forma

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \quad \text{em} \quad \mathbb{R}^2, \tag{25}$$

em que $\Delta_\Phi u = \operatorname{div}(\phi(|\nabla u|)\nabla u)$ e $\Phi : \mathbb{R} \to [0, +\infty)$ é uma N-função do tipo

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds \tag{26}$$

para uma $\phi \in C^1([0, +\infty), [0, +\infty))$ tal que:

(ϕ_1) $\phi(t) > 0$ e $(\phi(t)t)' > 0$ para qualquer $t > 0$.

(ϕ_2) Existem $l, m \in \mathbb{R}$ com $1 < l \leq m$ tais que

$$l - 1 \leq \frac{(\phi(t)t)'}{\phi(t)} \leq m - 1 \quad \text{para todo} \quad t > 0.$$

(ϕ_3) Existem constantes $c_1, c_2, \eta > 0$ e $s > 1$ satisfazendo

$$c_1 t^{s-1} \leq \phi(t)t \leq c_2 t^{s-1} \quad \text{para} \quad t \in (0, \eta).$$

(ϕ_4) ϕ é não-decrescente em $(0, +\infty)$.

Gostaríamos de salientar que no estudo de problemas elípticos quasilineares conduzidos pelo operador Φ-Laplaciano, as condições (ϕ_1)-(ϕ_2) são bem conhecidas e garantem que Φ e sua função complementar $\tilde{\Phi}$ são N-funções que verificam a chamada condição Δ_2 (ver o Apêndice A por um momento). Essas condições garantem que Φ se comporte de tal forma que o espaço de Orlicz-Sobolev associado a Φ seja reflexivo e separável.

Nos últimos anos, diante da necessidade de uma descrição matemática de problemas físicos, houve um número crescente de trabalhos envolvendo o operador Φ-Laplaciano Δ_Φ e sua teoria já está bastante desenvolvida. Como primeiro exemplo podemos considerar o caso

$$\Phi(t) = |t|^p, \quad t \in \mathbb{R}, \quad p \in (1, +\infty),$$

que está relacionado ao célebre operador p-Laplaciano que frequentemente aparece em modelos físicos, por exemplo, em fluidos newtonianos e não-newtonianos (ver [35, 36] e referências neles contidas). Motivado por exemplos concretos de equações decorrentes da mecânica dos fluidos e da teoria de plasticidade, Seregin e Fuchs em [46, 47] (ver também [45]) foram levados à minimização de integrais em que aparece o modelo logarítmico

$$\Phi(t) = |t|^p \ln(1 + |t|), \quad t \in \mathbb{R}, \quad p \in [1, +\infty),$$

que é uma N-função do tipo (24). Outro modelo de N-função na forma (24) que frequentemente surge em muitos campos da física e ciências relacionadas, como biofísica e design de reações químicas, é

$$\Phi(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q, \quad t \in \mathbb{R}, \quad 1 < p < q < +\infty.$$

O operador diferencial associado com esta N-função é conhecido como o operador (p, q)-Laplaciano e o protótipo destes modelos pode ser escrito na forma

$$u_t = -\Delta_\Phi + f(x, u).$$

Nesta configuração, a função u geralmente descreve uma concentração, Δ_Φ corresponde à difusão e $f(x, u)$ é o termo de reação que corresponde aos processos fonte e perda. Para um relato bastante abrangente, o leitor interessado pode começar referindo-se a [20, 41]. Finalmente, vale mencionar que a N-função dada por

$$\Phi(t) = (1 + t^2)^\gamma - 1, \quad t \in \mathbb{R}, \quad \gamma > 1,$$

aparece nos trabalhos [49, 50], onde os autores relatam que os estudos de equações quasilineares envolvendo o operador associado Δ_Φ são motivados por modelos de elasticidade não linear. Para outros exemplos de N-funções do tipo (24) e mais aplicações, recomendamos ao leitor que consulte [45, 48] e a bibliografia neles contida.

Esta tese é uma coletânea de artigos publicados e submetidos listados abaixo:

(P1) *Existence of saddle-type solutions for a class of quasilinear problems in* $\mathbb{R}^2$, Topol. Methods Nonlinear Anal., 61(2), 2023, 825-868. (com Claudianor Alves and Piero Montecchiari).

(P2) *Existence of heteroclinic and saddle type solutions for a class of quasilinear problems in whole* $\mathbb{R}^2$, Commun. Contemp. Math., 2022. (com Claudianor Alves and Piero Montecchiari).

(P3) *Existence of heteroclinic solutions for the prescribed curvature equation*, J. Differential Equations, 362, 2023, 484-513. (com Claudianor Alves).

(P4) *Heteroclinic solutions for some classes of prescribed mean curvature equations in whole* $\mathbb{R}^2$, preprint. (com Claudianor Alves).

(P5) *Saddle solutions for Allen-Cahn type equations involving the prescribed mean curvature operator*, in preparation.

Gostaríamos de enfatizar que cada artigo é apresentado como um capítulo e a exposição dos capítulos varia um pouco das apresentações dos artigos a fim de complementar os estudos performados lá.

Outro artigo que complementa esta tese é o seguinte:

(P6) *Uniqueness of heteroclinic solutions in a class of autonomous quasilinear ODE problems*, preprint. (com Claudianor Alves and Piero Montecchiari).

A seguir, descrevemos a organização desta tese e apresentamos um breve panorama dos temas estudados nos capítulos.

No Capítulo 1, apresentamos o artigo conjunto com os professores Claudianor Alves e Piero Montecchiari [16]. O principal objetivo deste capítulo é provar a existência de soluções do tipo sela para a seguinte classe de equações quasilineares

$$-\Delta_\Phi u + V'(u) = 0 \text{ em } \mathbb{R}^2, \tag{27}$$

em que o potencial V satisfaz as seguintes condições:

(V_1) $V(t) \geq 0$ para todo $t \in \mathbb{R}$ e $V(t) = 0 \Leftrightarrow t = -\alpha, \alpha$ para $\alpha > 0$.

(V_2) $V(-t) = V(t)$ para qualquer $t \in \mathbb{R}$.

(V_3) Existem $\delta_\alpha \in (0, \alpha)$ e $w_1, w_2 > 0$ tais que

$$w_1\Phi(|t-\alpha|) \leq V(t) \leq w_2\Phi(|t-\alpha|) \quad \forall t \in (\alpha - \delta_\alpha, \alpha + \delta_\alpha).$$

(V_4) Existem $\omega_1, \omega_2 > 0$ tais que

$$V'(t) \leq -\omega_1\phi(\omega_2|\alpha - t|)|\alpha - t|t \quad \forall t \in [0, \alpha].$$

(V_5) Existe $\delta_0 > 0$ tal que V' é crescente em $(\alpha - \delta_0, \alpha)$.

(V_6) Existem $\gamma, \epsilon > 0$ tais que

$$\tilde{\Phi}(V'(t)) \leq \gamma \Phi(|\alpha - t|), \quad \forall t \in (\alpha - \epsilon, \alpha).$$

Vale a pena mencionar que um exemplo importante de um potencial V que verifica as condições (V_1)-(V_6) é dado por

$$V(t) = \Phi(|t^2 - \alpha^2|), \quad t \in \mathbb{R},$$

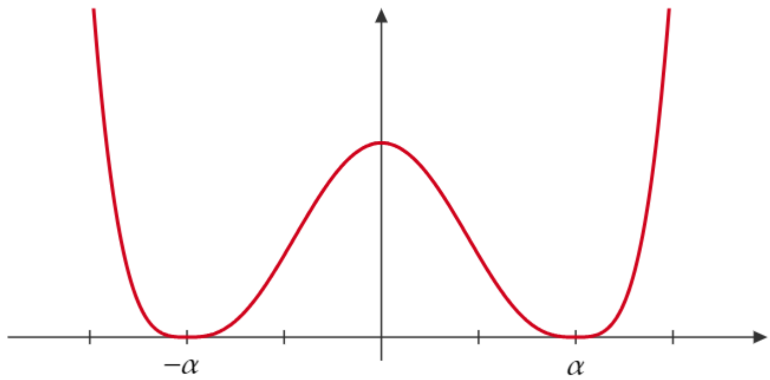

Figure 10: O potencial $V(t) = \Phi(|t^2 - \alpha^2|)$.

em que Φ é uma N-função da forma (26) verificando (ϕ_1)-(ϕ_2), que foi inspirada no clássico potencial de poço duplo de Ginzburg-Landau $V(t) = (t^2 - 1)^2$.

Nosso principal resultado envolvendo soluções do tipo sela é o seguinte:

Teorema 0.1 *Assuma (ϕ_1)-(ϕ_4) e (V_1)-(V_6). Então, para cada $j \geq 2$ existe $v_j \in C^{1,\gamma}_{loc}(\mathbb{R}^2)$ para algum $\gamma \in (0,1)$ tal que v_j é uma solução fraca de (27) satisfazendo*

(a) $0 < \tilde{v}_j(\rho, \theta) < \alpha$ *para qualquer* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *e* $\rho > 0$,

(b) $\tilde{v}_j(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_j(\rho, \frac{\pi}{2} - \theta)$ *para todo* $(\rho, \theta) \in [0, +\infty) \times \mathbb{R}$,

(c) $\tilde{v}_j(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_j(\rho, \theta)$ *para todo* $(\rho, \theta) \in [0, +\infty) \times \mathbb{R}$,

(d) $\tilde{v}_j(\rho, \theta) \to \alpha$ *quando* $\rho \to +\infty$ *para todo* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$,

em que $\tilde{v}_j(\rho, \theta) = v_j(\rho \cos(\theta), \rho \sin(\theta))$.

O item (d) do Teorema 0.1 é uma caracterização do comportamento assintótico de v_j, o que garante que para $k = 0, ..., 2j-1$ haja

$$\tilde{v}_j(\rho,\theta) \to (-\alpha)^{k+1} \quad \text{quando} \quad \rho \to +\infty \quad \text{sempre que} \quad \theta \in \left(\frac{\pi}{2} + k\frac{\pi}{j}, \frac{\pi}{2} + (k+1)\frac{\pi}{j}\right).$$

Portanto, a solução do tipo sela pode ser vista como uma transição de fase com interface cruzada.

Na prova do Teorema 0.1 é crucial provar a existência e unicidade da solução heteroclínica ímpar mínima para

$$-(\phi(|q'|)q')' + V'(q) = 0 \quad \text{em} \quad \mathbb{R}. \tag{28}$$

Depois disso, usamos as soluções heteroclínicas como suporte para caracterizar o comportamento assintótico da solução tipo sela para (27). A principal ferramenta utilizada é o método variacional em espaços de Orlicz-Sobolev, mais precisamente, técnica de minimização em um conjunto de funções admissíveis. A ideia é buscar mínimos do funcional ação

$$F(q) = \int_{\mathbb{R}} (\Phi(|q'|) + V(q))\, dt$$

sobre a classe

$$E_\Phi = \left\{ q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}) : q \text{ é ímpar q.t.p em } \mathbb{R} \right\}.$$

Denotando por K_Φ o conjunto de mínimos de F em E_Φ, temos o seguinte resultado:

Teorema 0.2 *Assuma (ϕ_1)-(ϕ_3) e (V_1)-(V_6). Então, existe um único $q \in K_\Phi$ tal que é uma solução fraca de (28) sendo heteroclínica de $-\alpha$ para α, isto é,*

$$q(t) \to -\alpha \quad \textit{quando} \quad t \to -\infty \quad \textit{e} \quad q(t) \to \alpha \quad \textit{quando} \quad t \to +\infty.$$

Além disso, $q \in C^{1,\gamma}_{loc}(\mathbb{R})$ para algum $\gamma \in (0,1)$ e satisfaz as seguintes propriedades:

(a) $q(t) = -q(-t)$ para qualquer $t \in \mathbb{R}$,

(b) $0 < q(t) < \alpha$ para todo $t > 0$,

(c) q é crescente em $\mathbb{R}$,

(d) $q'(t) \to 0$ quando $t \to \pm\infty$,

(e) q' é não-crescente em $[0, +\infty)$,

(f) $q'(t) > 0$ para qualquer $t \in \mathbb{R}$.

Gostaríamos de ressaltar que o Teorema 0.1 complementa o estudo feito em [3], pois naquele artigo os autores consideraram o operador Laplaciano, enquanto em nosso estudo consideramos uma grande classe de equações envolvendo operadores quaselineares. No entanto, é importante mencionar que algumas estimativas encontradas em [3] não podem ser utilizadas aqui, como por exemplo alguns princípios de máximo, regularidade C^2 para o operador Laplaciano bem como existência e unicidade de solução para equações diferenciais ordinárias de segunda ordem. Aqui, foi necessário desenvolver novas estimativas usando, por exemplo, uma desigualdade do tipo Harnack, regularidade $C^{1,\alpha}$ por Lieberman [67] e um novo resultado de unicidade para uma classe de equações diferenciais ordinárias conduzidas por operador quasilinear.

Agora, alguns resultados envolvendo soluções heteroclínicas e suas generalizações serão discutidos brevemente. Para problemas unidimensionais, gostaríamos de citar os artigos de Rabinowitz [80, 81] e Gavioli e Sanchez [56] e suas referências, em que o leitor pode encontrar resultados interessantes sobre a existência de soluções heteroclínicas para problemas relacionados. Outras generalizações do estudo de soluções do tipo heteroclínicas foram feitas em dimensões superiores, ver por exemplo Rabinowitz [79], Alves [13], Rabinowitz e Stredulinsky [82]. Relacionado a sistema elíptico citamos o artigo de Byeon, Montecchiari e Rabinowitz [27]. Na literatura também encontramos alguns artigos que estudam a existência de solução heteroclínica para classes de problemas quaselineares, veja por exemplo Feliz [71–73] e para uma versão vetorial, recomendamos o artigo por Ruan [86]. Por fim, para um relato recente sobre soluções heteroclínicas envolvendo o operador Laplaciano fracionário, encaminhamos o leitor para [30, 31] onde os autores mostraram a existência e unicidade do seguinte problema

$$(-\partial^2_{zz})^s q + q^3 - q = 0 \ \text{ em } \ \mathbb{R}, \quad q(0) = 0, \quad \lim_{t\to\pm\infty} q(t) = \pm 1, \quad q' > 0.$$

A existência de soluções heteroclínicas para problemas de dimensões superiores foi explorada por Alves, Ambrosio e Torres Ledesma [14].

No capítulo 2 apresentamos o artigo [17], que é um trabalho conjunto com os professores Claudianor Alves e Piero Montecchiari. Neste capítulo, estudando a existência de soluções heteroclínicas e do tipo sela fracas relacionadas da versão não autônoma da equação (27), que é dada por

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \quad \text{em} \quad \mathbb{R}^2, \tag{29}$$

em que $A : \mathbb{R}^2 \to \mathbb{R}$ satisfaz

(A_1) A é uma função contínua e $A(x,y) > 0$ para cada $(x,y) \in \mathbb{R}^2$,

(A_2) $A(x,y) = A(-x,y) = A(x,-y)$ para todo $(x,y) \in \mathbb{R}^2$,

(A_3) $A(x,y) = A(x+1,y) = A(x,y+1)$ para qualquer $(x,y) \in \mathbb{R}^2$,

(A_4) $A(x,y) = A(y,x)$ para todo $(x,y) \in \mathbb{R}^2$.

Um interessante modelo para A é dado por

$$A(x,y) = \cos(2\pi x)\cos(2\pi y) + c \text{ com } c > 1.$$

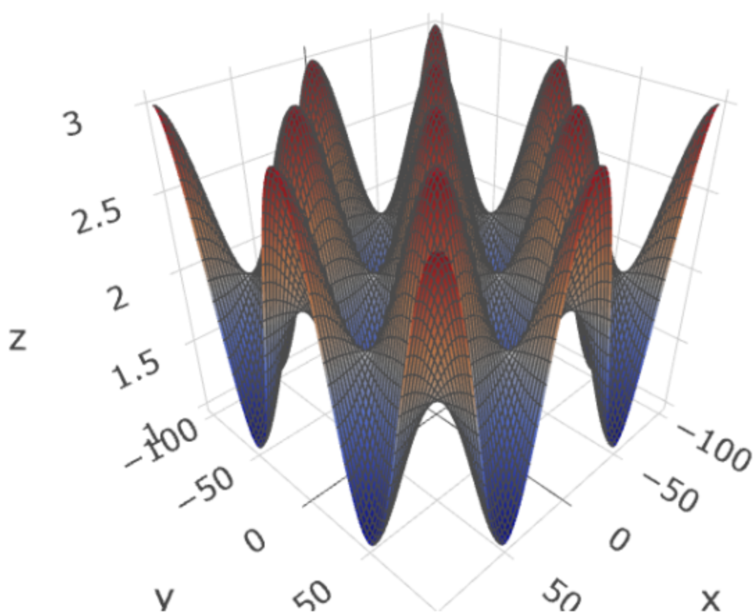

Figure 11: Gráfico de $A(x,y) = \cos(2\pi x)\cos(2\pi y) + 2$.

Neste capítulo, usamos métodos variacionais relacionados aos introduzidos em [4] e [16] para estabelecer a existência de (minimal) soluções do tipo heteroclínica de $-\alpha$ a α de (29), ou seja, soluções fracas $v \in C^{1,\beta}_{\text{loc}}(\mathbb{R}^2)$ que são 1-periódicas na variável y tais que

$$v(x,y) \to -\alpha \text{ quando } x \to -\infty \text{ e } v(x,y) \to \alpha \text{ quando } x \to +\infty \text{ uniformemente em } y \in \mathbb{R}.$$

Além disso, tomamos emprestadas algumas ideias desenvolvidas em [4] e [79] para procurar mínimos do funcional ação

$$I(u) = \int_{\mathbb{R}} \int_0^1 (\Phi(|\nabla u|) + A(x,y)V(u))\, dydx,$$

sobre a classe

$$E_\Phi(\alpha) = \left\{ u \in W^{1,\Phi}_{\text{loc}}(\mathbb{R} \times [0,1]) :\ 0 \leq u(x,y) \leq \alpha \text{ para } x > 0 \text{ e } u \text{ é ímpar em } x \right\},$$

em que $W^{1,\Phi}_{\text{loc}}(\mathbb{R} \times [0,1])$ denota o espaço Orlicz-Sobolev usual. Denotando por $K_\Phi(\alpha)$ o conjunto de mínimos de I em $E_\Phi(\alpha)$, mostramos que $K_\Phi(\alpha)$ não é vazio e é constituído por soluções (minimal) do tipo heteroclínica de (29). As propriedades de minimalidade destas soluções do tipo heteroclínica nos permitem, num segundo passo, construir um quadro variacional inspirado no introduzido em [4] para detectar a existência de soluções do tipo sela de (29), caracterizando seu o comportamento assintótico. Mais precisamente, temos os seguintes resultados:

Teorema 0.3 *Assuma (ϕ_1)-(ϕ_3), $V \in C^1(\mathbb{R}, \mathbb{R})$, (V_1)-(V_3) e (A_1)-(A_3). Então, existe $v \in C^{1,\beta}_{loc}(\mathbb{R}^2)$ para algum $\beta \in (0,1)$ tal que v é uma solução fraca de (29) que verifica o seguinte:*

(a) $v(x,y) = -v(-x,y)$ para todo $(x,y) \in \mathbb{R}^2$,

(b) $v(x,y) = v(x,y+1)$ para qualquer $(x,y) \in \mathbb{R}^2$,

(c) $0 < v(x,y) < \alpha$ para cada $x > 0$ e $y \in \mathbb{R}$.

Além disso, v é uma solução heteroclínica de $-\alpha$ a α.

Teorema 0.4 *Assuma (ϕ_1)-(ϕ_4), $V \in C^1(\mathbb{R}, \mathbb{R})$, (V_1)-(V_4) e (A_1)-(A_4). Então, existe $v \in C^{1,\beta}_{loc}(\mathbb{R}^2)$ para algum $\beta \in (0,1)$ tal que v é uma solução fraca de (29) que verifica o seguinte:*

(a) $0 < v(x,y) < \alpha$ sobre o primeiro quadrante em $\mathbb{R}^2$,

(b) $v(x,y) = -v(-x,y) = -v(x,-y)$ para todo $(x,y) \in \mathbb{R}^2$,

(c) $v(x,y) = v(y,x)$ para qualquer $(x,y) \in \mathbb{R}^2$,

(d) Existe $u_0 \in K_\Phi(\alpha)$ tal que $\|v - \tau_j u_0\|_{L^\infty(\mathbb{R}\times[j,j+1])} \to 0$ quando $j \to +\infty$,

em que $\tau_j u_0(x,y) = u_0(x, y-j)$ *para qualquer* $(x,y) \in \mathbb{R}^2$.

O item (d) do Teorema 0.4 caracteriza o comportamento assintótico de v. Ele garante que ao longo de direções paralelas aos eixos coordenados a solução do tipo sela seja assintótica ao conjunto heteroclínico minimal $K_\Phi(\alpha)$. Isso implica que ao longo de qualquer direção não paralela aos eixos coordenados v é assintótico no infinito para $\pm\alpha$ e, portanto, a solução do tipo sela pode ser vista como uma solução de transição de fase com interface cruzada.

Gostaríamos de pontuar que os Teoremas 0.3 e 0.4 melhoram os resultados do Capítulo 1 não apenas pelo fato de que a função $A(x,y)$ pode não ser constante, mas também porque, ao contrário do Capítulo 1, as suposições (V_5) e (V_6) não são necessárias. Além disso, notamos que embora a abordagem variacional seja inspirada naquela usada em [4], muitas ferramentas usadas no contexto do Laplaciano clássico, como por exemplo alguns princípios máximos, regularidade C^2, teoremas de existência e unicidade local, não estão mais disponíveis na estrutura atual. Para mostrar que (29) admite soluções do tipo transição foi necessário desenvolver novas estimativas baseadas nas desigualdades do tipo Harnack encontradas em [91] e em resultados sobre regularidade $C^{1,\alpha}$ para problemas quaselineares conforme obtido por Lieberman em [67].

Hoje em dia é um fato bem conhecido que o operador de curvatura média prescrita

$$div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \tag{30}$$

tem sido extensivamente estudado nos últimos anos, devido à estreita ligação com a teoria da capilaridade [43]. Após os trabalhos pioneiros de Young [92], Laplace [66] e Gauss [51] no início do século 18 sobre a curvatura média de uma superfície capilar, muito já foi produzido na literatura e é difícil e exaustivo mensurar aqui a vastidão de aplicações físicas envolvendo o operador (30), porém para o leitor interessado neste assunto, poderíamos citar aqui alguns problemas que aparecem em transporte ótimo [25] e nas superfícies mínimas [58]. Além disso, (30) também aparece em alguns problemas envolvendo processos de reação-difusão que ocorrem frequentemente em uma ampla variedade de configurações físicas e biológicas. Por exemplo, em [65], Kurganov e Rosenau observaram que quando a saturação da difusão é incorporada a esses processos, pode causar um impacto profundo na morfologia das transições que conectam os estados de equilíbrio, como agora não só equilíbrios descontínuos tornam-se permissíveis, mas ondas viajantes podem surgir em seu lugar. Uma classe específica de tais processos é modelada pela seguinte equação

$$u_t = div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - aV'(u), \tag{31}$$

em que a função de reação V é o clássico potencial de poço duplo de Ginzburg-Landau e a é uma constante. O impacto da difusão saturada em processos de difusão de reação foi investigado por eles na reta e no plano.

Conforme indicado no parágrafo anterior, [65] forneceu uma significativa motivação física para o estudo das equações da forma (31) tendo como principal objetivo a existência e classificações das soluções do tipo transição, ou seja, soluções inteiras de (31) que são assintóticos em diferentes direções aos estados de equilíbrio do sistema. Nesse sentido, Bonheure, Obersnel e Omari in [23] investigaram a existência de uma solução heteroclínica para a equação unidimensional

$$-\left(\frac{q'}{\sqrt{1+(q')^2}}\right)' + a(t)V'(q) = 0 \quad \text{em} \quad \mathbb{R}, \tag{32}$$

procurando mínimos de um funcional ação sobre um subconjunto convexo de $BV_{\text{loc}}(\mathbb{R})$ feito de todas as funções satisfazendo uma condição assintótica no infinito, em que os autores consideraram como usual V um potencial de poço duplo com mínimos em $t = \pm 1$ com a função a assintótica a uma função periódica positiva, ou seja, $a \in L^\infty(\mathbb{R})$ com $0 < \operatorname{ess\,inf}_{t\in\mathbb{R}} a(t)$ e existe $a^* \in L^\infty(\mathbb{R})$ τ-periódico, para algum $\tau > 0$, tal que $a(t) \leq a^*(t)$ quase em todos os lugares em $\mathbb{R}$ satisfazendo

$$\operatorname*{ess\,lim}_{|t|\to+\infty} (a^*(t) - a(t)) = 0.$$

No capítulo 3 apresentamos o trabalho [14], no qual é um artigo conjunto com o professor Claudianor Alves. Este capítulo trata da existência e propriedades qualitativas das soluções heteroclínicas da equação de curvatura prescrita (32). A ideia básica é truncar o operador de curvatura média para construir uma estrutura variacional inspirado naquele introduzido em [74] no espaço Orlicz-Sobolev $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$, isto é, para obter uma equação auxiliar da forma

$$-(\phi(|q'|)q')' + a(t)V'(q) = 0 \quad \text{em} \quad \mathbb{R} \tag{33}$$

em que $\phi : (0, +\infty) \to (0, +\infty)$ é uma função C^1 que verifica (ϕ_1)-(ϕ_3), a fim de estabelecer a existência de solução heteroclínica para (32) no caso em que a função a pertença à seguinte classe de funções

Classe 11: $a \in L^\infty(\mathbb{R})$ é uma função par não negativa que satisfaz

$$0 < a_0 := \inf_{t \geq M} a(t) \quad \text{para algum} \quad M > 0.$$

Ao longo deste capítulo, dizemos que uma função q é uma *solução heteroclínica de* $-\alpha$ *a* α para (32) ((33)) se $q \in C^{1,\beta}_{\text{loc}}(\mathbb{R})$ para algum $\beta \in (0,1)$ e satisfaz a equação (32) ((33)) para todo $t \in \mathbb{R}$, e Além disso,

$$\lim_{t \to -\infty} q(t) = -\alpha, \quad \lim_{t \to +\infty} q(t) = \alpha \quad \text{e} \quad \lim_{t \to +\pm\infty} q'(t) = 0.$$

Nosso principal resultado neste capítulo é o seguinte:

Teorema 0.5 *Assuma que a pertence à Classe 11,* $V \in C^2(\mathbb{R}, \mathbb{R})$*,* (V_1)*-*(V_2) *e que*

(V_7) *(i)* $V''(\pm\alpha) > 0$*.*

(ii) Existem $\tilde{\alpha} > 0$ *e* $C = C(\tilde{\alpha}) > 0$ *tais que* $\sup_{|t| \in [0,\alpha]} |V'(t)| \leq C$ *para todo* $\alpha \in (0, \tilde{\alpha})$*.*

Então, para cada $L > 0$ *existe* $\alpha_0 > 0$ *tal que para cada* $\alpha \in (0, \alpha_0)$ *a equação* (32) *possui uma solução heteroclínica* q_α *de* $-\alpha$ *a* α *satisfazendo:*

(a) $q_\alpha(t) = -q_\alpha(-t)$ *para todo* $t \in \mathbb{R}$*,*

(b) $0 < q_\alpha(t) < \alpha$ *para todo* $t > 0$*,*

(c) $|q'_\alpha(t)| < \sqrt{L}$ *para qualquer* $t \in \mathbb{R}$*.*

A suposição (V_7)-(ii) é uma condição uniforme nos potenciais V que depende de $\alpha > 0$ e uma classe de tais potenciais do tipo Ginzburg-Landau para os quais (V_1)-(V_2) e (V_7) são todos satisfeitos é

$$V(t) = (t^2 - \alpha^2)^2, \quad \alpha > 0,$$

que inclui o potencial duplo clássico de Ginzburg-Landau quando $\alpha = 1$. O leitor está convidado a ver que o teorema acima é verdadeiro para os potenciais do tipo Ginzburg-Landau, por exemplo, quando α é pequeno.

Nosso segundo resultado principal é o seguinte:

Teorema 0.6 *Assuma* (ϕ_1)*-*(ϕ_2)*,* $V \in C^1(\mathbb{R}, \mathbb{R})$*,* (V_1)*-*(V_3) *e que a pertence à Classe 11. Então a equação* (33) *tem uma solução heteroclínica de* $-\alpha$ *a* α *satisfazendo* *(a)* $q(t) = -q(-t)$ *para qualquer* $t \in \mathbb{R}$*,*

(b) $0 \leq q(t) \leq \alpha$ *para todo* $t > 0$*.*

Além disso, levando em consideração as suposições (ϕ_3) *e*

(V_8*) Existem* $d_1, d_2 > 0$ *e* $\lambda > 0$

$$|V'(t)| \leq d_1\phi(d_2|t-\alpha|)|t-\alpha| \text{ para todo } t \in [\alpha-\lambda, \alpha+\lambda],$$

então as desigualdades em (b) são estritas.

Além disso, o caso clássico $\Phi(t) = \dfrac{t^2}{2}$ corresponde à equação (24), e neste caso o Teorema 0.6 pode ser escrito da seguinte maneira

Teorema 0.7 *Assuma* $V \in C^2(\mathbb{R}, \mathbb{R})$*,* (V_1)*-*(V_2)*,* $(V_5)-(i)$ *e que a pertence à Classe 11. Então a equação (24) tem uma solução heteroclínica de* $-\alpha$ *a* α *em* $C^2(\mathbb{R})$ *tal que*

(a) $q(t) = -q(-t)$ *para qualquer* $t \in \mathbb{R}$*,*

(b) $0 < q(t) < \alpha$ *para todo* $t > 0$*.*

O Teorema 0.6 vale para todo $\alpha > 0$ e complementa o estudo feito na Seção 1.1 do Capítulo 1, porque $a = 1$ lá. Além disso, o Teorema 0.7 também complementa alguns artigos sobre o estudo de soluções heteroclínicas, porque aqui estamos considerando uma nova classe de funções a que permite ser nula em um intervalo simétrico compacto em $\mathbb{R}$, e o Teorema 0.6 complementa o estudo feito em [23], pois nesse artigo os autores consideraram o caso $\inf_{\mathbb{R}} a(t) > 0$ e aplicaram métodos variacionais no espaço $BV_{loc}(\mathbb{R})$, enquanto aqui usamos métodos variacionais nos espaços de Orlicz-Sobolev adaptando para o nosso caso algumas ideias encontradas em [74], $\inf_{\mathbb{R}} a(t) = 0$ e provamos alguns resultados envolvendo a unicidade de solução heteroclínica para (32) quando $a(t)$ é constante.

No capítulo 4, apresentamos outro artigo conjunto com o professor Claudianor [15]. O principal objetivo deste capítulo é usar métodos variacionais para mostrar a existência de soluções heteroclínicas para a equação de curvatura média prescrita do tipo

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + A(\epsilon x, y)V'(u) = 0 \quad \text{em} \quad \mathbb{R}^2 \tag{34}$$

levando em consideração diferentes condições geométricas sobre a função $A:\mathbb{R}^2\to\mathbb{R}$ com $\epsilon>0$. Ao longo do Capítulo 4, entendemos por solução heteroclínica uma função u que é solução fraca de (34) e tem a seguinte propriedade assintótica no infinito

$$u(x,y)\to\alpha \text{ quando } x\to-\infty \text{ e } u(x,y)\to\beta \text{ quando } x\to+\infty \text{ uniformemente em } y\in\mathbb{R},$$

em que α e β são mínimos globais de $V:\mathbb{R}\to\mathbb{R}$ que satisfazem as seguintes suposições:

($\tilde{V}_1$) $V\in C^1(\mathbb{R},\mathbb{R})$.

($\tilde{V}_2$) $\alpha<\beta$ e $V(\alpha)=V(\beta)=0$.

($\tilde{V}_3$) $V(t)\geq 0$ para qualquer $t\in\mathbb{R}$ e $V(t)>0$ para todo $t\in(\alpha,\beta)$.

($\tilde{V}_4$) Existem $\lambda>0$ e $C(\lambda)>0$ tais que $\sup\limits_{t\in(\alpha,\beta)}|V'(t)|\leq C(\lambda)$ quando $\max\{|\alpha|,|\beta|\}\in(0,\lambda)$.

Gostaríamos de salientar que a condição ($\tilde{V}_4$) é uniforme em relação às raízes α e β dos potenciais V e uma classe de tais potenciais do tipo Ginzburg-Landau para os quais ($\tilde{V}_1$)-($\tilde{V}_4$) são satisfeitos é

$$V(t)=(t-\alpha)^2(t-\beta)^2. \tag{35}$$

Além disso, quando $\alpha=-\beta$, outra classe de potenciais V do tipo Sine-Gordon pode ser dada por

$$V(t)=\beta+\beta\cos\left(\frac{t\pi}{\beta}\right). \tag{36}$$

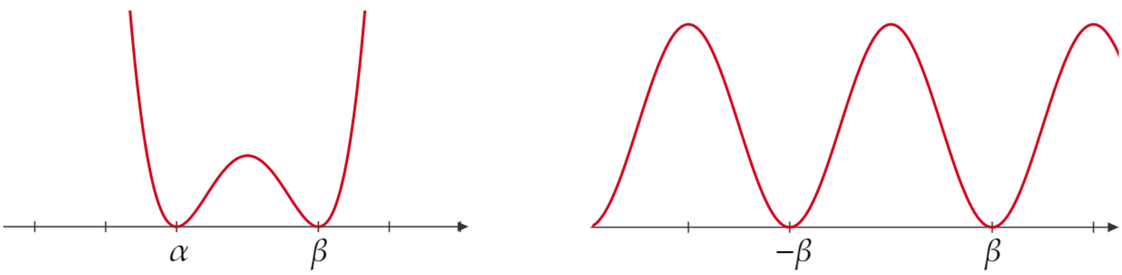

Figure 12: Os potenciais $V(t)=(t-\alpha)^2(t-\beta)^2$ e $V(t)=\beta+\beta\cos\left(\frac{t\pi}{\beta}\right)$ respectivamente.

Esses tipos de potenciais surgem em vários campos da Física Matemática, por exemplo em modelos de transições de fase em ligas metálicas binárias e propagação de deslocamentos em cristais, respectivamente, em que o protótipo desses modelos pode ser representado por equações estacionárias do tipo Allen-Cahn (21). Geralmente a introdução de um fator $A(x)$ pode ser usada para estudar materiais não homogêneos. Para uma discussão mais profunda dessas aplicações, nós encaminhamos o leitor interessado para [11, 44].

A seguir, associados à função A assumimos as suposições:

$(\tilde{A}_1)$ A é contínua e existe $A_0 > 0$ tal que $A(x, y) \geq A_0$ para todo $(x, y) \in \mathbb{R}^2$.

$(\tilde{A}_2)$ $A(x, y) = A(x, -y)$ para todo $(x, y) \in \mathbb{R}^2$.

$(\tilde{A}_3)$ $A(x, y) = A(x, y + 1)$ para qualquer $(x, y) \in \mathbb{R}^2$.

Agora vamos citar as classes de A que iremos considerar nesta tese.

Classe A: A satisfaz $(\tilde{A}_1)$-$(\tilde{A}_3)$ e é 1-periódica na variável x.

Classe B: A satisfaz $(\tilde{A}_1)$-$(\tilde{A}_3)$ e existe uma função contínua $A_p : \mathbb{R}^2 \to \mathbb{R}$, que é 1-periódica em x, satisfazendo $A(x, y) < A_p(x, y)$ para todo $(x, y) \in \mathbb{R}^2$ e

$$|A(x, y) - A_p(x, y)| \to 0 \text{ quando } |(x, y)| \to +\infty.$$

Classe C: A satisfaz $(\tilde{A}_1)$-$(\tilde{A}_3)$ e

$$\inf_{\mathbb{R}^2} A(x, y) \leq \sup_{y \in [0,1]} A(0, y) < \liminf_{|(x,y)| \to +\infty} A(x, y) = A_\infty < +\infty.$$

Classe D: A satisfaz $(\tilde{A}_2)$-$(\tilde{A}_3)$, é uma função contínua não negativa, par em x, $A \in L^\infty(\mathbb{R}^2)$ e existe $K > 0$ tal que

$$\inf_{|x| \geq K, \; y \in [0,1]} A(x, y) > 0.$$

Gostaríamos de destacar que algumas dessas condições são bem conhecidas no contexto do operador laplaciano. Por exemplo, uma condição como a Classe A foi estudada por Rabinowitz [79] para mostrar a existência de solução heteroclínica para uma classe de equações diferenciais parciais de segunda ordem na qual ele inclui a equação da forma

$$-\Delta u + A(x, y)V'(u) = 0 \quad \text{em} \quad \Omega, \tag{37}$$

em que o conjunto Ω é um domínio cilíndrico em $\mathbb{R}^n$ dado por $\Omega = \mathbb{R} \times D$ com D sendo conjunto aberto limitado em $\mathbb{R}^{n-1}$ tal que $\partial D \in C^1$. Na literatura também encontramos trabalhos interessantes que estudam a equação (37) no caso em que $A(x, y)$ é periódico em todas as variáveis quando $\Omega = \mathbb{R}^2$, veja por exemplo Rabinowitz and Stredulinsky [82] e Alessio, Gui and Montecchiari [4]. Relacionado às Classes B e C citamos um artigo de Alves [13], em que o autor estabeleceu a existência de soluções clássicas para (37) sobre um domínio cilíndrico que são heteroclínicas na variável x. Finalmente, a Classe D foi introduzida em [14]. Os principais resultados deste capítulo podem ser apresentados da seguinte forma.

Teorema 0.8 *Assuma $(\tilde{V}_1)$-$(\tilde{V}_4)$, $\epsilon = 1$ e que A pertence à Classe A ou B. Dado $L > 0$ existe $\delta > 0$ tal que se $\max\{|\alpha|, |\beta|\} \in (0, \delta)$ então a equação (34) possui uma solução heteroclínica $u_{\alpha,\beta}$ de α a β em $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, para algum $\gamma \in (0, 1)$, satisfazendo*

(a) $u_{\alpha,\beta}$ é 1-periódica em y.

(b) $\alpha \le u_{\alpha,\beta}(x, y) \le \beta$ para qualquer $(x, y) \in \mathbb{R}^2$.

(c) $\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}$.

Além disso, se $V \in C^2(\mathbb{R}, \mathbb{R})$ então as desigualdades em (b) são estritas.

Teorema 0.9 *Assuma $(\tilde{V}_1)$-$(\tilde{V}_4)$ e que A pertence à Classe C. Existe $\epsilon_0 > 0$ tal que para cada $\epsilon \in (0, \epsilon_0)$ e $L > 0$ existe $\delta > 0$ tal que se $\max\{|\alpha|, |\beta|\} \in (0, \delta)$ então a equação (34) possui uma solução heteroclínica $u_{\alpha,\beta}$ de α a β em $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, para algum $\gamma \in (0, 1)$, verificando*

(a) $u_{\alpha,\beta}$ é 1-periódica em y.

(b) $\alpha \le u_{\alpha,\beta}(x, y) \le \beta$ para qualquer $(x, y) \in \mathbb{R}^2$.

(c) $\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}$.

Além disso, se $V \in C^2(\mathbb{R}, \mathbb{R})$ ocorre então as desigualdades em (b) são estritas.

Exigindo um pouco mais do potencial V podemos relaxar as condições sobre a função A para garantir a existência de uma solução heteroclínica para (34), como diz o seguinte resultado.

Teorema 0.10 *Assuma $V \in C^2(\mathbb{R}, \mathbb{R})$, $(\tilde{V}_2)$-$(\tilde{V}_4)$ com $\alpha = -\beta$, $\epsilon = 1$ e que A pertence à Classe D. Além disso, assuma (V_2) e*

($\tilde{V}_5$) $V''(-\beta), V''(\beta) > 0$.

Então, para cada $L > 0$ *existe* $\delta > 0$ *tal que se* $\beta \in (0, \delta)$ *então a equação* (34) *possui uma solução heteroclínica* u_β *de* $-\beta$ *a* β *em* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$*, para algum* $\gamma \in (0,1)$*, verificando*

(a) $u_\beta(x, y) = -u_\beta(-x, y)$ *para qualquer* $(x, y) \in \mathbb{R}^2$.

(b) $u_\beta(x, y) = u_\beta(x, y + 1)$ *para todo* $(x, y) \in \mathbb{R}^2$.

(c) $0 < u_\beta(x, y) < \beta$ *para* $x > 0$.

(d) $\|\nabla u_\beta\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$.

O leitor é convidado a ver que os teoremas acima são verdadeiros para os potenciais de Ginzburg-Landau (35) e Sine-Gordon (36) quando as raízes α e β têm uma pequena distância entre elas.

Motivados pelas ideias do Capítulo 3, na prova dos teoremas acima, truncamos o operador diferencial envolvido em (37) de tal maneira que o novo operador pode ser visto como um operador quasilinear na forma de divergência. Por esta razão, como primeiro passo no presente capítulo, estudamos equações quaselineares da forma

$$-\Delta_\Phi u + A(\epsilon x, y)V'(u) = 0 \quad \text{em} \quad \mathbb{R}^2, \tag{38}$$

em que Φ é uma N-função da forma (26) com $\phi : (0, +\infty) \to (0, +\infty)$ sendo uma função C^1 verificando as condições (ϕ_1)-(ϕ_3). As soluções de (38) são encontradas como mínimos do funcional ação

$$I(w) = \sum_{j \in \mathbb{Z}} \left(\int_0^1 \int_j^{j+1} (\Phi(|\nabla w|) + A(\epsilon x, y)V(w))\, dxdy \right)$$

sobre a classe de funções admissíveis

$$\Gamma_\Phi(\alpha,\beta) = \left\{ w \in W^{1,\Phi}_{\text{loc}}(\mathbb{R} \times (0,1)) : \tau_k w \to \alpha(\beta) \text{ em } L^\Phi((0,1) \times (0,1)) \text{ quando } k \to -\infty(+\infty) \right\}.$$

Nossos resultados envolvendo a equação quaselinear (38) são apresentados abaixo:

Teorema 0.11 *Assuma* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$, $\epsilon = 1$ *e que* A *pertence à Classe A ou B. Então a equação* (38) *tem uma solução heteroclínica de* α *a* β *em* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *para algum* $\gamma \in (0, 1)$ *tal que*

(a) $u(x, y) = u(x, y + 1)$ *para qualquer* $(x, y) \in \mathbb{R}^2$.

(b) $\alpha \leq u(x,y) \leq \beta$ *para todo* $(x,y) \in \mathbb{R}^2$.

Além disso, levando em consideração as suposições (ϕ_3) *e*

$(\tilde{V}_6)$ *Existem* $d_1, d_2, d_3, d_4 > 0$ *e* $\lambda > 0$ *tal que*

$$|V'(t)| \leq d_1\phi(d_2|t-\beta|)|t-\beta| \text{ para todo } t \in [\beta-\lambda, \beta+\lambda]$$

e

$$|V'(t)| \leq d_3\phi(d_4|t-\alpha|)|t-\alpha| \text{ para todo } t \in [\alpha-\lambda, \alpha+\lambda],$$

então as desigualdades em (b) são estritas.

Teorema 0.12 *Assuma* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$ *e que A pertence à Classe C. Então, existe uma constante* $\epsilon_0 > 0$ *tal que para cada* $\epsilon \in (0, \epsilon_0)$ *a equação (38) tem uma solução heteroclínica de* α *a* β *em* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *para algum* $\gamma \in (0,1)$ *tal que*

(a) $u(x,y) = u(x,y+1)$ *para qualquer* $(x,y) \in \mathbb{R}^2$.

(b) $\alpha \leq u(x,y) \leq \beta$ *para todo* $(x,y) \in \mathbb{R}^2$.

Além disso, assumindo (ϕ_3) *e* $(\tilde{V}_6)$ *temos que as desigualdades em (b) são estritas.*

Teorema 0.13 *Assuma* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$ *e* (V_2) *com* $\alpha = -\beta$, $\epsilon = 1$ *e que A pertence à Classe D. Considere também a seguinte suposição*

$(\tilde{V}_7)$ *Existem* $\mu > 0$ *e* $\theta \in (0, \beta)$ *tais que*

$$\mu\Phi(|t-\beta|) \leq V(t), \quad \forall t \in (\beta-\theta, \beta+\theta).$$

Então, a equação (38) possui uma solução heteroclínica u de $-\beta$ *a* β *em* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *para alguma* $\gamma \in (0,1)$ *tal que*

(a) $u(x,y) = -u(-x,y)$ *para qualquer* $(x,y) \in \mathbb{R}^2$.

(b) $u(x,y) = u(x,y+1)$ *para todo* $(x,y) \in \mathbb{R}^2$.

(c) $0 \leq u(x,y) \leq \beta$ *para qualquer* $x > 0$ *e* $y \in \mathbb{R}$.

Além disso, se (ϕ_3) *e* $(\tilde{V}_6)$ *ocorrem então as desigualdades em (c) são estritas.*

Aqui vale a pena mencionar que um exemplo de potencial V que satisfaz as condições $(\tilde{V}_1)$-$(\tilde{V}_7)$ é dado por

$$V(t) = \Phi(|(t-\alpha)(t-\beta)|), \tag{39}$$

em que Φ é uma N-função do tipo (26) verificando (ϕ_1)-(ϕ_2). Além disso, o caso clássico $\Phi(t) = \dfrac{t^2}{2}$ corresponde ao operador Laplaciano, e neste caso, como estamos considerando uma nova classe de funções A, podemos reescrever o Teorema 0.13 da seguinte forma

Teorema 0.14 *Assuma $\alpha = -\beta$, $V \in C^2(\mathbb{R}, \mathbb{R})$, $(\tilde{V}_2)$-$(\tilde{V}_3)$, $(\tilde{V}_5)$, (V_2) e que A pertence à Classe D. Então a equação (37) com $\Omega = \mathbb{R}^2$ possui uma solução heteroclínica (clássica) u de $-\beta$ a β tal que*

(a) $u(x,y) = -u(-x,y)$ para qualquer $(x,y) \in \mathbb{R}^2$.

(b) $u(x,y) = u(x,y+1)$ para todo $(x,y) \in \mathbb{R}^2$.

(c) $0 < u(x,y) < \beta$ para qualquer $x > 0$ e $y \in \mathbb{R}$.

Apontamos agora algumas interações de nossos resultados com outros trabalhos já conhecidos na literatura. Por exemplo, os Teoremas 0.8, 0.9 e 0.10 complementam o estudo realizado em [14] e [23], porque nesses artigos os autores consideraram a equação unidimensional (12), enquanto tratamos de (14) e investigamos a existência de uma solução heteroclínica de (14) para outras classes de funções A. Além disso, os Teoremas 0.11 e 0.12 complementam os resultados obtidos em [13], porque naquele artigo o autor considerou o operador Laplaciano enquanto aqui consideramos uma grande classe de operadores quaselineares.

Finalmente, no Capítulo 5 apresentamos o artigo [62], que combina os argumentos desenvolvidos nos capítulos anteriores para estudar a existência e propriedades qualitativas de soluções de sela para algumas classes de equações de curvatura média prescritas como segue

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + A(x,y)V'(u) = 0 \quad \text{em} \quad \mathbb{R}^2. \tag{40}$$

Os principais teoremas deste capítulo estão listados abaixo.

Teorema 0.15 *Assuma $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2), (V_7), e (A_1)-(A_4). Dado $L > 0$ existe $\delta > 0$ tal que se $\alpha \in (0, \delta)$ então a equação de curvatura média prescrita (40) possui uma solução fraca $v_{\alpha,L}$ em $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, para algum $\gamma \in (0,1)$, satisfazendo as seguintes propriedades:*

(a) $0 < v_{\alpha,L}(x,y) < \alpha$ *no primeiro quadrante em* $\mathbb{R}^2$,

(b) $v_{\alpha,L}(x,y) = -v_{\alpha,L}(-x,y) = -v_{\alpha,L}(x,-y)$ *para todo* $(x,y) \in \mathbb{R}^2$,

(c) $v_{\alpha,L}(x,y) = v_{\alpha,L}(y,x)$ *para qualquer* $(x,y) \in \mathbb{R}^2$,

(d) $v_{\alpha,L}(x,y) \to \alpha$ *quando* $x \to \pm\infty$ *e* $y \to \pm\infty$,

(e) $v_{\alpha,L}(x,y) \to -\alpha$ *quando* $x \to \mp\infty$ *e* $y \to \pm\infty$,

(f) $\|\nabla v_{\alpha,L}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$.

Quando $A(x,y)$ é uma constante positiva, obtemos um número infinito de soluções do tipo sela geometricamente distintas para a equação (40). Este fato é relatado no seguinte resultado.

Teorema 0.16 *Assuma* $V \in C^2(\mathbb{R},\mathbb{R})$, *(*$V_1$*)-(*$V_2$*), (*$V_7$*), e que* $A(x,y)$ *é uma constante positiva. Então, dado* $L > 0$ *existe* $\delta > 0$ *tal que se* $\alpha \in (0,\delta)$ *então para cada* $j \geq 2$ *a equação de curvatura média prescrita (40) possui uma solução fraca* $v_{\alpha,L,j}$ *em* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, *para algum* $\gamma \in (0,1)$, *satisfazendo*

(a) $0 < \tilde{v}_{\alpha,L,j}(\rho,\theta) < \alpha$ *para qualquer* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *e* $\rho > 0$,

(b) $\tilde{v}_{\alpha,L,j}(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_{\alpha,L,j}(\rho, \frac{\pi}{2} - \theta)$ *para todo* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$,

(c) $\tilde{v}_{\alpha,L,j}(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_{\alpha,L,j}(\rho,\theta)$ *para todo* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$,

(d) $\tilde{v}_{\alpha,L,j}(\rho,\theta) \to (-\alpha)^{k+1}$ *quando* $\rho \to +\infty$ *sempre que* $\theta \in \left(\frac{\pi}{2} + k\frac{\pi}{j}, \frac{\pi}{2} + (k+1)\frac{\pi}{j}\right)$ *para* $k = 0, \ldots, 2j-1$,

(e) $\|\nabla v_{\alpha,L,j}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$,

em que $\tilde{v}_{\alpha,L,j}(\rho,\theta) = v_{\alpha,L,j}(\rho\cos(\theta), \rho\sin(\theta))$.

As soluções $v_{\alpha,L,j}$ descritas no teorema acima são caracterizadas pelo fato de que, ao longo de diferentes direções paralelas às linhas finais, elas são uniformemente assintóticas para $\pm\alpha$ e tais soluções podem serem apropriadamente denominados "soluções pizzas". Além disso, para provar os Teoremas 0.15 e 0.16 foi necessário estender os resultados dos Capítulos 1 e 2 sobre soluções de sela para uma classe maior de N-funções e o leitor interessado pode consultar imediatamente a Seção 5.1.

No Apêndice A, escrevemos alguns resultados envolvendo espaços de Orlicz e Orlicz-Sobolev para leitores não familiarizados com o assunto. Tais resultados são cruciais para uma boa compreensão deste trabalho.

Esta tese termina com o Apêndice B, em que detalhamos algumas propriedades sobre uma classe de potenciais de poço duplo, que foram frequentemente mencionados ao longo do texto.

Para finalizar esta introdução, gostaríamos de salientar que outros resultados interessantes desta tese não foram listados aqui, porém, o leitor interessado poderá encontrar tais resultados ao longo dos capítulos.

CHAPTER 1

SADDLE-TYPE SOLUTIONS FOR AUTONOMOUS QUASILINEAR EQUATIONS IN $\mathbb{R}^2$

In this chapter, we will show the existence of infinite saddle-type solutions for autonomous quasilinear elliptic equations of the form

$$-\Delta_\Phi u + V'(u) = 0 \text{ in } \mathbb{R}^2, \tag{1.1}$$

where $\Delta_\Phi u = \operatorname{div}(\phi(|\nabla u|)\nabla u)$, $\Phi : \mathbb{R} \to [0, +\infty)$ is an N-function of the form (6) satisfying (ϕ_1)-(ϕ_4) and $V : \mathbb{R} \to \mathbb{R}$ is a double-well potential with minima at $t = \pm\alpha$ satisfying $V \in C^2(\mathbb{R})$ and (V_1)-(V_6). An important prototype of V is the model $V(t) = \Phi(|t^2 - \alpha^2|)$. It is important to point out that a solution v of (1.1) is said to be saddle-type when v is a weak solution of (1.1) in $C^{1,\beta}_{\text{loc}}(\mathbb{R}^2)$ for some $\beta \in (0, 1)$ and the function $\tilde{v}(\rho, \theta) = v(\rho \cos(\theta), \rho \sin(\theta))$ satisfies for a certain $j \in \mathbb{N}$ the equalities below

$$\tilde{v}(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}(\rho, \frac{\pi}{2} - \theta) \text{ and } \tilde{v}(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}(\rho, \theta), \text{ for all } (\rho, \theta) \in [0, +\infty) \times \mathbb{R}.$$

In other words, a saddle solution is antisymmetric with respect to the half-line $\theta = \frac{\pi}{2}$ and $\frac{\pi}{j}$ is an antiperiodic in the angle variable. Moreover, the characterization of the asymptotic behavior of v is given by

$$\tilde{v}(\rho, \theta) \to \alpha \text{ as } \rho \to +\infty \text{ for any } \theta \in \left[\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2}\right).$$

To fulfill our objective in this chapter, we strongly resorted to a deep and detailed study of the quasilinear ordinary differential equation of the type

$$-\left(\phi(|q'|)q'\right)' + V'(q) = 0 \ \text{ in } \ \mathbb{R}. \tag{1.2}$$

We will see, in our argument, that this one-dimensional study applies an important rule to find and characterize the asymptotic behavior of saddle-type solutions.

1.1 Heteroclinic solution on $\mathbb{R}$

The purpose of this section is to use arguments from the calculus of variations to find a solution to equation (1.2), as well as to investigate issues such as uniqueness of the solution, qualitative properties, exponential estimates at infinity, and compactness properties.

1.1.1 The Cauchy problem

To begin with, let us consider the differential equation (1.2) provided with the following conditions

$$q(0) = q_0 \ \text{ and } \ q'(0) = q_0', \tag{1.3}$$

where q_0 and q_0' are real numbers. The conditions in (1.3) are called *initial conditions* and the problem

$$\begin{cases} (\phi(|q'(t)|)q'(t))' = V'(q(t)) \ \ t \in \mathbb{R}, \\ q(0) = q_0, \\ q'(0) = q_0', \end{cases} \tag{CP}$$

is said *Cauchy problem*. Here we would like to point out that when there is $q \in C^{1,\gamma}_{\text{loc}}(\mathbb{R})$, for some $\gamma \in (0,1)$, satisfying equation (1.2) punctually and checking the initial conditions (1.3) we simply say that q is a *solution to the Cauchy problem* (CP). In general, in the study of Cauchy problem, some questions arise, such as the existence of a solution q, its domain of definition and uniqueness when the solution exists. The following is our uniqueness result on the Cauchy problem (CP).

Theorem 1.1 *Assume that there exists a solution q for (CP) such that there are positive constants r and ρ satisfying:*

(a) $q'(t) \geq \rho$ *for any* $t \in (-r, r)$.

(b) $q \in W^{1,\infty}(\mathbb{R})$.

Then, q is unique in $(-r, r)$.

Proof. According to item (b), let us fix $L > 0$ such that

$$|q'(t)| \leq L \text{ for all } t \in \mathbb{R}.$$

Now, suppose that u is another solution of (CP). Setting

$$w(t) = \phi(|q'(t)|)q'(t) - \phi(|u'(t)|)u'(t), \quad t \in \mathbb{R},$$

a direct computation gives

$$w'(t) = \psi(t) \quad \text{and} \quad w(0) = 0,$$

where

$$\psi(t) = V'(q(t)) - V'(u(t)), \quad t \in \mathbb{R}.$$

Consequently,

$$w(t) = \int_0^t w'(s)ds = \int_0^t \psi(s)ds \quad \text{for} \quad t > 0$$

and

$$|w(t)| \leq t \max_{s \in [0,t]} |\psi(s)|, \quad t > 0. \tag{1.4}$$

On the other hand, as $V \in C^2(\mathbb{R})$, from item (b),

$$|\psi(t)| = |V'(q(t)) - V'(u(t))| \leq K|q(t) - u(t)|, \quad \forall t \in \mathbb{R},$$

for some $K > 0$. Hence, using the equality $q(0) = q_0 = u(0)$,

$$|\psi(t)| \leq K \int_0^t |q'(s) - u'(s)|ds \quad \forall t > 0. \tag{1.5}$$

Now, given $t \in (0, r)$, the item (a) ensures that $u'(t), q'(t) > 0$. Then, assuming that $q'(t) \leq u'(t)$ and using (ϕ_1),

$$\phi(u'(t))u'(t) - \phi(q'(t))q'(t) = \int_{q'(t)}^{u'(t)} (\phi(s)s)'ds \geq K_t|q'(t) - u'(t)|,$$

where

$$K_t = \inf_{q'(t) \le s \le u'(t)} (\phi(s)s)'.$$

Thereby, by definition of w,

$$|w(t)| \ge K_t |q'(t) - u'(t)| \quad \text{for} \quad t \in (0, r).$$

By (ϕ_2),

$$(\phi(t)t)' \ge \frac{l-1}{L}\phi(t)t \quad \text{for any} \quad t \in (0, L),$$

Consequently, by item (a),

$$(\phi(s)s)' \ge \frac{l-1}{L}\phi(q'(t))q'(t) \ge \frac{l-1}{L}\phi(\rho)\rho \quad \forall s \in [q'(t), u'(t)],$$

implying that $K_t \ge a$ where $a = \frac{l-1}{L}\phi(\rho)\rho$ for any $t \in (0, r)$, and so,

$$|q'(t) - u'(t)| \le \frac{1}{a}|w(t)|, \quad \forall t \in (0, r). \tag{1.6}$$

Gathering (1.5) and (1.6) we get

$$|\psi(t)| \le \frac{K}{a}\int_0^t |w(s)|ds, \quad \forall t \in (0, r)$$

that combines with (1.4) to provide

$$|w(t)| \le \frac{K}{a}t\int_0^t |w(s)|ds, \quad \forall t \in (0, r).$$

Fixing $A = \frac{K}{a}$ and $\chi(t) = \frac{w(t)}{t}$ for $t \in (0, r)$, we find

$$|\chi(t)| \le A\int_0^t |w(s)|ds = A\int_0^\epsilon |w(s)|ds + A\int_\epsilon^t s|\chi(s)|ds,$$

for any $0 < \epsilon < t < r$. Now, it follows from Gronwall's inequality (see [78, Theorem 1.2.2]) that

$$|\chi(t)| \le \left(A\int_0^\epsilon |w(s)|ds\right) e^{A\int_\epsilon^t sds} \quad \forall t \in (0, r).$$

Taking $\epsilon \to 0$ we find $w(t) = 0$ for each $t \in (0, r)$, and so, $\phi(q'(t))q'(t) = \phi(u'(t))u'(t)$, but since $\phi(t)t$ is increasing on $(0, +\infty)$ and $q(0) = u(0)$ we conclude that $q = u$ in $(0, r)$. The same argument works for $t \in (-r, 0)$, and the proof is completed. ∎

1.1.2 Existence of minimal solution

Our goal in this section is to use a minimization technique to find a heteroclinic solution from $-\alpha$ to α for quasilinear elliptic equation (1.2). In order to find odd solutions, let's consider the following class

$$E_\Phi = \left\{ u \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}) : u(t) = -u(-t) \text{ a.e. in } \mathbb{R} \right\},$$

where $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$ denotes the usual Orlicz-Sobolev space. Now, the functional[1] associated $F : W^{1,\Phi}_{\text{loc}}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ with equation (1.2) is given by

$$F(u) = \int_{\mathbb{R}} \mathcal{L}(u)dt, \quad \text{where} \quad \mathcal{L}(u) = \Phi(|u'|) + V(u).$$

First, we are going to show that the functional F is bounded from below. Indeed, by the definitions of Φ and V, $\mathcal{L}(u) \geq 0$ for all $u \in W^{1,\Phi}_{\text{loc}}(\mathbb{R})$, and so,

$$F(u) \geq 0 \quad \text{for any} \quad u \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}),$$

from where it follows that F is bounded from below. Since the main idea of this section is to show that F has a minimum on E_Φ, the question now is whether there exists $u \in E_\Phi$ such that $F(u) < +\infty$. To see this, it is simple to note that the function $\varphi_\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_\alpha(t) = \begin{cases} -\alpha, & \text{if} \quad t \leq -\alpha, \\ t, & \text{if} \quad -\alpha \leq t \leq \alpha, \\ \alpha, & \text{if} \quad t \geq \alpha, \end{cases} \tag{1.7}$$

belongs to E_Φ and satisfies $F(\varphi_\alpha) < +\infty$. Hence,

$$c_\Phi = \inf_{u \in E_\Phi} F(q)$$

is well defined.

With these preliminaries, let us now prove some estimates to show that the infimum of F on E_Φ is assumed.

Lemma 1.1 *If $u \in E_\Phi$ and $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$, then*

$$\Phi(|u(t_1) - u(t_2)|) \leq \frac{\xi_1(t_2 - t_1)}{t_2 - t_1} \int_{t_1}^{t_2} \Phi(|u'(t)|)dt,$$

where ξ_1 was fixed in Lemma A.2.

[1]The term functional is used to designate a real function whose field of definition is a subset of some space of functions.

Proof. First, from [26, Theorem 8.2],

$$|u(t_2) - u(t_1)| = \left| \int_{t_1}^{t_2} u'(t)dt \right|,$$

and since Φ is even we obtain that

$$\Phi(|u(t_2) - u(t_1)|) = \Phi\left(\int_{t_1}^{t_2} u'(t)dt \right). \tag{1.8}$$

Therefore, due to Jensen's inequality [87, Theorem 3.3] we get

$$\Phi\left(\int_{t_1}^{t_2} u'(t)dt \right) \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \Phi((t_2 - t_1)u'(t))dt. \tag{1.9}$$

Thereby, combining estimates (1.8) and (1.9), one has

$$\Phi(|u(t_2) - u(t_1)|) \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \Phi((t_2 - t_1)u'(t))dt$$

that combines with Lemma A.2 to give

$$\Phi(|u(t_1) - u(t_2)|) \leq \frac{\xi_1(t_2 - t_1)}{t_2 - t_1} \int_{t_1}^{t_2} \Phi(|u'(t)|)dt,$$

and the lemma follows. ■

As a consequence, we obtain the following result.

Lemma 1.2 *Let $r > 0$. If $u \in E_\Phi$ satisfies $V(u(t)) \geq r$ for any $t \in (t_1, t_2) \subset [0, +\infty)$, then there is $\mu_r > 0$ independent of t_1 and t_2 such that*

$$\int_{t_1}^{t_2} \mathcal{L}(u)dt \geq \frac{t_2 - t_1}{\xi_1(t_2 - t_1)} \Phi(|u(t_1) - u(t_2)|) + r(t_2 - t_1) \geq \mu_r h\left(\Phi(|u(t_1) - u(t_2)|)\right),$$

where $h(t) = \min\{t^{\frac{1}{l}}, t^{\frac{1}{m}}\}$.

Proof. Let $u \in E_\Phi$ satisfying $V(u(t)) \geq r > 0$ for any $t \in (t_1, t_2)$. Thanks to Lemma 1.1,

$$\int_{t_1}^{t_2} \mathcal{L}(u)dt \geq \frac{t_2 - t_1}{\xi_1(t_2 - t_1)} \Phi(|u(t_1) - u(t_2)|) + r(t_2 - t_1).$$

On the other hand, we know that $\xi_1(t_2 - t_1) = \max\{(t_2 - t_1)^l, (t_2 - t_1)^m\}$. Thus, if $\xi_1(t_2 - t_1) = (t_2 - t_1)^m$, then

$$\int_{t_1}^{t_2} \mathcal{L}(u)dt \geq \frac{1}{m} \left[\frac{1}{(t_2 - t_1)^{\frac{m-1}{m}}} \left(\Phi(|u(t_2) - u(t_1)|)\right)^{\frac{1}{m}} \right]^m + \frac{m-1}{m} \left(r^{\frac{m-1}{m}} (t_2 - t_1)^{\frac{m-1}{m}} \right)^{\frac{m}{m-1}}.$$

Consequently, employing Young's inequality for the conjugate exponents m and $\frac{m}{m-1}$ we obtain

$$\int_{t_1}^{t_2} \mathcal{L}(u)dt \geq r^{\frac{m-1}{m}} \Phi(|u(t_2) - u(t_1)|)^{\frac{1}{m}}.$$

Now, if $\xi_1(t_2-t_1)=(t_2-t_1)^l$, a similar argument works to prove that

$$\int_{t_1}^{t_2}\mathcal{L}(u)dt \geq r^{\frac{l-1}{l}}\Phi(|u(t_2)-u(t_1)|)^{\frac{1}{l}}.$$

Setting $\mu_r=\min\{r^{\frac{l-1}{l}},r^{\frac{m-1}{m}}\}$ and $h(t)=\min\{t^{\frac{1}{l}},t^{\frac{1}{m}}\}$, we arrive at the inequality below

$$\int_{t_1}^{t_2}\mathcal{L}(u)dt \geq \mu_r h\left(\Phi(|u(t_2)-u(t_1)|)\right),$$

which is precisely the assertion of the lemma. ■

To fulfill our purpose in this section, hereafter, given $\delta>0$ we will fix a single real number $\lambda_\delta>0$ such that

$$\lambda_\delta<\min\left\{1,\mu_{\underline{w}\Phi\left(\frac{\delta}{2}\right)}h\left(\Phi\left(\frac{\delta}{2}\right)\right)\right\}, \tag{1.10}$$

where $\mu_{\underline{w}\Phi\left(\frac{\delta}{2}\right)}$ is given according to the Lemma 1.1 with $r=\underline{w}\Phi\left(\frac{\delta}{2}\right)$. Moreover, from (V_1) and (V_3) we claim that there are $\underline{w},\overline{w}>0$ satisfying

$$\underline{w}\Phi(|t-\alpha|)\leq V(t)\leq \overline{w}\Phi(|t-\alpha|), \quad \forall t\in[0,\alpha+\delta_\alpha], \tag{1.11}$$

where $\delta_\alpha>0$ was given in (V_3). Indeed, by (V_1) and the fact that $\Phi(t)=0$ if, and only if $t=0$, it follows that the function $\frac{V(t)}{\Phi(|t-\alpha|)}$ is continuous and strictly positive in $[0,\alpha-\delta_\alpha]$. Hence, there are $b_1,b_2>0$ such that

$$b_1\Phi(|t-\alpha|)\leq V(t)\leq b_2\Phi(|t-\alpha|) \quad \forall t\in[0,\alpha-\delta_\alpha].$$

Now (1.11) follows by taking $\underline{w}=\min\{\alpha_1,w_1\}$ and $\overline{w}=\max\{\alpha_2,w_2\}$, where w_1 and w_2 were given in (V_3).

With this in mind, we prove the following result.

Lemma 1.3 *If $u\in E_\Phi$ and $\delta\in(0,\delta_\alpha]$ such that $F(u)\leq c_\Phi+\lambda_\delta$, then $\|u\|_{L^\infty(\mathbb{R})}\leq\alpha+\delta$.*

Proof. Let $u\in E_\Phi$ and assume for the sake of contradiction that there exists $t_0\in\mathbb{R}$ such that $u(t_0)>\alpha+\delta$. As u is odd, we can assume without loss of generality $t_0>0$. Thus, since $u(0)=0$ and u is continuous, there exist $t_1,\sigma,\tau\in\mathbb{R}$ with $0<t_1<\sigma<\tau<t_0$ satisfying

$$u(t_1)=\alpha,\ u(\sigma)=\alpha+\frac{\delta}{2},\ u(\tau)=\alpha+\delta,\ \text{and}\ \alpha+\frac{\delta}{2}\leq u(t)\leq\alpha+\delta\ \forall t\in(\sigma,\tau).$$

Consequently, from (1.11),

$$\underline{w}\Phi\left(\frac{\delta}{2}\right)\leq V(u(t)),\quad \forall t\in(\sigma,\tau).$$

According to Lemma 1.2, there is $\mu_{\underline{w}\Phi(\frac{\delta}{2})} > 0$ such that

$$\int_{\sigma}^{\tau} \mathcal{L}(u)dt \geq \mu_{\underline{w}\Phi(\frac{\delta}{2})} h(\Phi(|u(\sigma) - u(\tau)|)) = \mu_{\underline{w}\Phi(\frac{\delta}{2})} h\left(\Phi\left(\frac{\delta}{2}\right)\right). \tag{1.12}$$

On the other hand, as $u(t_1) = \alpha$, the function given by

$$\tilde{u}(t) = \begin{cases} -\alpha, & \text{if} \quad t \leq -t_1, \\ u(t), & \text{if} \quad -t_1 \leq t \leq t_1, \\ \alpha, & \text{if} \quad t \geq t_1 \end{cases}$$

belongs to E_Φ, and hence, $c_\Phi \leq F(\tilde{u})$, or equivalently,

$$c_\Phi \leq \int_{-t_1}^{t_1} \mathcal{L}(u)dt. \tag{1.13}$$

From (1.10)-(1.13),

$$c_\Phi + \lambda_\delta \geq F(u) \geq \int_{-t_1}^{t_1} \mathcal{L}(u)dt + \int_{\sigma}^{\tau} \mathcal{L}(u)dt > c_\Phi + \lambda_\delta,$$

a contradiction. This concludes the proof. ■

In other words, Lemma 1.3 states that when the energy of u with respect to F is sufficiently close to the minimum energy c_Φ, then the L^∞-norm of u is less than or equal to $\alpha + \delta$. It follows from this fact that elements of E_Φ that have minimum energy in relation to the functional F have L^∞-norm less than or equal to α.

Corollary 1.1 *If $u \in E_\Phi$ such that $F(u) = c_\Phi$, then $\|u\|_{L^\infty(\mathbb{R})} \leq \alpha$.*

Proof. Indeed, given any $\delta \in (0, \delta_\alpha]$, we have that $F(u) \leq c_\Phi + \lambda_\delta$, and so, thanks to Lemma 1.3, $\|u\|_{L^\infty(\mathbb{R})} \leq \alpha + \delta$. Therefore, taking the limit $\delta \to 0$ we get $\|u\|_{L^\infty(\mathbb{R})} \leq \alpha$, and the result follows. ■

As an important consequence of Lemma 1.3, we also derive the following result that characterizes the asymptotic behavior of functions $u \in E_\Phi$ that have energy with respect to F close to the minimum energy.

Lemma 1.4 *Let $u \in E_\Phi$ and $\delta \in (0, \delta_\alpha]$ such that $F(u) \leq c_\Phi + \lambda_\delta$. Then,*

$$|u(t)| \to \alpha \quad as \quad |t| \to +\infty.$$

Proof. First of all, we claim that

$$\liminf_{t \to +\infty} \Phi(||u(t)| - \alpha|) = 0.$$

Indeed, if this limit does not hold, there are $t_0, r > 0$ satisfying

$$\Phi(||u(t)| - \alpha|) \geq r \quad \forall t > t_0,$$

which combined with Lemma 1.3, (V_2) and (1.11) yields

$$\underline{w}r \leq \underline{w}\Phi(||u(t)| - \alpha|) \leq V(u(t)) \quad \forall t > t_0,$$

from where it follows that

$$F(u) \geq \underline{w}r(t - t_0) \quad \text{for all} \quad t > t_0,$$

and taking the limit $t \to +\infty$, it follows that $F(u) = +\infty$, which is impossible. Next we are going to show that

$$\limsup_{t \to +\infty} \Phi(||u(t)| - \alpha|) = 0.$$

Assume by contradiction that

$$\limsup_{t \to +\infty} \Phi(||u(t)| - \alpha|) > 0.$$

From this, there is $r > 0$ such that

$$\limsup_{t \to +\infty} \Phi(||u(t)| - \alpha|) > 2r. \tag{1.14}$$

In what follows, let us fix $\epsilon > 0$ satisfying $2^{1-m} > \epsilon$. By continuity of the functions Φ and u, we can find a sequence of disjoint intervals (σ_i, τ_i) with $0 < \sigma_i < \tau_i < \sigma_{i+1} < \tau_{i+1}$, $i \in \mathbb{N}$, and $\sigma_i \to +\infty$ as $i \to +\infty$ such that for each i,

$$\Phi(||u(\sigma_i)| - \alpha|) = r\epsilon, \;\; \Phi(||u(\tau_i)| - \alpha|) = r \;\text{ and }\; r\epsilon \leq \Phi(||u(t)| - \alpha|) \leq r \, \forall t \in [\sigma_i, \tau_i], \tag{1.15}$$

and so, from (1.11) together with (1.15) yields $V(u(t)) \geq \underline{w}r\epsilon$ for every $t \in [\sigma_i, \tau_i]$ and for all $i \in \mathbb{N}$. From this, by Lemma 1.2 there exists $\mu_{\underline{w}\epsilon r} > 0$ such that

$$\int_{\sigma_i}^{\tau_i} \mathcal{L}(u)dt \geq \mu_{\underline{w}\epsilon r} h(\Phi(|u(\tau_i) - u(\sigma_i)|), \quad \forall i \in \mathbb{N}. \tag{1.16}$$

On the other hand, the reader can easily verify the following elementary inequality through item-(a) of Lemma A.8

$$\Phi(|t - s|) \geq 2^{1-m}\Phi(|t|) - \Phi(|s|), \quad \forall t, s \in \mathbb{R}.$$

Now, combining the inequality above with (1.15), we get

$$\Phi(|u(\tau_i) - u(\sigma_i)|) \geq 2^{1-m}\Phi(||u(\tau_i)| - \alpha|) - \Phi(||u(\sigma_i)| - \alpha|) = (2^{1-m} - \epsilon)r,$$

from where it follows from (1.16) that

$$F(u) \geq \sum_{i=1}^{+\infty} \int_{\sigma_i}^{\tau_i} \mathcal{L}(u)dt \geq \sum_{i=1}^{+\infty} \mu_{\underline{w}\epsilon r} h\left((2^{1-m} - \epsilon)r\right) = +\infty,$$

which contradicts the fact that $F(u) < +\infty$, and the proof is over. ■

The following result better characterizes the behavior of functions on E_Φ that have energy close to the minimum energy.

Corollary 1.2 *Let $u \in E_\Phi$ and $\delta \in (0, \delta_\alpha]$ such that $F(u) \leq c_\Phi + \lambda_\delta$. Then,*

$$\lim_{t\to+\infty} u(t) = \alpha \quad or \quad \lim_{t\to+\infty} u(t) = -\alpha.$$

Proof. According to Lemma 1.4, $|u(t)| \to \alpha$ as $|t| \to +\infty$. Then, if the corollary is not true, there should be two sequences (t_n) and (s_n) of positive real numbers such that $t_n, s_n \to +\infty$ as $n \to +\infty$ satisfying the following property

$$u(t_n) \to \alpha \quad \text{and} \quad u(s_n) \to -\alpha.$$

Thereby, there exists $n_0 \in \mathbb{N}$ such that $u(s_n) < 0 < u(t_n)$ for all $n \geq n_0$. By continuity of u there is $z_n \in (t_n, s_n)$ or $z_n \in (s_n, t_n)$ with $u(z_n) = 0$ for any $n \geq n_0$ and $z_n \to +\infty$, which is a contradiction. ■

Because Lemma 1.3, we easily derive the following compactness result in $L^\infty_{\text{loc}}(\mathbb{R})$.

Lemma 1.5 *Let $(u_n) \subset E_\Phi$ and $\delta \in (0, \delta_\alpha]$ such that $F(u_n) \leq c_\Phi + \lambda_\delta$ for all $n \in \mathbb{N}$. Then, there exists $u \in E_\Phi$ such that, along a subsequence,*

$$u_n \rightharpoonup u \ in \ W^{1,\Phi}_{loc}(\mathbb{R}) \quad and \quad u_n \to u \ in \ L^\infty_{loc}(\mathbb{R}).$$

Moreover, $F(u) \leq \liminf_{n\to+\infty} F(u_n)$.

Proof. First, since $F(u_n) \leq c_\Phi + \lambda_\delta$ for any $n \in \mathbb{N}$, then Lemma 1.3 ensures that

$$\|u_n\|_{L^\infty(\mathbb{R})} \leq \alpha + \delta \quad \text{for all} \quad n \in \mathbb{N}.$$

Thereby, for each $R > 0$,

$$\int_{-R}^{R} \Phi(|u_n|)dt \leq 2R\Phi(\alpha + \delta) \quad \text{and} \quad \int_{-R}^{R} \Phi(|u_n'|)dt \leq c_\Phi + \lambda_\delta \quad \forall n \in \mathbb{N}.$$

From this, it is easily seen that (u_n) is bounded[2] in $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$. Now, in view of $\Phi \in \Delta_2$, $W^{1,\Phi}(D)$ is reflexive Banach spaces whenever D is an open and bounded set in $\mathbb{R}$, and hence, a classical diagonal argument yields that there is $u \in W^{1,\Phi}_{\text{loc}}(\mathbb{R})$ and a subsequence of (u_n), still denoted (u_n), such that

$$u_n \rightharpoonup u \text{ in } W^{1,\Phi}_{\text{loc}}(\mathbb{R}) \text{ and } u_n \to u \text{ in } L^{\infty}_{\text{loc}}(\mathbb{R}).$$

Consequently, by the pointwise convergence we get $u(t) = -u(-t)$ for all $t \in \mathbb{R}$, and so, $u \in E_\Phi$. Finally, it is easy to check by weak lower semicontinuity that $F(u) \leq \liminf_{n\to+\infty} F(u_n)$, and the lemma is proved. ■

To continue our study in search of the existence of a heteroclinic solution to equation (1.2), we would like to highlight the notion of a weak solution of equation (1.2). So, a function $q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R})$ is said to be a *weak solution* of (1.2) if it satisfies the following relation

$$\int_{\mathbb{R}} (\phi(|q'|)q'\psi' + V'(q)\psi)\, dt = 0 \quad \text{for all} \quad \psi \in X^{1,\Phi}_0(\mathbb{R}), \tag{1.17}$$

where

$$X^{1,\Phi}_0(\mathbb{R}) = \left\{ w \in W^{1,\Phi}(\mathbb{R}) \text{ with } w(t) = 0 \text{ for } |t| \geq R \text{ for some } R > 0 \right\}. \tag{1.18}$$

Moreover, we will introduce the following class

$$K_\Phi = \{u \in E_\Phi : F(u) = c_\Phi\},$$

which consists of the minimum points of F that are odd. Next, our goal is to apply a direct method of the Calculus of Variations to show that K_Φ is not empty and that every element of K_Φ is a weak solution of (1.2).

Theorem 1.2 *It holds that K_Φ is not empty, and any $q \in K_\Phi$ is a weak solution of* (1.2) *such that $q \in C^{1,\gamma}_{loc}(\mathbb{R})$ for some $\gamma \in (0,1)$. Moreover,*

$$-\left(\phi(|q'(t)|)q'(t)\right)' + V'\left(q(t)\right) = 0 \quad \textit{for all} \quad t \in \mathbb{R}. \tag{1.19}$$

Proof. The first step is to consider a minimizing sequence $(u_n) \subset E_\Phi$ for F. So, since $F(u_n) \to c_\Phi$, one has $F(u_n) \leq c_\Phi + \epsilon$ for all n sufficiently large and for some $\epsilon > 0$ small

[2]We say that a sequence is bounded in $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$ if it is bounded in $W^{1,\Phi}(D)$ for every open and bounded set D in $\mathbb{R}$.

enough. Consequently, invoking Lemma 1.5, there are a subsequence of (u_n), still denoted by (u_n), and $q \in E_\Phi$ such that

$$u_n \rightharpoonup q \ \text{ in } \ W^{1,\Phi}_{\text{loc}}(\mathbb{R}) \quad \text{and} \quad u_n \to q \ \text{ in } \ L^\infty_{\text{loc}}(\mathbb{R}),$$

from where it follows that $F(q) \leq c_\Phi$, and so, $F(q) = c_\Phi$. We claim that q is a weak solution of (1.2). To see this, given $\psi \in X_0^{1,\Phi}(\mathbb{R})$ we can write ψ as being the sum of an even function and an odd function, that is, $\psi(t) = \psi_o(t) + \psi_e(t)$, where

$$\psi_e(t) = \frac{1}{2}(\psi(t) + \psi(t)) \quad \text{and} \quad \psi_o(t) = \frac{1}{2}(\psi(t) - \psi(t)).$$

Now by item-(b) of Lemma A.8, for each $s > 0$ we have that

$$\Phi(|q' + s\psi'|) - \Phi(|q' + s\psi_o'|) \geq \phi(|q' + s\psi_o'|)(q' + s\psi_o')(s\psi_e'). \tag{1.20}$$

On the other hand, since $F(q) = c_\Phi$ and $\psi \in X_0^{1,\Phi}(\mathbb{R})$, a direct computation shows that $F(q + s\psi), F(q + s\psi_o) < +\infty$, because for $|t|$ sufficiently large we must have $s\psi(t) = s\psi_o(t) = 0$. Thereby, from (1.20),

$$\begin{aligned} F(q + s\psi) - F(q + s\psi_o) \geq s \int_{\mathbb{R}} \phi(|q' + s\psi_o'|) q' \psi_e' dt + s^2 \int_{\mathbb{R}} \phi(|q' + s\psi_o'|) \psi_o' \psi_e' dt \\ + \int_{\mathbb{R}} (V(q + s\psi) - V(q + s\psi_o))\, dt. \end{aligned} \tag{1.21}$$

As functions $\phi(|q' + s\psi_o'|) q' \psi_e'$ and $\phi(|q' + s\psi_o'|) \psi_o' \psi_e'$ are odd,

$$\int_{\mathbb{R}} \phi(|q' + s\psi_o'|) q' \psi_e' dt = \int_{\mathbb{R}} \phi(|q' + s\psi_o'|) \psi_o' \psi_e' dt = 0, \tag{1.22}$$

and thus, substituting (1.22) into (1.21), we get

$$F(q + s\psi) - F(q + s\psi_o) \geq \int_{\mathbb{R}} (V(q + s\psi) - V(q + s\psi_o)) dt. \tag{1.23}$$

Since $q + s\psi_o \in E_\Phi$ we obtain that $F(q) = c_\Phi \leq F(q + s\psi_o)$, and therefore,

$$F(q + s\psi) - F(q) \geq \int_{\mathbb{R}} (V(q + s\psi) - V(q + s\psi_o)) dt,$$

from where it follows that

$$\begin{aligned} \int_{\mathbb{R}} (\phi(|q'|) q' \psi' + V'(q)\psi) dt &= \lim_{s \to 0^+} \frac{F(q + s\psi) - F(q)}{s} \\ &\geq \lim_{s \to 0^+} \int_{\mathbb{R}} \frac{V(q + s\psi) - V(q + s\psi_o)}{s} dt \\ &\geq \lim_{s \to 0^+} \int_{\mathbb{R}} \left(\frac{V(q + s\psi) - V(q)}{s} - \frac{V(q + s\psi_o) - V(q)}{s} \right) dt \\ &\geq \int_{\mathbb{R}} V'(q)(\psi - \psi_o) dt = \int_{\mathbb{R}} V'(q) \psi_e dt, \end{aligned} \tag{1.24}$$

Now, as $V'(q)\psi_e$ is an odd function we derive that

$$\int_{\mathbb{R}} (\phi(|q'|)q'\psi' + V'(q)\psi)dt \geq 0 \quad \forall \psi \in X_0^{1,\Phi}(\mathbb{R}),$$

which guarantees that q is a weak solution of (1.2). Moreover, the condition (ϕ_2) allows us to use [67, Theorem 1.7] to find $\gamma \in (0,1)$ such that $q \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R})$. Finally, in order to prove (1.19), the fact that q is a weak solution implies that

$$(\phi(|q'(t)|)q'(t))' = V'(q(t)) \text{ almost everywhere } t \in \mathbb{R}. \tag{1.25}$$

Indeed, considering $\varphi \in C_0^\infty(\mathbb{R})$, a calculation shows that

$$0 = \int_{\mathbb{R}} (\phi(|q'(t)|)q'(t)\varphi'(t) + V'(q(t))\varphi(t))\,dt = -\int_{\mathbb{R}} \left((\phi(|q'(t)|)q'(t))' - V'(q(t))\right)\varphi(t)dt,$$

and so applying [26, Corollary 4.24] we get (1.25). Now, Lemma A.6 combined with (1.25) yields $\phi(|q'|)q' \in W^{1,\tilde{\Phi}}_{\mathrm{loc}}(\mathbb{R})$ and therefore from (A.1) we conclude that $\phi(|q'|)q' \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. Next, by [26, Theorem 8.2] and the fact that $V'(q)$ is continuous it is easy to see that the equality (1.19) occurs for every $t \in \mathbb{R}$. This finishes the proof. ■

We would like to end this subsection by stating that the minimum energy of the functional F cannot be zero, that is, $c_\Phi > 0$. Indeed, if $c_\Phi = 0$ then since the minimum energy is achieved by the elements of K_Φ, for $q \in K_\Phi$ one has

$$\int_{\mathbb{R}} (\Phi(|q'|) + V(q))\,dt = 0,$$

from where it follows that $q = \alpha$ or $q = -\alpha$ on $\mathbb{R}$, which is a contradiction.

1.1.3 Qualitative properties

The purpose of this subsection is to approach, from the variational point of view, several qualitative properties of minimal solutions of (1.2) to better understand the geometry of these solutions. We start by showing the following result.

Lemma 1.6 *If $q \in K_\Phi$, then*

(a) $q(t) > 0$ or $q(t) < 0$ for all $t > 0$.

(b) $-\alpha < q(t) < \alpha$ for any $t \in \mathbb{R}$.

Proof. Let be $q \in K_\Phi$. To show item (a), we first claim that $q(t) = 0$ if, and only if $t = 0$. In fact, assume by absurd that there is $t_0 \neq 0$ such that $q(t_0) = 0$. So, without lost of generality, we can assume that $t_0 > 0$. Consequently, setting

$$Q(t) = \begin{cases} q(t+t_0), & \text{if} \quad t \geq 0, \\ -Q(-t), & \text{if} \quad t \leq 0, \end{cases}$$

we immediately obtain that $Q \in E_\Phi$ and

$$c_\Phi \leq F(Q) = \int_{\mathbb{R}} \mathcal{L}(Q)dt = 2\int_{t_0}^{+\infty} \mathcal{L}(q)dt = F(q) - \int_{-t_0}^{t_0} \mathcal{L}(q)dt = c_\Phi - \int_{-t_0}^{t_0} \mathcal{L}(q)dt,$$

leading to

$$\int_{-t_0}^{t_0} \mathcal{L}(q)dt = 0.$$

By the properties on the potential V, we must have $q = \alpha$ or $q = -\alpha$ on $(-t_0, t_0)$, which is impossible, because q is odd. Therefore, this information combined with the regularity of q implies $q > 0$ or $q < 0$ on $(0, +\infty)$, proving item (a). To finish the proof, by item (a) we can assume that $q(t) > 0$ for any $t > 0$. Now, assume by contradiction that there is $t_0 \in (0, +\infty)$ such that $q(t_0) = \alpha$. Said that, let us consider $r > t_0$ and $R > 0$ satisfying

$$R > \max\left\{\|q'\|_{L^\infty([0,r])}, \eta\right\},$$

where $\eta > 0$ was given in (ϕ_3). Now, we consider $\tilde{\phi} : (0, +\infty) \to (0, +\infty)$ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & \text{if} \quad 0 < t \leq R, \\ \dfrac{\phi(R)t^{s-2}}{R^{s-2}}, & \text{if} \quad R \leq t, \end{cases}$$

where $s > 1$ was also fixed in (ϕ_3). Thanks to (ϕ_3), a simple computation implies that there are $\gamma_1, \gamma_2 > 0$ dependent on the constants δ, R, s, c_1 and c_2 such that

$$\tilde{\phi}(t)t \leq \gamma_1 t^{s-1} \quad \text{and} \quad \tilde{\phi}(t)t^2 \geq \gamma_2 t^s \quad \text{for all} \quad t \geq 0. \tag{1.26}$$

Using the function $\tilde{\phi}$, let us also consider the scalar measurable function $G : \mathbb{R}^3 \to \mathbb{R}$ by

$$G(t, u, p) = \frac{\tilde{\phi}(|p|)p}{\gamma_2}.$$

From (1.26), it is easy to check that

$$|G(t, u, p)| \leq \frac{\gamma_1}{\gamma_2}|p|^{s-1} \quad \text{and} \quad pG(t, u, p) \geq |p|^s \quad \text{for all} \quad (t, u, p) \in \mathbb{R}^3.$$

In what follows, we will also consider the scalar measurable function $B : \mathbb{R}^3 \to \mathbb{R}$ given by

$$B(t,u,p) = \frac{V'(\alpha - u)}{\gamma_2}.$$

Combining (ϕ_3) with (V_7), a straightforward calculation ensures that for each $M > 0$ there is $C_M > 0$ such that

$$|B(t,u,p)| \le C_M |u|^{s-1} \quad \text{for all} \quad (t,u,p) \in \mathbb{R} \times (-M,M) \times \mathbb{R}. \tag{1.27}$$

To prove the above inequality, we are going to analyze by cases. Indeed, given $M > 0$ let $\epsilon \in (0, \min\{M, \eta, \tau\})$, where η and τ were given in (ϕ_3) and (V_4) respectively. On the other hand, from (V_4) there are $\tilde{\omega}_1, \tilde{\omega}_2 > 0$ such that

$$|V'(t)| \le \tilde{\omega}_1 \phi(\tilde{\omega}_2|\alpha - t|)|\alpha - t||t| \ \text{ for all } \ t \in [0, \alpha + \tau].$$

So the first case is if $|u| \le \dfrac{\epsilon}{\tilde{\omega}_2 + 1}$, and in this case we have that $\alpha - u \in [\alpha - \epsilon, \alpha + \epsilon]$. Consequently,

$$|V'(\alpha - u)| \le \tilde{\omega}_1 \phi(\tilde{\omega}_2|u|)|u||\alpha - u| \le (\alpha + \epsilon)\frac{\tilde{\omega}_1}{\tilde{\omega}_2}\phi((\tilde{\omega}_2 + 1)|u|)(\tilde{\omega}_2 + 1)|u|.$$

Now, since $(\tilde{\omega}_2 + 1)|u| \in (0, \eta)$ we obtain by (ϕ_3) that

$$|V'(\alpha - u)| \le (\alpha + \epsilon)\frac{\tilde{\omega}_1}{\tilde{\omega}_2} c_2 (\tilde{\omega}_2 + 1)^{s-1}|u|^{s-1}. \tag{1.28}$$

Now, for the last case, if $|u| \in [\epsilon/(\tilde{\omega}_2 + 1), M]$ then by continuity it's easy to see that there exists $\tilde{C} = \tilde{C}(M) > 0$ such that

$$|V'(\alpha - u)| \le \tilde{C}_M |u|^{s-1} \quad \forall |u| \in [\epsilon/(\tilde{\omega}_2 + 1), M]\,. \tag{1.29}$$

Finally, combining inequality (1.28) and (1.29) we get estimate (1.27). All this is necessary to guarantee that the functions G and B fulfill the assumptions of the Harnack type inequality found in [91, Theorem 1.1]. Having that in mind, setting $w(t) = \alpha - q(t)$ for $t \in \mathbb{R}$, we infer that w is a weak solution of the quasilinear equation

$$G'(t, w, w') + B(t, w, w') = 0 \ \text{ in } \ [0, r],$$

where G' is the derivative of $G(t, w(t), w'(t))$ at t. Employing the Harnack-type inequality mentioned above, we deduce that $w = 0$ in $[0, r]$, that is, $q = \alpha$ in $[0, r]$, which contradicts the fact that $q(0) = 0$. Therefore, $0 < q(t) < \alpha$ for $t > 0$. The proof is completed noting that q is odd. ■

Since the functional F is even, that is, $F(-u) = F(u)$ for any $u \in E_\Phi$, then if $q \in K_\Phi$ then $-q \in K_\Phi$. From now on, for the sake of simplicity, we denote by q^+ a function $q \in K_\Phi$ satisfying

$$\lim_{t \to +\infty} q(t) = \alpha$$

and by q^- if

$$\lim_{t \to +\infty} q(t) = -\alpha.$$

Trivially, $q^- = -q^+$.

Lemma 1.7 *Let $q \in K_\Phi$. Then, q^+ is increasing on $\mathbb{R}$ and q^- is decreasing on $\mathbb{R}$.*

Proof. We will first prove that q^+ is increasing on $\mathbb{R}$. If not, then we can assume without loss of generality that there are $t_1, t_2 \in (0, +\infty)$ with $t_1 < t_2$ and $q^+(t_2) \leq q^+(t_1)$. So, by Intermediate Value Theorem there exists $t_0 \in (0, t_1]$ verifying $q^+(t_0) = q^+(t_2)$. Now, setting the function

$$Q^+(t) = \begin{cases} q^+(t), & \text{if} \quad 0 \leq t \leq t_0 \\ q^+(t + t_2 - t_0), & \text{if} \quad t_0 \leq t \\ -Q^+(-t), & \text{if} \quad t \leq 0, \end{cases}$$

one gets $Q^+ \in E_\Phi$ and

$$c_\Phi \leq F(Q^+) = 2 \int_0^{t_0} \mathcal{L}(q^+)dt + 2 \int_{t_0}^{+\infty} \mathcal{L}(q^+(t + t_2 - t_0))dt = F(q^+) - 2 \int_{t_0}^{t_2} \mathcal{L}(q^+)dt$$

implying that

$$\int_{t_1}^{t_2} \mathcal{L}(q^+)dt = 0,$$

and so by the assumptions on V and Φ, we can infer that $q^+(t) = \alpha$ for all $t \in (t_1, t_2)$, which contradicts the Lemma 1.6. The lemma follows using the fact that q^+ is odd. ■

Lemma 1.8 *Let $q \in K_\Phi$. Then, ${q^+}'$ is non-increasing on $[0, +\infty)$ and ${q^-}'$ is non-decreasing on $[0, +\infty)$.*

Proof. It suffices to show that ${q^+}'$ is non-increasing on $[0, +\infty)$. Now, let us first notice that $q^+(t) \in [0, \alpha)$ for any $t \geq 0$, and so, (V_4) provides

$$V'(q^+(t)) \leq 0 \quad \text{for each} \quad t \in [0, +\infty).$$

By (1.19) we get the inequality

$$\left(\phi(|{q^+}'(t)|){q^+}'(t)\right)' \leq 0 \quad \text{for all} \quad t \in [0, +\infty),$$

from which it follows that the function $t \in [0, +\infty) \mapsto \phi(|{q^+}'(t)|){q^+}'(t)$ is non-increasing. Invoking Lemma 1.7, ${q^+}'(t) \geq 0$ in $\mathbb{R}$, and therefore, $t \in [0, +\infty) \mapsto \phi({q^+}'(t)){q^+}'(t)$ is non-increasing. To complete the proof, assume by contraction that ${q^+}'$ is not non-increasing on $[0, +\infty)$. Thereby, there are $t_1, t_2 \in [0, +\infty)$ such that $0 \leq t_1 < t_2$ and $0 \leq {q^+}'(t_1) < {q^+}'(t_2)$. Consequently,

$$\phi\left({q^+}'(t_2)\right){q^+}'(t_2) \leq \phi\left({q^+}'(t_1)\right){q^+}'(t_1).$$

On the other hand, by (ϕ_1),

$$\phi\left({q^+}'(t_1)\right){q^+}'(t_1) < \phi\left({q^+}'(t_2)\right){q^+}'(t_2),$$

which is a contradiction. This ends the proof. ■

Corollary 1.3 *If $q \in K_\Phi$, then the following inequality holds true ${q^+}' > 0$ on $\mathbb{R}$.*

Proof. As consequence of Lemma 1.7, ${q^+}'(t) \geq 0$ for all $t \in \mathbb{R}$. Arguing by contraction, assume that there exists $t_0 > 0$ such that ${q^+}'(t_0) = 0$. From Lemma 1.8,

$${q^+}'(t) \leq {q^+}'(t_0) = 0 \quad \text{for all} \quad t \geq t_0,$$

and hence, ${q^+}' = 0$ on $(t_0, +\infty)$, that is, q^+ is constant on $(t_0, +\infty)$. But $q^+(t) \to \alpha$ as $t \to +\infty$, and therefore, $q^+ = \alpha$ on $(t_0, +\infty)$, which contradicts Lemma 1.6. Now, the results follows by using the fact that ${q^+}'$ is even. ■

Corollary 1.4 *If $q \in K_\Phi$, then $\lim_{|t| \to +\infty} q'(t) = 0$. In particular, $q' \in L^\infty(\mathbb{R})$.*

Proof. Since $F(q) < +\infty$,

$$\int_{\mathbb{R}} \Phi(|q'(t)|)dt < +\infty,$$

and consequently,

$$\liminf_{|t| \to +\infty} \Phi(|q'(t)|) = 0.$$

Now, we claim that

$$\liminf_{|t| \to +\infty} |q'(t)| = 0.$$

Indeed, if this is not the case, there exist $\epsilon > 0$ and $t_0 > 0$ such that $|q'(t)| \geq \epsilon$ for any $t \geq t_0$. From (ϕ_1), the function Φ is increasing on $(0, +\infty)$, then

$$\Phi(|q'(t)|) \geq \Phi(\epsilon) \quad \text{for all} \quad t \geq t_0,$$

which is impossible. Assuming without loss of generality that $q = q^+$ and invoking Lemma 1.8, we have that $q^{+\prime}$ is non-increasing on $[0, +\infty)$, then

$$\lim_{t \to +\infty} q^{+\prime}(t) = \liminf_{t \to +\infty} q^{+\prime}(t) = 0.$$

Since $q^{+\prime}$ is even, we deduce that $\lim_{|t| \to +\infty} q^{+\prime}(t) = 0$. ■

The first consequence of Theorem 1.1 is a result of comparing among the elements of K_Φ with the same asymptotic behavior at infinite.

Lemma 1.9 *(Comparison Lemma) If $q_1^+, q_2^+ \in K_\Phi$, then*

$$q_1^+(t) \geq q_2^+(t) \quad \textit{or} \quad q_1^+(t) \leq q_2^+(t), \quad \forall t \geq 0.$$

Proof. If $q_1^+ \neq q_2^+$, then there is $t_0 > 0$ such that $q_1^+(t_0) \neq q_2^+(t_0)$, and so, we can assume that $q_1^+(t_0) > q_2^+(t_0)$. In what follows, we define the functions

$$\varphi(t) = \begin{cases} q_1^+(t), & \text{if} \quad q_1^+(t) > q_2^+(t) \quad \text{and} \quad t \geq 0, \\ q_2^+(t), & \text{if} \quad q_2^+(t) \geq q_1^+(t) \quad \text{and} \quad t \geq 0, \\ -\varphi(-t), & \text{if} \quad t < 0 \end{cases}$$

and

$$\psi(t) = \begin{cases} q_2^+(t), & \text{if} \quad q_1^+(t) > q_2^+(t) \quad \text{and} \quad t \geq 0, \\ q_1^+(t), & \text{if} \quad q_2^+(t) \geq q_1^+(t) \quad \text{and} \quad t \geq 0, \\ -\psi(-t), & \text{if} \quad t < 0 \end{cases}$$

that clearly belong to E_Φ. Consequently,

$$\begin{aligned} 2c_\Phi \leq F(\varphi) + F(\psi) &= \int_{\mathbb{R}} \mathcal{L}(\varphi)dt + \int_{\mathbb{R}} \mathcal{L}(\psi)dt \\ &= 2\int_{\{q_1^+ > q_2^+\}} \mathcal{L}(q_1^+)dt + 2\int_{\{q_2^+ \geq q_1^+\}} \mathcal{L}(q_2^+)dt + 2\int_{\{q_1^+ > q_2^+\}} \mathcal{L}(q_2^+)dt + 2\int_{\{q_2^+ \geq q_1^+\}} \mathcal{L}(q_1^+)dt \\ &= \int_{\mathbb{R}} \mathcal{L}(q_1^+)dt + \int_{\mathbb{R}} \mathcal{L}(q_2^+)dt = F(q_1^+) + F(q_2^+) = 2c_\Phi, \end{aligned}$$

from where it follows that $F(\varphi) = F(\psi) = c_\Phi$, and therefore, $\varphi, \psi \in K_\Phi$. Now, since $q_1^+(t_0) > q_2^+(t_0)$, by continuity there exists $\epsilon > 0$ such that

$$q_1^+(t) > q_2^+(t), \quad \forall t \in (-\epsilon + t_0, t_0 + \epsilon).$$

We claim that

$$q_1^+(t) \geq q_2^+(t) \text{ for all } t \geq t_0. \tag{1.30}$$

Indeed, suppose by contradiction that there is $t_1 > t_0$ such that $q_2^+(t_1) > q_1^+(t_1)$ and fix $r > 0$ satisfying $t_0 + r > t_1$. Thus, setting $\tilde{\varphi}(t) = \varphi(t + t_0)$ and $\tilde{q_1^+}(t) = q_1^+(t + t_0)$ it is easily seen that $\tilde{\varphi}$ and $\tilde{q_1^+}$ satisfy the Cauchy problem

$$\begin{cases} (\phi(|u'(t)|)u'(t))' = V'(u(t)), & t \in \mathbb{R}, \\ u(0) = q_1^+(t_0), \\ u'(0) = {q_1^+}'(t_0). \end{cases}$$

Thanks to Corollaries 1.1, 1.3 and 1.4, $\tilde{\varphi}$ and $\tilde{q_1^+}$ satisfy the items (a) and (b) of Theorem 1.1 on $[0, r)$, and hence, $q_1^+ = \varphi$ on $(t_0, t_0 + r)$. In particular, $q_1^+(t_1) = \varphi(t_1) = q_2^+(t_1)$, which is impossible. Therefore, inequality (1.30) is valid. To complete the proof, suppose by contradiction that there is $t_2 \in (0, t_0)$ such that $q_1^+(t_2) < q_2^+(t_2)$. Similar to what was previously developed, taking $s > 0$ such that $t_2 \in (t_0, t_0 - s)$, it can be shown that $\varphi = q_1^+$ on $(t_0 - s, t_0)$. Then, in particular, $q_1^+(t_2) = \varphi(t_2) = q_2^+(t_2)$, a contradiction, and the lemma follows. ■

We will end this subsection by presenting some results that will be crucial in the development of this thesis.

Lemma 1.10 *Let $q \in E_\Phi$ and $\delta \in (0, \delta_\alpha]$ such that $F(q) \leq c_\Phi + \lambda_\delta$. Then,*

$$q - \alpha \in W^{1,\Phi}([0, +\infty)) \quad or \quad q + \alpha \in W^{1,\Phi}([0, +\infty)).$$

Proof. By Corollary 1.2,

$$q(t) \to \alpha \text{ or } q(t) \to -\alpha \text{ as } t \to +\infty.$$

In what follows, we will first analyze the case $q(t) \to \alpha$ as $t \to +\infty$. Thus, there exists $t_0 > 0$ such that

$$q(t) \in (\alpha - \delta, \alpha + \delta) \text{ for any } t \geq t_0.$$

From (1.11),

$$\begin{aligned} \int_{t_0}^{+\infty} (\Phi(|q'(t)|) + \Phi(|q(t) - \alpha|))\, dt &\leq \int_{t_0}^{+\infty} \left(\Phi(|q'(t)|) + \frac{1}{w} V(q(t)) \right) dt \\ &\leq \max\left\{1, \frac{1}{w}\right\} F(q) \\ &\leq \max\left\{1, \frac{1}{w}\right\} (c_\Phi + \lambda_\delta) < +\infty. \end{aligned}$$

Therefore, as $q \in W^{1,\Phi}_{loc}(\mathbb{R})$ and $\Phi \in \Delta_2$, one has $q - \alpha \in W^{1,\Phi}([0, +\infty))$. Proceeding in a similar way, the case $q(t) \to -\alpha$ as $t \to +\infty$ yields $q + \alpha \in W^{1,\Phi}([0, +\infty))$, which completes the proof. ■

As an immediate consequence we have the following corollary.

Corollary 1.5 *If $q_1, q_2 \in K_\Phi$, then $q_1^+ - q_2^+ \in W^{1,\Phi}(\mathbb{R})$.*

Finally, we have the following result.

Lemma 1.11 *If $q \in K_\Phi$ then*

$$\int_{\mathbb{R}} (\phi(|q'|)q'\varphi' + V'(q)\varphi)\, dt = 0 \quad \textit{for all} \quad \varphi \in W^{1,\Phi}(\mathbb{R}). \tag{1.31}$$

Proof. Let us first note that given $\varphi \in W^{1,\Phi}(\mathbb{R})$ there is a sequence $(\varphi_n) \subset C_0^\infty(\mathbb{R})$ such that $\varphi_n \to \varphi$ in $W^{1,\Phi}(\mathbb{R})$ because $\Phi \in \Delta_2$. Now, since $q \in K_\Phi$ we get

$$\int_{\mathbb{R}} (\phi(|q'|)q'\varphi_n' + V'(q)\varphi_n)\, dt = 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

On the other hand, according to Lemma A.6 and (V_6), it is easy to check that $\phi(|q'|)q' \in L^{\tilde{\Phi}}(\mathbb{R})$ and $V'(q) \in L^{\tilde{\Phi}}(\mathbb{R})$. The information above allows us to make a direct application of Holder's inequality to obtain equality (1.31), and the lemma follows. ■

1.1.4 Uniqueness of the minimal solution

In this subsection, we will show that the set K_Φ has only two elements, namely

$$K_\Phi = \{q^+, q^-\}.$$

As a direct consequence, we will conclude that in K_Φ the problem

$$\begin{cases} -(\phi(|q'|)q')' + V'(q) = 0 \quad \text{in} \quad \mathbb{R}, \\ \lim\limits_{t \to +\infty} q(t) = \alpha \end{cases}$$

has a unique solution.

Lemma 1.12 *The set K_Φ has exactly two elements.*

Proof. We saw in Theorem 1.2 that there is $q \in K_\Phi$ and consequently $q^+, q^- \in K_\Phi$. We will now show that K_Φ has only these two elements. Indeed, considering any $Q \in K_\Phi$, we intend to show that $Q^+ = q^+$ on $\mathbb{R}$. So, by Lemma 1.9,

$$q^+(t) \geq Q^+(t) \quad \text{or} \quad q^+(t) \leq Q^+(t), \quad \forall t \geq 0.$$

We can assume without loss of generality that

$$q^+(t) \geq Q^+(t), \quad \forall t \geq 0. \tag{1.32}$$

Now fix $\delta \in (0, \delta_0)$ sufficiently small such that we may take $t_1 > 0$ satisfying $q^+(t_1) = \alpha - \delta$, where δ_0 was given in (V_5). Also, we may fix $\tau \in \mathbb{R}$ satisfying $Q^+(t_1 + \tau) = \alpha - \delta$. We claim that $\tau \geq 0$. In fact, if $\tau < 0$, then as Q^+ is increasing on $\mathbb{R}$ it follows by (1.32) that

$$\alpha - \delta = Q^+(t_1 + \tau) < Q^+(t_1) \leq q^+(t_1) = \alpha - \delta,$$

which is absurd. Next, setting

$$Q^+_\tau(t) = Q^+(t + \tau) \;\text{ for }\; t \in \mathbb{R},$$

one has $Q^+_\tau(t_1) = \alpha - \delta = q^+(t_1)$. Consequently, considering the functions

$$\varphi_1(t) = \begin{cases} \max\left\{(q^+ - Q^+_\tau)(t), 0\right\}, & \text{if } \; t \geq t_1 \\ 0, & \text{if } \; t < t_1 \end{cases}$$

and

$$\varphi_2(t) = \begin{cases} \max\left\{(Q^+_\tau - q^+)(t), 0\right\}, & \text{if } \; t \geq t_1 \\ 0, & \text{if } \; t < t_1 \end{cases}$$

we have from Lemma 1.5 that $\varphi_1, \varphi_2 \in W^{1,\Phi}(\mathbb{R})$. Thus, by Lemma 1.11,

$$\int_{\mathbb{R}} \left(\phi(|q^{+\prime}|)q^{+\prime}\varphi_1' - \phi(|Q^{+\prime}_\tau|)Q^{+\prime}_\tau\varphi_1'\right) dt = \int_{\mathbb{R}} \left(V'(Q^+_\tau) - V'(q^+)\right)\varphi_1 dt.$$

In this way, putting $P_1 = \{t \in \mathbb{R} : q^+(t) \geq Q_\tau(t)\}$, by (V_5) one gets

$$\int_{P_1 \cap (t_1, +\infty)} \left(\phi(|q^{+\prime}|)q^{+\prime} - \phi(|Q^{+\prime}_\tau|)Q^{+\prime}_\tau\right)\left(q^{+\prime} - Q^{+\prime}_\tau\right) dt \leq 0. \tag{1.33}$$

Using item (c) of Lemma A.8 in estimation (1.33) we obtain $q^{+\prime} = Q^{+\prime}_\tau$ on $P_1 \cap (t_1, +\infty)$. On the other hand, a similar argument shows that

$$\int_{P_2 \cap (t_1, +\infty)} \left(\phi(|Q^{+\prime}_\tau|)Q^{+\prime}_\tau - \phi(|q^{+\prime}|)q^{+\prime}\right)\left(Q^{+\prime}_\tau - q^{+\prime}\right) dt \leq 0,$$

where $P_2 = \{t \in \mathbb{R} : q^+(t) \leq Q_\tau(t)\}$. Then, using Lemma A.8-$(c)$ again, we conclude that $q^{+\prime} = Q^{+\prime}_\tau$ on $P_2 \cap (t_1, +\infty)$. Since $P_1 \cup P_2 = (t_1, +\infty)$ and $Q^+_\tau(t_1) = q^+(t_1)$ we infer that

$$q^+(t) = Q^+_\tau(t) \text{ for any } t \in (t_1, +\infty).$$

Now, we define the following set

$$X = \left\{ y > 0 : q^+(t) = Q^+_\tau(t) \text{ for any } t \geq y \right\}.$$

Clearly, $X \neq \emptyset$ because $t_1 \in X$. Setting $y_0 = \inf X$, our next goal is to prove that $y_0 = 0$. Indeed, if not, then there exists $\epsilon > 0$ such that $y_0 - \epsilon > 0$, and so by Corollary 1.3, ${q^+}'(t) > 0$ for any $t \in [y_0 - \epsilon, y_0 + \epsilon]$. Since $\tau \geq 0$, ${Q^+_\tau}'(t) > 0$ for all $t \in [y_0 - \epsilon, y_0 + \epsilon]$, and hence there is $\rho > 0$ verifying

$${q^+}'(t), {Q^+_\tau}'(t) \geq \rho, \quad \forall t \in [y_0 - \epsilon, y_0 + \epsilon].$$

According to Theorem 1.1, $q^+ = Q^+_\tau$ in $[y_0 - \epsilon, y_0 + \epsilon]$, and consequently $y_0 - \epsilon \in X$, which is impossible. Thereby, $q^+(t) = Q^+_\tau(t)$ for any $t \in (0, +\infty)$. To complete the proof, it is sufficient to show that $\tau = 0$. Indeed, as $q^+(0) = 0$, and $Q^+(t) = 0$ if and only if $t = 0$, we obtain $0 = q^+(0) = Q^+_\tau(0) = Q^+(\tau)$, showing that $\tau = 0$, and the proof is complete. ■

Putting together all the information so far, we get the following result.

Theorem 1.3 *Assume (ϕ_1)-(ϕ_3) and (V_1)-(V_6). Then, there exists a unique $q \in K_\Phi$ such that it is a weak solution of (1.2) being heteroclinic from $-\alpha$ to α, i.e*

$$q(t) \to -\alpha \quad as \quad t \to -\infty \quad and \quad q(t) \to \alpha \quad as \quad t \to +\infty.$$

Moreover, $q \in C^{1,\gamma}_{loc}(\mathbb{R})$ for some $\gamma \in (0,1)$ and satisfies the following properties

(a) $q(t) = -q(-t)$ for any $t \in \mathbb{R}$,

(b) $0 < q(t) < \alpha$ for all $t > 0$,

(c) q is increasing on $\mathbb{R}$,

(d) $q'(t) \to 0$ as $t \to \pm\infty$,

(e) q' is non-increasing on $[0, +\infty)$,

(f) $q'(t) > 0$ for any $t \in \mathbb{R}$.

1.1.5 Compactness properties

The main objective of this subsection is to prove the below proposition that establishes the strong convergence for minimizing sequences of F on E_Φ.

Proposition 1.1 *If $(q_n) \subset E_\Phi$ is such that $F(q_n) \to c_\Phi$, then there exists $q \in K_\Phi$ such that, along a subsequence,*

$$\|q_n - q\|_{W^{1,\Phi}(\mathbb{R})} \to 0.$$

To prove the above proposition, we will consider the following subset of E_Φ defined by

$$\tilde{E}_\Phi(\alpha) = \left\{ w \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}) : w \text{ is odd a.e. in } \mathbb{R} \text{ and } w - \alpha \in W^{1,\Phi}\left([0,+\infty)\right) \right\}.$$

Moreover, let us also consider the following real number

$$\tilde{c}_\Phi = \inf_{w \in \tilde{E}_\Phi(\alpha)} F(w).$$

It is very important to point out that $\tilde{E}_\Phi(\alpha) \neq \emptyset$, because the function φ_α given in (1.7) clearly belongs to $\tilde{E}_\Phi(\alpha)$. Moreover, it is plain that if $w \in \tilde{E}_\Phi(\alpha)$, then $w + \alpha \in W^{1,\Phi}\left((-\infty, 0]\right)$, and that if $w_1, w_2 \in \tilde{E}_\Phi(\alpha)$, then $w_1 - w_2 \in W^{1,\Phi}(\mathbb{R})$. Have this in mind, we are able to define on $\tilde{E}_\Phi(\alpha)$ the metric $\rho : \tilde{E}_\Phi(\alpha) \times \tilde{E}_\Phi(\alpha) \to [0, +\infty)$ given by

$$\rho(w_1, w_2) = \|w_1 - w_2\|_{W^{1,\Phi}(\mathbb{R})}.$$

A direct computation guarantees that $(\tilde{E}_\Phi(\alpha), \rho)$ is a complete metric space.

The following result establishes an important relation between c_Φ and $\tilde{c}_\Phi$.

Lemma 1.13 *It holds that $\tilde{c}_\Phi = c_\Phi$.*

Proof. Clearly, $c_\Phi \leq \tilde{c}_\Phi$. We are going to show that $\tilde{c}_\Phi \leq c_\Phi$. For this, let $(q_n) \subset E_\Phi$ be a sequence such that $F(q_n) \to c_\Phi$. So, given $\delta \in (0, \delta_\alpha]$, there is $n_0 \in \mathbb{N}$ such that

$$F(q_n) \leq c_\Phi + \lambda_\delta \quad \text{for all} \quad n \geq n_0.$$

From Lemma 1.10,

$$q_n - \alpha \in W^{1,\Phi}([0,+\infty)) \quad \text{or} \quad q_n + \alpha \in W^{1,\Phi}([0,+\infty)) \quad \forall n \geq n_0.$$

Now, as $F(-q_n) = F(q_n)$, replacing q_n by $-q_n$ if necessary, we can assume that $q_n - \alpha \in W^{1,\Phi}([0,+\infty))$ for any $n \geq n_0$, and hence, $(q_n) \subset \tilde{E}_\Phi(\alpha)$, from where it follows that

$$\tilde{c}_\Phi \leq c_\Phi + \lambda_\delta.$$

Consequently, letting $\delta \to 0$ we infer that $\tilde{c}_\Phi \leq c_\Phi$, and therefore $c_\Phi = \tilde{c}_\Phi$. ■

It is important to note that the argument contained in the proof of the previous lemma guarantees that given a minimizing sequence (q_n) for F on E_Φ, that is $F(q_n) \to c_\Phi$, we may assume without loss of generality that $(q_n) \subset \tilde{E}_\Phi(\alpha)$.

To continue our analysis, we say that a sequence (q_n) is a $(PS)_d$ *sequence for* F, with $d \in \mathbb{R}$, if $(q_n) \subset \tilde{E}_\Phi(\alpha)$ satisfies

$$F(q_n) \to d \quad \text{and} \quad \|F'(q_n)\|_* \to 0 \quad \text{as} \quad n \to +\infty,$$

where

$$\|F'(w)\|_* = \sup\left\{F'(w)\psi : \psi \in X_0^{1,\Phi}(\mathbb{R}) \text{ and } \|\psi\|_{W^{1,\Phi}(\mathbb{R})} \leq 1\right\}.$$

Lemma 1.14 *If $(q_n) \subset E_\Phi$ and $F(q_n) \to c_\Phi$, then there exists a sequence $(p_n) \subset \tilde{E}_\Phi(\alpha)$ such that (p_n) is a $(PS)_{c_\Phi}$ sequence for F and*

$$\|q_n - p_n\|_{W^{1,\Phi}(\mathbb{R})} \leq \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Proof. From the above discussion, we can assume that $(q_n) \subset \tilde{E}_\Phi(\alpha)$. As $(\tilde{E}_\Phi(\alpha), \rho)$ is a complete metric space, we can employ the Ekeland's Variational Principle found in [93] to find a sequence $(p_n) \subset \tilde{E}_\Phi(\alpha)$ satisfying:

(a) $F(p_n) \leq F(q_n)$ for all $n \in \mathbb{N}$,

(b) $\rho(p_n, q_n) \leq \frac{1}{n}$ for any $n \in \mathbb{N}$,

(c) $F(p_n) - F(u) < \frac{1}{n}\|p_n - u\|_{W^{1,\Phi}(\mathbb{R})}$ for each $u \in \tilde{E}_\Phi(\alpha)$ with $u \neq p_n$.

Now, let $\psi \in X_0^{1,\Phi}(\mathbb{R})$ and write

$$\psi = \psi_o + \psi_e,$$

where ψ_o is odd and ψ_e is even. It is easily seen that $p_n + t\psi_o \in \tilde{E}_\Phi(\alpha)$ for any $n \in \mathbb{N}$ and $t > 0$. Thus, by item (c),

$$\begin{aligned}\frac{F(p_n + t\psi) - F(p_n)}{t} &= \frac{F(p_n + t\psi) - F(p_n + t\psi_o)}{t} + \frac{F(p_n + t\psi_o) - F(p_n)}{t} \\ &\geq \frac{F(p_n + t\psi) - F(p_n + t\psi_o)}{t} - \frac{1}{n}\|\psi_o\|_{W^{1,\Phi}(\mathbb{R})}.\end{aligned}$$

Arguing as in the proof of Theorem 1.2 we obtain that

$$F'(p_n)\psi \geq -\frac{1}{n}\|\psi_o\|_{W^{1,\Phi}(\mathbb{R})} \quad \forall n \in \mathbb{N}. \tag{1.34}$$

Claim: $\|\psi_o\|_{W^{1,\Phi}(\mathbb{R})} \leq \|\psi\|_{W^{1,\Phi}(\mathbb{R})}$.

Indeed, for any $\lambda > 0$,

$$\int_{\mathbb{R}} \Phi\left(\frac{|\psi_o(t)|}{\lambda}\right) dt = \int_{\mathbb{R}} \Phi\left(\frac{|\psi(t) - \psi(-t)|}{2\lambda}\right) dt$$
$$\leq \frac{1}{2}\int_{\mathbb{R}} \Phi\left(\frac{|\psi(t)|}{\lambda}\right) dt + \frac{1}{2}\int_{\mathbb{R}} \Phi\left(\frac{|\psi(-t)|}{\lambda}\right) dt = \int_{\mathbb{R}} \Phi\left(\frac{|\psi(t)|}{\lambda}\right) dt.$$

Taking $\lambda = \|\psi\|_{L^\Phi(\mathbb{R})}$,

$$\int_{\mathbb{R}} \Phi\left(\frac{|\psi_o(t)|}{\|\psi\|_{L^\Phi(\mathbb{R})}}\right) dt \leq \int_{\mathbb{R}} \Phi\left(\frac{|\psi(t)|}{\|\psi\|_{L^\Phi(\mathbb{R})}}\right) dt \leq 1,$$

which leads to $\|\psi_o\|_{L^\Phi(\mathbb{R})} \leq \|\psi\|_{L^\Phi(\mathbb{R})}$. Similarly, $\|{\psi_o}'\|_{L^\Phi(\mathbb{R})} \leq \|\psi'\|_{L^\Phi(\mathbb{R})}$. Thereby,

$$\|\psi_o\|_{W^{1,\Phi}(\mathbb{R})} \leq \|\psi\|_{W^{1,\Phi}(\mathbb{R})},$$

which proves the claim.

Hence, using the above claim in (1.34), one finds

$$F'(p_n)\psi \geq -\frac{1}{n}\|\psi\|_{W^{1,\Phi}(\mathbb{R})} \quad \forall \psi \in X_0^{1,\Phi}(\mathbb{R}).$$

Replacing ψ by $-\psi$,

$$|F'(p_n)\psi| \leq \frac{1}{n}\|\psi\|_{W^{1,\Phi}(\mathbb{R})} \quad \forall \psi \in X_0^{1,\Phi}(\mathbb{R}).$$

Thereby, $\|F'(p_n)\|_* \to 0$ as $n \to +\infty$. On the other hand, by (a) and Lemma 1.13,

$$c_\Phi = \tilde{c}_\Phi \leq F(p_n) \leq F(q_n) = c_\Phi + o_n(1),$$

showing that $F(p_n) \to c_\Phi$. Therefore, (p_n) is a $(PS)_{c_\Phi}$ sequence for F, and the lemma is proved. ■

From now on, we will always consider $(q_n) \subset E_\Phi$ and $(p_n) \subset \tilde{E}_\Phi(\alpha)$ as in the last lemma. So, (p_n) is also bounded in $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$. In fact, by Lemma 1.14, for each $L > 0$, one has

$$\|p_n\|_{W^{1,\Phi}([-L,L])} \leq \|p_n - q_n\|_{W^{1,\Phi}([-L,L])} + \|q_n\|_{W^{1,\Phi}([-L,L])} \leq \frac{1}{n} + \|q_n\|_{W^{1,\Phi}([-L,L])}.$$

Now, since (q_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$, it follows that (p_n) also is bounded in $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$. Consequently, for some subsequence, there exists $q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R})$ satisfying

$$p_n \rightharpoonup q \quad \text{in} \quad W^{1,\Phi}_{\text{loc}}(\mathbb{R}), \tag{1.35}$$

$$p_n \to q \quad \text{in} \quad L^{\Phi}_{\text{loc}}(\mathbb{R}), \tag{1.36}$$

$$p_n \to q \quad \text{in} \quad L^{1}_{\text{loc}}(\mathbb{R}) \tag{1.37}$$

and

$$p_n \to q \quad \text{in} \quad L^{\infty}_{\text{loc}}(\mathbb{R}). \tag{1.38}$$

Lemma 1.15 *There is a subsequence of (p_n), still denoted by itself, such that*

$$p_n{}'(t) \to q'(t) \text{ almost everywhere in } \mathbb{R}.$$

Proof. Given $L > 0$, let us consider $\psi \in C_0^\infty(\mathbb{R})$ satisfying

$$0 \le \psi \le 1, \quad \psi \equiv 1 \text{ in } [-L, L] \text{ and } \operatorname{supp}(\psi) \subset [-L-1, L+1].$$

According to item (c) of the Lemma A.8,

$$\begin{aligned} 0 &\le \int_{-L}^{L} \left(\phi(|p_n'|)p_n' - \phi(|q'|)q'\right)(p_n' - q')dt \\ &\le \int_{-L-1}^{L+1} \psi(t)\left(\phi(|p_n'|)p_n' - \phi(|q'|)q'\right)(p_n' - q')dt \\ &\le \int_{-L-1}^{L+1} \psi(t)\phi(|p_n'|)p_n'(p_n' - q')dt - \int_{-L-1}^{L+1} \psi(t)\phi(|q'|)q'(p_n{}' - q')dt. \end{aligned} \tag{1.39}$$

Setting the linear functional $f : W^{1,\Phi}([-L-1, L+1]) \to \mathbb{R}$ by

$$f(v) = \int_{-L-1}^{L+1} \psi\phi(|q'|)q'v'dt,$$

we have that it is continuous, because $\phi(|q'|)q' \in L^{\tilde{\Phi}}([-L-1, L+1])$ via Lemma A.6, and so, by Hölder's inequality

$$\begin{aligned} \left| \int_{-L-1}^{L+1} \psi\phi(|q'|)q'v'dt \right| &\le \int_{-L-1}^{L+1} |\phi(|q'|)q'||v'|dt \\ &\le 2\|\phi(|q'|)q'\|_{L^{\tilde{\Phi}}([-L-1,L+1])}\|v\|_{W^{1,\Phi}([-L-1,L+1])}, \end{aligned}$$

for any $v \in W^{1,\Phi}([-L-1, L+1])$. Thus, (1.35) asserts that $f(p_n - q) \to 0$, or equivalently,

$$\int_{-L-1}^{L+1} \psi\phi(|q'|)q'(p_n{}' - q')dt \to 0. \tag{1.40}$$

On the other hand, using again the Lemma A.6 and the boundedness of (p_n) in $W^{1,\Phi}_{\mathrm{loc}}(\mathbb{R})$, there is $C > 0$ such that

$$\int_{-L-1}^{L+1} \tilde{\Phi}(\phi(|p_n'|)p_n')dt \leq 2^m \int_{-L-1}^{L+1} \Phi(|p_n'|)dt \leq C, \quad \forall n \in \mathbb{N},$$

implying that $(\phi(|p_n'|)p_n')$ is bounded in $L^{\tilde{\Phi}}([-L-1, L+1])$. So, by (1.36) and Hölder's inequality,

$$\int_{-L-1}^{L+1} (p_n - q)\phi(|p_n'|)p_n'\psi' dt \to 0. \tag{1.41}$$

Now, considering the sequence (ψp_n), by (1.38), passing to a subsequence if necessary, we can assume that

$$\psi p_n \rightharpoonup \psi q \ \text{ in } \ W^{1,\Phi}([-L-1, L+1]) \quad \text{and} \quad \psi(t)p_n(t) \to \psi(t)q(t) \ \text{ a.e. } \ \mathbb{R}.$$

Consequently,

$$V'(p_n(t))\psi(t)(p_n(t) - q(t)) \to 0 \text{ almost everywhere in } [-L-1, L+1]$$

and by (1.37) there exist $h \in L^1([-L-1, L+1])$ and $M > 0$ such that, along a subsequence,

$$|V'(p_n)\psi(p_n - q)| \leq M|\psi|(h + |q|) \in L^1([-L-1, L+1]).$$

Applying the Lebesgue's dominated convergence theorem we obtain

$$\int_{-L-1}^{L+1} V'(p_n)(\psi p_n - \psi q)dt \to 0. \tag{1.42}$$

As ψ has compact support, $\psi(p_n - q) \in X_0^{1,\Phi}(\mathbb{R})$ for any $n \in \mathbb{N}$. We would like point out that

$$F'(p_n)(\psi p_n - \psi q) \to 0. \tag{1.43}$$

In fact, just note that

$$|F'(p_n)(\psi p_n - \psi q)| \leq \|F'(p_n)\|_* \|\psi p_n - \psi q\|_{W^{1,\Phi}(\mathbb{R})},$$

(ψp_n) is bounded in $W^{1,\Phi}(\mathbb{R})$ and (p_n) is a $(PS)_{c_\Phi}$ sequence for F. Recalling that

$$F'(p_n)(\psi p_n - \psi q) = \int_{-L-1}^{L+1} \phi(|p_n'|)p_n'(\psi p_n - \psi q)' dt + \int_{-L-1}^{L+1} V'(p_n)(\psi p_n - \psi q)dt,$$

it follows from (1.42) and (1.43) that

$$\int_{-L-1}^{L+1} \phi(|q_n'|)p_n'(\psi p_n - \psi q)' dt \to 0. \tag{1.44}$$

Now, since

$$(\psi p_n - \psi q)' = \psi p_n' + p_n\psi' - \psi q' - q\psi',$$

one gets

$$\int_{-L-1}^{L+1} \psi\phi(|p_n'|)p_n'(p_n'-q')dt = \int_{-L-1}^{L+1} \phi(|p_n'|)p_n'(\psi p_n - \psi q)'dt - \int_{-L-1}^{L+1} (p_n-q)\phi(|p_n'|)p_n'\psi' dt. \tag{1.45}$$

From (1.41), (1.44) and (1.45),

$$\int_{-L-1}^{L+1} \psi\phi(|p_n'|)p_n'(p_n'-q')dt \to 0. \tag{1.46}$$

Finally, from (1.40), (1.46) and (1.39),

$$\int_{-L}^{L} (\phi(|p_n'|)p_n' - \phi(|q'|)q')(p_n'-q')dt \to 0.$$

This limit combined with the Lemma A.8-(c) leads to, along a subsequence,

$$\left(\phi(|p_n'|)p_n' - \phi(|q'|)q'\right)(p_n'-q') \to 0 \ \text{ a.e. in } \ [-L, L].$$

Applying a result found in Dal Maso and Murat [32], we infer that

$$p_n'(t) \to q'(t) \ \text{ a.e. in } \ [-L, L].$$

As $L > 0$ is arbitrary, there is a subsequence of (p_n), still denoted by itself, such that $p_n'(t) \to q'(t)$ almost everywhere in $\mathbb{R}$, finishing the proof of the lemma. ■

Now, we are ready to prove Proposition 1.1.

Proof of Proposition 1.1.

According to Lemma 1.14 there exists a sequence $(p_n) \subset \tilde{E}_\Phi(\alpha)$ with $F(p_n) \to c_\Phi$ such that

$$\|q_n - p_n\|_{W^{1,\Phi}(\mathbb{R})} \le \frac{1}{n} \ \ \forall n \in \mathbb{N}, \tag{1.47}$$

and so, there is $q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R})$ satisfy (1.35)-(1.38). Now observe that

$$\|q_n - q\|_{L^\Phi([-L,L])} \le \frac{1}{n} + \|p_n - q\|_{L^\Phi([-L,L])} \ \ \forall L > 0. \tag{1.48}$$

From (1.36), q is the punctual limit of (q_n), and hence $q \in E_\Phi$ and $F(q) = c_\Phi$, that is, $q \in K_\Phi$. We conclude from (1.35) and (1.38) that

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(|q'|)dt &\le \liminf_n \int_{\mathbb{R}} \Phi(|p_n'|)dt \\
&\le \limsup_n \int_{\mathbb{R}} \Phi(|p_n'|)dt = \limsup_n \left(F(p_n) - \int_{\mathbb{R}} V(p_n)dt\right) \\
&\le F(q) - \int_{\mathbb{R}} V(q)dt = \int_{\mathbb{R}} \Phi(|q'|)dt,
\end{aligned}$$

that is,

$$\int_{\mathbb{R}} \Phi(|p_n'|)dt \to \int_{\mathbb{R}} \Phi(|q'|)dt. \tag{1.49}$$

By the convexity of Φ,

$$\frac{\Phi(|p_n'(t)|)}{2} + \frac{\Phi(|q'(t)|)}{2} - \Phi\left(\frac{|p_n'(t) - q'(t)|}{2}\right) \geq 0, \quad \forall t \in \mathbb{R}.$$

Thereby, the Lemma 1.15 together with Fatou's Lemma and (1.49) leads to

$$\begin{aligned}\int_{\mathbb{R}} \Phi(|q'|)dt &\leq \liminf_n \int_{\mathbb{R}} \left(\frac{\Phi(|p_n'|)}{2} + \frac{\Phi(|q'|)}{2} - \Phi\left(\frac{|p_n' - q'|}{2}\right)\right) dt \\ &\leq \int_{\mathbb{R}} \Phi(|q'|)dt - \limsup_n \int_{\mathbb{R}} \Phi\left(\frac{|p_n' - q'|}{2}\right) dt,\end{aligned}$$

that is,

$$\int_{\mathbb{R}} \Phi\left(\frac{|p_n' - q'|}{2}\right) dt \to 0.$$

As $\Phi \in \Delta_2$, the last limit yields

$$\|p_n' - q'\|_{L^\Phi(\mathbb{R})} \to 0$$

that combines with (1.47) to give

$$\|q_n' - q'\|_{L^\Phi(\mathbb{R})} \to 0. \tag{1.50}$$

On the other hand, from (1.49),

$$\int_{\mathbb{R}} V(q_n)dt = F(q_n) - \int_{\mathbb{R}} \Phi(|q_n'|)dt \to F(q) - \int_{\mathbb{R}} \Phi(|q'|)dt = \int_{\mathbb{R}} V(q)dt,$$

that is,

$$\int_{\mathbb{R}} V(q_n)dt \to \int_{\mathbb{R}} V(q)dt. \tag{1.51}$$

Now, using the fact that $F(q_n) \to c_\Phi$ combined with Lemma 1.3 it follows that given $\delta \in (0, \delta_\alpha)$ there is $n_1 \in \mathbb{N}$ such that

$$\|q_n\|_{L^\infty(\mathbb{R})} \leq \alpha + \delta \quad \forall n \geq n_1.$$

Thus, since $q \in K_\Phi$ and $V \in C^1(\mathbb{R}, \mathbb{R})$, there exists $M > 0$ such that

$$|V(q_n(t)) - V(q(t))| \leq M|q_n(t) - q(t)| \quad \forall t \in \mathbb{R}.$$

Thereby, by (1.38), $V(q_n) \to V(q)$ uniformly in $[-L, L]$ for any $L > 0$, and consequently,

$$\int_{-L}^{L} V(q_n)dt \to \int_{-L}^{L} V(q)dt. \tag{1.52}$$

Gathering (1.51) and (1.52),

$$\int_{L}^{+\infty} V(q_n)dt \to \int_{L}^{+\infty} V(q)dt \quad \forall L > 0.$$

Hereafter, we assume without loss of generality that $q = q^+$. Since $V(q^+) \in L^1(\mathbb{R})$, given $\epsilon > 0$ there exists $L \geq L_{\delta_1}$ such that

$$\int_{L}^{+\infty} V(q^+)dt < \frac{\epsilon}{2}.$$

Furthermore, for such values of L, there is $n_0 \in \mathbb{N}$ such that

$$\int_{L}^{+\infty} V(q_n)dt < \epsilon \quad \forall n \geq n_0.$$

Consequently, employing (1.11), it is easily seen that

$$\int_{L}^{+\infty} \Phi(|\alpha - q^+|)dt < \frac{\epsilon}{2\underline{w}} \quad \text{and} \quad \int_{L}^{+\infty} \Phi(|\alpha - q_n|)dt < \frac{\epsilon}{\underline{w}} \quad \forall n \geq n_0.$$

Finally, as $q_n \to q^+$ in $L^\Phi([-L, L])$,

$$\int_{\mathbb{R}} \Phi(|q_n - q^+|)dt = o_n(1) + 2\int_{L}^{+\infty} \Phi(|q_n - q^+|)dt, \tag{1.53}$$

and so, the Lemma A.8-(a) provides

$$2\int_{L}^{+\infty} \Phi(|q_n - q^+|)dt \leq 2^m \int_{L}^{+\infty} \big(\Phi(|q_n - \alpha|) + \Phi(|q^+ - \alpha|)\big)\, dt \leq \left(2^m \frac{1}{\underline{w}} + 2^{m-1}\frac{1}{\underline{w}}\right)\epsilon, \tag{1.54}$$

for each $n \geq n_0$. As $\epsilon > 0$ is arbitrary, it follows from (1.53) and (1.54),

$$\int_{\mathbb{R}} \Phi(|q_n - q^+|)dt \to 0.$$

Using the fact that $\Phi \in \Delta_2$, we arrive at

$$\|q_n - q^+\|_{L^\Phi(\mathbb{R})} \to 0. \tag{1.55}$$

Now, the proposition follows from (1.50) and (1.55). ■

1.1.6 Exponential estimates

In this subsection, we study exponential-type estimates for the heteroclinic solutions of (1.2) in K_Φ, which will play an important role in finding a saddle-type solution for equation (1.1). To fulfill our objective, given $d \in (0, \alpha)$ let us define the following real number

$$l_d = \frac{c_\Phi + 1}{\omega_d}, \quad \text{where} \quad \omega_d = \min_{|s| \leq \alpha - d} V(s) > 0.$$

Next, we list some useful technical lemmas in the development of this subsection.

Lemma 1.16 *Let $q \in E_\Phi$. If $\int_{t_1}^{t_2} \mathcal{L}(q)dt < c_\Phi + 1$ and $|q(t)| \le \alpha - d$ for any $t \in (t_1, t_2)$ with $t_1 < t_2$, then $l_d > t_2 - t_1$.*

Proof. Just note that

$$c_\Phi + 1 > \int_{t_1}^{t_2} \mathcal{L}(q)dt \ge \int_{t_1}^{t_2} V(q)dt \ge \int_{t_1}^{t_2} \min_{|s|\le 1-d} V(s)\, dt,$$

from where it follows that $c_\Phi + 1 > \omega_d(t_2 - t_1)$, and hence $l_d > t_2 - t_1$. ■

From now on, given $\delta \in (0, \delta_\alpha]$ we fix $d(\delta) \in (0, \delta/2)$ such that

$$2(1 + \overline{w})\Phi(d(\delta)) < \lambda_\delta, \tag{1.56}$$

where λ_δ was given in (1.10). Moreover, we also denote $L_\delta = l_{d(\delta)}$.

Lemma 1.17 *Let $q \in E_\Phi$, $\delta \in (0, \delta_\alpha]$ and $L \ge L_\delta$ such that $\int_{-L}^{L} \mathcal{L}(q)dt \le c_\Phi + \lambda_\delta$. Then, there exists $t_+ \in (0, L_\delta)$ such that*

$$|q(t_+)| \ge \alpha - d(\delta) \quad \textit{and} \quad c_\Phi - \lambda_\delta \le \int_{-t_+}^{t_+} \mathcal{L}(q)dt.$$

Proof. By (1.10),

$$\int_{-L_\delta}^{L_\delta} \mathcal{L}(q)dt \le \int_{-L}^{L} \mathcal{L}(q)dt \le c_\Phi + \lambda_\delta < c_\Phi + 1.$$

We claim that there is $t_+ \in (0, L_\delta)$ such that

$$|q(t_+)| \ge \alpha - d(\delta).$$

If not, then $|q(t)| < \alpha - d(\delta)$ for all $t \in (0, L_\delta)$, and by Lemma 1.16, $L_\delta > L_\delta$, which is absurd. Possibly considering the function $-q$, it is not restrictive to assume that $q(t_+) \ge \alpha - d(\delta)$. Furthermore, we can assume that $\alpha \ge q(t_+) \ge \alpha - d(\delta)$ and set

$$\tilde{q}(t) = \begin{cases} q(t), & \text{if} \quad 0 \le t \le t_+ \\ q(t_+) + (\alpha - q(t_+))(t - t_+), & \text{if} \quad t_+ \le t \le t_+ + 1 \\ \alpha, & \text{if} \quad t > t_+ + 1 \\ -\tilde{q}(-t), & \text{if} \quad t < 0. \end{cases}$$

Since

$$\alpha - \frac{\delta_\alpha}{2} \le \alpha - d(\delta) \le \tilde{q}(t) \le \alpha \;\; \text{for any} \;\; t \in [t_+, t_+ + 1],$$

then by (1.11),

$$V(\tilde{q}(t)) \le \overline{w}\Phi(|\alpha - \tilde{q}(t)|) \le \overline{w}\Phi(d(\delta)) \;\; \forall t \in [t_+, t_+ + 1].$$

A direct estimate gives us that

$$\int_{t_+}^{t_++1} \mathcal{L}(\tilde{q})dt \leq (1+\overline{w})\Phi(d(\delta)).$$

Hence, as $\tilde{q} \in E_\Phi$, the symmetry of $\tilde{q}$ and the choice of $d(\delta)$ in (1.56) lead to

$$\begin{aligned} c_\Phi \leq F(\tilde{q}) &= \int_{-t_+}^{t_+} \mathcal{L}(q)dt + 2\int_{t_+}^{t_++1} \mathcal{L}(\tilde{q})dt \\ &\leq \int_{-t_+}^{t_+} \mathcal{L}(q)dt + 2(1+\overline{w})\Phi(d(\delta)) \leq \int_{-t_+}^{t_+} \mathcal{L}(q)dt + \lambda_\delta, \end{aligned}$$

and this is precisely the assertion of the lemma. ■

The last lemma permits to derive the following result.

Lemma 1.18 *Let $q \in E_\Phi$, $\delta \in (0, \delta_\alpha]$ and $L \geq L_\delta$ such that $\int_{-L}^{L} \mathcal{L}(q)dt \leq c_\Phi + \lambda_\delta$. Then,*

$$|q(t)| \geq \alpha - \delta \ \ \textit{for any} \ \ t \in [L_\delta, L].$$

Proof. Arguing by contradiction, assume that there exists $x \in [L_\delta, L]$ such that $|q(x)| < \alpha - \delta$. According to Lemma 1.17, there is $t_+ \in (0, L_\delta)$ such that

$$|q(t_+)| \geq \alpha - d(\delta) \ \text{ and } \ c_\Phi - \lambda_\delta \leq \int_{-t_+}^{t_+} \mathcal{L}(q)dt.$$

Assuming without loss of generality that

$$q(t_+) \geq \alpha - d(\delta) \geq \alpha - \frac{\delta}{2}.$$

So the continuity of q ensures that there exist $\sigma, \tau \in \mathbb{R}$ such that $0 < t_+ < \sigma < \tau < x$ and

$$q(\sigma) = \alpha - \frac{\delta}{2}, \ q(\tau) = \alpha - \delta \text{ and } \alpha - \delta \leq q(t) \leq \alpha - \frac{\delta}{2} \ \ \forall t \in (\sigma, \tau).$$

Hence, from (1.11), one has

$$\underline{w}\Phi\left(\frac{\delta}{2}\right) \leq \underline{w}\Phi(|\alpha - q(t)|) \leq V(q(t)) \ \ \forall t \in (\sigma, \tau),$$

and therefore by Lemma 1.2 there exists $\mu_{\underline{w}\Phi(\frac{\delta}{2})} > 0$ satisfying

$$\int_\sigma^\tau \mathcal{L}(q)dt \geq \mu_{\underline{w}\Phi(\frac{\delta}{2})} h\left(\Phi(|q(\tau) - q(\sigma)|)\right) = \mu_{\underline{w}\Phi(\frac{\delta}{2})} h\left(\Phi\left(\frac{\delta}{2}\right)\right).$$

By definition of λ_δ in (1.10),

$$c_\Phi + \lambda_\delta \geq \int_{-L}^{L} \mathcal{L}(q)dt \geq \int_{-t_+}^{t_+} \mathcal{L}(q)dt + 2\int_\sigma^\tau \mathcal{L}(q)dt \geq c_\Phi - \lambda_\delta + 2h\left(\Phi\left(\frac{\delta}{2}\right)\right)\mu_{\underline{w}\Phi(\frac{\delta}{2})} > c_\Phi + \lambda_\delta,$$

which is impossible, and the proof is over. ■

As immediate consequence of the last lemma is the corollary below.

Corollary 1.6 *Let $q \in E_\Phi$ and $\delta \in (0, \delta_\alpha]$ such that $F(q) \leq c_\Phi + \lambda_\delta$. Then,*

$$|q(t)| \geq \alpha - \delta \quad \textit{for any} \quad t \geq L_\delta.$$

To continue our study, for each $L > 0$ let's consider the following class of admissible functions

$$E_L = \left\{u \in W^{1,\Phi}((-L, L)) : u(t) = -u(-t) \text{ for all } t \in (-L, L)\right\}$$

and the functional $F_L : W^{1,\Phi}((-L, L)) \to \mathbb{R} \cup \{+\infty\}$ given by

$$F_L(u) = \int_{-L}^{L} \mathcal{L}(u)dt.$$

By a direct computation, F_L is lower semicontinuous with respect to the weak $W^{1,\Phi}((-L, L))$ topology and bounded from below. We define

$$c_L := \inf_{u \in E_L} F_L(u).$$

Since $F_L(0) < +\infty$, c_L is well defined, and moreover, $c_L \leq c_\Phi$ for all $L > 0$. From now on, let us also consider

$$K_L = \{u \in E_L : F_L(u) = c_L\}.$$

Now, proceeding analogously as in the proofs of Theorem 1.2, Lemma 1.3 and Corollary 1.1 we get the following lemmas.

Lemma 1.19 *For every $L > 0$, one has $K_L \neq \emptyset$. Moreover, if $q \in K_L$ then $q \in C^1((-L, L))$ and is a weak solution of*

$$\begin{cases} -(\phi(|q'|)q')' + V'(q) = 0 \quad \textit{in} \quad (-L, L) \\ q'(\pm L) = 0. \end{cases}$$

Lemma 1.20 *Let $L > 0$, $q \in E_L$ and $\delta \in (0, \delta_\alpha]$ such that $F_L(q) \leq c_\Phi + \lambda_\delta$. Then,*

$$\|q\|_{L^\infty((-L,L))} \leq \alpha + \delta.$$

In particular, if $q \in K_L$ then $\|q\|_{L^\infty((-L,L))} \leq \alpha$.

Now we want to show an estimate of the exponential type involving the function $q \in K_L$.

Lemma 1.21 *For each $L \geq L_{\delta_\alpha}$, there exists $q \in K_L$ such that*

$$\alpha - \delta_\alpha \leq q(t) \ \ \textit{for all} \ \ t \in [L_{\delta_\alpha}, L].$$

Moreover, there are $\theta_1, \theta_2 > 0$ such that

$$0 \leq \alpha - q(t) \leq \theta_1 e^{-\theta_2 t} \ \ \textit{for all} \ \ t \in [L_{\delta_\alpha}, L]. \tag{1.57}$$

Proof. Note that if $L \geq L_{\delta_\alpha}$ and $q \in K_L$, then $F_L(q) = c_L \leq c_\Phi < c_\Phi + \lambda_{\delta_\alpha}$. Then, by Lemma 1.18,

$$|q(t)| \geq \alpha - \delta_\alpha \ \ \text{for all} \ \ t \in [L_{\delta_\alpha}, L].$$

Assuming without loss of generality that $q(t) \geq \alpha - \delta_\alpha$ for any $t \in [L_{\delta_\alpha}, L]$, we can define the function

$$\tilde{q}(t) = \begin{cases} q(t), & \text{if} \quad t \in [L_{\delta_\alpha}, L] \\ q(2L - t), & \text{if} \quad t \in [L, 2L - L_{\delta_\alpha}], \end{cases}$$

and $v(t) = \alpha - \tilde{q}(t)$ for $t \in [L_{\delta_\alpha}, 2L - L_{\delta_\alpha}]$. From Lemma 1.20 and (V_4),

$$V'(q) \leq -\omega_1(\alpha - \delta_\alpha)\phi(\omega_2 v)(v) \ \ \text{in} \ \ [L_{\delta_\alpha}, L].$$

So, if $\psi \in W_0^{1,\Phi}([L_{\delta_\alpha}, 2L - L_{\delta_\alpha}])$, $\psi \geq 0$ and $\psi(t) = \psi(2L - t)$ for any $t \in [L_{\delta_\alpha}, L]$, then an easy computation shows that

$$\int_{L_{\delta_1}}^{2L - L_{\delta_\alpha}} \left(\phi(|v'|)v'\psi' + \frac{\omega_1(\alpha - \delta_\alpha)}{\omega_2}\phi(\omega_2 v)\omega_2 v\psi \right) dt \leq 0.$$

Now, to complete the estimation of the exponential type (1.57), let us consider the real function

$$\zeta(t) = \delta_\alpha \frac{\cosh(a(t - L))}{\cosh(a(L - L_{\delta_\alpha}))}, \ \ t \in \mathbb{R},$$

for some constant $a > 0$. Now, for values of $t \in \mathbb{R}$ such that $|\zeta'(t)| > 0$ it is easily seen that

$$(\phi(|\zeta'(t)|)\zeta'(t))' = \phi(|\zeta'(t)|)\zeta''(t) + \phi'(|\zeta'(t)|)|\zeta'(t)|\zeta''(t).$$

We note that the equality $\zeta''(t) = a^2\zeta(t)$ for any $t \in \mathbb{R}$ together with (ϕ_2) yields

$$(\phi(|\zeta'(t)|)\zeta'(t))' \leq ma^2\phi(|\zeta'(t)|)\zeta(t).$$

For the case that $\zeta'(t) = 0$, an easy verification shows that

$$(\phi(|\zeta'(t)|)\zeta'(t))' = \phi(0)\zeta''(t) \leq ma^2\phi(0)\zeta(t) = ma^2\phi(|\zeta'(t)|)\zeta(t).$$

In both cases,

$$(\phi(|\zeta'(t)|)\zeta'(t))' \leq ma^2\phi(|\zeta'(t)|)\zeta(t) \quad \forall t \in \mathbb{R}.$$

Moreover, since $|\zeta'(t)| \leq a\zeta(t)$ for any $t \in \mathbb{R}$, fixing $a < \omega_2$ and using (ϕ_4), we derive that $\phi(|\zeta'(t)|) \leq \phi(\omega_2\zeta(t))$. Therefore,

$$-(\phi(|\zeta'(t)|)\zeta'(t))' + ma\phi(\omega_2\zeta(t))\omega_2\zeta(t) \geq 0, \quad \forall t \in \mathbb{R}.$$

A direct computation gives

$$\int_{L_{\delta_\alpha}}^{2L-L_{\delta_\alpha}} (\phi(|\zeta'|)\zeta'\psi' + ma\phi(\omega_2\zeta)\omega_2\zeta\psi)\,dt \geq 0.$$

On the other hand, by definition of v and ζ,

$$v(L_{\delta_\alpha}) \leq \delta_\alpha = \zeta(L_{\delta_\alpha}) \;\text{ and }\; v(2L - L_{\delta_\alpha}) \leq \delta_\alpha = \zeta(2L - L_{\delta_\alpha}).$$

We may now take $a > 0$ sufficiently small such that

$$\lambda(a) := ma < \frac{(\alpha - \delta_\alpha)\omega_1}{\omega_2}.$$

Have this in mind and considering $\psi_* = (v - \zeta)^+$, we have $\psi_*(t) = \psi_*(2L - t)$ for each $t \in [L_{\delta_\alpha}, L]$, and so,

$$\int_{L_{\delta_\alpha}}^{2L-L_{\delta_\alpha}} ((\phi(|v'|)v' - \phi(|\zeta'|)\zeta')\,{\psi_*}' + \lambda(a)(\phi(\omega_2 v)\omega_2 v - \phi(\omega_2\zeta)\omega_2\zeta)\psi_*)\,dt \leq 0.$$

Putting

$$P = \{t \in [L_{\delta_\alpha}, 2L - L_{\delta_\alpha}] : v(t) \geq \zeta(t)\},$$

it follows by Lemma A.8-(c) that $\psi_* = 0$ in P, that is,

$$v(t) \leq \zeta(t) \;\text{ for any }\; t \in [L_{\delta_\alpha}, 2L - L_{\delta_\alpha}].$$

Since

$$\frac{e^t}{2} \leq \cosh(t) \leq e^t \;\text{ for all }\; t \geq 0,$$

we find

$$0 \leq \alpha - q(t) \leq 2e^{a(L_{\delta_\alpha} - t)} \quad \forall t \in [L_{\delta_1}, L],$$

and the lemma follows. ■

Thanks to Lemma 1.21, the next lemma establishes a better characterization of the behavior of the function $L \mapsto c_L$.

Lemma 1.22 *The function $L \mapsto c_L$ is monotone increasing with $c_L \to c_\Phi$ as $L \to +\infty$. Moreover, there are θ and β positive real numbers such that*

$$0 \leq c_\Phi - c_L \leq \theta e^{-\beta L} \quad \forall L > 0.$$

Proof. To prove the monotonicity of $L \mapsto c_L$, fix $L_1 \leq L_2$ and let $q \in K_{L_2}$. Of course, $q \in K_{L_1}$, and so, $c_{L_1} \leq F_{L_1}(q) \leq F_{L_2}(q) = c_{L_2}$. Let us prove now the exponential estimate. By Lemma 1.21, for $L > L_{\delta_\alpha}$ there exists $q \in K_L$ such that $\alpha - \delta_\alpha \leq q(t)$ for any $t \in [L_{\delta_\alpha}, L]$. Setting

$$\tilde{q}(t) = \begin{cases} q(t), & \text{if} \quad 0 \leq t \leq L, \\ q(L) + (\alpha - q(L))(t - L), & \text{if} \quad L \leq t \leq L+1, \\ \alpha, & \text{if} \quad t > L+1, \\ -\tilde{q}(-t), & \text{if} \quad t < 0, \end{cases}$$

it is easy to see that $\tilde{q} \in E_\Phi$, and then $c_\Phi \leq F(\tilde{q})$. Moreover, by symmetry,

$$F(\tilde{q}) = c_L + 2\int_L^{L+1} \mathcal{L}(\tilde{q})dt,$$

leading to

$$c_\Phi - c_L \leq 2\int_L^{L+1} \mathcal{L}(\tilde{q})dt.$$

Now, since

$$\alpha - \delta_\alpha \leq q(L) \leq \tilde{q}(t) \leq \alpha \quad \text{for any} \quad t \in [L, L+1],$$

by (1.11),

$$V(\tilde{q}(t)) \leq \overline{w}\Phi(|\alpha - q(L)|) \quad \text{for all} \quad t \in [L, L+1].$$

According to Lemma 1.21,

$$\int_L^{L+1} \mathcal{L}(\tilde{q})dt = \int_L^{L+1} \Phi(|\alpha - q(L)|)dt + \int_L^{L+1} V(\tilde{q})dt \leq \Phi(\theta_1 e^{-\theta_2 L}) + \overline{w}\Phi(\theta_1 e^{-\theta_2 L}).$$

Taking L sufficiently large, the Lemma A.2 implies that

$$\int_L^{L+1} \mathcal{L}(\tilde{q})dt \leq \max\{1, \overline{w}\}\Phi(\theta_1)e^{-l\theta_2 L},$$

from where it follows that

$$c_\Phi - c_L \leq 2\max\{1, \overline{w}\}\Phi(\theta_1)e^{-l\theta_2 L}.$$

Therefore, it is possible to find real numbers $\theta, \beta > 0$ satisfying precisely the assertion of the lemma. ■

Our next lemma establishes in some sense a compactness property concerning to family of functionals F_L.

Lemma 1.23 *Let $(L_n) \subset \mathbb{R}$ with $L_n \to +\infty$ and $q_n \in E_{L_n}$ such that $F_{L_n}(q_n) - c_{L_n} \to 0$ as $n \to +\infty$. Then, there exists $q \in K_\Phi$ verifying, along a subsequence,*

$$\|q_n - q\|_{W^{1,\Phi}((-L_n,L_n))} \to 0 \text{ as } n \to +\infty.$$

Proof. Let (δ_n) be a sequence with $\delta_n \in (0, \delta_\alpha]$ for all $n \in \mathbb{N}$ and $\delta_n \to 0$ as $n \to +\infty$. Since $L_n \to +\infty$ and $F_{L_n}(q_n) - c_{L_n} \to 0$, we can assume without loss of generality that

$$L_n \geq L_{\delta_n} \quad \text{and} \quad F_{L_n}(q_n) < c_{L_n} + \lambda_{\delta_n}, \quad \forall n \in \mathbb{N},$$

where λ_{δ_n} was given in (1.10). Thereby, by Lemma 1.18, $|q_n(L_n)| \geq \alpha - \delta_n$. Assuming, up to reflection, that $q_n(L_n) \geq \alpha - \delta_n$, we set

$$\tilde{q}_n(t) = \begin{cases} q_n(t), & \text{if} \quad 0 \leq t \leq L_n \\ q_n(L_n) + (\alpha - q_n(L_n))(t - L_n), & \text{if} \quad L_n \leq t \leq L_n + 1 \\ \alpha, & \text{if} \quad L_n + 1 \leq t \\ -\tilde{q}_n(-t), & \text{if} \quad t \leq 0. \end{cases}$$

Note that $(\tilde{q}_n) \subset E_\Phi$ and hence $F(\tilde{q}_n) \geq c_\Phi$ for any $n \in \mathbb{N}$. According to Lemma 1.20,

$$\alpha - \delta_n \leq q_n(L_n) \leq \alpha + \delta_n,$$

and so, by (1.11),

$$V(\tilde{q}_n(t)) \leq \overline{w}\Phi(\delta_n), \quad \forall t \in [L_n, L_n + 1].$$

Therefore, we get the inequality

$$F(\tilde{q}_n) = F_{L_n}(q_n) + 2\int_{L_n}^{L_n+1} \mathcal{L}(\tilde{q}_n)dt < c_{L_n} + \lambda_{\delta_n} + 2\Phi(\delta_n) + 2\overline{w}\Phi(\delta_n).$$

Since $c_{L_n} \to c_\Phi$ and $\lambda_{\delta_n} \to 0$ as $\delta_n \to 0$, we have $F(\tilde{q}_n) \to c_\Phi$. Then, by Proposition 1.1, there exists $q \in K_\Phi$ such that, along a subsequence, $\|\tilde{q}_n - q\|_{W^{1,\Phi}(\mathbb{R})} \to 0$. In particular, as $\Phi \in \Delta_2$,

$$\|q_n - q\|_{W^{1,\Phi}((-L_n,L_n))} \to 0 \text{ as } n \to +\infty,$$

and the proof is over. ■

By the previous lemma we obtain in particular that for each $r > 0$ there exist $\nu_r > 0$ and $M_r > 0$ such that for all $L > M_r$,

$$\text{if } u \in E_L \text{ satisfies } \|u - q\|_{W^{1,\Phi}((-L,L))} > r, \ \forall q \in K_\Phi, \text{ then } F_L(u) - c_L > \nu_r. \tag{1.58}$$

1.2 Saddle solutions on $\mathbb{R}^2$

In this section, we collect the results obtained earlier to find saddle-type solutions for equation (1.1).

1.2.1 Construction of solution on a infinite triangular set

To formulate the minimization problem of this subsection, let us first fix $j \in \mathbb{N}$ with $j \geq 2$ and $a_j = \tan(\frac{\pi}{2j})$. Moreover, given $y > 0$, with abuse of notation, let us denote

$$I_y := (-a_j y, a_j y), \quad E_y := E_{a_j y}, \quad c_y := c_{a_j y} \quad \text{and} \quad F_y(q) := F_{a_j y}(q) \quad \text{for} \quad q \in E_y.$$

That said, we define

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 : x \in I_y \quad \text{and} \quad y > 0\right\}$$

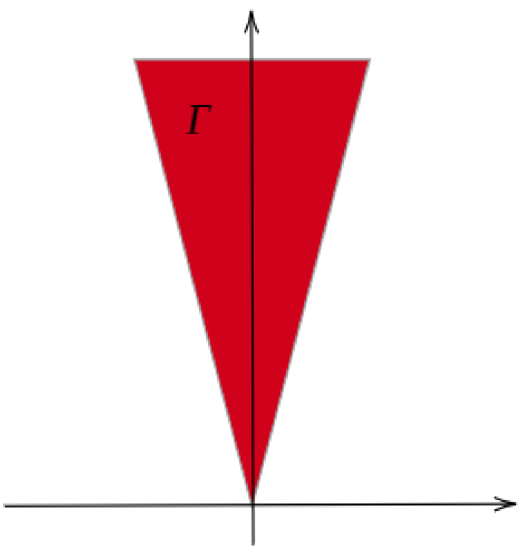

Figure 1.1: Geometric illustration of Γ.

and the class

$$E_\infty(\alpha) = \left\{w \in W^{1,\Phi}_{\text{loc}}(\Gamma) : \; w \text{ is odd in } x \text{ and } 0 \leq w(x, y) \leq \alpha \text{ for } x > 0\right\}.$$

Hereafter, without loss of generality, for each $y > 0$ fixed and $w \in E_\infty(\alpha)$, $w(\cdot, y)$ designates the real function in $x \in I_y$ and so $w(\cdot, y) \in E_y$. Finally, we may define the functional $J : W^{1,\Phi}_{\rm loc}(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ by

$$J(w) = \int_0^{+\infty} \left(\int_{I_y} \tilde{\mathcal{L}}(w)dx - c_y \right) dy,$$

where

$$\tilde{\mathcal{L}}(w) = \Phi(|\nabla w|) + V(w), \quad w \in W^{1,\Phi}_{\rm loc}(\Gamma).$$

Invoking Lemma A.7,

$$J(w) \geq 0 \ \text{ for any } \ w \in E_\infty(\alpha),$$

from where it follows that J is bounded from below on $E_\infty(\alpha)$.

Lemma 1.24 *Setting $u^+(x,y) = q^+(x)$ for any $(x,y) \in \Gamma$, we have that $u^+ \in E_\infty(\alpha)$ and $J(u^+) < +\infty$, where q^+ was given in Theorem 1.3.*

Proof. Trivially, $u^+ \in E_\infty(\alpha)$. Also, since u^+ is independent of the variable y, this yields that $\partial_y u^+(x,y) = 0$. Then,

$$J(u^+) = \int_0^{+\infty} \left(F_y(q^+) - c_y\right) dy \leq \int_0^{+\infty} (c_\Phi - c_y)dy \leq \theta \int_0^{+\infty} e^{-\beta y} dy < +\infty,$$

via by Lemma 1.22 and the proof is finished. ■

According to the above lemma, the below number

$$d_\infty := \inf_{w \in E_\infty(\alpha)} J(w)$$

is well defined.

In what follows, if $(u_n) \subset W^{1,\Phi}_{\rm loc}(\Gamma)$ and $u \in W^{1,\Phi}_{\rm loc}(\Gamma)$, we write $u_n \rightharpoonup u$ in $W^{1,\Phi}_{\rm loc}(\Gamma)$ to denote that $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Omega)$ for any Ω relatively compact in Γ. Hence, we obtain the following result.

Lemma 1.25 *Let $(u_n) \subset W^{1,\Phi}_{loc}(\Gamma)$ and $u \in W^{1,\Phi}_{loc}(\Gamma)$ be such that $u_n \rightharpoonup u$ in $W^{1,\Phi}_{loc}(\Gamma)$. Then*

$$J(u) \leq \liminf_{n \to +\infty} J(u_n).$$

Proof. For any fixed $R > 0$, it is easy to check that

$$\int_0^R \int_{I_y} \tilde{\mathcal{L}}(u)dxdy \leq \liminf_{n \to +\infty} \int_0^R \int_{I_y} \tilde{\mathcal{L}}(u_n)dxdy.$$

Consequently,

$$\int_0^R \left(\int_{I_y} \tilde{\mathcal{L}}(u)dx - c_y \right) dy \leq \liminf_{n \to +\infty} \int_0^R \left(\int_{I_y} \tilde{\mathcal{L}}(u_n)dx - c_y \right) dy.$$

Then, if $J(u) = +\infty$ we also have $\liminf_{n\to+\infty} J(u_n) = +\infty$. If otherwise $J(u) < +\infty$, then given any $\epsilon > 0$ there exists $R > 0$ such that

$$\int_R^{+\infty} \left(\int_{I_y} \tilde{\mathcal{L}}(u)dx - c_y \right) dy \leq \epsilon.$$

Therefore,

$$J(u)-\epsilon \leq \int_0^R \left(\int_{I_y} \tilde{\mathcal{L}}(u)dx - c_y \right) dy \leq \liminf_{n\to+\infty} \int_0^R \left(\int_{I_y} \tilde{\mathcal{L}}(u_n)dx - c_y \right) dy \leq \liminf_{n\to+\infty} J(u_n),$$

and the lemma follows since ϵ is arbitrary. ■

For each interval $(\sigma,\tau) \subset \mathbb{R}_+$, let us fix $Q_{(\sigma,\tau)} = I_\sigma \times (\sigma,\tau) \subset \Gamma$ and

$$J_{(\sigma,\tau)}(u) = \int_\sigma^\tau \left(\int_{I_y} \tilde{\mathcal{L}}(u)dx - c_y \right) dy.$$

Thereby, for each $u \in E_\infty(\alpha)$ we have $u \in W^{1,\Phi}(Q_{(\sigma,\tau)})$. Making an argument similar to the proof of Lemma 1.1 we get the inequality

$$\int_{I_\sigma} \Phi(|u(x,\tau) - u(x,\sigma)|)dx \leq \frac{\xi_1(|\tau-\sigma|)}{|\tau-\sigma|} \int_{Q_{(\sigma,\tau)}} \Phi(|\partial_y u|)dxdy. \tag{1.59}$$

Given any bounded interval $I \subset \mathbb{R}$ and $u \in E_\infty(\alpha)$, there exists $y_0 > 0$ such that $I \subset I_{y_0}$ and by Fubini's Theorem, $u(\cdot,y) \in W^{1,\Phi}(I)$ almost everywhere in $y \in (y_0,+\infty)$. As a direct consequence of (1.59) we obtain the following lemma.

Lemma 1.26 *Let $I \subset \mathbb{R}$ be a bounded interval and $u \in E_\infty(\alpha)$ with $J(u) < +\infty$. Then, there is $y_0 > 0$ such that the function $y \in (y_0,+\infty) \mapsto u(\cdot,y) \in L^\Phi(I)$ is uniformly continuous a.e. in $(y_0,+\infty)$.*

Proof. Let $y_0 > 0$ be such that $I \subset I_{y_0}$. As $J(u) < +\infty$, Lemma A.7 together with (1.59) shows that

$$\int_I \Phi(|u(x,y_1) - u(x,y_2)|)dx \leq \frac{\xi_1(|y_1-y_2|)}{|y_1-y_2|} J(u), \quad \forall y_1,y_2 \in (y_0,+\infty).$$

From this, given $\epsilon > 0$ there is $\delta > 0$ verifying

$$\int_I \Phi(|u(x,y_1) - u(x,y_2)|)dx < \epsilon \ \text{ for } \ |y_1-y_2| < \delta \ \text{ with } \ y_1,y_2 \in (y_0,+\infty).$$

According to Lemma A.2,

$$\xi_0\left(\|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^\Phi(I)}\right) \leq \int_I \Phi(|u(x,y_1) - u(x,y_2)|)dx < \epsilon \ \text{ for } \ |y_1-y_2| < \delta,$$

showing that

$$|y_1 - y_2| < \delta \Rightarrow \|u(\cdot, y_1) - u(\cdot, y_2)\|_{L^\Phi(I)} < \xi_0^{-1}(\epsilon),$$

and the lemma follows. ■

The following result is a crucial estimate to finish our study in this chapter.

Lemma 1.27 *Let $r > 0$ and $u \in E_\infty(\alpha)$ such that $F_y(u(\cdot, y)) - c_y \geq r$ a.e. in $y \in (\sigma, \tau) \subset (0, +\infty)$. Then, there is $\mu_r > 0$ independent of σ and τ satisfying*

$$\begin{aligned} J_{(\sigma,\tau)}(u) &\geq \frac{|\sigma - \tau|}{\xi_1(|\sigma - \tau|)} \int_\sigma^\tau \Phi(|u(x, \tau) - u(x, \sigma)|)dx + r|\sigma - \tau| \\ &\geq \mu_r h\left(\int_\sigma^\tau \Phi(|u(x, \tau) - u(x, \sigma)|)dx\right), \end{aligned}$$

where h was given in Lemma 1.2.

Proof. First of all, note that by Lemma A.7 we can derive the following inequality

$$J_{(\sigma,\tau)}(u) \geq \int_{Q_{(\sigma,\tau)}} \Phi(|\partial_y u|)dxdy + \int_\sigma^\tau (F_y(u(\cdot, y)) - c_y)\, dy.$$

According to (1.59),

$$J_{(\sigma,\tau)}(u) \geq \frac{|\sigma - \tau|}{\xi_1(|\sigma - \tau|)} \int_{I_\sigma} \Phi(|u(x, \tau) - u(x, \sigma)|)dx + r|\sigma - \tau|.$$

Arguing as in the proof of Lemma 1.2 we have the existence of a constant $\mu_r > 0$ independent of σ and τ such that

$$J_{(\sigma,\tau)}(u) \geq \mu_r h\left(\int_{I_\sigma} \Phi(|u(x, \tau) - u(x, \sigma)|)dx\right),$$

which completes the proof. ■

The result below allows us to characterize the asymptotic behavior of the functions $u \in E_\infty(\alpha)$ such that $J(u) < +\infty$.

Lemma 1.28 *If $u \in E_\infty(\alpha)$ and $J(u) < +\infty$, then there exists $q \in K_\Phi$ such that fixing any bounded interval $I \subset \mathbb{R}$, we have*

$$\|u(\cdot, y) - q\|_{L^\Phi(I)} \to 0 \quad as \quad y \to +\infty.$$

Proof. Since $J(u) < +\infty$ and $F_y(u(\cdot, y)) - c_y \geq 0$ for almost every $y > 0$, we obtain that

$$\liminf_{y \to +\infty} (F_y(u(\cdot, y)) - c_y) = 0.$$

Thus, there exists an increasing sequence $y_n \to +\infty$ such that $F_{y_n}(u(\cdot, y_n)) - c_{y_n} \to 0$. By Lemma 1.23, there exists $q \in K_\Phi$ verifying, along a subsequence,

$$\|u(\cdot, y_n) - q\|_{W^{1,\Phi}((-y_n, y_n))} \to 0 \text{ as } n \to +\infty.$$

Fixing any bounded interval $I \subset \mathbb{R}$ it follows that there is $n_0 \in \mathbb{N}$ satisfying $I \subset (-y_n, y_n)$ for all $n \geq n_0$, and so

$$\|u(\cdot, y_n) - q\|_{L^\Phi(I)} \to 0.$$

Possibly considering the function $-u$, it is not restrictive to assume that

$$\|u(\cdot, y_n) - q^+\|_{L^\Phi(I)} \to 0.$$

To finish the proof, we must show that

$$\|u(\cdot, y) - q^+\|_{L^\Phi(I)} \to 0 \text{ as } y \to +\infty.$$

Indeed, arguing by contradiction, assume that there exists $r > 0$ such that

$$\limsup_{y \to +\infty} \|u(\cdot, y) - q^+\|_{W^{1,\Phi}(I)} > 2r.$$

Using the Lemma 1.26, there exists a sequence of intervals (σ_n, τ_n), with

$$0 < \sigma_n < \tau_n < \sigma_{n+1} < \tau_{n+1}$$

and $\sigma_n \to +\infty$ as $n \to +\infty$ such that

(i) $r \leq \|u(\cdot, y) - q^+\|_{L^\Phi(I)} \leq 2r$ for all $y \in [\sigma_n, \tau_n]$,

(ii) $\|u(\cdot, \sigma_n) - q^+\|_{L^\Phi(I)} = r$,

(iii) $\|u(\cdot, \tau_n) - q^+\|_{L^\Phi(I)} = 2r$.

Due to triangular inequality,

$$\|u(\cdot, \tau_n) - u(\cdot, \sigma_n)\|_{L^\Phi(I)} \geq r, \quad \forall n \in \mathbb{N}. \tag{1.60}$$

Now, we note that there exists $\epsilon > 0$ such that

$$\int_I \Phi\left(|u(\cdot, \tau_n) - u(\cdot, \sigma_n)|\right) dx \geq \epsilon, \quad \forall n \in \mathbb{N}. \tag{1.61}$$

In fact, we proceed by contradiction and suppose that there is a sequence $(i_n) \subset \mathbb{N}$ satisfying

$$\int_I \Phi\left(|u(\cdot,\tau_{i_n}) - u(\cdot,\sigma_{i_n})|\right) dx \to 0 \quad \text{as} \quad n \to +\infty.$$

As $\Phi \in \Delta_2$, the above limit implies that

$$\|u(\cdot,\tau_{i_n}) - u(\cdot,\sigma_{i_n})\|_{L^\Phi(I)} \to 0,$$

which contradicts (1.60). On the other hand, note that we can consider r small such that

$$r < \frac{\|q^+\|_{L^\Phi(I)}}{2},$$

and hence,

$$\|u(\cdot,y) - q^-\|_{L^\Phi(I)} \geq \|q^+ - q^-\|_{L^\Phi(I)} - \|u(\cdot,y) - q^+\|_{L^\Phi(I)} \geq 2\|q^+\|_{L^\Phi(I)} - 2r \geq \|q^+\|_{L^\Phi(I)},$$

that is,

$$\|u(\cdot,y) - q^-\|_{L^\Phi(I)} \geq \|q^+\|_{L^\Phi(I)} \ \text{ for each } \ y \in (\sigma_n,\tau_n).$$

In short,

$$\|u(\cdot,y) - q\|_{L^\Phi(I)} \geq r \ \text{ for all } \ q \in K_\Phi = \{q^+,q^-\} \ \text{ and } \ y \in (\sigma_n,\tau_n).$$

Consequently, by (1.58), there are $\nu_r > 0$ and $n_0 \in \mathbb{N}$ such that

$$F_L(u(\cdot,y)) - c_L > \nu_r \text{ for any } y \in (\sigma_n,\tau_n) \text{ and } n \geq n_0.$$

Invoking Lemma 1.27, there exists $\mu_r > 0$ satisfying

$$J_{(\sigma_n,\tau_n)}(u) \geq \mu_r h\left(\int_I \Phi\left(|u(\cdot,\tau_n) - u(\cdot,\sigma_n)|\right) dx\right) \quad \forall n \geq n_0$$

that combined with (1.61) provides

$$J(u) \geq \sum_{n\geq n_0} J_{(\sigma_n,\tau_n)}(u) \geq \mu_r \sum_{n \geq n_0} h(\epsilon),$$

which is absurd, because $J(u) < +\infty$. ■

Lemma 1.29 *There exists $\overline{u} \in E_\infty(\alpha)$ such that $J(\overline{u}) = d_\infty$ and*

$$\overline{u}(\cdot,y) \to q^+ \ \textit{in} \ \ L^\Phi_{loc}(\mathbb{R}) \ \ \textit{as} \ \ y \to +\infty.$$

Proof. Let $(u_n) \subset E_\infty(\alpha)$ be a minimizing sequence for J. It is not difficult to see that fixing any $r > 0$ and

$$\Gamma_r = \Gamma \cap \left\{(x,y) \in \mathbb{R}^2 \,:\, y < r\right\},$$

then (u_n) is a bounded sequence on $W^{1,\Phi}(\Gamma_r)$. Indeed, since $\|u_n\|_{L^\infty(\Gamma)} \leq \alpha$ for all $n \in \mathbb{N}$, we have

$$\int_{\Gamma_r} \Phi(|u_n|)dxdy \leq \Phi(\alpha)|\Gamma_r| < +\infty \quad \forall\, n \in \mathbb{N}.$$

Moreover, there is $M > 0$ such that $J(u_n) \leq M$ for all $n \in \mathbb{N}$. Thereby, for each $r > 0$ fixed,

$$\int_{\Gamma_r} \Phi(|\nabla u_n|)dxdy \leq J(u_n) + \int_0^r c_y dy \leq M + rc_\Phi < +\infty \quad \forall n \in \mathbb{N},$$

and our claim follows, because $\Phi \in \Delta_2$. Furthermore, by a classical diagonal argument, there exists $\overline{u} \in W^{1,\Phi}_{\rm loc}(\Gamma)$ and a subsequence of (u_n), still denoted (u_n), such that

$$u_n \rightharpoonup \overline{u} \text{ in } W^{1,\Phi}_{\rm loc}(\Gamma) \quad \text{and} \quad u_n(x,y) \to \overline{u}(x,y) \text{ a.e. in } \Gamma.$$

By pointwise convergence, one has $\overline{u}(x,y) = -\overline{u}(-x,y)$ for almost every $(x,y) \in \Gamma$ and $0 \leq \overline{u}(x,y) \leq \alpha$ for almost every $x \geq 0$, and hence, $\overline{u} \in E_\infty$. Furthermore, $J(\overline{u}) = d_\infty$, via Lemma 1.25. Finally, from Lemma 1.28 it is possible to conclude that $\overline{u}(\cdot,y) \to q^+$ in $L^\Phi_{\rm loc}(\mathbb{R})$ as $y \to +\infty$, and the lemma follows. ■

Setting

$$K_\infty(\alpha) = \{w \in E_\infty(\alpha) : J(w) = d_\infty\},$$

we have by the previous lemma that $K_\infty(\alpha) \neq \emptyset$. Repeating the arguments used in the proof of Theorem 1.2, it is possible to prove the following result.

Lemma 1.30 *If $\overline{u} \in K_\infty(\alpha)$, then for any $\psi \in W^{1,\Phi}_{loc}(\mathbb{R}^2)$ with ψ compact support in $\mathbb{R}^2$ we have*

$$\iint_\Gamma \left(\phi(|\nabla \overline{u}|)\nabla \overline{u} \nabla \psi + V'(\overline{u})\psi\right) dydx = 0.$$

As a consequence of Lemma 1.30, $\overline{u} \in K_\infty(\alpha)$ is weak solution of

$$-\Delta_\Phi w + V'(w) = 0 \quad \text{in} \quad \Gamma.$$

Elliptic regularity theory implies that $\overline{u}$ belongs to $C^{1,\beta}_{\rm loc}(\Gamma)$ for some $\beta \in (0,1)$. We can now proceed analogously to the proof of Lemma 1.6 to obtain that

$$0 < \overline{u}(x,y) < \alpha \text{ for all } (x,y) \in \Gamma.$$

1.2.2 Existence of saddle-type solutions

This section is devoted to the proof of the following theorem.

Theorem 1.4 *Assume (ϕ_1)-(ϕ_4) and (V_1)-(V_6). For each $j \geq 2$ there exists $v_j \in C^{1,\gamma}_{loc}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$ such that v_j is a weak solution of (1.1) satisfying*

(a) $0 < \tilde{v}_j(\rho,\theta) < \alpha$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *and* $\rho > 0$,

(b) $\tilde{v}_j(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_j(\rho, \frac{\pi}{2} - \theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$,

(c) $\tilde{v}_j(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_j(\rho,\theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$,

(d) $\tilde{v}_j(\rho,\theta) \to \alpha$ *as* $\rho \to +\infty$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$,

where $\tilde{v}_j(\rho,\theta) = v_j(\rho\cos(\theta), \rho\sin(\theta))$.

Proof. The existence of v_j will be done via a recursive reflection of the function $\overline{u} : \Gamma \to \mathbb{R}$ given by Lemma 1.29. First of all, setting $\theta_j = \frac{\pi}{j}$, with $j \geq 2$, we consider the rotation matrix

$$T_j = \begin{pmatrix} \cos(\theta_j) & \sin(\theta_j) \\ -\sin(\theta_j) & \cos(\theta_j) \end{pmatrix}.$$

Thus, putting $\Gamma^0 = \Gamma$, we designate $\Gamma^i = T_j^i(\Gamma)$ for $i = 0,1,2,...,2j-1$, i.e., Γ^i is the $i\theta_j$-rotated de Γ. Consequently,

$$\mathbb{R}^2 = \bigcup_{i=0}^{2j-1} \Gamma^i, \quad T_j^{-i}(\Gamma^i) = \Gamma, \quad \text{and} \quad int(\Gamma^i) \cap int(\Gamma^j) = \emptyset \quad \text{for} \quad i \neq j.$$

Now, we define the function $v_j : \mathbb{R}^2 \to \mathbb{R}$ by

$$v_j(x,y) = (-1)^i \overline{u}\left(T_j^{-i}(x,y)\right) \quad \forall (x,y) \in \Gamma^i.$$

Note that $v_j|_{\Gamma^i}$ is the reflection of $v_j|_{\Gamma^{i-1}}$ with respect to the axis separating Γ^{i-1} from Γ^i, for any $i = 1,2,3,...,2j-1$. From the properties of the reflection operator, $v_j \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}^2)$. Moreover, we note that if $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ has compact support in $\mathbb{R}^2$, then $\psi \circ T_j^i \in W^{1,\Phi}(\mathbb{R}^2)$ also has compact support in $\mathbb{R}^2$, because T_j^i is a linear operator. Consequently, by Lemma 1.30,

$$\int_{\Gamma^i} (\phi(|\nabla v_j|)\nabla v_j \nabla\psi + V'(v_j)\psi)dydx$$
$$= (-1)^i \int_{\Gamma} (\phi(|\nabla \overline{u}|)\nabla\overline{u}\nabla(\psi \circ T_j^i) + V'(\overline{u})(\psi \circ T_j^i))dydx = 0.$$

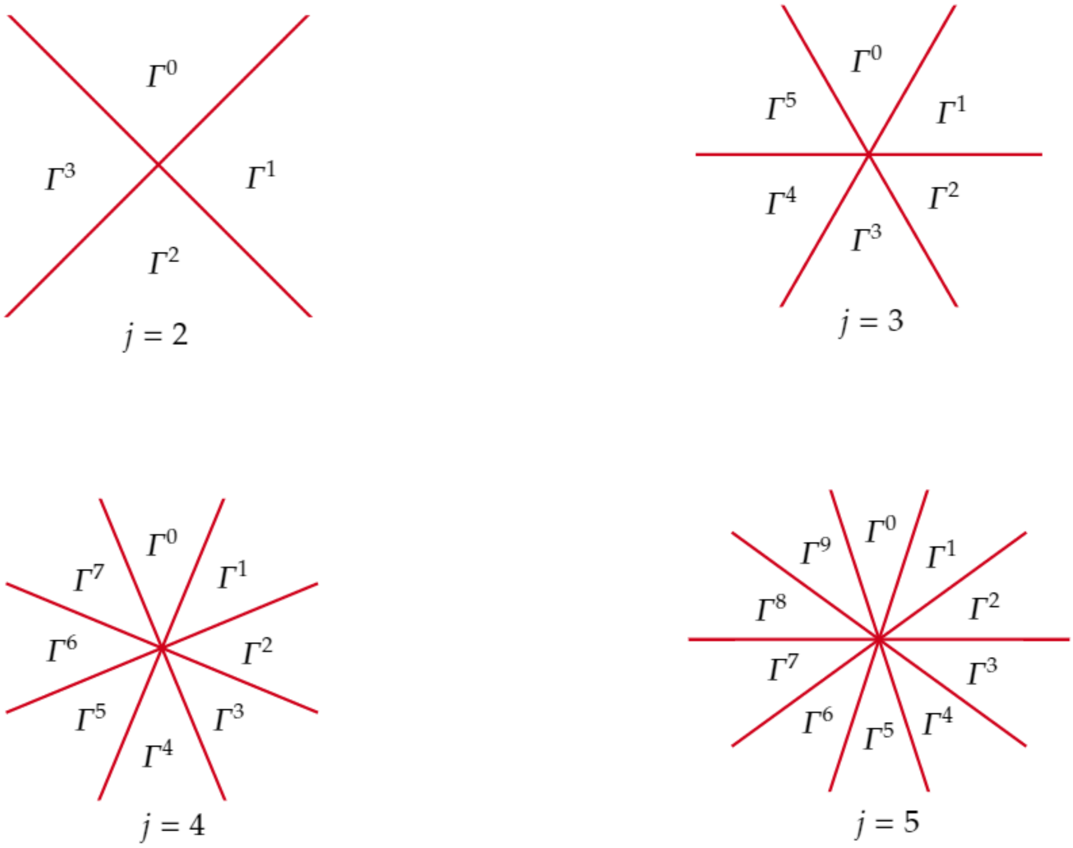

Figure 1.2: Geometric illustration of sets Γ^i for $j = 2, 3, 4, 5$.

Thus, for any $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$,

$$\int_{\mathbb{R}^2} (\phi(|\nabla v_j|)\nabla v_j \nabla\psi + V'(v_j)\psi) dydx$$
$$= \sum_{i=0}^{2j-1} \int_{\Gamma^i} (\phi(|\nabla v_j|)\nabla v_j \nabla\psi + V'(v_j)\psi) dydx = 0.$$

Hence, v_j is a weak solution of equation (1.1) and by regularity arguments $v \in C^{1,\gamma}_{\rm loc}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$. Moreover, setting

$$\tilde{v}_j(\rho,\theta) = v_j(\rho\cos(\theta), \rho\sin(\theta)),$$

since $\overline{u}(x,y) = -\overline{u}(-x,y)$ for all $(x,y) \in \Gamma$, a direct computation shows that v_j checks the conditions (a)-(c) of Theorem 1.4. To complete the proof, we are going to prove that v_j satisfies item (d). Assume by contradiction that there are $\theta_0 \in [\frac{\pi}{2} - \frac{\pi}{2\theta}, \frac{\pi}{2})$, $\eta_0 > 0$ and a sequence $(\rho_n) \subset (0, +\infty)$ with $\rho_n \to +\infty$ such that

$$|\alpha - \tilde{v}_j(\rho_n, \theta_0)| \geq 2\eta_0 \quad \forall n \in \mathbb{N}.$$

Setting $(x_n, y_n) = (\rho_n \cos(\theta_0), \rho_n \sin(\theta_0))$ we have that $(x_n, y_n) \in \Gamma$, and therefore $\tilde{v}_j(\rho_n, \theta_0) = \overline{u}(x_n, y_n)$. Furthermore,

$$\alpha - \overline{u}(x_n, y_n) \geq 2\eta_0 \quad \forall n \in \mathbb{N}. \tag{1.62}$$

Now, fixing $\eta > 0$, by Mean Value Theorem, there exists $(x_0, y_0) \in [(x_n, y_n), (x, y)]$ such that

$$|v_j(x,y) - v_j(x_n, y_n)| \leq |\nabla v_j(x_0, y_0)|\eta, \quad (x,y) \in B_\eta(x_n, y_n). \tag{1.63}$$

On the other hand, as $\|v_j\|_{L^\infty(\mathbb{R}^2)} \leq \alpha$ we have that there is $\Lambda_1 > 0$ such that $|V'(v_j(x,y))| \leq \Lambda_1$ for all $(x,y) \in \mathbb{R}^2$. Fixed $\tilde{\eta} > 0$, by [67, Theorem 1.7], there exists $C > 0$, that only depends on $\tilde{\eta}$, such that

$$\|v_j\|_{C^1(B_\eta(x_n, y_n))} \leq C, \quad \forall n \in \mathbb{N} \ \text{ and } \ \forall \eta \in (0, \tilde{\eta}).$$

Thus, taking η such that $C\eta < \eta_0$, (1.63) leads to

$$|v_j(x_n, y_n) - v_j(x,y)| \leq C\eta < \eta_0, \quad \forall (x,y) \in B_\eta(x_n, y_n).$$

Moreover, we note that there exists $n_0 \in \mathbb{N}$ such that $B_\eta(x_n, y_n) \subset \Gamma$ for any $n \geq n_0$. Then,

$$|\overline{u}(x_n, y_n) - \overline{u}(x,y)| < \eta_0, \quad \forall (x,y) \in B_\eta(x_n, y_n) \ \text{ and } \ \forall n \geq n_0. \tag{1.64}$$

Consequently, from (1.62) and (1.64),

$$|\alpha - \overline{u}(x,y)| \geq \eta_0, \quad \forall (x,y) \in B_\eta(x_n, y_n) \ \text{ and } \ \forall n \geq n_0. \tag{1.65}$$

Claim 1.1 *There exists $\eta_1 > 0$ such that*

$$\|\overline{u}(\cdot, y) - q^+\|_{W^{1,\Phi}(I_y)} \geq \eta_1 \ \textit{ for all } \ y \in \bigcup_{n \geq n_0} \left(y_n - \frac{\eta}{2}, y_n + \frac{\eta}{2}\right).$$

In fact, we proceed by contradiction and suppose that for some subsequence there is a $z_n \in (y_n - \frac{\eta}{2}, y_n + \frac{\eta}{2})$ satisfying $\|\overline{u}(\cdot, z_n) - q^+\|_{W^{1,\Phi}(I_{y_n})} \to 0$. Thanks to Lemma A.5-$(c)$,

$$\|\overline{u}(\cdot, z_n) - q^+\|_{L^\infty(I_{y_n})} \to 0.$$

Consequently, there exists $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and

$$|\overline{u}(x, z_n) - q^+(x)| < \frac{\eta_0}{4} \quad \forall x \in I_{y_n} \quad \text{and} \quad \forall n \geq n_1.$$

In particular, for $(x, z_n) \in B_\eta(x_n, y_n)$ we have that $x \to +\infty$ because $x_n \to +\infty$, and hence since $q^+(x) \to \alpha$ as $x \to +\infty$ we derive that

$$|\overline{u}(x, z_n) - \alpha| \leq |\overline{u}(x, z_n) - q^+(x)| + |q^+(x) - \alpha| < \frac{\eta_0}{2}$$

for $(x, z_n) \in B_\eta(x_n, y_n)$ with $n \geq n_1$ sufficiently large, which contradicts (1.65).

Claim 1.2 *There are $\eta_2 > 0$ and $\overline{n}_0 \geq n_0$ such that*

$$\|\overline{u}(\cdot, y) - q^-\|_{W^{1,\Phi}(I_y)} \geq \eta_2 \quad \textit{for all} \quad y \in \bigcup_{n \geq \overline{n}_0} \left(y_n - \frac{\eta}{2}, y_n + \frac{\eta}{2}\right).$$

Indeed, by Lemma 1.29, it follows that fixing any bounded interval $I \subset \mathbb{R}$ we have $\overline{u}(\cdot, y) \to q^+$ in $L^\Phi(I)$ as $y \to +\infty$. Then, taking y sufficiently large such $I \subset I_y$ and $\|\overline{u}(\cdot, y) - q^+\|_{L^\Phi(I)} < \|q^+\|_{L^\Phi(I)}$ we conclude that

$$\|\overline{u}(\cdot, y) - q^-\|_{W^{1,\Phi}(I_y)} \geq \|\overline{u}(\cdot, y) - q^-\|_{L^\Phi(I)} \geq \|q^+ - q^-\|_{L^\Phi(I)} - \|\overline{u}(\cdot, y) - q^+\|_{L^\Phi(I)} \geq \|q^+\|_{L^\Phi(I)},$$

which is enough to prove the claim.

Finally, gathering Claims 1.1 and 1.2 with (1.58), there is $\mu > 0$ such that

$$F_y(\overline{u}(\cdot, y)) - c_y > \mu \quad \text{for all} \quad y \in \bigcup_{n \geq \overline{n}_0} \left(y_n - \frac{\eta}{2}, y_n + \frac{\eta}{2}\right).$$

By Lemma 1.27,

$$J(\overline{u}) \geq \sum_{n \geq \overline{n}_0} J_{(y_n - \frac{\eta}{2}, y_n + \frac{\eta}{2})}(\overline{u}) \geq \sum_{n \geq \overline{n}_0} \mu\eta,$$

which gives rise to the contradiction $J(\overline{u}) = +\infty$, and the proof is complete. ■

Note that the characterization of the asymptotic behavior of the saddle-type solution $v_j(x, y)$ between two contiguous nodal lines given in item (d) of Theorem 1.4 is of the exponential type, that is, the function $\alpha \pm v_j$ has an exponential decay at infinity. Furthermore, this asymptotic behavior represents a multiple transition between the pure phases $+\alpha$ and $-\alpha$ with cross interface.

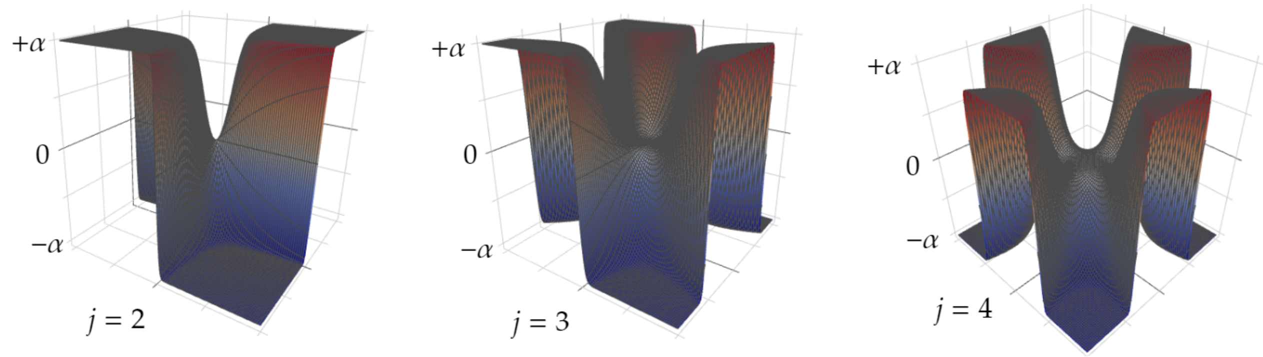

Figure 1.3: Geometric illustration of saddle type solutions v_j for $j = 2, 3, 4$.

1.3 Final remarks

In this last section we will make some additional comments in order to complement the study carried out in the previous sections. First, introducing the inhomogeneous factor $a(t)$ bounded on $\mathbb{R}$ into equation (1.2) we can obtain a result similar to Theorem 1.1 and in this case we write the following

Theorem 1.5 *Assume $a \in L^\infty(\mathbb{R})$ and that there is a solution $q \in C^{1,\gamma}_{loc}(\mathbb{R})$, for some $\gamma \in (0,1)$, for the following quasilinear Cauchy problem*

$$\begin{cases} (\phi(|q'(t)|)q'(t))' = a(t)V'(q(t)) \quad t \in \mathbb{R}, \\ q(0) = q_0, \\ q'(0) = q_0', \end{cases}$$

such that there are $r, \rho > 0$ satisfying

(a) $q'(t) \geq \rho$ for any $t \in (-r, r)$.

(b) $q \in W^{1,\infty}(\mathbb{R})$.

Then, q is unique in $(-r, r)$.

The above theorem can be refined when the graph of the derivative of $\phi(t)t$, $t > 0$, is above the line $y = c$ for some $c > 0$. For more details on this fact, see Section 3.4.

We saw throughout the chapter that the study of the uniqueness of the minimal heteroclinic solution for the problem

$$-(\phi(|q'|)q') + V'(q) = 0 \ \text{ in } \ \mathbb{R}, \quad q(0) = 0, \quad \lim_{t \to \pm\infty} q(t) = \pm\alpha \tag{1.66}$$

was crucial for the construction of a saddle-type solution in $\mathbb{R}^2$. An interesting question is whether the uniqueness of (1.66) persists over the set of arbitrary heteroclinic solutions. The answer to this question is positive and was answered in the recent work [18] by Alves, Isneri and Montecchiari. In that paper, the authors studied the existence, uniqueness and various qualitative properties for (1.66) and showed that Theorem 1.5 applies an important rule in the arguments. Moreover, when V is the Φ-double well potential

$$V(t) = \Phi(|t^2 - \alpha^2|)$$

they provide a comparison, related to the parameters l and m given in (ϕ_2), between the unique solution q of (1.66) and dilations of the unique solution $q_+(t) = \alpha \tanh(\alpha t)$ of classical logistic problem

$$-q'' + 2q^3 - 2\alpha^2 q = 0 \ \text{ in } \ \mathbb{R}, \ \ q(0) = 0, \ \ \lim_{t\to\pm\infty} q(t) = \pm\alpha.$$

This result is refined when

$$\phi(t) = t^{p-2} \ \text{ and } \ V(t) = \frac{|t^2 - \alpha^2|^p}{p}$$

providing an explicit solution depending on the hyperbolic tangent, that is,

$$q(t) = \alpha \tanh\left(\frac{\alpha t}{\sqrt[p]{p-1}}\right) \ \text{ whenever } \ p \in (1, +\infty).$$

For further details on the analytical properties of problem (1.66) we refer the reader to [18].

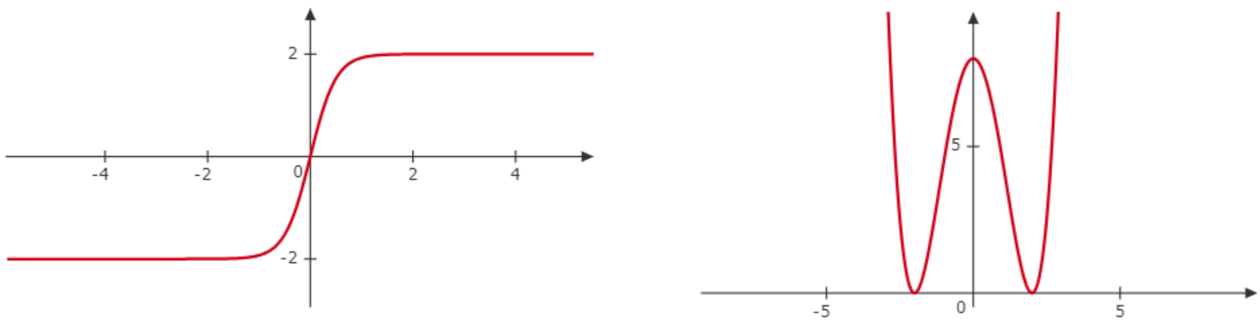

Figure 1.4: The graph of $q_+(t) = \alpha \tanh(\alpha t)$ and $V(t) = \dfrac{(t^2 - \alpha^2)^2}{2}$ with $\alpha = 2$.

CHAPTER 2

SADDLE SOLUTIONS FOR NON-AUTONOMOUS QUASILINEAR EQUATIONS IN $\mathbb{R}^2$

In this chapter, we use variational methods to show the existence of a solution to the non-autonomous quasilinear elliptic equation of the form

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{2.1}$$

where $A \in C(\mathbb{R}^2, \mathbb{R})$ is positive, even, periodic and symmetric with respect to the plane diagonal $x_2 = x_1$. Throughout the chapter, we will assume conditions (ϕ_1)-(ϕ_4) on ϕ and (V_1)-(V_4) on V. The class of equations listed above includes the case where $A = 1$ on $\mathbb{R}^2$, which was covered in Chapter 1. However, the method presented here differs from the one treated there. Our main result includes the case of the potential

$$V(t) = \Phi(|t^2 - \alpha|), \quad t \in \mathbb{R},$$

which was modeled on the classical double well Ginzburg-Landau potential.

2.1 Heteroclinic Solutions on $\mathbb{R}^2$

We start this section by studying the existence of heteroclinic solutions for equation (2.1), then properties of the type of compactness and exponential estimates of these solutions.

2.1.1 Existence of minimal solution on the strip $\mathbb{R} \times (0,1)$

In this section, we will establish the existence of (minimal) heteroclinic type solutions from $-\alpha$ to α for (2.1). To begin with, for

$$\Omega_0 = \mathbb{R} \times [0,1]$$

let us consider the class

$$E_\Phi(\alpha) = \Big\{ w \in W^{1,\Phi}_{\mathrm{loc}}(\Omega_0) : \ 0 \le w(x,y) \le \alpha \text{ for } x > 0 \text{ and } w \text{ is odd in } x \Big\}.$$

In the sequel, $I : W^{1,\Phi}_{\mathrm{loc}}(\Omega_0) \to \mathbb{R} \cup \{+\infty\}$ designates the functional given by

$$I(w) = \int_{\Omega_0} \left(\Phi(|\nabla w|) + A(x,y)V(w)\right) dydx.$$

A direct computation shows that

$$u_n \rightharpoonup u \ \text{ in } \ W^{1,\Phi}_{\mathrm{loc}}(\Omega_0) \Rightarrow I(u) \le \liminf_{n \to +\infty} I(u_n). \tag{2.2}$$

Hereinafter, the expression $u_n \rightharpoonup u$ in $W^{1,\Phi}_{\mathrm{loc}}(\Omega_0)$ means that

$$u_n \rightharpoonup u \ \text{ in } \ W^{1,\Phi}([L,R] \times [0,1]) \ \text{ for every } \ R, L \in \mathbb{R} \ \text{ with } \ L < R.$$

Setting

$$\mathcal{L}(w) = \Phi\left(|\nabla w|\right) + A(x,y)V(w) \ \text{ for } \ w \in W^{1,\Phi}_{\mathrm{loc}}(\Omega_0),$$

it follows from the definitions of Φ, V and A that

$$\mathcal{L}(w) \ge 0, \quad \forall w \in W^{1,\Phi}_{\mathrm{loc}}(\Omega_0),$$

and so, the functional I is bounded from below. Moreover, it is easy to check that the function $\varphi_\alpha : \Omega_0 \to \mathbb{R}$ defined by

$$\varphi_\alpha(x,y) = \begin{cases} \alpha, & \text{if } \ x > \alpha & \text{and } \ y \in [0,1], \\ x, & \text{if } \ -\alpha \le x \le \alpha & \text{and } \ y \in [0,1], \\ -\alpha, & \text{if } \ x < -\alpha & \text{and } \ y \in [0,1] \end{cases} \tag{2.3}$$

belongs to $E_\Phi(\alpha)$ with $I(\varphi_\alpha) < +\infty$. Therefore,

$$c_\Phi(\alpha) := \inf_{w \in E_\Phi(\alpha)} I(w)$$

is well defined.

To end this introductory part, from now on, for each $x \in \mathbb{R}$ fixed and $u \in E_\Phi(\alpha)$, we will identify $u(x,\cdot)$ as being a real function in $y \in [0,1]$. For each $y \in [0,1]$ fixed, we will also identify $u(\cdot,y)$ as being a real function in $x \in \mathbb{R}$. Employing Fubini's Theorem,

$$u(x,\cdot) \in W^{1,\Phi}(0,1) \ \text{ a.e. in } \ x \in \mathbb{R} \ \text{ and } \ u(\cdot,y) \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}) \ \text{ a.e. in } \ y \in [0,1].$$

Moreover, since the functions in $E_\Phi(\alpha)$ have L^∞-norm less than or equal to α, without loss of generality, we can make a modification on function V, by assuming that it satisfies the following:

$$V(t) = V(2\alpha) \quad \text{for} \quad |t| \geq 2\alpha. \tag{2.4}$$

Finally, hereafter, we will denote this new modification of V by itself. Moreover, according to (A_1)-(A_4),

$$0 < \min_{\mathbb{R}^2} A(x,y) \leq A(x,y) \leq \max_{\mathbb{R}^2} A(x,y) < +\infty.$$

So, in what follows,

$$\underline{A} = \min_{\mathbb{R}^2} A(x,y) \ \text{ and } \ \overline{A} = \max_{\mathbb{R}^2} A(x,y).$$

Now, we prove an important estimate that will be used often in this chapter.

Lemma 2.1 *Let $w \in E_\Phi(\alpha)$. If $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, then*

$$\int_0^1 \Phi\left(|w(x_2,y) - w(x_1,y)|\right) dy \leq \frac{\xi_1\left(|x_1 - x_2|\right)}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi(|w_x|) dx dy,$$

where ξ_1 was given in Lemma A.2.

Proof. First of all note that from Lemma A.5, $w \in W^{1,l}_{\text{loc}}(\Omega_0)$, and hence, by [26, Theorem 8.2],

$$|w(x_2,y) - w(x_1,y)| = \left| \int_{x_1}^{x_2} w_x(x,y) dx \right|.$$

As Φ is even,

$$\Phi\left(|w(x_2,y) - w(x_1,y)|\right) = \Phi\left(\int_{x_1}^{x_2} w_x(x,y) dx \right). \tag{2.5}$$

Invoking Jensen's Inequality given in [87, Theorem 3.3],

$$\Phi\left(\int_{x_1}^{x_2} w_x(x,y) dx \right) \leq \frac{1}{|x_1 - x_2|} \int_{x_1}^{x_2} \Phi((x_2 - x_1) w_x(x,y)) dx, \tag{2.6}$$

and hence, by (2.5) and (2.6),

$$\int_0^1 \Phi\left(|w(x_2,y) - w(x_1,y)|\right) dy \leq \frac{1}{|x_1 - x_2|} \int_0^1 \int_{x_1}^{x_2} \Phi\left((x_2 - x_1) w_x(x,y)\right) dx dy.$$

According to Lemma A.2,

$$\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right) dy \leq \frac{\xi_1(|x_1-x_2|)}{|x_1-x_2|}\int_0^1\int_{x_1}^{x_2}\Phi\left(w_x(x,y)\right) dxdy,$$

and the lemma follows. ■

As a consequence of the last lemma, we obtain the following result.

Corollary 2.1 *If $w \in E_\Phi(\alpha)$ and $I(w) < +\infty$, then*

(a) The function $x \in \mathbb{R} \mapsto w(x,\cdot) \in L^\Phi(0,1)$ is uniformly continuous a.e..

(b) The function $x \in \mathbb{R} \mapsto \|w(x,\cdot)-\alpha\|_{L^\Phi(0,1)}$ is continuous a.e..

Proof. Let be $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$. Since Φ is an increasing function on $(0,+\infty)$ and $|\partial_x w| \leq |\nabla w|$, then the Lemma 2.1 ensures that

$$\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right) dy \leq \frac{\xi_1(|x_1-x_2|)}{|x_1-x_2|}\int_0^1\int_{x_1}^{x_2}\Phi(|\nabla w|) dxdy,$$

and so,

$$\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right) dy \leq I(w)\max\left\{|x_1-x_2|^{l-1}, |x_1-x_2|^{m-1}\right\}.$$

From this, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right) dy < \epsilon \ \text{ for } \ |x_1-x_2| < \delta.$$

The last inequality combined with Lemma A.2 gives

$$\xi_0\left(\|w(x_2,\cdot)-w(x_1,\cdot)\|_{L^\Phi(0,1)}\right) < \epsilon \ \text{ for } \ |x_1-x_2| < \delta.$$

Therefore,

$$|x_1-x_2| < \delta \Rightarrow \|w(x_2,\cdot)-w(x_1,\cdot)\|_{L^\Phi(0,1)} < \xi_0^{-1}(\epsilon),$$

finishing the proof of (a). The item (b) follows from (a), because we have the inequality below

$$\left|\|w(x_2,\cdot)-\alpha\|_{L^\Phi(0,1)} - \|w(x_1,\cdot)-\alpha\|_{L^\Phi(0,1)}\right| \leq \|w(x_2,\cdot)-w(x_1,\cdot)\|_{L^\Phi(0,1)}.$$

This completes the proof. ■

Another important consequence of Lemma 2.1 is the following result.

Lemma 2.2 *If $w \in E_\Phi(\alpha)$ satisfies*

$$\|w(x,\cdot) - \alpha\|_{W^{1,\Phi}(0,1)} \geq r \text{ a.e. in } x \in (x_1, x_2) \subset [0, +\infty),$$

for some $r > 0$, then there exists $\mu_r > 0$ independent of x_1 and x_2 satisfying

$$\begin{aligned}\int_{x_1}^{x_2} \int_0^1 \mathcal{L}(w) dy dx &\geq \frac{|x_2 - x_1|}{2\xi_1(|x_2 - x_1|)} \int_0^1 \Phi(|w(x_2, y) - w(x_1, y)|) dy + (2\mu_r)^{\frac{m}{m-1}} |x_2 - x_1| \\ &\geq \mu_r h\left(\int_0^1 \Phi(|w(x_2, y) - w(x_1, y)|) dy\right),\end{aligned}$$

where $h(t) = \min\left\{t^{\frac{1}{l}}, t^{\frac{1}{m}}\right\}$.

Proof. In what follows, we are going to work with the functional $F : W^{1,\Phi}(0,1) \to \mathbb{R}$ defined by

$$F(v) = \int_0^1 \left(\frac{1}{2}\Phi(|v'|) + \underline{A}V(v)\right) dy.$$

We claim that for any sequence $(v_n) \subset W^{1,\Phi}(0,1)$ with $0 \leq v_n(y) \leq \alpha$ for all $y \in (0,1)$ and $F(v_n) \to 0$ as $n \to +\infty$, we must have

$$\|v_n - \alpha\|_{W^{1,\Phi}(0,1)} \to 0 \quad as \quad n \to +\infty.$$

Indeed, first we note that the limit $F(v_n) \to 0$ gives

$$\int_0^1 \Phi(|v_n'|) dy \to 0 \quad \text{and} \quad \int_0^1 V(v_n) dy \to 0, \quad as \quad n \to +\infty. \tag{2.7}$$

Thus, since $0 \leq v_n(y) \leq \alpha$ for every $y \in (0,1)$, (1.11) ensures that

$$\int_0^1 \Phi(|v_n - \alpha|) dy \leq \frac{1}{\underline{w}} \int_0^1 V(v_n) dy \quad \forall n \in \mathbb{N}.$$

Consequently,

$$\int_0^1 \Phi(|v_n - \alpha|) dy \to 0 \quad as \quad n \to +\infty.$$

The limits above together with (2.7) and the fact that $\Phi \in \Delta_2$ yield

$$\|v_n - \alpha\|_{W^{1,\Phi}(0,1)} \to 0 \quad \text{as} \quad n \to +\infty,$$

which proves the claim. Thereby, if $v \in W^{1,\Phi}(0,1)$, $0 \leq v \leq \alpha$ in $(0,1)$ and $\|v - \alpha\|_{W^{1,\Phi}(0,1)} \geq r$, then there exists $\mu_r \in (0, 1/2)$ such that

$$F(v) \geq (2\mu_r)^{\frac{m}{m-1}}.$$

Now, if $w \in E_\Phi(\alpha)$, we know that $0 \le w(x,\cdot) \le \alpha$ on $(0,1)$ for almost every $x > 0$, and so, as $\|w(x,\cdot)-\alpha\|_{W^{1,\Phi}(0,1)} \ge r$ a.e. in (x_1,x_2), we must have

$$F\left(w(x,\cdot)\right) \ge (2\mu_r)^{\frac{m}{m-1}} \quad \text{a.e. in} \quad x \in (x_1,x_2),$$

which leads to

$$\begin{aligned}\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx &= \int_{x_1}^{x_2}\int_0^1 \left(\Phi\left(|\nabla w|\right) + A(x,y)V(w)\right)dydx \\ &\ge \frac{1}{2}\int_{x_1}^{x_2}\int_0^1 \Phi\left(|\partial_x w|\right)dydx + \int_{x_1}^{x_2}\int_0^1 \left(\frac{1}{2}\Phi\left(|\partial_y w|\right) + \underline{A}V(w)\right)dydx \\ &\ge \frac{1}{2}\int_{x_1}^{x_2}\int_0^1 \Phi\left(|\partial_x w|\right)dydx + \int_{x_1}^{x_2} F\left(w(x,\cdot)\right)dx \\ &\ge \frac{1}{2}\int_{x_1}^{x_2}\int_0^1 \Phi\left(|\partial_x w|\right)dydx + (2\mu_r)^{\frac{m}{m-1}}|x_2-x_1|.\end{aligned}$$

Thanks to Lemma 2.1,

$$\begin{aligned}\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx &\ge \frac{|x_1-x_2|}{2\xi_1(|x_1-x_2|)}\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy + (2\mu_r)^{\frac{m}{m-1}}|x_2-x_1| \\ &\ge \frac{|x_1-x_2|}{2\xi_1(|x_1-x_2|)}\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy + 2^{\frac{m}{m-1}-1}\mu_r^{\frac{m}{m-1}}|x_2-x_1|.\end{aligned}$$

Recalling that $\xi_1\left(|x_2-x_1|\right) = \max\left\{|x_2-x_1|^l, |x_2-x_1|^m\right\}$, we will consider the cases $\xi_1\left(|x_2-x_1|\right) = |x_2-x_1|^m$ and $\xi_1\left(|x_2-x_1|\right) = |x_2-x_1|^l$. If $\xi_1\left(|x_2-x_1|\right) = |x_2-x_1|^m$,

$$\begin{aligned}&\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx \ge \frac{1}{2|x_1-x_2|^{m-1}}\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy + 2^{\frac{m}{m-1}-1}\mu_r^{\frac{m}{m-1}}|x_2-x_1| \\ &\ge \frac{1}{2m}\left[\frac{1}{|x_1-x_2|^{\frac{m-1}{m}}}\left(\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy\right)^{\frac{1}{m}}\right]^m + \frac{m-1}{2m}\left(2\mu_r|x_2-x_1|^{\frac{m-1}{m}}\right)^{\frac{m}{m-1}}.\end{aligned}$$

Using Young's inequality for the conjugate exponents m and $\frac{m}{m-1}$, we find

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx \ge \frac{1}{2}\left[\frac{1}{|x_1-x_2|^{\frac{m-1}{m}}}\left(\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy\right)^{\frac{1}{m}} 2\mu_r|x_2-x_1|^{\frac{m-1}{m}}\right],$$

that is,

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx \ge \mu_r\left(\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy\right)^{\frac{1}{m}}. \tag{2.8}$$

If $\xi_1\left(|x_1-x_2|\right) = |x_1-x_2|^l$, a similar argument works to prove that

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx \ge \frac{1}{2l}\left[\frac{1}{|x_1-x_2|^{\frac{l-1}{l}}}\left(\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy\right)^{\frac{1}{l}}\right]^l + (2\mu_r)^{\frac{m}{m-1}}|x_2-x_1|.$$

Now, since $l \leq m$ and $0 < 2\mu_r < 1$, one has $(2\mu_r)^{\frac{l}{l-1}} \leq (2\mu_r)^{\frac{m}{m-1}}$. Therefore,

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx \geq \frac{1}{2l}\left[\frac{1}{|x_1-x_2|^{\frac{l-1}{l}}}\left(\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy\right)^{\frac{1}{l}}\right]^l + (2\mu_r)^{\frac{l}{l-1}}|x_2-x_1|.$$

Employing again Young's inequality, we derive

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx \geq \mu_r\left(\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy\right)^{\frac{1}{l}}. \tag{2.9}$$

From (2.8) and (2.9),

$$\int_{x_1}^{x_2}\int_0^1 \mathcal{L}(w)dydx \geq \mu_r h\left(\int_0^1 \Phi\left(|w(x_2,y)-w(x_1,y)|\right)dy\right),$$

where $h(t) = \min\left\{t^{\frac{1}{l}}, t^{\frac{1}{m}}\right\}$, which is precisely the assertion of the lemma. ∎

The next result characterizes the asymptotic behavior of functions $w \in E_\Phi(\alpha)$ with $I(w) < +\infty$.

Lemma 2.3 *If $w \in E_\Phi(\alpha)$ and $I(w) < +\infty$, then*

$$\|w(x,\cdot)-\alpha\|_{L^\Phi(0,1)} \to 0 \quad as \quad x \to +\infty \quad and \quad \|w(x,\cdot)+\alpha\|_{L^\Phi(0,1)} \to 0 \quad as \quad x \to -\infty.$$

Proof. To begin with, we claim that

$$\liminf_{x\to+\infty}\int_0^1 \Phi\left(|w(x,y)-\alpha|\right)dy = 0. \tag{2.10}$$

Indeed, if the limit does not hold, then there are $r > 0$ and $x_1 > 0$ satisfying

$$\int_0^1 \Phi\left(|w(x,y)-\alpha|\right)dy \geq r, \quad \forall x > x_1.$$

So, the properties of Φ together with Lemma A.2 guarantee that

$$\begin{aligned} r &\leq \xi_1\left(\|w(x,\cdot)-\alpha\|_{W^{1,\Phi}(0,1)}\right)\int_0^1 \Phi\left(\frac{|w(x,y)-\alpha|}{\|w(x,\cdot)-\alpha\|_{W^{1,\Phi}(0,1)}}\right)dy \\ &\leq \xi_1\left(\|w(x,\cdot)-\alpha\|_{W^{1,\Phi}(0,1)}\right)\int_0^1 \Phi\left(\frac{|w(x,y)-\alpha|}{\|w(x,\cdot)-\alpha\|_{L^\Phi(0,1)}}\right)dy \\ &\leq \xi_1\left(\|w(x,\cdot)-\alpha\|_{W^{1,\Phi}(0,1)}\right), \end{aligned}$$

that is,

$$\|w(x,\cdot)-\alpha\|_{W^{1,\Phi}(0,1)} \geq \xi_1^{-1}(r) := r_1 \text{ for all } x > x_1.$$

The last inequality permits to apply Lemma 2.2 to get $\mu_{r_1} > 0$ satisfying

$$I(w) \geq \int_{x_1}^{x}\int_0^1 \mathcal{L}(w)dydx \geq (2\mu_{r_1})^{\frac{m}{m-1}}(x-x_1).$$

Taking the limit of $x \to +\infty$ we infer that $I(w) = +\infty$, which is absurd, and (2.10) is proved. Now, as $\Phi \in \Delta_2$, the limit in (2.10) is equivalent to

$$\liminf_{x\to+\infty} \|w(x,\cdot) - \alpha\|_{L^\Phi(0,1)} = 0. \tag{2.11}$$

Next we are going to show that

$$\limsup_{x\to+\infty} \|w(x,\cdot) - \alpha\|_{L^\Phi(0,1)} = 0. \tag{2.12}$$

To see why, assume by contradiction that

$$\limsup_{x\to+\infty} \|w(x,\cdot) - \alpha\|_{L^\Phi(0,1)} > 0.$$

Then, there exists $r > 0$ such that

$$\limsup_{x\to+\infty} \|w(x,\cdot) - \alpha\|_{L^\Phi(0,1)} > 2r. \tag{2.13}$$

By Corollary 2.1, we can assume that the function $x \in \mathbb{R} \mapsto \|w(x,\cdot) - \alpha\|_{L^\Phi(0,1)}$ is continuous in $\mathbb{R}$. So, according to (2.11) and (2.13), there is a sequence of disjoint intervals (σ_i, τ_i) with $0 < \sigma_i < \tau_i < \sigma_{i+1} < \tau_{i+1}$, $i \in \mathbb{N}$, and $\sigma_i \to +\infty$ as $i \to +\infty$ such that for each i,

$$r \le \|w(x,\cdot) - \alpha\|_{L^\Phi(0,1)} \le 2r \quad \text{for} \quad x \in [\sigma_i, \tau_i]$$

and

$$\|w(\sigma_i,\cdot) - \alpha\|_{L^\Phi(0,1)} = r \quad \text{and} \quad \|w(\tau_i,\cdot) - \alpha\|_{L^\Phi(0,1)} = 2r.$$

Due to triangular inequality,

$$\|w(\tau_i,\cdot) - w(\sigma_i,\cdot)\|_{L^\Phi(0,1)} \ge r \quad \forall i \in \mathbb{N}, \tag{2.14}$$

from where it follows that there exists $\epsilon > 0$ such that

$$\int_0^1 \Phi\left(|w(\tau_i,\cdot) - w(\sigma_i,\cdot)|\right) dy \ge \epsilon, \quad \forall i \in \mathbb{N}. \tag{2.15}$$

In fact, arguing by contradiction, let us suppose that there is a sequence $(i_n) \subset \mathbb{N}$ satisfying

$$\int_0^1 \Phi\left(|w(\tau_{i_n},\cdot) - w(\sigma_{i_n},\cdot)|\right) dy \to 0 \quad \text{as} \quad n \to +\infty.$$

Since $\Phi \in \Delta_2$, the above limit implies that

$$\|w(\tau_{i_n},\cdot) - w(\sigma_{i_n},\cdot)\|_{L^\Phi(0,1)} \to 0 \quad \text{as} \quad n \to +\infty,$$

which contradicts (2.14). Consequently, by Lemma 2.2 there exists $\mu_r > 0$ such that

$$I(w) \geq \sum_{i=1}^{+\infty} \int_{\sigma_i}^{\tau_i} \int_0^1 \mathcal{L}(w) dy dx \geq \sum_{i=1}^{+\infty} \mu_r h \left(\int_0^1 \Phi\left(|w(\tau_i, \cdot) - w(\sigma_i, \cdot)| \right) dy \right)$$

that combined with (2.15) provides

$$I(w) \geq \mu_r \sum_{i=1}^{+\infty} h(\epsilon),$$

which is absurd, because $I(w) < +\infty$. Now, the lemma follows from (2.11) and (2.12). ■

Our next result is a key point in our approach, because it establishes the existence of heteroclinic solution for a class of problem defined on the strip $\Omega_0 = \mathbb{R} \times [0, 1]$, which will be used to prove in the next subsection the existence of a heteroclinic solution from $-\alpha$ to α for (2.1).

Theorem 2.1 *There exists $u \in E_\Phi(\alpha)$ such that $I(u) = c_\Phi(\alpha)$. Moreover, u is a weak solution to the quasilinear elliptic problem*

$$\begin{cases} -\Delta_\Phi u + A(x, y)V'(u) = 0 \quad in \quad \Omega_0 \\ \dfrac{\partial u}{\partial \eta}(x, y) = 0, \quad on \quad \partial\Omega_0. \end{cases} \tag{P}$$

Proof. Let $(u_n) \subset E_\Phi(\alpha)$ be a minimizing sequence for I. It is straightforward to check that (u_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$. Then, by a classical diagonal argument, there are a subsequence of (u_n), still denoted by (u_n), and $u \in W^{1,\Phi}_{\text{loc}}(\Omega_0)$ verifying

$$u_n \rightharpoonup u \;\text{ in }\; W^{1,\Phi}_{\text{loc}}(\Omega_0) \quad \text{and} \quad u_n(x, y) \to u(x, y) \;\text{ a.e. in }\; \Omega_0.$$

By the pointwise convergence, it is plain that

$$u(x, y) = -u(-x, y) \;\text{ a.e. in } \Omega_0 \;\text{ and }\; 0 \leq u(x, y) \leq \alpha \;\text{ for }\; x \geq 0,$$

from where it follows that $u \in E_\Phi(\alpha)$. Hence, from (2.2) we may conclude $I(u) = c_\Phi(\alpha)$. To complete the proof, it is sufficient to show that

$$\int_{\Omega_0} \left(\phi\left(|\nabla u| \right) \nabla u \nabla \psi + A(x, y)V'(u)\psi \right) dy dx \geq 0 \quad \forall \psi \in X^{1,\Phi}_0(\Omega_0),$$

where

$$X^{1,\Phi}_0(\Omega_0) = \left\{ w \in W^{1,\Phi}(\Omega_0) \text{ with } w(x, y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0 \right\}. \tag{2.16}$$

Given $\psi \in X_0^{1,\Phi}(\Omega_0)$, we can write

$$\psi(x,y) = \psi_o(x,y) + \psi_e(x,y),$$

where

$$\psi_e(x,y) = \frac{\psi(x,y)+\psi(-x,y)}{2} \quad \text{and} \quad \psi_o(x,y) = \frac{\psi(x,y)-\psi(-x,y)}{2}.$$

Note that ψ_o is odd in x and ψ_e is even in x. From this, for $t > 0$ we set

$$\varphi(x,y) = \begin{cases} u(x,y) + t\psi_o(x,y), & \text{if} \quad x \geq 0 \text{ and } u(x,y) + t\psi_o(x,y) \geq 0 \\ -u(x,y) - t\psi_o(x,y), & \text{if} \quad x \geq 0 \text{ and } u(x,y) + t\psi_o(x,y) \leq 0 \\ -\varphi(-x,y) & \text{if} \quad x < 0, \end{cases}$$

from where it follows that φ is odd in the variable x and $\varphi(x,y) \geq 0$ for $x \geq 0$. Moreover, from (V_2), $I(\varphi) = I(u + t\psi_o)$. Next, putting

$$\tilde{\varphi}(x,y) = \max\{-\alpha, \min\{\alpha, \varphi(x,y)\}\} \quad \text{for } (x,y) \in \Omega_0,$$

a direct computation shows that $\tilde{\varphi} \in E_\Phi(\alpha)$ with

$$|\nabla\tilde{\varphi}(x,y)| \leq |\nabla(u + t\psi_o)(x,y)|, \quad \forall (x,y) \in \Omega_0.$$

Furthermore, from (V_1)-(V_2),

$$V(\tilde{\varphi}(x,y)) \leq V((u + t\psi_o)(x,y)), \quad \forall (x,y) \in \Omega_0.$$

Therefore,

$$I(u + t\psi_o) = I(\varphi) \geq I(\tilde{\varphi}) \geq c_\Phi(\alpha) = I(u). \tag{2.17}$$

On the other hand, according to the Lemma A.8-(b),

$$\Phi(|\nabla(u+t\psi)|) - \Phi(|\nabla(u+t\psi_o)|) \geq \phi(|\nabla(u+t\psi_o)|)\nabla(u+t\psi_o)\nabla(t\psi_e),$$

and so,

$$\begin{aligned} \int_{\Omega_0} (\Phi(|\nabla(u+t\psi)|) - &\Phi(|\nabla(u+t\psi_o)|))dxdy \\ &\geq \int_{\Omega_0} \phi(|\nabla(u+t\psi_o)|)\left(t\nabla u\nabla\psi_e + t^2\nabla\psi_o\nabla\psi_e\right)dxdy. \end{aligned} \tag{2.18}$$

Since $I(u) = c_\Phi(\alpha)$ and $\psi \in X_0^{1,\Phi}(\Omega_0)$, we see that $I(u+t\psi), I(u+t\psi_o) < +\infty$, because for $|x|$ sufficiently large we must have

$$u(x,y) + t\psi(x,y) = u(x,y) \text{ and } u(x,y) + t\psi_o(x,y) = u(x,y).$$

Thus,

$$I(u+t\psi)-I(u+t\psi_o)=\int_{\Omega_0}(\Phi(|\nabla(u+t\psi)|)-\Phi(|\nabla(u+t\psi_o)|))\,dxdy + \int_{\Omega_0}A(x,y)\,(V(u+t\psi)-V(u+t\psi_o))\,dxdy,$$

and by (2.18),

$$\begin{aligned} I(u+t\psi)-I(u+t\psi_o)\geq t\int_{\Omega_0}&\phi(|\nabla(u+t\psi_o)|)\nabla u\nabla\psi_e dxdy \\ &+t^2\int_{\Omega_0}\phi(|\nabla(u+t\psi_o)|)\nabla\psi_o\nabla\psi_e dxdy \\ &+\int_{\Omega_0}A(x,y)\,(V(u+t\psi)-V(u+t\psi_o))\,dxdy. \end{aligned} \tag{2.19}$$

It is easily seen that the functions $\phi(|\nabla(u+t\psi_o)|)\nabla u\nabla\psi_e$ and $\phi(|\nabla(u+t\psi_o)|)\nabla\psi_o\nabla\psi_e$ are odd in the variable x, and so,

$$\int_{\Omega_0}\phi(|\nabla(u+t\psi_o)|)\nabla u\nabla\psi_e dxdy=\int_{\Omega_0}\phi(|\nabla(u+t\psi_o)|)\nabla\psi_o\nabla\psi_e dxdy=0. \tag{2.20}$$

Substituting (2.20) into (2.19), we infer that

$$I(u+t\psi)-I(u+t\psi_o)\geq\int_{\Omega_0}A(x,y)\,(V(u+t\psi)-V(u+t\psi_o))\,dxdy$$

that combines with (2.17) to give

$$I(u+t\psi)-I(u)\geq\int_{\Omega_0}A(x,y)\,(V(u+t\psi)-V(u+t\psi_o))\,dxdy,$$

and so,

$$\begin{aligned} \int_{\Omega_0}(\phi(|\nabla u|)\nabla u\nabla\psi+A(x,y)V'(u)\psi)dxdy &= \lim_{t\to 0^+}\frac{I(u+t\psi)-I(u)}{t} \\ &\geq \lim_{t\to 0^+}\int_{\Omega_0}A(x,y)\frac{V(u+t\psi)-V(u+t\psi_o)}{t}dxdy \\ &\geq \lim_{t\to 0^+}\int_{\Omega_0}A(x,y)\left(\frac{V(u+t\psi)-V(u)}{t}-\frac{V(u+t\psi_o)-V(u)}{t}\right)dxdy \\ &\geq \int_{\Omega_0}A(x,y)V'(u)(\psi-\psi_o)dxdy=\int_{\Omega_0}A(x,y)V'(u)\psi_e dxdy. \end{aligned}$$

Since the function $A(x,y)V'(u)\psi_e$ is odd in x, it follows that

$$\int_{\Omega_0}(\phi(|\nabla u|)\nabla u\nabla\psi+A(x,y)V'(u)\psi)\,dxdy\geq 0,$$

which completes the proof. ■

2.1.2 Existence of solutions on $\mathbb{R}^2$

We will show in this subsection how the study of the previous subsection helps us to find a heteroclinic solution for (2.1). For this, let us consider

$$K_\Phi(\alpha) = \{u \in E_\Phi(\alpha) : I(u) = c_\Phi(\alpha)\}.$$

Invoking Theorem 2.1, $K_\Phi(\alpha) \neq \emptyset$. In the sequel, for each $u \in K_\Phi(\alpha)$, we will show that there is a function $v \in K_\Phi(\alpha)$ depending on u such that

$$v(x,0) = v(x,1) \text{ for any } x \in \mathbb{R}.$$

To prove this, we define

$$E_{\Phi,p}(\alpha) = \{w \in E_\Phi(\alpha) : w(x,0) = w(x,1) \text{ a.e. in } x \in \mathbb{R}\}$$

and

$$c_{\Phi,p}(\alpha) = \inf_{w \in E_{\Phi,p}(\alpha)} I(w).$$

The next lemma establishes an important relation between $c_\Phi(\alpha)$ and $c_{\Phi,p}(\alpha)$.

Lemma 2.4 *It holds that $c_{\Phi,p}(\alpha) = c_\Phi(\alpha)$. Moreover, given $u \in K_\Phi(\alpha)$ there exists $v \in K_\Phi(\alpha)$, depending on u, such that*

$$v(x,0) = v(x,1) \quad \textit{for all} \quad x \in \mathbb{R}.$$

Proof. Since $E_{\Phi,p}(\alpha) \subset E_\Phi(\alpha)$, $c_\Phi(\alpha) \leq c_{\Phi,p}(\alpha)$. Now we are going to prove that $c_{\Phi,p}(\alpha) \leq c_\Phi(\alpha)$. To see this, given $w \in E_\Phi(\alpha)$, we write

$$I(w) = J_1(w) + J_2(w),$$

where

$$J_1(w) = \int_{\mathbb{R}} \int_0^{\frac{1}{2}} \mathcal{L}(w)dydx \quad \text{and} \quad J_2(w) = \int_{\mathbb{R}} \int_{\frac{1}{2}}^1 \mathcal{L}(w)dydx.$$

Let $u \in K_\Phi(\alpha)$. So, if $J_1(u) \leq J_2(u)$, let us consider the function

$$v(x,y) = \begin{cases} u(x,y), & \text{if} \quad 0 \leq y \leq \frac{1}{2}, \\ u(x,1-y), & \text{if} \quad \frac{1}{2} \leq y \leq 1 \end{cases}$$

that belongs to $E_{\Phi,p}(\alpha)$. From (A_2)-(A_3),

$$J_2(v) = J_1(v) = J_1(u),$$

and hence,

$$I(v) = J_1(v) + J_2(v) = 2J_1(u) \leq J_1(u) + J_2(u) = I(u),$$

showing that $c_{\Phi,p}(\alpha) \leq c_\Phi(\alpha)$. For that reason, $c_{\Phi,p}(\alpha) = c_\Phi(\alpha)$ and $I(v) = c_\Phi(\alpha)$ with $v(x,0) = v(x,1)$ for every $x \in \mathbb{R}$. On the other hand, if $J_2(u) \leq J_1(u)$, we consider

$$\tilde{v}(x,y) = \begin{cases} u(x, 1-y), & \text{if} \quad 0 \leq y \leq \frac{1}{2} \\ u(x,y), & \text{if} \quad \frac{1}{2} \leq y \leq 1. \end{cases}$$

By a similar argument, $\tilde{v} \in E_{\Phi,p}(\alpha)$ and $J_1(\tilde{v}) = J_2(\tilde{v}) = J_2(u)$, from where it follows that $c_{\Phi,p}(\alpha) = c_\Phi(\alpha)$, proving the desired result. ■

The Lemma 2.4 shows that the set

$$K_{\Phi,p}(\alpha) = \{w \in K_\Phi(\alpha) : \; w(x,0) = w(x,1) \text{ for all } x \in \mathbb{R}\}$$

is non empty. We would like point out that if $w \in K_{\Phi,p}(\alpha)$, then it can extend periodicity on $\mathbb{R}^2$ with period 1. Hereafter, the elements of $K_{\Phi,p}(\alpha)$ will be considered extended in whole $\mathbb{R}^2$.

Now, we are ready to prove our main theorem of this section.

Theorem 2.2 *Assume* (ϕ_1)-(ϕ_3), $V \in C^1(\mathbb{R},\mathbb{R})$, ($V_1$)-($V_3$) *and* ($A_1$)-($A_3$). *Then, there exists* $v \in C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that* v *is a weak solution of* (2.1) *that verifies the following*

(a) $v(x,y) = -v(-x,y)$ *for all* $(x,y) \in \mathbb{R}^2$,

(b) $v(x,y) = v(x,y+1)$ *for any* $(x,y) \in \mathbb{R}^2$,

(c) $0 < v(x,y) < \alpha$ *for each* $x > 0$ *and* $y \in \mathbb{R}$.

Moreover, v *is a heteroclinic solution from* $-\alpha$ *to* α, *that is,*

$$v(x,y) \to -\alpha \;\; \textit{as} \;\; x \to -\infty \;\; \textit{and} \;\; v(x,y) \to \alpha \;\; \textit{as} \;\; x \to +\infty \;\; \textit{uniformly in} \;\; y \in \mathbb{R}.$$

Proof. Let $v \in K_{\Phi,p}(\alpha)$. Then (a) and (b) are immediate. According to the proof of Theorem 2.1,

$$\int_{\Omega_0} (\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi)\, dydx = 0 \quad \forall \psi \in X_0^{1,\Phi}(\Omega_0),$$

where $X_0^{1,\Phi}(\Omega_0)$ was given in (2.16). In the sequel, we fix $\Omega_1 = \mathbb{R} \times [1,2]$,

$$E_1 = \left\{ w \in W^{1,\Phi}_{\text{loc}}(\Omega_1) : w(x,y) = -w(-x,y), \;\; x \in \mathbb{R}, \;\; \text{and } 0 \leq w(x,y) \leq \alpha \text{ for } x > 0 \right\},$$

the functional $I^1 : W^{1,\Phi}_{\text{loc}}(\Omega_1) \to \mathbb{R} \cup \{+\infty\}$ given by

$$I^1(w) = \int_{\Omega_1} \mathcal{L}(w) dydx,$$

and the real number

$$c^1 = \inf_{w \in E_1} I^1(w).$$

It is easily seen that $c_\Phi(\alpha) = c^1$ and

$$\int_{\Omega_1} (\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi)\, dydx = 0 \quad \forall \psi \in X^{1,\Phi}_0(\Omega_1),$$

where

$$X^{1,\Phi}_0(\Omega_1) = \left\{u \in W^{1,\Phi}(\Omega_1) \text{ with } u(x,y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0\right\}.$$

From this, a straightforward computation ensures that

$$\int_{\mathbb{R}\times[0,2]} (\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi)\, dydx = 0 \quad \forall \psi \in X^{1,\Phi}_0(\mathbb{R} \times [0,2]),$$

where

$$X^{1,\Phi}_0(\mathbb{R} \times [0,2]) = \left\{u \in W^{1,\Phi}(\mathbb{R} \times [0,2]) \text{ with } u(x,y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0\right\}.$$

A similar argument works to prove that

$$\int_{\mathbb{R}\times[i,k]} (\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi)\, dydx = 0,$$

for all $i, k \in \mathbb{Z}$ with $i < k$ e for any $\psi \in X^{1,\Phi}_0(\mathbb{R} \times [i,k])$ where

$$X^{1,\Phi}_0(\mathbb{R} \times [i,k]) = \left\{u \in W^{1,\Phi}(\mathbb{R} \times [i,k]) \text{ with } u(x,y) = 0 \text{ for } |x| \geq L \text{ for some } L > 0\right\}.$$

So, since k and i are arbitrary, we get

$$\int_{\mathbb{R}^2} (\phi(|\nabla v|)\nabla v \nabla \psi + A(x,y)V'(v)\psi)\, dydx = 0,$$

for any $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$. By [67, Theorem 1.7] there exist $\gamma \in (0,1)$ and $M > 0$ such that $v \in C^{1,\gamma}_{\text{loc}}(\mathbb{R}^2)$ with

$$\|v\|_{C^{1,\gamma}_{\text{loc}}(\mathbb{R}^2)} \leq M.$$

Next, we will show now that v is a heteroclinic solution from $-\alpha$ to α. To do this, given $n \in \mathbb{N}$, we set

$$v_n(x,y) = v(x+n,y) \quad \forall (x,y) \in [0,1] \times [0,1].$$

Thereby, (v_n) is bounded in $C^{1,\gamma}([0,1]\times[0,1])$, and so there exists $v_0 \in C^1([0,1]\times[0,1])$ and a subsequence (v_{n_j}) of (v_n) such that

$$v_{n_j} \to v_0 \text{ in } C^1([0,1]\times[0,1]).$$

In particular, for $x\in[0,1]$ fixed,

$$v_{n_j}(x,\cdot) \to v_0(x,\cdot) \text{ as } j\to+\infty \text{ uniformly in } y\in[0,1].$$

On the other hand, according to Lemma 2.3,

$$v_{n_j}(x,\cdot)\to\alpha \text{ in } L^{\Phi}(0,1) \text{ as } j\to+\infty.$$

Passing to a subsequence if necessary,

$$v_{n_j}(x,y)\to\alpha \text{ for almost every } y\in[0,1],$$

and hence, $v_0=\alpha$ on $[0,1]\times[0,1]$. Thus,

$$v_{n_j}(x,y)\to\alpha \text{ as } j\to+\infty \text{ uniformly in } y\in[0,1],$$

and consequently,

$$v(x,y)\to\alpha \text{ as } x\to+\infty \text{ uniformly in } y\in[0,1].$$

Since v is 1-periodic in the variable y and odd in the variable x,

$$v(x,y)\to-\alpha \text{ as } x\to-\infty \text{ and } v(x,y)\to\alpha \text{ as } x\to+\infty \text{ uniformly in } y\in\mathbb{R}.$$

Finally, adapting the same arguments explored in Lemma 1.6 let us conclude that

$$0<v(x,y)<\alpha \text{ for all } x>0 \text{ and } y\in\mathbb{R},$$

and the proof is complete. ■

If $u\in K_{\Phi}(\alpha)$, then we can extend u by periodicity on $\mathbb{R}^2$ with period 2 in y satisfying the equation (2.1). Indeed, defining the function

$$\tilde{u}(x,y)=\begin{cases} u(x,y), & \text{if } (x,y)\in\mathbb{R}\times[0,1],\\ u(x,2-y), & \text{if } (x,y)\in\mathbb{R}\times[1,2],\end{cases}$$

we have that

$$\tilde{u}(x,0)=\tilde{u}(x,2) \quad\text{and}\quad \frac{\partial\tilde{u}}{\partial\eta}(x,0)=0=\frac{\partial\tilde{u}}{\partial\eta}(x,2).$$

Now, we extend $\tilde{u}$ by periodicity to whole $\mathbb{R}^2$ by setting $\overline{u} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\overline{u} = \tilde{u} \quad \text{in} \quad \mathbb{R} \times [0, 2]$$

and

$$\overline{u}(x, y) = \tilde{u}(x, y - 2k),$$

where $y \in \mathbb{R}$ and $k \in \mathbb{Z}$ is the only integer such that $0 \leq y - 2k < 2$. From now on, without loss of generality, we can assume that $u \in K_\Phi(\alpha)$ is a periodic function with period 2 in the variable y.

Arguing as in the proof of Theorem 2.2, we derive the following result.

Theorem 2.3 *Assume (ϕ_1)-(ϕ_3), $V \in C^1(\mathbb{R}, \mathbb{R})$, ($V_1$)-($V_3$) and ($A_1$)-($A_3$). If $u \in K_\Phi(\alpha)$, then u is a weak solution of* (2.1) *in $C^{1,\gamma}_{loc}(\mathbb{R}^2, \mathbb{R})$ for some $\gamma \in (0, 1)$ that verifies the following*

(a) $u(x, y) = -u(-x, y)$ for all $(x, y) \in \mathbb{R}^2$,

(b) $u(x, y) = u(x, y + 2)$ for each $(x, y) \in \mathbb{R}^2$,

(c) $0 < u(x, y) < \alpha$ for any $x > 0$ and $y \in \mathbb{R}$.

Moreover, u is a heteroclinic solution from $-\alpha$ to α, i.e.

$$u(x, y) \to -\alpha \quad \text{as} \quad x \to -\infty \quad \text{and} \quad u(x, y) \to \alpha \quad \text{as} \quad x \to +\infty \quad \text{uniformly in} \quad y \in \mathbb{R}.$$

Remark 2.1 *If $\Phi(t) = \frac{|t|^2}{2}$, the operator Δ_Φ is the Laplacian operator, and in this case, using a local unique theorem for elliptic equations it is possible to prove that Theorems 2.2 and 2.3 are essentially the same, because every 2-periodic solution of*

$$-\Delta u + A(x, y)V'(u) = 0 \quad in \quad \mathbb{R}^2$$

is exactly 1-periodic solution, for more details see [4, Lemma 2.4] or [79, Proposition 2.18]. Here, since we are working with a large class of operator we were not able to prove that these theorems are equal.

Remark 2.2 *Here we would like to point out that Theorems 2.2 and 2.3 are valid for the p-Laplacian operator with $1 < p < +\infty$, because condition (ϕ_4) and the fact that ϕ was defined at $t = 0$ were not necessary.*

2.1.3 Compactness properties

In this section, for our purposes, we need to establish strong convergence for minimizing sequences of I on $E_\Phi(\alpha)$, as indicated below.

Proposition 2.1 *Let $(u_n) \subset E_\Phi(\alpha)$ with $I(u_n) \to c_\Phi(\alpha)$. Then, there exists $u_0 \in K_\Phi(\alpha)$ such that, along a subsequence,*

$$\|u_n - u_0\|_{W^{1,\Phi}(\Omega_0)} \to 0.$$

To better characterize the compactness properties of I, for each $L \in (0, +\infty]$ we set

$$\Omega_{0,L} = (-L, L) \times [0, 1]$$

and

$$I_{0,L}(w) = \iint_{\Omega_{0,L}} \mathcal{L}(w)dydx \quad \text{for} \quad w \in W^{1,\Phi}(\Omega_{0,L}).$$

Note that $\Omega_{0,+\infty} = \Omega_0$, $I_{0,+\infty} = I$ and that $I_{0,L}$ is also well defined on $E_\Phi(\alpha)$ being weakly lower semicontinuous with respect to the $W^{1,\Phi}(\Omega_{0,L})$ topology. Moreover, given $u \in E_\Phi(\alpha)$, we can identify $u|_{\Omega_{0,L}}$ with u itself, and so if $0 < L_1 < L_2$, one has

$$I_{0,L_1}(u) \le I_{0,L_2}(u) \le I(u) \quad \forall u \in E_\Phi(\alpha).$$

From now on, by Lemma A.5-(c), we can fix $\Lambda > 0$ satisfying

$$\|w\|_{L^\infty(0,1)} \le \Lambda \|w\|_{W^{1,\Phi}(0,1)} \quad \forall w \in W^{1,\Phi}(0, 1). \tag{2.21}$$

Moreover, given $\delta > 0$, one defines

$$\lambda_\delta = 2^{m+1}\delta^l + \overline{A} \max_{|s-\alpha| \le \Lambda\delta} V(s) \quad \text{and} \quad l_\delta = \frac{c_\Phi(\alpha) + 1}{(2\mu_\delta)^{\frac{m}{m-1}}}, \tag{2.22}$$

where $\mu_\delta > 0$ is given according to Lemma 2.2.

The next lemma is crucial to prove a compactness result involving the functional I.

Lemma 2.5 *There exists $\delta_0 \in \left(0, \frac{\delta_\alpha}{2}\right)$ such that, for any $\delta \in (0, \delta_0)$, if $u \in E_\Phi(\alpha)$, $L \in (l_\delta + 1, +\infty]$ and $I_{0,L}(u) \le c_\Phi(\alpha) + \lambda_\delta$, then the following hold:*

(a) There exists $x_+ \in (0, l_\delta)$ verifying

$$\|u(x_+, \cdot) - \alpha\|_{W^{1,\Phi}(0,1)} < \delta.$$

(b) For x_+ given in (a) we have

$$\int_{x_+}^{L}\int_0^1 (\Phi(|\nabla u|) + A(x,y)V(u))\,dydx \le \frac{3}{2}\lambda_\delta.$$

(c) For each $x \in (x_+, L)$,

$$\|u(x,\cdot) - \alpha\|_{L^\Phi(0,1)} \le \delta_\alpha.$$

Proof. First note that $\lambda_\delta \to 0$ as $\delta \to 0$. Thus, we can fix $\delta_0 \in (0, \delta_\alpha/2)$ satisfying

$$\lambda_\delta < \min\left\{1, \frac{2}{3}\mu_{\frac{\delta_\alpha}{2}}\left(\frac{\delta_\alpha}{2}\right)^{\frac{m}{l}}\right\} \quad \forall\delta \in (0,\delta_0), \tag{2.23}$$

where $\delta_\alpha > 0$ was defined in (V_3) and $\mu_{\frac{\delta_\alpha}{2}}$ is given by Lemma 2.2 in correspondence to $r = \frac{\delta_\alpha}{2}$. Let $u \in E_\Phi(\alpha)$, $L \in (l_\delta + 1, +\infty]$ and $\delta \in (0, \delta_0)$ with $I_{0,L}(u) \le c_\Phi(\alpha) + \lambda_\delta$. Assuming that (a) is false, we deduce

$$\|u(x,\cdot) - \alpha\|_{W^{1,\Phi}(0,1)} \ge \delta \quad \forall x \in (0, l_\delta).$$

According to Lemma 2.2, there exists $\mu_\delta > 0$ such that

$$I_{0,L}(u) \ge \int_0^{l_\delta}\int_0^1 \mathcal{L}(u)dydx \ge (2\mu_\delta)^{\frac{m}{m-1}} l_\delta = c_\Phi(\alpha) + 1 > c_\Phi(\alpha) + \lambda_\delta,$$

which is a contradiction. Therefore, there is $x_+ \in (0, l_\delta)$ checking item (a). To prove (b), let us consider

$$\tilde{u}(x,y) = \begin{cases} u(x,y), & \text{if } \ 0 \le x \le x_+ & \text{and } \ y \in [0,1], \\ (\alpha - u(x_+,y))(x - x_+) + u(x_+,y), & \text{if } \ x_+ \le x \le x_+ + 1 & \text{and } \ y \in [0,1], \\ \alpha, & \text{if } \ x_+ + 1 \le x & \text{and } \ y \in [0,1], \\ -\tilde{u}(-x,y), & \text{if } \ x < 0 & \text{and } \ y \in [0,1]. \end{cases}$$

Thereby, $\tilde{u} \in E_\Phi(\alpha)$ and $c_\Phi(\alpha) \le I(\tilde{u}) = I_{0,x_++1}(\tilde{u})$. Moreover,

$$\partial_x\tilde{u}(x,y) = \alpha - u(x_+,y) \text{ and } \partial_y\tilde{u}(x,y) = (x_+ + 1 - x)\partial_y u(x_+,y) \text{ in } (x_+, x_+ + 1)\times[0,1].$$

Using the fact that Φ is increscing on $(0,+\infty)$ and Lemma A.8-(a), it is possible to show that

$$\Phi(|\nabla\tilde{u}|) \le 2^m\Phi(|\alpha - u(x_+,y)|) + 2^m\Phi(|\partial_y u(x_+,y)|) \ \text{ on } \ (x_+, x_+ + 1)\times[0,1],$$

from where it follows that

$$\begin{aligned}\int_{x_+}^{x_++1}\int_0^1 \mathcal{L}(\tilde{u})dydx \le 2^m\int_{x_+}^{x_++1}\int_0^1 (\Phi(|\alpha - u(x_+,y)|) + \Phi(|\partial_y u(x_+,y)|))dydx \\ + \int_{x_+}^{x_++1}\int_0^1 A(x,y)V(\tilde{u})dydx.\end{aligned} \tag{2.24}$$

Now, applying Lemma A.2,

$$\int_0^1 \Phi(|\alpha - u(x_+, y)|)dy \le \xi_1 \left(\|\alpha - u(x_+, \cdot)\|_{L^\Phi(0,1)}\right) \le \xi_1(\delta) = \delta^l. \tag{2.25}$$

A similar argument works to prove that

$$\int_0^1 \Phi(|\partial_y u(x_+, y)|)dy \le \delta^l. \tag{2.26}$$

Gathering (2.24) with (2.25) and (2.26), we obtain

$$\int_{x_+}^{x_++1}\int_0^1 \mathcal{L}(\tilde{u})dydx \le 2^{m+1}\delta^l + \overline{A}\int_{x_+}^{x_++1}\int_0^1 V(\tilde{u})dydx.$$

By item (a) and (2.21),

$$\|\tilde{u}(x, \cdot) - \alpha\|_{L^\infty(0,1)} \le \Lambda\delta \quad \forall x \in (x_+, x_+ + 1),$$

and hence

$$\int_{x_+}^{x_++1}\int_0^1 \mathcal{L}(\tilde{u})dydx \le 2^{m+1}\delta^l + \overline{A}\max_{|s-\alpha|\le\Lambda\delta} V(s) = \lambda_\delta. \tag{2.27}$$

Now, since

$$I_{0,L}(\tilde{u}) = I_{0,x_+}(u) + 2\int_{x_+}^L\int_0^1 \mathcal{L}(\tilde{u})dydx = I_{0,L}(u) + 2\int_{x_+}^L\int_0^1 \mathcal{L}(\tilde{u})dydx - 2\int_{x_+}^L\int_0^1 \mathcal{L}(u)dydx,$$

and $c_\Phi(\alpha) \le I_{0,L}(\tilde{u})$ follows from (2.27) that

$$\int_{x_+}^L\int_0^1 \mathcal{L}(u)dydx \le \frac{3}{2}\lambda_\delta,$$

which proves (b). Finally, if (c) does not hold, we should find $\theta \in (x_+, L)$ satisfying

$$\|u(\theta, \cdot) - \alpha\|_{L^\Phi(0,1)} > \delta_\alpha.$$

Recalling that by (a),

$$\|u(x_+, \cdot) - \alpha\|_{L^\Phi(0,1)} < \frac{\delta_\alpha}{2},$$

the Corollary 2.1 together with Intermediate Value Theorem guarantees the existence of $\sigma \in (x_+, \theta)$ such that

$$\|u(\theta, \cdot) - u(\sigma, \cdot)\|_{L^\Phi(0,1)} \ge \frac{\delta_\alpha}{2} \ \text{ and } \ \|u(x, \cdot) - \alpha\|_{L^\Phi(0,1)} \ge \frac{\delta_\alpha}{2} \ \ \forall x \in (\sigma, \theta).$$

Invoking Lemma 2.2,

$$\int_{x_+}^L\int_0^1 \mathcal{L}(u)dydx \ge \mu_{\frac{\delta_\alpha}{2}} h\left(\int_0^1 \Phi(|u(\theta, y) - u(\sigma, y)|)dy\right).$$

On the other hand, from Lemma A.2,

$$\int_0^1 \Phi(|u(\theta,y)-u(\sigma,y)|)dy \geq \xi_0\left(\|u(\theta,\cdot)-u(\sigma,\cdot)\|_{L^\Phi(0,1)}\right) \geq \xi_0\left(\frac{\delta_\alpha}{2}\right).$$

Taking $\delta_\alpha > 0$ small if necessary and by definition of function h we get the inequality below

$$\int_{x_+}^L \int_0^1 \mathcal{L}(u)dydx \geq \mu_{\frac{\delta_\alpha}{2}}\left(\frac{\delta_\alpha}{2}\right)^{\frac{m}{l}}$$

that combines with (b) to give

$$\mu_{\frac{\delta_\alpha}{2}}\left(\frac{\delta_\alpha}{2}\right)^{\frac{m}{l}} \leq \frac{3}{2}\lambda_\delta,$$

which contradicts (2.23), and the lemma follows. ■

From Lemma 2.5, we obtain in particular the following result.

Lemma 2.6 *For all $\epsilon > 0$ there are $\overline{\lambda}_\epsilon > 0$ and $\bar{l}_\epsilon > 0$ such that if $u \in E_\Phi(\alpha)$ and $I(u) \leq c_\Phi(\alpha) + \overline{\lambda}_\epsilon$, then $u - \alpha \in W^{1,\Phi}\left((\bar{l}_\epsilon, +\infty)\times(0,1)\right)$ and*

$$\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \left(\Phi(|u-\alpha|)+\Phi(|\nabla u|)\right)dydx \leq \epsilon.$$

Proof. By definition of λ_δ, see (2.22), we know that $\lambda_\delta \to 0$ as $\delta \to 0$. Thereby, given $\epsilon > 0$ we can choose $\delta_0 \in (0, \delta_\alpha/2)$ satisfying

$$\frac{3}{2}\lambda_\delta \leq \frac{\epsilon}{\max\left\{1, \frac{1}{\underline{A}\ \underline{w}}\right\}} \quad \forall \delta \in (0,\delta_0),$$

where $\underline{w}$ was given in (1.11). Denoting $\overline{\lambda}_\epsilon = \lambda_\delta$, $\bar{l}_\epsilon = l_\delta$ and $L = +\infty$, it follows from Lemma 2.5 that

$$\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \left(\Phi(|\nabla u|)+A(x,y)V(u)\right)dydx \leq \frac{3}{2}\lambda_\delta \leq \frac{\epsilon}{\max\left\{1, \frac{1}{\underline{A}\ \underline{w}}\right\}}. \tag{2.28}$$

According to (1.11),

$$\begin{aligned}\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \left(\Phi(|u-\alpha|)+\Phi(|\nabla u|)\right)dydx &\leq \int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \Phi(|\nabla u|)dydx + \int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \frac{1}{\underline{w}}V(u)dydx \\ &\leq \int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \Phi(|\nabla u|)dydx + \frac{1}{\underline{w}\ \underline{A}}\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 A(x,y)V(u)dydx \\ &\leq \max\left\{1, \frac{1}{\underline{A}\ \underline{w}}\right\}\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \left(\Phi(|\nabla u|)+A(x,y)V(u)\right)dydx.\end{aligned} \tag{2.29}$$

From (2.28) and (2.29), $u-\alpha \in W^{1,\Phi}\left((\bar{l}_\epsilon,+\infty)\times(0,1)\right)$ with

$$\int_{\bar{l}_\epsilon}^{+\infty}\int_0^1 \left(\Phi(|u-\alpha|)+\Phi(|\nabla u|)\right)dydx \leq \epsilon,$$

and this is precisely the assertion of the lemma. ■

In order to continue our analysis, we will fix the following set

$$\tilde{E}_\Phi(\alpha)=\left\{w\in W^{1,\Phi}_{\rm loc}(\Omega_0): w \text{ is odd in } x \text{ and } w-\alpha\in W^{1,\Phi}([0,+\infty)\times[0,1])\right\}$$

and the real number

$$\tilde{c}_\Phi(\alpha)=\inf_{w\in\tilde{E}_\Phi(\alpha)} I(w).$$

It is very important to point out that $\tilde{E}_\Phi(\alpha)\neq\emptyset$, because the function φ_α given in (2.3) belongs to $\tilde{E}_\Phi(\alpha)$. Moreover, it is easy to check that if $w\in\tilde{E}_\Phi(\alpha)$, then $w+\alpha\in W^{1,\Phi}((-\infty,0]\times[0,1])$, and that if $w_1,w_2\in\tilde{E}_\Phi(\alpha)$, then $w_1-w_2\in W^{1,\Phi}(\Omega_0)$. Have this in mind, we are able to define on $\tilde{E}_\Phi(\alpha)$ the metric $\rho:\tilde{E}_\Phi(\alpha)\times\tilde{E}_\Phi(\alpha)\to[0,+\infty)$ given by

$$\rho(w_1,w_2)=\|w_1-w_2\|_{W^{1,\Phi}(\Omega_0)}.$$

A direct computation guarantees that $(\tilde{E}_\Phi(\alpha),\rho)$ is a complete metric space.

The next lemma shows that the numbers $c_\Phi(\alpha)$ and $\tilde{c}_\Phi(\alpha)$ are equal.

Lemma 2.7 *It holds that $\tilde{c}_\Phi(\alpha)=c_\Phi(\alpha)$. Moreover, if $(u_n)\subset E_\Phi(\alpha)$ and $I(u_n)\to c_\Phi(\alpha)$, then there exists $n_0\in\mathbb{N}$ such that $u_n\in\tilde{E}_\Phi(\alpha)$ for any $n\geq n_0$. Therefore, (u_n) is a minimizing sequence for I on $\tilde{E}_\Phi(\alpha)$.*

Proof. Let $(u_n)\subset E_\Phi(\alpha)$ be a sequence with $I(u_n)\to c_\Phi(\alpha)$. Thus, given $\epsilon>0$ there is $n_0\in\mathbb{N}$ verifying $I(u_n)\leq c_\Phi(\alpha)+\epsilon$ for any $n\geq n_0$. By Lemma 2.6, there exists $\bar{l}_\epsilon>0$ such that $u_n-\alpha\in W^{1,\Phi}((\bar{l}_\epsilon,+\infty)\times[0,1])$ for all $n\geq n_0$, and hence,

$$u_n-\alpha\in W^{1,\Phi}([0,+\infty)\times[0,1]) \quad \forall n\geq n_0.$$

From this, $u_n\in\tilde{E}_\Phi(\alpha)$ and $\tilde{c}_\Phi(\alpha)\leq I(u_n)$ for each $n\geq n_0$. Taking the limit of $n\to+\infty$, we get $\tilde{c}_\Phi(\alpha)\leq c_\Phi(\alpha)$. Now, let us consider $(v_n)\subset\tilde{E}_\Phi(\alpha)$ with $I(v_n)\to\tilde{c}_\Phi(\alpha)$ and

$$\overline{v}_n(x,y)=\begin{cases}\alpha, & \text{if } \; v_n(x,y)\geq\alpha\\ v_n(x,y), & \text{if } \; -\alpha\leq v_n(x,y)\leq\alpha\\ -\alpha, & \text{if } \; v_n(x,y)\leq-\alpha.\end{cases}$$

From the properties of Φ, V and $\overline{v}_n$, $I(\overline{v}_n) \le I(v_n)$ for every $n \in \mathbb{N}$. Setting

$$\tilde{v}_n(x,y) = \begin{cases} \overline{v}_n(x,y), & \text{if } \ \overline{v}_n \ge 0 \text{ and } x > 0 \\ -\overline{v}_n(x,y), & \text{if } \ \overline{v}_n \le 0 \text{ and } x > 0 \\ -\overline{v}_n(-x,y), & \text{if } \ x \le 0, \end{cases}$$

it is easy to see that $(\tilde{v}_n) \subset E_\Phi(\alpha)$ and $I(\tilde{v}_n) = I(\overline{v}_n)$ for each $n \in \mathbb{N}$. Therefore,

$$c_\Phi(\alpha) \le I(\tilde{v}_n) = I(\overline{v}_n) \le I(v_n) = \tilde{c}_\Phi(\alpha) + o_n(1).$$

Taking the limit of $n \to +\infty$ we obtain $c_\Phi(\alpha) \le \tilde{c}_\Phi(\alpha)$, from where it follows that $c_\Phi(\alpha) = \tilde{c}_\Phi(\alpha)$. Finally, if $(u_n) \subset E_\Phi(\alpha)$ and $I(u_n) \to c_\Phi(\alpha)$, then we already know that there is $n_0 \in \mathbb{N}$ such that $u_n \in \tilde{E}_\Phi(\alpha)$ for $n \ge n_0$, and as $c_\Phi(\alpha) = \tilde{c}_\Phi(\alpha)$, we deduce that (u_n) is a minimizing sequence for I on $\tilde{E}_\Phi(\alpha)$. ■

In the sequel, we say that a sequence (u_n) is a $(PS)_d$ *sequence for* I, with $d \in \mathbb{R}$, if $(u_n) \subset \tilde{E}_\Phi(\alpha)$ such that

$$I(u_n) \to d \quad \text{and} \quad \|I'(u_n)\|_* \to 0 \quad \text{as} \quad n \to +\infty,$$

where

$$\|I'(w)\|_* = \sup\left\{ I'(w)\psi : \psi \in X_0^{1,\Phi}(\Omega_0) \text{ and } \|\psi\|_{W^{1,\Phi}(\Omega_0)} \le 1 \right\}.$$

Lemma 2.8 *If $(u_n) \subset E_\Phi(\alpha)$ and $I(u_n) \to c_\Phi(\alpha)$, then there is a sequence $(w_n) \subset \tilde{E}_\Phi(\alpha)$ such that (w_n) is a $(PS)_{c_\Phi(\alpha)}$ sequence for I and*

$$\|u_n - w_n\|_{W^{1,\Phi}(\Omega_0)} \le \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Proof. Let $(u_n) \subset E_\Phi(\alpha)$ with $I(u_n) \to c_\Phi(\alpha)$. As $(\tilde{E}_\Phi(\alpha), \rho)$ is a complete metric space, we can employ the Ekeland's Variational Principle to find a sequence $(w_n) \subset \tilde{E}_\Phi(\alpha)$ satisfying:

(a) $I(w_n) \le I(u_n)$ for any $n \in \mathbb{N}$,

(b) $\rho(w_n, u_n) \le \dfrac{1}{n}$ for all $n \in \mathbb{N}$,

(c) $I(w_n) - I(w) < \dfrac{1}{n}\|w_n - w\|_{W^{1,\Phi}(\Omega_0)}$ for each $w \in \tilde{E}_\Phi(\alpha)$ with $w \ne w_n$.

Now, given $\psi \in X_0^{1,\Phi}(\Omega_0)$ we can write $\psi = \psi_o + \psi_e$, where ψ_o is odd in the variable x and ψ_e is ever in x. It is easily seen that $w_n + t\psi_o \in \tilde{E}_\Phi(\alpha)$ for all $n \in \mathbb{N}$ and $t > 0$. From (c),

$$\begin{aligned} I(w_n + t\psi) - I(w_n) &= I(w_n + t\psi) - I(w_n + t\psi_o) + I(w_n + t\psi_o) - I(w_n) \\ &\ge I(w_n + t\psi) - I(w_n + t\psi_o) - \frac{1}{n}\|t\psi_o\|_{W^{1,\Phi}(\Omega_0)}, \end{aligned}$$

or equivalently,

$$\frac{I(w_n+t\psi)-I(w_n)}{t} \geq \frac{I(w_n+t\psi)-I(w_n+t\psi_o)}{t} - \frac{1}{n}\|\psi_o\|_{W^{1,\Phi}(\Omega_0)}.$$

Arguing as in the proof of Theorem 2.1,

$$I'(w_n)\psi \geq -\frac{1}{n}\|\psi_o\|_{W^{1,\Phi}(\Omega_0)}. \tag{2.30}$$

Here we would like point out that the same arguments found in Lemma 1.14 work to show that

$$\|\psi_o\|_{W^{1,\Phi}(\Omega_0)} \leq \|\psi\|_{W^{1,\Phi}(\Omega_0)}. \tag{2.31}$$

From (2.30)-(2.31) and replacing ψ by $-\psi$, one gets

$$|I'(w_n)\psi| \leq \frac{1}{n}\|\psi\|_{W^{1,\Phi}(\Omega_0)}.$$

Thereby,

$$\|I'(w_n)\|_* \to 0 \quad \text{as} \quad n \to +\infty.$$

Finally, from Lemma 2.7 and (a),

$$c_\Phi(\alpha) = \tilde{c}_\Phi(\alpha) \leq I(w_n) \leq I(u_n) = c_\Phi(\alpha) + o_n(1),$$

showing that $I(w_n) \to c_\Phi(\alpha)$. Therefore, (w_n) is a $(PS)_{c_\Phi(\alpha)}$ sequence for I, and the lemma is proved. ■

From now on, we consider $(u_n) \subset E_\Phi(\alpha)$ and $(w_n) \subset \tilde{E}_\Phi(\alpha)$ as in the last lemma. So, (w_n) is also bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$. Indeed, for each $L > 0$ the Lemma 2.8 ensures that

$$\|w_n\|_{W^{1,\Phi}(\Omega_{0,L})} \leq \|w_n - u_n\|_{W^{1,\Phi}(\Omega_{0,L})} + \|u_n\|_{W^{1,\Phi}(\Omega_{0,L})} \leq \frac{1}{n} + \|u_n\|_{W^{1,\Phi}(\Omega_{0,L})}.$$

Since (u_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$, it follows that (w_n) also is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$. Then, for some subsequence, there is $u_0 \in W^{1,\Phi}_{\text{loc}}(\Omega_0)$ verifying

$$w_n \rightharpoonup u_0 \quad \text{in} \quad W^{1,\Phi}_{\text{loc}}(\Omega_0), \tag{2.32}$$

$$w_n \to u_0 \quad \text{in} \quad L^{\Phi}_{\text{loc}}(\Omega_0), \tag{2.33}$$

$$w_n \to u_0 \quad \text{in} \quad L^{1}_{\text{loc}}(\Omega_0) \tag{2.34}$$

and

$$w_n(x,y) \to u_0(x,y) \;\; \text{a.e. in} \;\; \Omega_0. \tag{2.35}$$

Lemma 2.9 *There exists a subsequence of (w_n), still denoted by itself, such that*

$$\nabla w_n(x,y) \to \nabla u_0(x,y) \text{ a.e. in } \Omega_0.$$

Proof. Given $L > 0$, let us consider $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfying

$$0 \le \psi \le 1, \quad \psi \equiv 1 \text{ in } \Omega_{0,L} \text{ and } \operatorname{supp}(\psi) \subset \Omega_{0,L+1}.$$

From Lemma A.8-(c),

$$\begin{aligned}0 &\le \int_{\Omega_{0,L}} (\phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0)(\nabla w_n - \nabla u_0)dydx \\ &\le \int_{\Omega_{0,L+1}} \psi(\phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0)(\nabla w_n - \nabla u_0)dydx \\ &\le \int_{\Omega_{0,L+1}} \psi\phi(|\nabla w_n|)\nabla w_n(\nabla w_n - \nabla u_0)dydx - \int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0(\nabla w_n - \nabla u_0)dydx.\end{aligned} \tag{2.36}$$

Setting the linear functional $f : W^{1,\Phi}(\Omega_{0,L+1}) \to \mathbb{R}$ given by

$$f(v) = \int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0 \nabla v dydx,$$

we have that it is continuous, because $\phi(|\nabla u_0|)\nabla u_0 \in L^{\tilde{\Phi}}(\Omega_{0,L+1})$ via Lemma A.6, and so, by Hölder's inequality

$$\left| \int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0 \nabla v dydx \right| \le 2\|\phi(|\nabla u_0|)\nabla u_0\|_{L^{\tilde{\Phi}}(\Omega_{0,L+1})} \|v\|_{W^{1,\Phi}(\Omega_{0,L+1})},$$

for all $v \in W^{1,\Phi}(\Omega_{0,L+1})$. Therefore, (2.32) asserts that $f(w_n - u_0) \to 0$, or equivalently,

$$\int_{\Omega_{0,L+1}} \psi\phi(|\nabla u_0|)\nabla u_0(\nabla w_n - \nabla u_0)dydx \to 0. \tag{2.37}$$

Using again the Lemma A.6 and the boundedness of (w_n) in $W^{1,\Phi}_{\text{loc}}(\Omega_0)$, there is $C > 0$ such that

$$\int_{\Omega_{0,L+1}} \tilde{\Phi}(\phi(|\nabla w_n|)\nabla w_n)dydx \le C \quad \forall n \in \mathbb{N},$$

implying that $(\phi(|\nabla w_n|)\nabla w_n)$ is bounded in $L^{\tilde{\Phi}}(\Omega_{0,L+1})$. So, by (2.33) and Hölder's inequality,

$$\int_{\Omega_{0,L+1}} (w_n - u_0)\phi(|\nabla w_n|)\nabla w_n \nabla\psi dydx \to 0. \tag{2.38}$$

Now, considering the sequence (ψw_n) we have that $(\psi w_n) \subset W^{1,\Phi}(\Omega_0)$, because ψ has compact support, and by (2.35), passing to a subsequence if necessary, we can assume that

$$\psi w_n \rightharpoonup \psi u_0 \quad \text{in} \quad W^{1,\Phi}(\Omega_{0,L+1}) \quad \text{and} \quad \psi w_n \to \psi u_0 \quad \text{a.e.} \quad \Omega_0.$$

Consequently,

$$A(x,y)V'(w_n(x,y))(\psi(x,y)w_n(x,y) - \psi(x,y)u_0(x,y)) \to 0 \text{ a.e. in } \Omega_{0,L+1}.$$

From (2.4) and (2.34), there exist $h \in L^1(\Omega_{0,L+1})$ and $\beta > 0$ such that, along a subsequence,

$$|A(x,y)V'(w_n)(\psi w_n - \psi u_0)| \leq \beta \overline{A}|\psi|(h + |u_0|) \in L^1(\Omega_{0,L+1}).$$

Applying the Lebesgue's Dominated Convergence Theorem we obtain

$$\int_{\Omega_{0,L+1}} A(x,y)V'(w_n)(\psi w_n - \psi u_0)dydx \to 0. \tag{2.39}$$

Finally, we would like point out that

$$I'(w_n)(\psi w_n - \psi u_0) \to 0. \tag{2.40}$$

In fact, just note that

$$|I'(w_n)(\psi w_n - \psi u_0)| \leq \|I'(w_n)\|_* \|\psi w_n - \psi u_0\|_{W^{1,\Phi}(\Omega_0)},$$

$(\psi w_n) \subset X^{1,\Phi}(\Omega_0)$ is a bounded sequence in $W^{1,\Phi}(\Omega_0)$ and (w_n) is a $(PS)_c$ sequence for I. Recalling that

$$\begin{aligned} I'(w_n)(\psi w_n - \psi u_0) = \int_{\Omega_{0,L+1}} & \phi(|\nabla w_n|)\nabla w_n \nabla(\psi w_n - \psi u_0)dydx \\ & + \int_{\Omega_{0,L+1}} A(x,y)V'(w_n)(\psi w_n - \psi u_0)dydx, \end{aligned}$$

from where it follows by (2.39) and (2.40) that

$$\int_{\Omega_{0,L+1}} \phi(|\nabla w_n|)\nabla w_n \nabla(\psi w_n - \psi u_0)dydx \to 0. \tag{2.41}$$

Since $\nabla(\psi w_n - \psi u_0) = \psi \nabla w_n + w_n \nabla \psi - \psi \nabla u_0 - u_0 \nabla \psi$, we also have

$$\begin{aligned} \int_{\Omega_{0,L+1}} \psi\phi(|\nabla w_n|)\nabla w_n(\nabla w_n - \nabla u_0)dydx = & \int_{\Omega_{0,L+1}} \phi(|\nabla w_n|)\nabla w_n \nabla(\psi w_n - \psi u_0)dydx \\ & - \int_{\Omega_{0,L+1}} (w_n - u_0)\phi(|\nabla w_n|)\nabla w_n \nabla \psi dydx. \end{aligned} \tag{2.42}$$

From (2.38), (2.41) and (2.42),

$$\int_{\Omega_{0,L+1}} \psi\phi(|\nabla w_n|)\nabla w_n(\nabla w_n - \nabla u_0)dydx \to 0. \tag{2.43}$$

Finally, from (2.37), (2.43) and (2.36),

$$\int_{\Omega_{0,L}} (\phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0)(\nabla w_n - \nabla u_0)dydx \to 0.$$

This limit combined with the Lemma A.8-(c) leads to, along a subsequence,

$$\langle \phi(|\nabla w_n|)\nabla w_n - \phi(|\nabla u_0|)\nabla u_0, \nabla w_n - \nabla u_0\rangle \to 0 \;\text{ a.e. in }\; \Omega_{0,L}.$$

Applying a result found in Dal Maso and Murat [32], we infer that

$$\nabla w_n(x,y) \to \nabla u_0(x,y) \text{ a.e. in } \Omega_{0,L}.$$

As $L > 0$ is arbitrary, there exists a subsequence of (w_n), still denoted by itself, such that

$$\nabla w_n(x,y) \to \nabla u_0(x,y) \text{ for almost everywhere in } \Omega_0,$$

finishing the proof of the lemma. ∎

Finally, we are in a position to prove our best result of this subsection, namely Proposition 2.1.

Proof of Proposition 2.1.

Let $(u_n) \subset E_\Phi(\alpha)$ with $I(u_n) \to c_\Phi(\alpha)$. Invoking Lemma 2.8 there is a sequence $(w_n) \subset \tilde{E}_\Phi(\alpha)$ with $I(w_n) \to c_\Phi(\alpha)$ and

$$\|u_n - w_n\|_{W^{1,\Phi}(\Omega_0)} \leq \frac{1}{n} \quad \forall n \in \mathbb{N}. \tag{2.44}$$

Hence there exits $u_0 \in W^{1,\Phi}_{\text{loc}}(\Omega_0)$ satisfying (2.32)-(2.35). Moreover,

$$\|u_n - u_0\|_{L^\Phi(\Omega_{0,L})} \leq \frac{1}{n} + \|w_n - u_0\|_{L^\Phi(\Omega_{0,L})}, \quad \forall L > 0. \tag{2.45}$$

Thereby, by (2.33), u_0 is the punctual limit of (u_n), $u_0 \in E_\Phi(\alpha)$ and $I(u_0) = c_\Phi(\alpha)$, that is, $u_0 \in K_\Phi(\alpha)$. Now, arguing as in the proof of Proposition 1.1, one finds

$$\|\nabla w_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \to 0.$$

From (2.44),

$$\|\nabla u_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \leq \frac{1}{n} + \|\nabla w_n - \nabla u_0\|_{L^\Phi(\Omega_0)},$$

implying that

$$\|\nabla u_n - \nabla u_0\|_{L^\Phi(\Omega_0)} \to 0. \tag{2.46}$$

Finally, according to Lemma 2.6, given $\epsilon > 0$, there are $l_\epsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{l_\epsilon}^{+\infty}\int_0^1 \Phi(|u_0-\alpha|)dydx \leq \frac{\epsilon}{2^m} \text{ and } \int_{l_\epsilon}^{+\infty}\int_0^1 \Phi(|u_n-\alpha|)dydx \leq \frac{\epsilon}{2^m} \ \forall n \geq n_0.$$

So, it is easy to see by Lemma A.8-(c) that

$$\int_{l_\epsilon}^{+\infty}\int_0^1 \Phi(|u_n-u_0|)dydx \leq 2^{m-1}\int_{l_\epsilon}^{+\infty}\int_0^1 (\Phi(|u_n-\alpha|)+\Phi(|u_0-\alpha|))dydx \leq \epsilon \ \ \forall n \geq n_0. \tag{2.47}$$

As $\Phi \in \Delta_2$, (2.45) together with (2.47) gives

$$\|u_n - u_0\|_{L^\Phi(\Omega_0)} \to 0. \tag{2.48}$$

Now, the lemma follows from (2.46) and (2.48). ∎

2.1.4 Exponential estimates

In this subsection, we intend to obtain some exponential estimates at infinity, as well as their consequences. To this end, given $j \in \mathbb{N} \cup \{0\}$ let us define the sets

$$\Omega_j = \mathbb{R} \times [j, j+1] \ \text{ and } \ T_j = \{(x,y) \in \Omega_j : \ |x| \leq y\}.$$

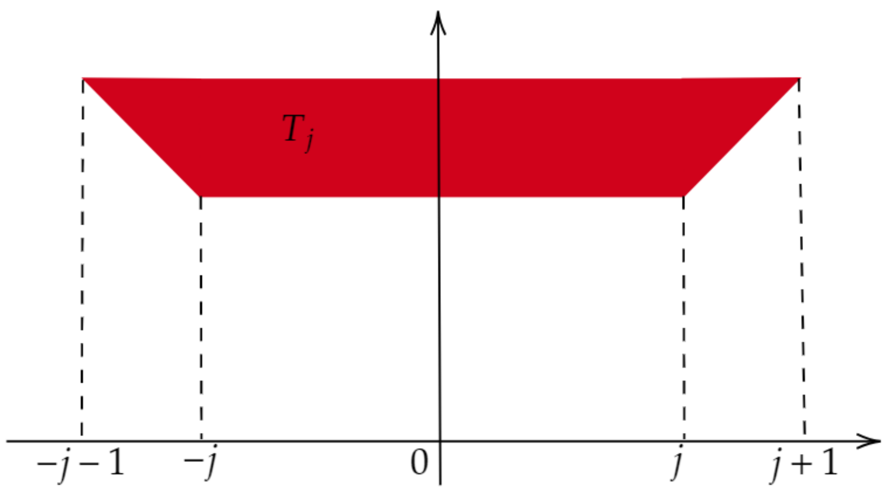

Figure 2.1: Geometric illustration of T_j.

Associated with sets above, we consider

$$E_j = \left\{w \in W^{1,\Phi}(T_j) : 0 \leq w(x,y) \leq \alpha \text{ for } x > 0 \text{ and } w \text{ is odd in } x\right\},$$

and the functional $I_j : W^{1,\Phi}(T_j) \to \mathbb{R} \cup \{+\infty\}$ given by

$$I_j(w) = \iint_{T_j} \mathcal{L}(w)dydx.$$

By a direct computation, we see that I_j is lower semicontinuous with respect to the weak topology of $W^{1,\Phi}(T_j)$ and bounded from below. Moreover, since $I_j(0) < +\infty$,

$$c_j := \inf_{w \in E_j} I_j(w)$$

is well defined. For each $j \in \mathbb{N} \cup \{0\}$ let us also consider

$$K_j = \{w \in E_j : I_j(w) = c_j\}.$$

Arguing as in the proof of Theorem 2.1, it is possible to prove the following result.

Lemma 2.10 *For every $j \in \mathbb{N} \cup \{0\}$, $K_j \neq \emptyset$. Moreover, if $u_j \in K_j$, then u_j is a weak solution in $C^{1,\beta}(T_j)$ for some $\beta \in (0,1)$ of*

$$-\Delta_\Phi u_j + A(x,y)V'(u_j) = 0 \quad in \quad T_j,$$

with $0 < u_j(x,y) < \alpha$ for $x > 0$,

$$\partial_y u_j(x,j) = 0 \quad for \quad |x| < j \quad and \quad \partial_y u_j(x,j+1) = 0 \quad for \quad |x| < j+1.$$

As immediate consequence of the last lemma is the corollary below.

Corollary 2.2 *For all $j \in \mathbb{N} \cup \{0\}$ we have $c_j \leq c_{j+1} < c_\Phi(\alpha)$.*

Proof. Invoking Lemma 2.10, for each $j \geq 0$ there exists $u_{j+1} \in K_{j+1}$. Now, considering the function

$$\overline{u}_j(x,y) = u_{j+1}(x,y+1) \text{ for } (x,y) \in T_j,$$

we see that $\overline{u}_j \in E_j$ and

$$c_j \leq I_j(\overline{u}_j) \leq I_{j+1}(u_{j+1}) = c_{j+1}.$$

Finally, from Theorem 2.2, there exists $v : \mathbb{R}^2 \to \mathbb{R}$ such that $v \in E_\Phi(\alpha)$ with $I(v) = c_\Phi(\alpha)$ and v is 1-periodic in the variable y. So, $v \in E_j$ for any $j \in \mathbb{N} \cup \{0\}$ and

$$c_j \leq I_j(v) < I(v) = c_\Phi(\alpha) \quad \forall j \in \mathbb{N} \cup \{0\},$$

showing the desired result. ■

If $j > 1$ and $u_j \in K_j$, then arguing as in the end of the previous subsection, u_j have an extension 2-periodic v_j in $(-j, j) \times \mathbb{R}$, i.e., there exists $v_j : (-j, j) \times \mathbb{R} \to \mathbb{R}$ that is 2-periodic in the variable y such that

$$v_j = u_j \text{ in } (-j, j) \times (j, j+1).$$

Moreover, v_j is a weak solution in $C^{1,\beta}_{\text{loc}}((-j, j)\times\mathbb{R}, \mathbb{R})$, for some $\beta \in (0, 1)$, of the equation

$$-\Delta_\Phi v_j + A(x, y)V'(v_j) = 0 \text{ in } (-j, j) \times \mathbb{R}.$$

An direct computation shows that

$$\int_{-j}^{j}\int_{j}^{j+1} \mathcal{L}(u_j)dydx = \int_{-j}^{j}\int_{0}^{1} \mathcal{L}(v_j)dydx. \tag{2.49}$$

From now on, given $u_j \in K_j$, with $j > 1$, let's fix v_j as above. Then, we have the following result.

Lemma 2.11 *There exists $L > 0$ such that for $j > L + \frac{1}{4}$, if $u_j \in K_j$ we must have*

$$|u_j(x, y) - \alpha| \le \delta_\alpha \quad \forall (x, y) \in T_j \text{ with } x \in \left(L, j - \frac{1}{4}\right),$$

where $\delta_\alpha > 0$ was given in (V_3).

Proof. Arguing by contradiction, assume that there is a sequence of indices $(j_n) \subset (0, +\infty)$ with $j_n \to +\infty$ such that for each j_n there exists $u_{j_n} \in K_{j_n}$ and points

$$(x_n, y_n) \in \left(0, j_n - \frac{1}{4}\right) \times (j_n, j_n + 1)$$

with $x_n \to +\infty$ satisfying

$$\alpha - \delta_\alpha > u_{j_n}(x_n, y_n) > 0. \tag{2.50}$$

Given $j > 1$, we fix the rectangles

$$Q_j = \left(-j + \frac{1}{8}, j - \frac{1}{8}\right) \times (j - 1, j + 2) \quad \text{and} \quad \tilde{Q}_j = \left(-j + \frac{1}{4}, j - \frac{1}{4}\right) \times (j, j + 1).$$

Now, taking $\eta_0 \in (0, \frac{1}{32})$ and $(x, y) \in \tilde{Q}_j$, it is clear that

$$B_{\eta_0}(x, y) \subset B_{2\eta_0}(x, y) \subset Q_j.$$

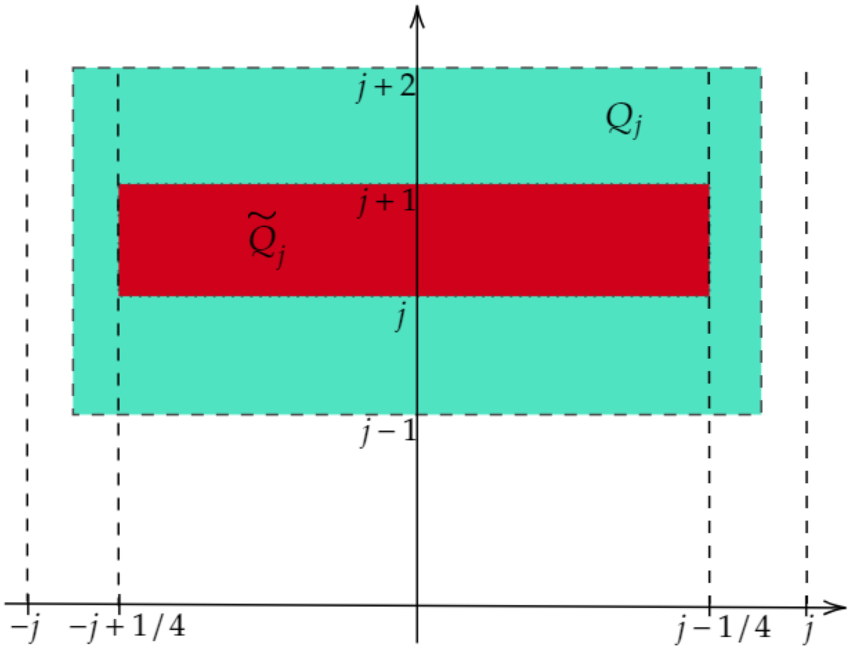

Figure 2.2: Sets Q_j and $\tilde{Q}_j$.

Defining the operator

$$B(x,y) = A(x,y)V'(v_j(x,y)) \text{ for } (x,y) \in Q_j,$$

there exists $\Lambda_1 > 0$ such that $|B(x,y)| \leq \Lambda_1$ for every $(x,y) \in Q_j$. So, since v_j is a weak solution of the equation

$$\Delta_\Phi w + B(x,y) = 0 \quad \text{in} \quad Q_j$$

with $\|v_j\|_{L^\infty(Q_j)} \leq \alpha$, it follows from [67, Theorem 1.7] that there is $C > 0$ such that

$$\|v_j\|_{C^1(\tilde{Q}_j)} \leq C \quad \forall j \in \mathbb{N}, \tag{2.51}$$

and so,

$$\|v_j\|_{C^1(B_{\eta_0}(x,y))} \leq C \quad \forall (x,y) \in \tilde{Q}_j.$$

From this, taking $\eta < \eta_0$ such that $C\eta < \delta_\alpha/2$ and invoking the Mean Value Theorem, we arrive at

$$|v_{j_n}(x,y) - v_{j_n}(x_n,y_n)| \leq C\eta < \frac{\delta_\alpha}{2} \quad \forall (x,y) \in B_\eta(x_n,y_n) \quad \text{and} \quad \forall n \in \mathbb{N}. \tag{2.52}$$

Thereby, from (2.50) and (2.52),

$$|\alpha - u_{j_n}(x,y)| \geq \frac{\delta_\alpha}{2} \quad \forall (x,y) \in B_\eta(x_n,y_n) \cap \tilde{Q}_{j_n},$$

leading to

$$\|\alpha - u_{j_n}(x,\cdot)\|_{L^\infty(j_n,j_n+1)} > \frac{\delta_\alpha}{2} \quad \forall x \in (x_n - \eta/2, x_n).$$

As the constants of embedding $W^{1,\Phi}(j_n, j_n+1) \hookrightarrow L^\infty(j_n, j_n+1)$ are independent of $n \in \mathbb{N}$, because such constants depend only on the length of the intervals (j_n, j_n+1), then there exists $r > 0$ such that

$$\|\alpha - u_{j_n}(x,\cdot)\|_{W^{1,\Phi}(j_n,j_n+1)} \geq r \quad \forall x \in (x_n - \eta/2, x_n).$$

Now, setting

$$\tilde{u}_{j_n}(x,y) = u_{j_n}(x, y + j_n), \quad \text{for} \quad (x,y) \in (-j_n, j_n) \times (0,1),$$

we obtain

$$\|\alpha - \tilde{u}_{j_n}(x,\cdot)\|_{W^{1,\Phi}(0,1)} \geq r \quad \forall x \in (x_n - \eta/2, x_n).$$

From Lemma 2.2, there exists $\mu_r > 0$ satisfying

$$\int_{x_n-\eta/2}^{x_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx \geq (2\mu_r)^{\frac{m}{m-1}} \frac{\eta}{2} \quad \forall n \in \mathbb{N}. \tag{2.53}$$

On the other hand, for each $n \in \mathbb{N}$ it is well known that

$$I_{0,j_n}(\tilde{u}_{j_n}) = \int_{-j_n}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx = \int_{-j_n}^{j_n} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n}) dy dx \leq I(u_{j_n}) = c_{j_n} < c_\Phi(\alpha).$$

Using the fact that $j_n \to +\infty$, it follows from the Lemma 2.5 that there are $x_+ > 0$ and $n_0 \in \mathbb{N}$ satisfying

$$\int_{x_+}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx < \frac{3}{2}\lambda_\delta \quad \forall n \geq n_0.$$

Next, we take λ_δ arbitrarily small of such way that

$$\int_{x_+}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx < (2\mu_r)^{\frac{m}{m-1}} \frac{\eta}{2} \quad \forall n \geq n_0.$$

Therefore, as $x_n \to +\infty$, increasing n_0 if necessary, we find

$$\int_{x_n-\eta/2}^{x_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx \leq \int_{x_+}^{x_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx \leq \int_{x_+}^{j_n} \int_0^1 \mathcal{L}(\tilde{u}_{j_n}) dy dx < (2\mu_r)^{\frac{m}{m-1}} \frac{\eta}{2},$$

for any $n \geq n_0$, which contradicts (2.53), and the proof is over. ■

In what follows, our goal is to get an estimate from above of the exponential type for $c_\Phi(\alpha) - c_L$. In order to do that, we fix the real function

$$\zeta(x) = \delta_\alpha \frac{\cosh\left(a\left(x - \frac{j-\frac{1}{4}+L}{2}\right)\right)}{\cosh\left(a\frac{j-\frac{1}{4}-L}{2}\right)} \quad \text{for} \quad x \in \mathbb{R},$$

where $L > 0$ was given in the Lemma 2.11 and for some constant $a > 0$ that will chose later. A simple computation provides $\zeta''(x) = a^2\zeta(x)$ for all $x \in \mathbb{R}$, which together with (ϕ_2) allow us to use the same idea found in Lemma 1.21 to show that

$$(\phi(|\zeta'(x)|)\zeta'(x))' \leq ma^2\phi(|\zeta'(x)|)\zeta(x) \ \ \forall x \in \mathbb{R}.$$

Since $|\zeta'(x)| \leq a\zeta(x)$ for each $x \in \mathbb{R}$, taking $a < \omega_2$ and using (ϕ_4), we get

$$\phi(|\zeta'(x)|) \leq \phi(\omega_2\zeta(x)) \ \text{ for every } x \ \in \mathbb{R},$$

and so,

$$-(\phi(|\zeta'(x)|)\zeta'(x))' + ma\phi(\omega_2\zeta(x))\omega_2\zeta(x) \geq 0 \ \ \forall x \in \mathbb{R}.$$

Therefore, if we define $w(x,y) = \zeta(x)$ for each $(x,y) \in \mathbb{R}^2$, then

$$-\Delta_\Phi w + ma\phi(\omega_2 w)\omega_2 w \geq 0 \ \text{ in } \ \mathbb{R}^2. \tag{2.54}$$

Now, fixing $u_j \in K_j$ satisfying Lemma 2.11 and setting the function

$$\nu(x,y) = \alpha - v_j(x,y), \ \ (x,y) \in (-j,j) \times \mathbb{R},$$

it follows from Lemma 2.11 that $0 < v_j(x,y) < \alpha$ for any $x \in (0,j)$, and so, since v_j is a periodic function in the variable y and continuous, there exists $b_j > 0$ verifying

$$0 < b_j \leq v_j(x,y) < \alpha \ \ \forall (x,y) \in \left[L, j - \frac{1}{4}\right] \times \mathbb{R}.$$

According to (V_4),

$$V'(v_j) \leq -\frac{\omega_1 b_j}{\omega_2}\phi(\omega_2\nu)(\omega_2\nu) \quad \text{in} \quad \left(L, j - \frac{1}{4}\right) \times \mathbb{R}. \tag{2.55}$$

In what follows, we take $a > 0$ sufficiently small such that $ma < \underline{A}\dfrac{b_j\omega_1}{\omega_2}$.

Claim 2.1 *Let $j_0 \in \mathbb{N}$ and $\psi \in X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0))$ with $\psi \geq 0$, where*

$$X_*^{1,\Phi}(\mathbb{R} \times (-j_0, j_0)) = \left\{ u \in W^{1,\Phi}(\mathbb{R} \times (-j_0, j_0)) \ \textit{with}\ u(x,y) = 0 \ \textit{for}\ x \notin \left(L, j - \frac{1}{4}\right) \right\},$$

then

$$\int_{\mathbb{R}} \int_{-j_0}^{j_0} (\phi(|\nabla \nu|)\nabla \nu \nabla \psi + ma\phi(\omega_2\nu)\omega_2\nu\psi)\, dydx \leq 0.$$

In fact, from (2.55) it may be concluded that

$$\begin{aligned}
\int_{\mathbb{R}}\int_{-j_0}^{j_0}(\phi(|\nabla\nu|)\nabla\nu\nabla\psi+ma\phi(\omega_2\nu)\omega_2\nu\psi)dydx \\
&= \int_L^{j-\frac{1}{4}}\int_{-j_0}^{j_0}(-\phi(|\nabla v_j|)\nabla v_j\nabla\psi+ma\phi(\omega_2\nu)\omega_2\nu\psi)dydx \\
&= \int_L^{j-\frac{1}{4}}\int_{-j_0}^{j_0}(A(x,y)V'(v_j)\psi+ma\phi(\omega_2\nu)\omega_2\nu\psi)dydx \\
&\leq \int_L^{j-\frac{1}{4}}\int_{-j_0}^{j_0}(A(x,y)V'(v_j)\psi+A(x,y)\frac{\omega_1 b_j}{\omega_2}\phi(\omega_2\nu)\omega_2\nu\psi)dydx \\
&\leq \int_L^{j-\frac{1}{4}}\int_{-j_0}^{j_0}(A(x,y)V'(v_j)\psi-A(x,y)V'(v_j)\psi)dydx = 0,
\end{aligned}$$

proving the Claim 2.1.

On the other hand, the definitions of ν and w together with Lemma 2.11 ensure that

$$\nu(x,y)\leq w(x,y) \ \text{ on } \ \left\{L, j-\frac{1}{4}\right\}\times\mathbb{R}. \tag{2.56}$$

Lemma 2.12 *It holds that $\nu(x,y)\leq w(x,y)$ in $(L,j-1/4)\times\mathbb{R}$.*

Proof. Suppose by contradiction that the lemma is false. Then, we can find $(x_1,y_1)\in(L,j-1/4)\times\mathbb{R}$ such that $\nu(x_1,y_1)>w(x_1,y_1)$. Let $j_0\in\mathbb{N}$ such that $(x_1,y_1)\in(L,j-1/4)\times(-j_0,j_0)$. Now, from (2.56) the function $\psi_*:\mathbb{R}\times(-j_0,j_0)\to\mathbb{R}$ given by

$$\psi_*(x,y)=\begin{cases}(\nu-w)^+(x,y), & \text{if } \ x\in(L,j-1/4)\\ 0, & \text{if } \ x\notin(L,j-1/4)\end{cases}$$

is well defined. Moreover, $\psi_*\in X_*^{1,\Phi}(\mathbb{R}\times(-j_0,j_0))$ and ψ_* is a non-negative continuous function. Therefore, according to Claim 2.1 and (2.54),

$$\int_{\mathbb{R}}\int_{-j_0}^{j_0}\left(\phi(|\nabla w|)\nabla w\nabla\psi_*+ma\phi(\omega_2 w)\omega_2 w\psi_*\right)dydx\geq 0$$

and

$$\int_{\mathbb{R}}\int_{-j_0}^{j_0}\left(\phi(|\nabla\nu|)\nabla\nu\nabla\psi_*+ma\phi(\omega_2\nu)\omega_2\nu\psi_*\right)dydx\leq 0,$$

which leads to

$$\iint_P\big((\phi(|\nabla\nu|)\nabla\nu-\phi(|\nabla w|)\nabla w)\nabla(\nu-w)+\kappa a^2(\phi(\nu)\nu-\phi(w)w)(\nu-w)\big)\,dydx\leq 0,$$

where

$$P=\{(x,y)\in\mathbb{R}\times(-j_0,j_0):\nu(x,y)\geq w(x,y)\}.$$

From Lemma A.8-(c), $\nu(x,y) \le w(x,y)$ for all $(x,y) \in (L, j-1/4) \times (-j_0, j_0)$, which is impossible. ■

Now, we are ready to prove an exponential estimate from above to $c_\Phi(\alpha) - c_j$.

Lemma 2.13 *There are $\theta_1, \theta_2 > 0$ such that*

$$0 < c_\Phi(\alpha) - c_j \le \theta_1 e^{-\theta_2 j} \quad \forall j \in \mathbb{N} \cup \{0\}.$$

In particular, $c_j \to c_\Phi(\alpha)$ as $j \to +\infty$.

Proof. First of all, we note that by Lemma 2.12,

$$|v_j(x,y) - \alpha| \le \delta_\alpha \frac{\cosh\left(a\left(x - \frac{j-\frac{1}{4}+L}{2}\right)\right)}{\cosh\left(a\frac{j-\frac{1}{4}-L}{2}\right)} \quad \forall (x,y) \in \left(L, j-\frac{1}{4}\right) \times \mathbb{R}.$$

Choosing $x_+ = \frac{j-\frac{1}{4}+L}{2}$, we have that

$$|v_j(x_+,y) - \alpha| \le \frac{\delta_\alpha}{\cosh\left(a\frac{j-\frac{1}{4}-L}{2}\right)} \quad \forall y \in \mathbb{R},$$

which implies

$$|v_j(x_+,y) - \alpha| \le 2\delta_\alpha e^{-\frac{a}{2}(j-\frac{1}{4}-L)} := \rho_j \;\text{ and }\; \Phi(|v_j(x_+,y)-\alpha|) \le \Phi(\rho_j) \;\; \forall y \in \mathbb{R}. \quad (2.57)$$

In the sequel, we fix j sufficiently large such that $x_+ + \rho_j \le j$ and

$$\tilde{v}_j(x,y) = \begin{cases} v_j(x,y), & \text{if } 0 \le x \le x_+ \text{ and } y \in \mathbb{R} \\ v_j(x_+,y) + \frac{1}{\rho_j}(x-x_+)(\alpha - v_j(x_+,y)), & \text{if } x_+ \le x \le x_+ + \rho_j \text{ and } y \in \mathbb{R} \\ \alpha, & \text{if } x_+ + \rho_j \le x \text{ and } y \in \mathbb{R} \\ -\tilde{v}_j(-x,y), & \text{if } x \le 0 \text{ and } y \in \mathbb{R}. \end{cases}$$

Hereafter, let us identify $\tilde{v}_j|_{\Omega_0}$ with the $\tilde{v}_j$ itself, and consequently $\tilde{v} \in E_\Phi(\alpha)$ and $c_\Phi(\alpha) \le I(\tilde{v})$. Now let us take a look at some important estimates for the end of the proof.

Claim 2.2 $|\partial_x \tilde{v}_j| \le 1$ *in* $(x_+, x_+ + \rho_j) \times \mathbb{R}$.

Indeed, note that $\partial_x \tilde{v}_j(x,y) = \frac{1}{\rho_j}(\alpha - v_j(x_+,y))$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$. From (2.57),

$$|\partial_x \tilde{v}_j(x,y)| \le \frac{1}{\rho_j}|\alpha - v_j(x_+,y)| \le 1, \quad \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

Claim 2.3 $|\partial_y \tilde{v}_j| \le 2C$ *in* $(x_+, x_+ + \rho_j) \times \mathbb{R}$*, where* $C > 0$ *was given in* (2.51).

By definition of $\tilde{v}_j$, $|\partial_y \tilde{v}_j(x,y)| \leq 2|\partial_y v_j(x_+,y)|$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$. Now, the definition of v_j combined with (2.51) leads to

$$|\partial_y \tilde{v}_j(x,y)| \leq 2C \ \ \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

Claim 2.4 *$A(x,y)V(\tilde{v}_j) \leq \overline{A}\overline{w}\Phi(\rho_j)$ in $(x_+, x_+ + \rho_j) \times \mathbb{R}$.*

From (1.11),

$$A(x,y)V(\tilde{v}_j(x,y)) \leq \overline{A}\overline{w}\Phi(|\tilde{v}_j(x,y) - \alpha|) \ \ \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R}.$$

Now, the definition of $\tilde{v}_j$ together with (2.57) yields

$$A(x,y)V(\tilde{v}_j(x,y)) \leq \overline{A}\overline{w}\Phi(|v_j(x_+,y) - \alpha|) \leq \overline{A}\overline{w}\Phi(\rho_j) \ \ \forall (x,y) \in (x_+, x_+ + \rho_j) \times \mathbb{R},$$

proving the Claim 2.4.

According to Claims 2.2, 2.3 and 2.4,

$$\begin{aligned}\int_{x_+}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx &\leq \int_{x_+}^{x_+ + \rho_j} \int_0^1 \left(2^m \Phi(|\partial_x \tilde{v}_j|) + 2^m \Phi(|\partial_y \tilde{v}_j|) + A(x,y)V(\tilde{v}_j)\right) dy dx \\ &\leq 2^m \Phi(1)\rho_j + 2^m \Phi(2C)\rho_j + \overline{A}\overline{w}\Phi(\rho_j)\rho_j.\end{aligned}$$

Now, since $\rho_j \to 0$ as $j \to +\infty$, there is a constant $\tilde{M} > 0$, independent of j and $\tilde{v}_j$ such that

$$\int_{x_+}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx \leq \tilde{M}\rho_j,$$

and so, by (2.49),

$$\begin{aligned}c_\Phi(\alpha) \leq I(\tilde{v}_j) = \int_{-x_+ - \rho_j}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx &\leq \int_{-j}^{j} \int_0^1 \mathcal{L}(v_j) dy dx + 2 \int_{x_+}^{x_+ + \rho_j} \int_0^1 \mathcal{L}(\tilde{v}_j) dy dx \\ &\leq \int_{-j}^{j} \int_{j+1}^{j} \mathcal{L}(u_j) dy dx + 2\tilde{M}\rho_j \leq I_j(u_j) + 2\tilde{M}\rho_j = c_j + 2\tilde{M}\rho_j,\end{aligned}$$

that is,

$$0 < c_\Phi(\alpha) - c_j \leq 4\tilde{M}\delta_\alpha e^{-\frac{a}{2}\left(j - \frac{1}{4} - L\right)},$$

for j sufficiently large. Therefore, it is possible to find real numbers $\theta_1, \theta_2 > 0$ satisfying precisely the assertion of the lemma. ■

Next, we establish further compactness property concerning the functionals I_{j_n}.

Lemma 2.14 *Let $j_n \to +\infty$ and $u_{j_n} \in E_{j_n}$ such that $I_{j_n}(u_{j_n}) - c_{j_n} \to 0$ as $n \to +\infty$. Then, there exists $u_0 \in K_\Phi(\alpha)$ verifying*

$$\|u_{j_n} - \tau_{j_n} u_0\|_{W^{1,\Phi}(T_{j_n})} \to 0 \ \text{as} \ n \to +\infty,$$

where $\tau_j u_0(x,y) = u_0(x, y - j)$ for all $j \in \mathbb{N}$.

Proof. Setting

$$w_{j_n}(x,y) = u_{j_n}(x, y + j_n), \text{ for } (x,y) \in (-j_n, j_n) \times [0,1],$$

it is easily seen that $I_{0,j_n}(w_{j_n}) \leq I_{j_n}(u_{j_n})$. Since $c_{j_n} < c_\Phi(\alpha)$ for all $n \in \mathbb{N}$ and $I_{j_n}(u_{j_n}) = c_{j_n} + o_n(1)$,

$$I_{0,j_n}(w_{j_n}) < c_\Phi(\alpha) + o_n(1) \quad \forall n \in \mathbb{N}. \tag{2.58}$$

We claim that for each $n \in \mathbb{N}$ there exists $x_{+,n} \in (\frac{j_n}{2}, j_n)$ satisfying

$$\alpha_n := \|w_{j_n}(x_{+,n}, \cdot) - \alpha\|_{W^{1,\Phi}(0,1)} \to 0 \text{ as } n \to +\infty.$$

Indeed, if the claim is not true, then there is $r > 0$ such that, for some subsequence,

$$\|w_{j_n}(x, \cdot) - \alpha\|_{W^{1,\Phi}(0,1)} \geq r \quad \forall x \in (\frac{j_n}{2}, j_n) \quad \text{and} \quad \forall n \in \mathbb{N}.$$

Invoking Lemma 2.2, there exists $\mu_r > 0$ verifying

$$I_{0,j_n}(w_{j_n}) \geq \int_{\frac{j_n}{2}}^{j_n} \int_0^1 \mathcal{L}(w_{j_n}) dy dx \geq (2\mu_r)^{\frac{m}{m-1}} \frac{j_n}{2}.$$

Taking j_n sufficiently large we have $I_{0,j_n}(w_{j_n}) > c_\Phi(\alpha) + o_n(1)$, contrary to (2.58), and the claim is proved. Without loss of generality, we can assume that $\alpha_n > 0$ for any $n \in \mathbb{N}$, and so we define the function $\tilde{w}_{j_n} : \Omega_0 \to \mathbb{R}$ by

$$\tilde{w}_{j_n}(x,y) = \begin{cases} w_{j_n}(x,y), & \text{if } \ 0 \leq x \leq x_{+,n} \\ w_{j_n}(x_{+,n}, y) + \frac{1}{\alpha_n}(x - x_{+,n})(\alpha - w_{j_n}(x_{+,n}, y)), & \text{if } \ x_{+,n} \leq x \leq x_{+,n} + \alpha_n \\ \alpha, & \text{if } \ x_{+,n} + \alpha_n \leq x \\ -\tilde{w}_{j_n}(-x, y), & \text{if } \ x \leq 0. \end{cases}$$

Thus, $\tilde{w}_{j_n} \in E_\Phi(\alpha)$ and

$$c_\Phi(\alpha) \leq I(\tilde{w}_{j_n}) = I_{0,x_{+,n}}(w_{j_n}) + 2\int_{x_{+,n}}^{x_{+,n}+\alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n}) dy dx. \tag{2.59}$$

On the other hand, from (2.21),

$$|\partial_x \tilde{w}_{j_n}| \leq \Lambda \; in \; (x_{+,n}, x_{+,n} + \alpha_n) \times (0,1) \quad \forall n \in \mathbb{N}. \tag{2.60}$$

Indeed, using (2.21), for each $(x,y) \in (x_{+,n}, x_{+,n} + \alpha_n) \times (0,1)$ we have

$$|\partial_x \tilde{w}_{j_n}(x,y)| = \frac{1}{\alpha_n}|\alpha - w_{j_n}(x_{+,n}, y)| \leq \frac{1}{\alpha_n}\|1 - w_{j_n}(x_{+,n}, \cdot)\|_{L^\infty(0,1)} \leq \Lambda \quad \forall n \in \mathbb{N}.$$

Moreover, an easy computation shows that

$$|\partial_y \tilde{w}_{j_n}(x,y)| \leq 2|\partial_y w_{j_n}(x_{+,n},y)| \quad \forall (x,y) \in (x_{+,n}, x_{+,n}+\alpha_n) \times (0,1). \tag{2.61}$$

Now, since $\alpha_n \to 0$ we can take n sufficiently large such that $\alpha_n < 1$, and for such values of n, the convexity of Φ ensures that

$$\begin{aligned}\int_0^1 \Phi(|\partial_y w_{j_n}(x_{+,n},y)|)dy &= \int_0^1 \Phi\left(\|\partial_y w_{j_n}(x_{+,n},\cdot)\|_{L^\Phi(0,1)} \frac{|\partial_y w_{j_n}(x_{+,n},y)|}{\|\partial_y w_{j_n}(x_{+,n},\cdot)\|_{L^\Phi(0,1)}}\right) dy \\ &\leq \|\partial_y w_{j_n}(x_{+,n},\cdot)\|_{L^\Phi(0,1)} \int_0^1 \Phi\left(\frac{|\partial_y w_{j_n}(x_{+,n},y)|}{\|\partial_y w_{j_n}(x_{+,n},\cdot)\|_{L^\Phi(0,1)}}\right) dy \\ &\leq \alpha_n,\end{aligned}$$

that is,

$$\int_0^1 \Phi(|\partial_y w_{j_n}(x_{+,n},y)|)dy \leq \alpha_n. \tag{2.62}$$

A similar argument works to prove that

$$A(x,y)V(\tilde{w}_{j_n}) \leq \overline{A}\overline{w}\Phi(|\alpha - w_{j_n}(x_{+,n},y)|) \ \text{ in } \ (x_{+,n}, x_{+,n}+\alpha_n) \times (0,1)$$

and

$$\int_0^1 \Phi(|\alpha - w_{j_n}(x_{+,n},y)|)dy \leq \alpha_n. \tag{2.63}$$

Therefore, we conclude from (2.60)-(2.63) that

$$\int_{x_{+,n}}^{x_{+,n}+\alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n})dydx \to 0 \ \text{ as } \ n \to +\infty. \tag{2.64}$$

According to (2.58), (2.59) and (2.64), $I(\tilde{w}_{j_n}) \to c_\Phi(\alpha)$. By Proposition 2.1, there exists $u_0 \in K_\Phi(\alpha)$ such that, along a subsequence,

$$\|\tilde{w}_{j_n} - u_0\|_{W^{1,\Phi}(\Omega_0)} \to 0.$$

As $\tilde{w}_{j_n}(x,y) = u_{j_n}(x, y+j_n)$ for $|x| \leq x_{+,n}$ and $y \in [0,1]$, we deduce

$$\|u_{j_n} - \tau_{j_n} u_0\|_{W^{1,\Phi}([-x_{+,n},x_{+,n}]\times[j_n,j_n+1])} \to 0 \ \text{ as } n \to +\infty. \tag{2.65}$$

By definition of $\tilde{w}_{j_n}$,

$$I(\tilde{w}_{j_n}) = \int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n})dydx + 2\int_{-x_{+,n}}^{x_{+,n}+\alpha_n} \int_0^1 \mathcal{L}(\tilde{w}_{j_n})dydx$$

that combines with (2.64) to provide

$$\int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n})dydx \to c_\Phi(\alpha). \tag{2.66}$$

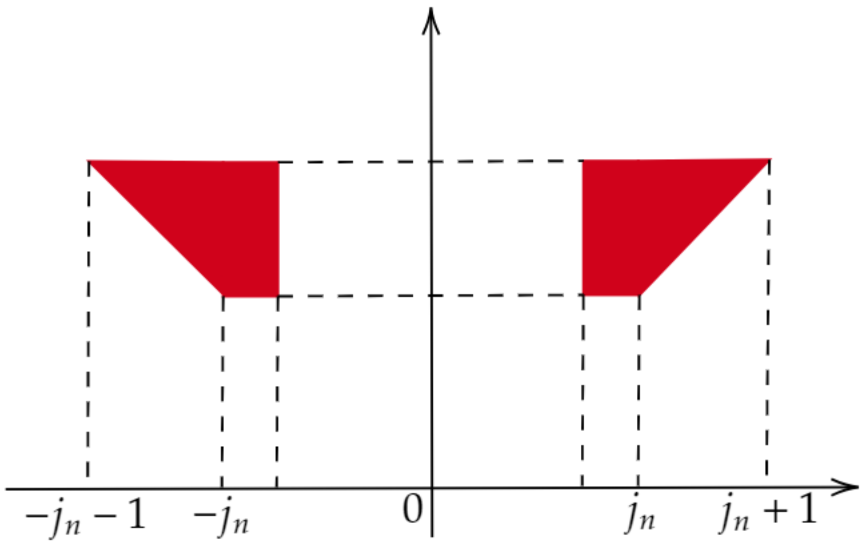

Figure 2.3: Geometric illustration of $R_{+,n}$.

Setting $R_{+,n} = T_{j_n} \setminus ([-x_{+,n}, x_{+,n}] \times [j_n, j_n + 1])$, we have

$$\iint_{R_{+,n}} \mathcal{L}(u_{j_n})dydx = I_{j_n}(u_{j_n}) - \int_{-x_{+,n}}^{x_{+,n}} \int_{j_n}^{j_n+1} \mathcal{L}(u_{j_n})dydx.$$

Now, the estimate $I_{j_n}(u_{j_n}) = c_{j_n} + o_n(1)$ together with (2.66) ensures that

$$\iint_{R_{+,n}} \mathcal{L}(u_{j_n})dydx \to 0. \tag{2.67}$$

On the other hand, from (1.11),

$$\begin{aligned}\iint_{R_{+,n}} (\Phi(|\nabla u_{j_n}|) + \Phi(|u_{j_n} - \alpha|))dydx &\leq \iint_{R_{+,n}} \left(\Phi(|\nabla u_{j_n}|) + \frac{1}{\underline{w}\ \underline{A}} A(x,y)V(u_{j_n})\right) dydx \\ &\leq \max\left\{1, \frac{1}{\underline{w}\ \underline{A}}\right\} \iint_{R_{+,n}} \mathcal{L}(u_{j_n})dydx.\end{aligned} \tag{2.68}$$

This combined with (2.67) leads to

$$\|u_{j_n} - \alpha\|_{W^{1,\Phi}(R_{+,n})} \to 0. \tag{2.69}$$

Finally, by Lemma 2.6, we also have that $\Phi(|\nabla u_0|), \Phi(|u_0 - \alpha|) \in L^1(\Omega_0)$, and so,

$$\iint_{R_{+,n}} \Phi(|\nabla \tau_{j_n} u_0|)dydx \to 0 \text{ and } \iint_{R_{+,n}} \Phi(|\tau_{j_n} u_0 - \alpha|)dydx \to 0.$$

As $\Phi \in \Delta_2$, these limits guarantee that

$$\|\tau_{j_n} u_0 - \alpha\|_{W^{1,\Phi}(R_{+,n})} \to 0. \tag{2.70}$$

Now the lemma follows from (2.65), (2.69) and (2.70). ■

2.2 Saddle solutions on $\mathbb{R}^2$

In this last section we will collect the results obtained previously to prove our main result (see for a moment the Theorem 2.4). The proof is constructive and makes use of variational arguments.

2.2.1 Construction of solution on a infinite triangular set

Let's study the existence of a solution to the following equation

$$-\Delta_\Phi w + A(x,y)V'(w) = 0 \quad \text{in} \quad \Gamma, \tag{2.71}$$

where Γ is the following triangular set on $\mathbb{R}^2$

$$\Gamma = \bigcup_{j=0}^{\infty} T_j.$$

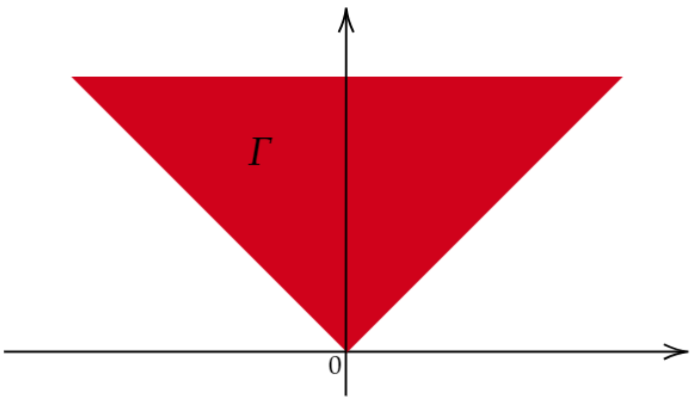

Figure 2.4: Geometric illustration of Γ.

Setting

$$E_\infty = \left\{ w \in W^{1,\Phi}_{\text{loc}}(\Gamma) : \; 0 \le w(x,y) \le \alpha \text{ for } x \ge 0 \text{ and } w \text{ is odd in } x \right\},$$

we infer that if $w \in E_\infty$ then $w|_{T_j} \in E_j$ for every $j \in \mathbb{N} \cup \{0\}$. Hereafter, let us identify $w|_{T_j}$ with w itself. With everything, we may define the functional $J : W^{1,\Phi}_{\text{loc}}(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ by

$$J(w) = \sum_{j=0}^{\infty} \left(I_j(w) - c_j \right).$$

Clearly, J is bounded from below on E_∞. Here, we would like point out that there exists $u \in E_\infty$ such that $J(u) < +\infty$. Indeed, from Theorem 2.2, there exists a function $u_* : \mathbb{R}^2 \to \mathbb{R}$ such that $u_* \in E_\infty$ with $I(u_*) = c_\Phi(\alpha)$. Invoking Lemma 2.13,

$$I_j(u_*) - c_j \leq I(u_*) - c_j = c_\Phi(\alpha) - c_j \leq \theta_1 e^{-\theta_2 j} \quad \forall j \in \mathbb{N} \cup \{0\}.$$

Thus,

$$J(u_*) = \sum_{j=0}^{\infty} (I_j(u_*) - c_j) \leq \theta_1 \sum_{j=0}^{\infty} e^{-\theta_2 j} < +\infty.$$

Consequently,

$$d_\infty := \inf_{w \in E_\infty} J(w)$$

is well defined.

In what follows, if $(u_n) \subset W^{1,\Phi}_{\text{loc}}(\Gamma)$ and $u \in W^{1,\Phi}_{\text{loc}}(\Gamma)$, we write $u_n \rightharpoonup u$ in $W^{1,\Phi}_{\text{loc}}(\Gamma)$ to denote that $u_n \rightharpoonup u$ in $W^{1,\Phi}(\Omega)$ for any Ω relatively compact in Γ. Here we would like point out that the same arguments found in Lemma 1.25 work to show that

$$u_n \rightharpoonup u \text{ in } W^{1,\Phi}_{\text{loc}}(\Gamma) \Rightarrow J(u) \leq \liminf_{n \to +\infty} J(u_n).$$

From this, we are ready to show the following result.

Lemma 2.15 *There exists $\overline{u} \in E_\infty$ such that $J(\overline{u}) = d_\infty$.*

Proof. Let $(w_n) \subset E_\infty$ be a minimizing sequence for J. Then there is $M > 0$ satisfying $J(w_n) \leq M$ for every $n \in \mathbb{N}$. Now, for each $k \in \mathbb{N}$ we define

$$\Gamma_k = \Gamma \cap \{y < k\}.$$

Consequently,

$$\iint_{\Gamma_k} \Phi(|\nabla w_n|) dy dx \leq \iint_{\Gamma_k} \mathcal{L}(w_n) dy dx \leq \sum_{j=0}^{k} I_j(w_n) \leq J(w_n) + \sum_{j=0}^{k} c_j \leq M + (k+1) c_\Phi(\alpha)$$

that together with $\|w_n\|_{L^\infty(\Gamma)} \leq \alpha$ ensures that (w_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Gamma)$. By a classical diagonal argument, for some subsequence, there exists $\overline{u} \in W^{1,\Phi}_{\text{loc}}(\Gamma)$ such that

$$w_n \rightharpoonup \overline{u} \;\text{ in }\; W^{1,\Phi}_{\text{loc}}(\Gamma) \;\text{ and }\; w_n(x,y) \to \overline{u}(x,y) \;\text{ a.e. in }\; \Gamma.$$

Next, by pointwise convergence, $\overline{u}(x,y) = -\overline{u}(-x,y)$ for almost every $(x,y) \in \Gamma$ and $0 \leq \overline{u}(x,y) \leq \alpha$ for almost every $(x,y) \in \Gamma$ with $x \geq 0$, that is, $\overline{u} \in E_\infty$. Moreover, $J(\overline{u}) = d_\infty$, which completes the proof. ■

Setting

$$K_\infty = \{w \in E_\infty : J(w) = d_\infty\},$$

we have by the previous lemma that $K_\infty \neq \emptyset$. Repeating the arguments used in the proof of Theorem 2.1, it is possible to prove the following result.

Lemma 2.16 *If $\overline{u} \in K_\infty$, then for any $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$ we have*

$$\iint_\Gamma (\phi(|\nabla \overline{u}|)\nabla \overline{u} \nabla \psi + A(x,y)V'(\overline{u})\psi)\, dydx = 0.$$

As a consequence of Lemma 2.16, if $\overline{u} \in K_\infty$ then $\overline{u}$ is weak solution of (2.71). Elliptic regularity theory yields that $\overline{u}$ is a solution in $C^{1,\beta}_{\text{loc}}(\Gamma)$ for some $\beta \in (0,1)$. Furthermore, arguing as in the proof of Theorem 2.2 we also have that

$$0 < \overline{u}(x,y) < \alpha \quad \text{for} \quad (x,y) \in \Gamma \quad \text{with} \quad x > 0.$$

2.2.2 Existence of saddle-type solution

Finally, in this subsection, we will show the existence of a saddle type solution for equation (2.1). The saddle solution will be obtained by recursive reflections of $\overline{u}$ on the faces of the triangular set Γ.

Theorem 2.4 *Assume (ϕ_1)-(ϕ_4), $V \in C^1(\mathbb{R},\mathbb{R})$, ($V_1$)-($V_4$) and ($A_1$)-($A_4$). Then, there is $v \in C^{1,\beta}_{loc}(\mathbb{R}^2)$ for some $\beta \in (0,1)$ such that v is a weak solution of (2.1) that verifies the following*

(a) $0 < v(x,y) < \alpha$ *on the fist quadrant in* $\mathbb{R}^2$,

(b) $v(x,y) = -v(-x,y) = -v(x,-y)$ *for all* $(x,y) \in \mathbb{R}^2$,

(c) $v(x,y) = v(y,x)$ *for any* $(x,y) \in \mathbb{R}^2$,

(d) There is $u_0 \in K_\Phi(\alpha)$ *such that* $\|v - \tau_j u_0\|_{L^\infty(\mathbb{R}\times[j,j+1])} \to 0$ *as* $j \to +\infty$,

where $\tau_j u_0(x,y) = u_0(x, y-j)$ *for all* $(x,y) \in \mathbb{R}^2$.

Proof. The existence of saddle-type solution v will be done via a recursive reflection of the function $\overline{u} : \Gamma \to \mathbb{R}$ given by Lemma 2.15. First of all, let us consider the rotation matrix

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

that is,

$$T(x,y) = (y,-x) \quad \text{for any} \quad (x,y) \in \mathbb{R}^2.$$

Setting $\Gamma^0 = \Gamma$, we designate $\Gamma^i = T^i(\Gamma)$ for $i = 0,1,2,3$, i.e., Γ^i is the $i\frac{\pi}{2}$-rotated de Γ. Consequently,

$$\mathbb{R}^2 = \bigcup_{i=0}^{3} \Gamma^i, \quad T^{-i}(\Gamma^i) = \Gamma \quad \text{and} \quad int(\Gamma^i) \cap int(\Gamma^j) = \emptyset \quad \text{for} \quad i \neq j.$$

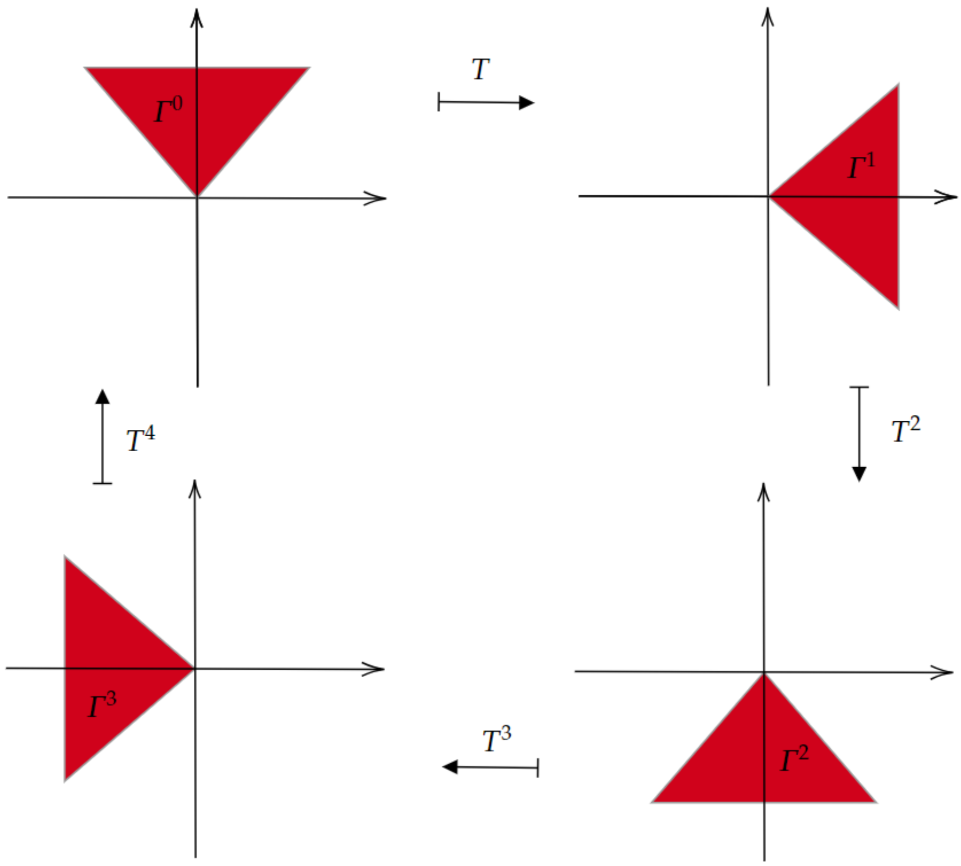

Figure 2.5: Geometric illustration of sets Γ^i.

Finally, we define the function $v : \mathbb{R}^2 \to \mathbb{R}$ by

$$v(x,y) = (-1)^i \overline{u}\left(T^{-i}(x,y)\right) \quad \forall (x,y) \in \Gamma^i.$$

Note that $v|_{\Gamma^i}$ is the reflection of $v|_{\Gamma^{i-1}}$ with respect to the axis separating Γ^{i-1} from Γ^i, for any $i = 1,2,3$. From the properties of the reflection operator, $v \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}^2)$. Now, we note that if $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$, then $\psi \circ T^i \in W^{1,\Phi}(\mathbb{R}^2)$ and has compact support in $\mathbb{R}^2$, because T^i is a linear operator. Moreover, from (A_4),

$$A\left(T^i(x,y)\right) = A(x,y) \quad \forall (x,y) \in \mathbb{R}^2.$$

Thus, invoking Lemma 2.16,

$$\int_{\Gamma^i}(\phi(|\nabla v|)\nabla v\nabla\psi + A(x,y)V'(v)\psi)dydx$$
$$= (-1)^i\int_{\Gamma}\left(\phi(|\nabla\overline{u}|)\nabla\overline{u}\nabla(\psi\circ T^i) + A(x,y)V'(\overline{u})(\psi\circ T^i)\right)dydx = 0.$$

Therefore, for any $\psi \in W^{1,\Phi}(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$,

$$\int_{\mathbb{R}^2}(\phi(|\nabla v|)\nabla v\nabla\psi + A(x,y)V'(v)\psi)dydx$$
$$= \sum_{i=0}^{3}\int_{\Gamma^i}(\phi(|\nabla v|)\nabla v\nabla\psi + A(x,y)V'(v)\psi)\,dydx = 0.$$

Furthermore, by regularity arguments, v is a weak solution of equation (2.1) in $C^{1,\beta}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\beta \in (0,1)$. A direct computation shows that v checks the conditions (a)-(c) of Theorem 2.4. To complete the proof, we are going to prove that v satisfies item (d). Since $J(v) = d_\infty < +\infty$, we must have $I_j(v) - c_j \to 0$ as $j \to +\infty$. By Lemma 2.14, there is $u_0 \in K_\Phi(\alpha)$ such that

$$\|v - \tau_j u_0\|_{W^{1,\Phi}(T_j)} \to 0 \quad \text{as} \quad j \to +\infty. \tag{2.72}$$

Now, we claim that

$$\|v - \tau_j u_0\|_{L^\infty(T_j)} \to 0 \quad \text{as} \quad j \to +\infty. \tag{2.73}$$

In fact, assume by contradiction that there exits $\epsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there are $j_n > n$ and $(x_n, y_n) \in T_{j_n}$ satisfying

$$|v(x_n,y_n) - \tau_{j_n}u_0(x_n,y_n)| \geq 3\epsilon_0.$$

From Mean Value Theorem, there is $\theta > 0$ sufficiently small such that

$$|\tau_{j_n}u_0(x,y) - \tau_{j_n}u_0(x_n,y_n)| \leq \epsilon_0 \quad \forall (x,y) \in B_\theta(x_n,y_n)\cap T_{j_n}$$

and

$$|v(x,y) - v(x_n,y_n)| \leq \epsilon_0 \quad \forall (x,y) \in B_\theta(x_n,y_n)\cap T_{j_n}.$$

Consequently,

$$\iint_{T_{j_n}}\Phi(|v - \tau_{j_n}u_0|)dydx \geq \Phi(\epsilon_0)\,|B_\theta(x_n,y_n)\cap T_{j_n}| \geq \beta_0 \quad \forall n \in \mathbb{N},$$

for some $\beta_0 > 0$. As $\Phi \in \Delta_2$, there is $r > 0$ such that

$$\|v - \tau_{j_n}u_0\|_{L^\Phi(T_{j_n})} \geq r \quad \forall n \in \mathbb{N},$$

which contradicts (2.72). Thereby, from (2.73), given $\epsilon > 0$ there is $j_0 > 0$ such that

$$|v(x,y) - \tau_j u_0(x,y)| < \frac{\epsilon}{2} \quad \forall (x,y) \in T_j \quad \text{and} \quad \forall j > j_0.$$

On the other hand, since $u_0(x,y) \to \alpha$ as $x \to +\infty$ uniformly in $y \in [0,1]$ we may take j_0 sufficiently large satisfying

$$|\tau_j u_0(x,y) - \alpha| < \frac{\epsilon}{2} \quad \forall (x,y) \in T_j \quad \text{with} \quad x > j_0 \text{ and } \ j \geq 0.$$

Therefore,

$$|v(x,y) - \alpha| < \epsilon \quad \forall x > j_0 \quad \text{and} \quad y > j_0.$$

A similar argument works to prove that

$$|v(x,y) + \alpha| < \epsilon \quad \forall x < -j_0 \quad \text{and} \quad y > j_0.$$

Gathering these estimates together with (2.73) we conclude the proof the theorem. ■

The above proof suggests the following behavior of the solution v.

Corollary 2.3 *Let v be given as in Theorem 2.4. Then, the following hold*

(a) $v(x,y) \to \alpha$ *as* $x \to +\infty$ *and* $y \to +\infty$,

(b) $v(x,y) \to -\alpha$ *as* $x \to -\infty$ *and* $y \to +\infty$,

(c) $v(x,y) \to -\alpha$ *as* $x \to +\infty$ *and* $y \to -\infty$,

(d) $v(x,y) \to \alpha$ *as* $x \to -\infty$ *and* $y \to -\infty$.

In other words, $v(x,y)$ is close to $+\alpha$ whenever $(x,y) \in \mathbb{R}^2$ is in one of the odd quadrants far enough away from the coordinate axes. Likewise, if (x,y) is in one of the even quadrants and far enough away from the coordinate axes, then $v(x,y)$ is close to $-\alpha$.

2.3 Final remarks

We would like to point out in this last section that although we have refined and adapted the variational procedure introduced in Chapter 1, the problem of the existence of a saddle solution for (2.1) in the case where $\Phi(t) = |t|^p$ with $p \in (1,2)$ is still an open question.

CHAPTER 3

HETEROCLINIC SOLUTION FOR THE PRESCRIBED CURVATURE EQUATION IN $\mathbb{R}$

In this chapter we use variational methods to establish the existence of heteroclinic solution for the prescribed mean curvature equation of the form

$$-\left(\frac{q'}{\sqrt{1+(q')^2}}\right)' + a(t)V'(q) = 0 \;\text{ in }\; \mathbb{R}, \tag{3.1}$$

where V is a double-well potential with minima at $t = \pm\alpha$ and $a \in L^\infty(\mathbb{R})$ is an even non-negative function with $0 < a_0 := \inf_{t \geq M} a(t)$ for some $M > 0$. Moreover, in the case where a is constant, for each initial conditions $q(0) = r_1$ and $q'(0) = r_2$, the uniqueness of the minimal heteroclinic type solutions for (3.1) has been proved. Our main effort here is to truncate the prescribed mean curvature operator and obtain an auxiliary quasilinear equation of the type

$$-(\phi(|q'|)q')' + a(t)V'(q) = 0 \;\text{ in }\; \mathbb{R}.$$

Afterwards, we discuss the existence of a heteroclinic solution q from $-\alpha$ to α of this auxiliary equation using minimization arguments and the qualitative qualities of this solution. Finally, we consider a control involving the root α and the graph of V to ensure that $\|q'\|_\infty$ is small, because this implies that q is a heteroclinic solution for (3.1). At this point, some estimates due to Lieberman [67] apply an important rule in our argument.

3.1 Existence of heteroclinic solutions for quasilinear equations

In this section, we will study the existence of a heteroclinic solution for the quasilinear equation of the form

$$-\left(\phi(|q'|)q'\right)' + a(t)V'(q) = 0 \ \text{ in } \ \mathbb{R} \tag{3.2}$$

by considering the conditions (ϕ_1)-(ϕ_3) on ϕ, $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_3) and (V_8) on V. This study will apply an important rule to find a heteroclinic solution for the prescribed mean curvature equation (3.1).

3.1.1 Existence of minimal solution

We begin remembering that from (V_1) and (V_3) there exist $\underline{w}, \overline{w} > 0$ such that

$$\underline{w}\Phi(|t-\alpha|) \leq V(t) \leq \overline{w}\Phi(|t-\alpha|) \quad \forall t \in [0, \alpha + \delta_\alpha]. \tag{3.3}$$

For this, see for a moment (1.11). From now on, we will consider the class of admissible functions

$$\Gamma(\alpha) = \left\{ q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}) : \lim_{t \to +\infty} q(t) = \alpha, \ q \text{ is odd and } q(t) \geq 0 \text{ for } t \geq 0 \right\}$$

and the energy functional $I : W^{1,\Phi}_{\text{loc}}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ given by

$$I(q) = \int_{\mathbb{R}} \left(\Phi(|q'|) + a(t)V(q)\right) dt.$$

According to the definitions of Φ, a and V,

$$I(q) \geq 0 \text{ for every } q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R}),$$

from where it follows that I is bounded from below. Moreover, the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} -\alpha, & \text{if } \ t \leq -\alpha, \\ t, & \text{if } \ -\alpha \leq t \leq \alpha, \\ \alpha, & \text{if } \ \alpha \leq t, \end{cases}$$

belongs to $\Gamma(\alpha)$ with $I(\psi) < +\infty$. Hence,

$$c(\alpha) = \inf_{q\in\Gamma(\alpha)} I(q)$$

is well defined.

With these preliminaries we can state and prove our first lemma.

Lemma 3.1 *Let $\epsilon \in (0,\alpha)$ and $t_1, t_2 \in \mathbb{R}$, $t_1 \neq t_2$, such that $\max\{t_1,t_2\} < -M$ or $\min\{t_1,t_2\} > M$. If $q \in W^{1,\Phi}_{loc}(\mathbb{R})$ has the property that*

$$q(t_1) = \alpha - \frac{\epsilon}{2} \text{ and } q(t_2) = \alpha - \epsilon \text{ or } q(t_1) = -\alpha + \frac{\epsilon}{2} \text{ and } q(t_2) = -\alpha + \epsilon,$$

then we have

$$\int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} (\Phi(|q'|) + a(t)V(q))\,dt \geq \mu_{\epsilon,a_0},$$

for some positive μ_{ϵ,a_0} independent of t_1 and t_2.

Proof. Assume $t_1 < t_2$. By continuity of q, without loss of generality, we can change t_1 and t_2 if necessary to get the inequalities below

$$\alpha - \epsilon \leq q(t) \leq \alpha - \frac{\epsilon}{2} \quad \text{or} \quad -\alpha + \frac{\epsilon}{2} \leq q(t) \leq -\alpha + \epsilon, \quad \forall t \in [t_1, t_2].$$

Repeating the same arguments found in Lemma 1.1, one has

$$\Phi(|q(t_1) - q(t_2)|) \leq \frac{\xi_1(t_2-t_1)}{t_2-t_1} \int_{t_1}^{t_2} \Phi(|q'(t)|)dt,$$

where ξ_1 was fixed in Lemma A.2. So,

$$\Phi\left(\frac{\epsilon}{2}\right) \frac{t_2-t_1}{\xi_1(t_2-t_1)} \leq \int_{t_1}^{t_2} \Phi(|q'(t)|)dt.$$

Since $t_2 < -M$ or $t_1 > M$,

$$\int_{t_1}^{t_2} (\Phi(|q'|) + a(t)V(q))\,dt \geq \Phi\left(\frac{\epsilon}{2}\right) \frac{t_2-t_1}{\xi_1(t_2-t_1)} + \theta_\epsilon a_0(t_2-t_1),$$

where

$$\theta_\epsilon = \min\left\{V(s) : \alpha - \epsilon \leq s \leq \alpha - \frac{\epsilon}{2} \quad \text{or} \quad -\alpha + \frac{\epsilon}{2} \leq s \leq -\alpha + \epsilon\right\} > 0.$$

Now, if $\xi_1(t_2-t_1) = (t_2-t_1)^m$,

$$\begin{aligned} \frac{\Phi\left(\frac{\epsilon}{2}\right)(t_2-t_1)}{\xi_1(t_2-t_1)} &+ \theta_\epsilon a_0(t_2-t_1) \\ &\geq \frac{1}{m}\left(\frac{\Phi\left(\frac{\epsilon}{2}\right)^{\frac{1}{m}}}{(t_2-t_1)^{\frac{m-1}{m}}}\right)^m + \frac{m-1}{m}\left((a_0\theta_\epsilon(t_2-t_1))^{\frac{m-1}{m}}\right)^{\frac{m}{m-1}}. \end{aligned}$$

Invoking Young's inequality for the conjugate exponents m and $\frac{m}{m-1}$,

$$\Phi\left(\frac{\epsilon}{2}\right)\frac{t_2-t_1}{\xi_1(t_2-t_1)}+\theta_\epsilon a_0(t_2-t_1)\geq (a_0\theta_\epsilon)^{\frac{m-1}{m}}\Phi\left(\frac{\epsilon}{2}\right)^{\frac{1}{m}}.$$

Similarly, if $\xi_1(t_2-t_1)=(t_2-t_1)^l$,

$$\Phi\left(\frac{\epsilon}{2}\right)\frac{t_2-t_1}{\xi_1(t_2-t_1)}+\theta_\epsilon a_0(t_2-t_1)\geq (a_0\theta_\epsilon)^{\frac{l-1}{l}}\Phi\left(\frac{\epsilon}{2}\right)^{\frac{1}{l}}.$$

Setting

$$\mu_{\epsilon,a_0}=\min\left\{(a_0\theta_\epsilon)^{\frac{m-1}{m}}\Phi\left(\frac{\epsilon}{2}\right)^{\frac{1}{m}},(a_0\theta_\epsilon)^{\frac{l-1}{l}}\Phi\left(\frac{\epsilon}{2}\right)^{\frac{1}{l}}\right\}>0,$$

we arrive at the inequality below

$$\int_{t_1}^{t_2}\left(\Phi(|q'|)+a(t)V(q)\right)dt\geq\mu_{\epsilon,a_0},$$

which completes the proof. ■

Applying Lemma 3.1, we can prove that the set

$$K(\alpha)=\{q\in\Gamma(\alpha):I(q)=c(\alpha)\}$$

is not empty. This fact is proved in the lemma below.

Lemma 3.2 *It holds that $K(\alpha)$ is not empty. Moreover, any $q\in K(\alpha)$ is a weak solution of (3.2) with $q\in C^{1,\beta}_{loc}(\mathbb{R})$ for some $\beta\in(0,1)$ and*

$$-\left(\phi(|q'(t)|)q'(t)\right)'+a(t)V'\left(q(t)\right)=0\quad\forall t\in\mathbb{R}.\tag{3.4}$$

Proof. Let $(q_n)\subset\Gamma(\alpha)$ be a minimizing sequence for I. We can assume without loss of generality that

$$0\leq q_n(t)\leq\alpha\text{ for all }t\geq 0.$$

Indeed, by setting the sequence

$$u_n(t)=\begin{cases}\min\{q_n(t),\alpha\}, & \text{if}\quad t\geq 0\\ -u_n(-t), & \text{if}\quad t\leq 0,\end{cases}$$

it is easy to check that $(u_n)\subset W^{1,\Phi}_{\text{loc}}(\mathbb{R})$, u_n is odd, $0\leq u_n(t)\leq\alpha$ for $t\geq 0$ and

$$|u_n(t)-\alpha|\leq|q_n(t)-\alpha|,\quad\forall t\geq 0.$$

Hence, $(u_n)\subset\Gamma(\alpha)$, and also, a direct computation implies that

$$I(u_n)\leq I(q_n),\quad\forall n\in\mathbb{N},$$

from where it follows that (u_n) is also a minimizing sequence for I on $\Gamma(\alpha)$ with $0 \leq u_n(t) \leq \alpha$ for any $t \geq 0$. Now, our next claim is that (q_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$. In fact, since $I(q_n) \to c(\alpha)$ then there exists $C > 0$ such that $I(q_n) \leq C$ for every $n \in \mathbb{N}$, and so, for each $L > 0$,

$$\int_{-L}^{L} \Phi(|q_n'|)dt \leq 2C, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

Moreover, the boundedness $\|q_n\|_{L^\infty(\mathbb{R})} \leq \alpha$ ensures that

$$\int_{-L}^{L} \Phi(|q_n|)dt \leq \Phi(\alpha)2L, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Therefore, as $\Phi \in \Delta_2$, (3.5) and (3.6) guarantee that (q_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\mathbb{R})$. A classical diagonal argument yields that there is $q \in W^{1,\Phi}_{\text{loc}}(\mathbb{R})$ and a subsequence of (q_n), still denoted by (q_n), such that

$$q_n \rightharpoonup q \text{ in } W^{1,\Phi}_{\text{loc}}(\mathbb{R}) \ \text{ and } \ q_n \to q \text{ in } L^\infty_{\text{loc}}(\mathbb{R}).$$

By the pointwise convergence, $q(t) = -q(-t)$ for every $t \in \mathbb{R}$ and $0 \leq q(t) \leq \alpha$ for any $t \geq 0$. Moreover, $I(q) \leq c(\alpha)$.

We claim that $q(t) \to \alpha$ as $t \to +\infty$. To prove the claim, let us first prove that

$$\liminf_{t\to+\infty} |q(t) - \alpha| = 0. \tag{3.7}$$

Indeed, if this limit does not hold, there are $t_0, r > 0$ satisfying

$$r \leq |q(t) - \alpha|, \quad \forall t \geq t_0.$$

So, since Φ is increasing on $(0, +\infty)$ and $q(t) \in [0, \alpha]$ for all $t \geq 0$, from (3.3),

$$\underline{w}\Phi(r) \leq \underline{w}\Phi(|q(t) - \alpha|) \leq V(q(t)), \quad \forall t \geq t_0.$$

Thereby, fixing $t_* > \max\{M, t_0\}$ one has

$$I(q) \geq \int_{t_*}^{t} a(t)V(q(t))dt \geq \underline{w}\Phi(r)a_0(t - t_*) \quad \forall t > t_*,$$

and so, taking the limit $t \to +\infty$ we get $I(q) = +\infty$, which is impossible. Next we are going to show that

$$\limsup_{t\to+\infty} |q(t) - \alpha| = 0. \tag{3.8}$$

Assume by contradiction that $\limsup\limits_{t\to+\infty} |q(t)-\alpha| > 0$. Thereby, there exists $r > 0$ such that

$$\limsup_{t\to+\infty} |q(t) - \alpha| > 2r.$$

In what follows, let us fix $\epsilon > 0$ satisfying $\epsilon < \min\{r, \alpha\}$. By continuity, we can find a sequence of disjoint intervals (σ_i, τ_i) with $0 < \sigma_i < \tau_i < \sigma_{i+1} < \tau_{i+1}$, $i \in \mathbb{N}$, and $\sigma_i \to +\infty$ as $i \to +\infty$ such that for each i,

$$|q(\sigma_i) - \alpha| = \frac{\epsilon}{2} \quad \text{and} \quad |q(\tau_i) - \alpha| = \epsilon,$$

that is,

$$q(\sigma_i) = \alpha - \frac{\epsilon}{2} \quad \text{and} \quad q(\tau_i) = \alpha - \epsilon,$$

because $|q(t)| \leq \alpha$ for any $t \in \mathbb{R}$. Now, since $\sigma_i \to +\infty$ as $i \to +\infty$, there is $i_0 \in \mathbb{N}$ such that $\sigma_i > M$ for each $i \geq i_0$. So, according to Lemma 3.1,

$$I(q) \geq \sum_{i=i_0}^{+\infty} \mu_{\epsilon, a_0} = +\infty,$$

which contradicts the fact that $I(q) < +\infty$. Therefore, from (3.7) and (3.8), it follows that $\lim_{t \to +\infty} q(t) = \alpha$, and hence $q \in \Gamma(\alpha)$ which implies $I(q) = c(\alpha)$, showing that the set $K(\alpha)$ is non-empty.

It remains to show that q is a weak solution of (3.2). To this end, given $\psi \in C_0^\infty(\mathbb{R})$, we can write $\psi(t) = \psi_o(t) + \psi_e(t)$, where

$$\psi_e(t) = \frac{\psi(t) + \psi(-t)}{2} \quad \text{and} \quad \psi_o(t) = \frac{\psi(t) - \psi(-t)}{2}.$$

Now, for $s > 0$ let us consider the function

$$\eta_o(t) = \begin{cases} q(t) + s\psi_o(t), & \text{if} \quad t \geq 0 \text{ and } q(t) + s\psi_o(t) \geq 0, \\ -q(t) - s\psi_o(t), & \text{if} \quad t \geq 0 \text{ and } q(t) + s\psi_o(t) \leq 0, \\ -\eta_o(-t) & \text{if} \quad t < 0. \end{cases}$$

A direct computation gives that $\eta_o \in \Gamma(\alpha)$. From (V_2),

$$I(q + s\psi_o) = I(\eta_o) \geq c(\alpha) = I(q). \tag{3.9}$$

On the other hand, according to Lemma A.8-(b),

$$\Phi(|q' + s\psi'|) - \Phi(|q' + s\psi_o'|) \geq \phi(|q' + s\psi_o'|)(q' + s\psi_o')(s\psi_e'),$$

which leads to

$$\begin{aligned} I(q + s\psi) - I(q + s\psi_o) \geq s \int_{\mathbb{R}} \phi(|q' + s\psi_o'|) q' \psi_e' dt + s^2 \int_{\mathbb{R}} \phi(|q' + s\psi_o'|) \psi_o' \psi_e' dt \\ + \int_{\mathbb{R}} a(t) \left(V(q + s\psi) - V(q + s\psi_o)\right) dt. \end{aligned} \tag{3.10}$$

Since functions $\phi(|q' + s\psi_o'|)q'\psi_e'$ and $\phi(|q' + s\psi_o'|)\psi_o'\psi_e'$ are odd, one has

$$\int_{\mathbb{R}} \phi(|q' + s\psi_o'|)q'\psi_e' dt = \int_{\mathbb{R}} \phi(|q' + s\psi_o'|)\psi_o'\psi_e' dt = 0. \tag{3.11}$$

Substituting (3.11) into (3.10), one gets

$$I(q + s\psi) - I(q + s\psi_o) \geq \int_{\mathbb{R}} a(t)(V(q + s\psi) - V(q + s\psi_o))dt.$$

This fact combined with (3.9) yields

$$I(q + s\psi) - I(q) \geq \int_{\mathbb{R}} a(t)(V(q + s\psi) - V(q + s\psi_o))dt,$$

from where it follows that

$$\begin{aligned}
\int_{\mathbb{R}} (\phi(|q'|)q'\psi' + a(t)V'(q)\psi)dt &\geq \lim_{s\to 0^+} \int_{\mathbb{R}} a(t)\frac{V(q + s\psi) - V(q + s\psi_o)}{s} dt \\
&\geq \lim_{s\to 0^+} \int_{\mathbb{R}} a(t)\left(\frac{V(q + s\psi) - V(q)}{s} - \frac{V(q + s\psi_o) - V(q)}{s}\right) dt \\
&\geq \int_{\mathbb{R}} a(t)V'(q)(\psi - \psi_o)dt = \int_{\mathbb{R}} a(t)V'(q)\psi_e dt = 0,
\end{aligned}$$

because $aV'(q)\psi_e$ is odd since a and V are even. Therefore, as ψ is arbitrary,

$$\int_{\mathbb{R}} (\phi(|q'|)q'\psi' + a(t)V'(q)\psi)\, dt = 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}).$$

Finally, in order to prove (3.4), we must prove that any weak solution q of (3.2) satisfies

$$(\phi(|q'|)q')' = a(t)V'(q) \text{ almost everywhere in } \mathbb{R}. \tag{3.12}$$

Using the fact that the right side of (3.12) is a continuous function, the Lemma A.6 together with (3.12) implies that $\phi(|q'|)q' \in W^{1,\tilde{\Phi}}_{\text{loc}}(\mathbb{R})$, and consequently from (A.1), $\phi(|q'|)q' \in W^{1,1}_{\text{loc}}(\mathbb{R})$. Now, by [26, Theorem 8.2] the equality (3.4) occurs for every $t \in \mathbb{R}$, and by [67, Theorem 1.7] there exists $\beta \in (0, 1)$ such that $q \in C^{1,\beta}_{\text{loc}}(\mathbb{R})$. This finishes the proof. ■

3.1.2 Qualitative properties

In this subsection we are interested in showing some qualitative properties for the minimal heteroclinic solution of (3.2).

The proof of Lemma 3.2 ensures that there exists $q \in K(\alpha)$ with $\|q\|_\infty \leq \alpha$. However, thanks to conditions (ϕ_3) and (V_8), we are able to show in the next lemma that $0 < q(t) < \alpha$ for any $t > 0$.

Lemma 3.3 *If $q \in K(\alpha)$ with $0 \leq q(t) \leq \alpha$ for all $t \geq 0$, then $0 < q(t) < \alpha$ for any $t > 0$. In particular, $|q(t)| < \alpha$ for every $t \in \mathbb{R}$.*

Proof. Let be $q \in K(\alpha)$ with $0 \leq q(t) \leq \alpha$ for any $t \geq 0$. Now, we first claim that $q(t) < \alpha$ for all $t > 0$. Indeed, assume for the sake of contradiction that there is $t_0 > 0$ such that $q(t_0) = \alpha$. Thus, let us consider the numbers $r > t_0$ and $R > 0$ satisfying

$$R > \max\left\{\|q'\|_{L^\infty([0,r])}, \eta\right\},$$

where $\eta > 0$ was given in (ϕ_3). Next, we define the function $\tilde{\phi} : (0, +\infty) \to (0, +\infty)$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & \text{if} \quad 0 < t \leq R, \\ \dfrac{\phi(R)t^{s-2}}{R^{s-2}}, & \text{if} \quad R \leq t, \end{cases}$$

where $s > 1$ was also fixed in (ϕ_3). From (ϕ_3), a simple computation implies that there exist $\gamma_1, \gamma_2 > 0$ dependent on the constants η, R, s, c_1 and c_2 such that

$$\tilde{\phi}(t)t \leq \gamma_1 t^{s-1} \text{ and } \tilde{\phi}(t)t^2 \geq \gamma_2 t^s \text{ for all } t \geq 0. \tag{3.13}$$

Now, let us consider the function $G : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$G(t, u, p) = \frac{\tilde{\phi}(|p|)p}{\gamma_2}.$$

From (3.13),

$$|G(t, u, p)| \leq \frac{\gamma_1}{\gamma_2}|p|^{s-1} \text{ and } pG(t, u, p) \geq |p|^s \text{ for all } (t, u, p) \in \mathbb{R}^3.$$

We will also consider the function $B : \mathbb{R}^3 \to \mathbb{R}$ given by

$$B(t, u, p) = \frac{a(t)V'(\alpha - u)}{\gamma_2}.$$

Combining (ϕ_3) with (V_8) and repeating the argument used in Lemma 1.6 we get that for each $M > 0$ there is $C_M > 0$ such that

$$|B(t, u, p)| \leq C_M|u|^{s-1} \quad \text{for all} \quad (t, u, p) \in \mathbb{R} \times (-M, M) \times \mathbb{R}.$$

Having that in mind, setting $w(t) = \alpha - q(t)$ for $t \in \mathbb{R}$, we infer that w is a weak solution of the quasilinear elliptic equation

$$G'(t, w, w') + B(t, w, w') = 0 \text{ in } [0, r],$$

where G' is the derivative of $G(t, w(t), w'(t))$ at t. Employing the Harnack-type inequality found in [91, Theorem 1.1] we get that $w(t) = 0$ for all $t \in [0, r]$, that is, $q(t) = \alpha$ for any $t \in [0, r]$, which contradicts the fact that $q(0) = 0$. The same argument works to prove that $q(t) > 0$ for any $t > 0$, and so, the proof is completed noting that q is odd. ■

It is important to note that the condition $a \in L^\infty(\mathbb{R})$ is crucial to ensure that $\lim_{|t|\to+\infty} q'(t) = 0$ whenever $q \in K(\alpha)$. To see this, let us first recall that $V'(\pm\alpha) = 0$, then from (V_1),

$$\lim_{|t|\to+\infty} V'(q(t)) = 0. \tag{3.14}$$

Lemma 3.4 *If $q \in K(\alpha)$, then $q'(t) \to 0$ as $|t| \to +\infty$.*

Proof. The fact that $q \in K(\alpha)$ implies that

$$\int_{\mathbb{R}} \Phi(|q'(t)|)dt < +\infty,$$

and consequently,

$$\liminf_{|t|\to+\infty} \Phi(|q'(t)|) = 0.$$

Since Φ is increasing on $(0, +\infty)$,

$$\liminf_{|t|\to+\infty} |q'(t)| = 0.$$

Now, our aim is to prove that

$$\limsup_{|t|\to+\infty} |q'(t)| = 0.$$

If this limit does not hold, then there exist $r > 0$ and a sequence $(t_n) \subset \mathbb{R}$ with $t_n \to +\infty$ satisfying

$$|q'(t_n)| \geq r, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

In what follows, we fix $d \in \mathbb{R}$ such that

$$2^m c(\alpha) < \tilde{\Phi}\left(\frac{\phi(r)r}{2}\right) d. \tag{3.16}$$

So, by continuity, given $t \in [t_n, t_n + d]$ there exists $s_n \in [t_n, t_n + d]$ in such a way that

$$|\phi(|q'(t)|)q'(t) - \phi(|q'(t_n)|)q'(t_n)| \leq d \left|(\phi(|q'(s_n)|)q'(s_n))'\right| = d \left|a(s_n)V'(q(s_n))\right|.$$

As $a \in L^\infty(\mathbb{R})$, the limit (3.14) guarantees that

$$|\phi(|q'(t)|)q'(t) - \phi(|q'(t_n)|)q'(t_n)| \to 0 \text{ as } n \to +\infty.$$

Therefore, there exists $n_0 \in \mathbb{N}$, which is uniform for all $t \in [t_n, t_n + d]$, satisfying

$$|\phi(|q'(t)|)q'(t) - \phi(|q'(t_n)|)q'(t_n)| < \frac{\phi(r)r}{2} \quad \forall n \geq n_0.$$

Thereby, by (ϕ_1) and (3.15),

$$\phi(r)r - \phi(|q'(t)|)|q'(t)| \leq \frac{\phi(r)r}{2} \quad \forall n \geq n_0,$$

that is,

$$\frac{\phi(r)r}{2} \leq \phi(|q'(t)|)|q'(t)| \ \text{ for } t \in [t_n, t_n + d] \text{ and } n \geq n_0.$$

Thanks to Lemmas A.2 and A.6,

$$\tilde{\Phi}\left(\frac{\phi(r)r}{2}\right) \leq \tilde{\Phi}(\phi(|q'(t)|)|q'(t)|) \leq 2^m \Phi(|q'(t)|) \ \text{ for all } t \in [t_n, t_n + d] \text{ and } n \geq n_0.$$

Finally, for $n \geq n_0$,

$$\tilde{\Phi}\left(\frac{\phi(r)r}{2}\right) d \leq 2^m \int_{t_n}^{t_n+d} \Phi(|q'(t)|)dt \leq 2^m c(\alpha),$$

which contradicts (3.16), and this finishes the proof. ■

To end this section, from the above considerations, it is easy to see that the following theorem follows directly from Lemmas 3.2, 3.3 and 3.4.

Theorem 3.1 *Assume* (ϕ_1)-(ϕ_2), $V \in C^1(\mathbb{R}, \mathbb{R})$, (V_1)-(V_3) *and that* a *belongs to Class 11. Then equation* (3.2) *has a heteroclinic solution from* $-\alpha$ *to* α *satisfying*

(a) $q(t) = -q(-t)$ *for any* $t \in \mathbb{R}$,

(b) $0 \leq q(t) \leq \alpha$ *for all* $t > 0$.

Moreover, taking into account the assumptions (ϕ_3) *and* (V_8) *then the inequalities in (b) are strict.*

In the particular case $\Phi(t) = \dfrac{t^2}{2}$ we can write the following result.

Theorem 3.2 *Assume* $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2), (V_5)-(i) *and that* a *belongs to Class 11. Then equation*

$$-q''(t) + a(t)V'(q(t)) = 0 \quad in \quad \mathbb{R} \tag{3.17}$$

has a heteroclinic solution from $-\alpha$ *to* α *in* $C^2(\mathbb{R})$ *such that*

(a) $q(t) = -q(-t)$ *for any* $t \in \mathbb{R}$,

(b) $0 < q(t) < \alpha$ *for all* $t > 0$.

3.2 Heteroclinic solution of the prescribed curvature equation

In this section, as a first step towards finding heteroclinic solutions of (3.1), we will make a truncation on the prescribed mean curvature operator to obtain an auxiliary ordinary differential equation. More precisely, for each $L > 0$, we consider the following quasilinear equation

$$-\left(\varphi_L\left(|q'|^2\right)q'\right)' + a(t)V'(q) = 0 \quad \text{in} \quad \mathbb{R}, \tag{$(AP)_L$}$$

where $\varphi_L : [0, +\infty) \to [0, +\infty)$ is the function defined by

$$\varphi_L(t) = \begin{cases} \dfrac{1}{\sqrt{1+t}}, & \text{if} \quad t \in [0, L], \\ x_L(t-L-1)^2 + y_L, & \text{if} \quad t \in [L, L+1], \\ y_L, & \text{if} \quad t \in [L+1, +\infty), \end{cases}$$

with

$$x_L = \frac{\sqrt{1+L}}{4(1+L)^2} \quad \text{and} \quad y_L = \frac{(4L+3)\sqrt{1+L}}{4(1+L)^2} = (4L+3)x_L.$$

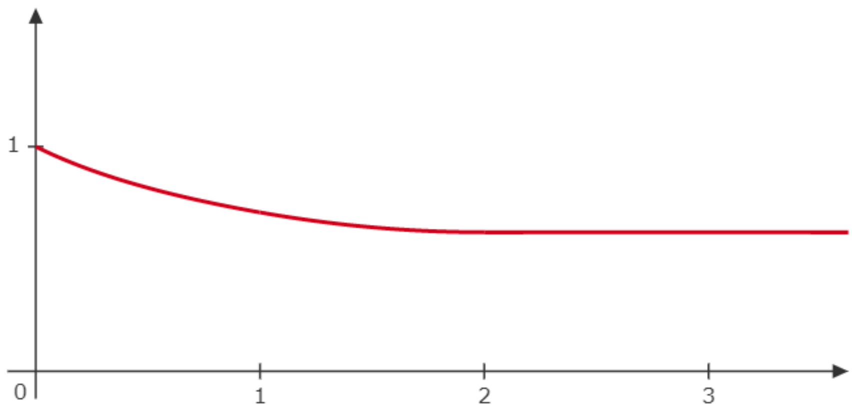

Figure 3.1: Graph of function φ_L with $L = 1$.

We would like to highlight that the methods and techniques used in the previous section do not apply directly to the function $\phi(t) = \frac{1}{\sqrt{1+t^2}}$ since it does not satisfy condition

(ϕ_2), because in this case we have

$$\frac{(\phi(t)t)'}{\phi(t)} \to 0 \quad \text{as} \quad t \to +\infty.$$

Therefore, the best idea here is to consider truncations of the differential operator involved in (3.1) to obtain equations $(AP)_L$, which coincides with the mean curvature operator on functions q such that $\|q'\|_{L^\infty(\mathbb{R})} \leq \sqrt{L}$.

3.2.1 The truncated prescribed mean curvature operator

In this subsection, we will prove some auxiliary results about the function φ_L that will be very useful throughout this chapter.

Lemma 3.5 *For each $L > 0$, the function φ_L satisfies the following properties*

(a) φ_L *is* C^1.

(b) $y_L \leq \varphi_L(t) \leq 1$ *for each* $t \geq 0$.

(c) $y_L t \leq A_L(t) \leq t$ *for any* $t \geq 0$, *where* $A_L(t) = \int_0^t \varphi_L(s)ds$ *for all* $t \geq 0$.

(d) The function $t \mapsto A_L(t^2)$ *is convex.*

(e) $(\varphi_L(t^2)\,t)' \geq 2x_L$ *for all* $t \geq 0$.

Proof. The items (a), (b) and (c) follow by straightforward computation. Now, in order to show (d), we can use a comparison argument like in [42, Lemma 2.2]. Next, we give the proof for the sake of completeness. Firstly, we note that

$$\varphi_L'(t) = \begin{cases} -\dfrac{1}{2\sqrt{(1+t)^3}}, & \text{if} \quad t \in [0, L], \\ 2x_L(t - L - 1), & \text{if} \quad t \in [L, L+1], \\ 0, & \text{if} \quad t \in [L+1, +\infty). \end{cases}$$

Moreover, putting $b_L(t) = \varphi_L(t) + 2t\varphi_L'(t)$ we get

$$b_L(t) = \begin{cases} \dfrac{1}{\sqrt{1+t}} - \dfrac{t}{\sqrt{(1+t)^3}}, & \text{if} \quad t \in [0, L], \\ x_L(t - L - 1)^2 + y_L + 4x_L t(t - L - 1), & \text{if} \quad t \in [L, L+1], \\ y_L, & \text{if} \quad t \in [L+1, +\infty). \end{cases}$$

From this,

$$\left(A_L(t^2)\right)'' = 2b_L(t^2) \quad \text{for} \quad t \geq 0. \tag{3.18}$$

Now, a direct computation shows that b_L is strictly decreasing in $[0, L]$, and so, $b_L(t) \geq b_L(L) = 4x_L$ for all $t \in [0, L]$. Our next claim is that

$$b_L(t) \geq 2x_L \text{ for any } t \in [L, L+1].$$

In fact, considering the real function

$$f_L(t) = x_L(t-L-1)^2 + y_L + 4x_L t(t-L-1)$$

it follows that f_L has a unique minimum at $t_L = \frac{3(L+1)}{5}$. Thereby, if $L \geq \frac{3}{2}$, then b_L is strictly increasing in $[L, L+1]$, and hence, $b_L(t) \geq b_L(L) = 4x_L$ for any $t \in [L, L+1]$. In the case $L < \frac{3}{2}$, we have that $t_L \in [L, L+1]$, and consequently, $b_L(t) \geq b_L(t_L)$ for all $t \in [L, L+1]$. A direct calculus shows that

$$b_L(t_L) = x_L\frac{4}{25}(L+1)^2 + (4L+3)x_L - x_L\frac{24}{25}(L+1)^2 = x_L(4L+3) - x_L\frac{4}{5}(L+1)^2,$$

that is,

$$b_L(t_L) = \frac{x_L}{5}\left(-4L^2 + 12L + 11\right) \geq \frac{11}{5}x_L,$$

which is our claim. Therefore, since $b_L(t) = y_L \geq 2x_L$ for each $t \geq L+1$, one gets

$$b_L(t) \geq 2x_L \text{ for all } t \geq 0, \tag{3.19}$$

and the item (d) follows from (3.18) and (3.19). Finally, to complete item (e), just note that from (3.18),

$$\left(\varphi_L\left(t^2\right)t\right)' = \frac{1}{2}\left(A_L(t^2)\right)'' \geq 2x_L \text{ for all } t \geq 0,$$

which completes the proof. ■

Our next step is to associate equation $(AP)_L$ with an N-function of the form (6). To this end, for each $L > 0$, we set the functions

$$\Phi_L(t) = \frac{1}{2}A_L(t^2) \quad \text{and} \quad \phi_L(t) = \varphi_L(t^2) \quad \text{for all} \quad t \in \mathbb{R},$$

which satisfies the equality below

$$\Phi_L(t) = \int_0^{|t|} \phi_L(s)s ds.$$

Thanks to Lemma 3.5, it is easy to check that Φ_L is an N-function. Moreover, from Lemma 3.5 we also deduce the following result that will be used later on.

Lemma 3.6 *For each $L > 0$, if $\Omega \subset \mathbb{R}^N$ is a domain, then $L^{\Phi_L}(\Omega) = L^2(\Omega)$. Moreover, the norm of $L^{\Phi_L}(\Omega)$ is equivalent to the norm of $L^2(\Omega)$.*

Proof. To see that, given $u \in L^2(\Omega)$, by Lemma 3.5-(c),

$$\int_\Omega \Phi_L(|u|)dx \leq \frac{1}{2}\int_\Omega |u|^2 dx.$$

This shows that $L^2(\Omega) \subset L^{\Phi_L}(\Omega)$. Conversely, if $u \in L^{\Phi_L}(\Omega)$ then there is $\lambda > 0$ such that

$$\int_\Omega \Phi_L\left(\frac{|u|}{\lambda}\right) dx < +\infty.$$

From Lemma 3.5-(c),

$$\frac{y_L}{2\lambda^2}\int_\Omega |u|^2 dx \leq \int_\Omega \Phi_L\left(\frac{|u|}{\lambda}\right) dx < +\infty,$$

which implies $u \in L^2(\Omega)$, and so, $L^{\Phi_L}(\Omega) = L^2(\Omega)$.

Now we are going to prove that the norm of $L^{\Phi_L}(\Omega)$ is equivalent to the norm of $L^2(\Omega)$. For $u \in L^2(\Omega)$ with $u \neq 0$,

$$\frac{y_L}{2}\int_\Omega \left|\frac{|u|}{\|u\|_{L^{\Phi_L}(\Omega)}}\right|^2 dx \leq \int_\Omega \Phi_L\left(\frac{|u|}{\|u\|_{L^{\Phi_L}(\Omega)}}\right) dx \leq 1,$$

from where it follows that

$$\frac{y_L}{2}\|u\|^2_{L^2(\Omega)} \leq \|u\|^2_{L^{\Phi_L}(\Omega)}. \tag{3.20}$$

On the other hand, for each $\epsilon > 0$ small enough,

$$1 < \int_\Omega \Phi_L\left(\frac{|u|}{\|u\|_{L^{\Phi_L}(\Omega)} - \epsilon}\right) dx \leq \frac{1}{2}\int_\Omega \left|\frac{|u|}{\|u\|_{L^{\Phi_L}(\Omega)} - \epsilon}\right|^2 dx,$$

which leads to

$$\left(\|u\|_{L^{\Phi_L}(\Omega)} - \epsilon\right)^2 < \frac{1}{2}\|u\|^2_{L^2(\Omega)},$$

and so, as ϵ is small,

$$\|u\|^2_{L^{\Phi_L}(\Omega)} \leq \frac{1}{2}\|u\|^2_{L^2(\Omega)}. \tag{3.21}$$

Now, the lemma follows from (3.20) and (3.21). ■

As a direct consequence of Lemma 3.6, for each $L > 0$ the space $L^{\Phi_L}(\Omega)$ is reflexive, which ensures that the N-functions Φ_L and $\tilde{\Phi}_L$ satisfy Δ_2-condition, where $\tilde{\Phi}_L$ is the complementary function associated with Φ_L. Moreover, ϕ_L satisfies conditions (ϕ_1)-(ϕ_3). Indeed, it is clear that by Lemma 3.5-(e) ϕ_L checks (ϕ_1) and by Lemma 3.5-(b) checks

(ϕ_3) with $s = 2$. Now, with direct computations one can get that there are real numbers $m_L, l_L > 1$ with $l_L \leq m_L$ such that

$$l_L - 1 \leq \frac{(\phi_L(t)t)'}{\phi_L(t)} \leq m_L - 1 \quad \text{for any} \quad t \geq 0,$$

showing that ϕ_L verifies (ϕ_2). This is evident in the following lemma.

Lemma 3.7 *It turns out that ϕ_L satisfies (ϕ_1)-(ϕ_3). Moreover, the best constants in (ϕ_2) are*

$$l_L = 1 + \inf_{t>0} \frac{(\phi_L(t)t)'}{\phi_L(t)} = \begin{cases} \frac{6L+6-L^2}{4L+4}, & \text{if} \quad L \leq 1, \\ \frac{7+4L}{4L+4}, & \text{if} \quad L > 1 \end{cases} \quad \text{and} \quad m_L = 1 + \sup_{t>0} \frac{(\phi_L(t)t)'}{\phi_L(t)} = 2.$$

Proof. Let us initially note that

$$\phi_L'(t) = \begin{cases} -\dfrac{t}{(\sqrt{1+t^2})^3}, & \text{if} \quad t \in [0, \sqrt{L}], \\ 4x_L(t^2 - L - 1)t, & \text{if} \quad t \in [\sqrt{L}, \sqrt{L+1}], \\ 0, & \text{if} \quad t \in [\sqrt{L+1}, +\infty) \end{cases}$$

and

$$\frac{(\phi_L(t)t)'}{\phi_L(t)} = 1 + \frac{\phi_L'(t)t}{\phi_L(t)} \quad \text{for all} \quad t \in \mathbb{R}. \tag{3.22}$$

It is easy to see that

$$\frac{(\phi_L(t)t)'}{\phi_L(t)} = 1 \quad \text{for all} \quad t \in (\sqrt{L+1}, +\infty). \tag{3.23}$$

Moreover, when $t \in [0, \sqrt{L}]$,

$$\frac{\phi_L'(t)t}{\phi_L(t)} = \frac{-t^2}{1+t^2},$$

from which it follows that this function is decreasing, and thereby,

$$\frac{1}{L+1} \leq \frac{(\phi_L(t)t)'}{\phi_L(t)} \leq 1 \quad \text{for all} \quad t \in [0, \sqrt{L}]. \tag{3.24}$$

Now, for $t \in [\sqrt{L}, \sqrt{L+1}]$, we see that

$$\frac{\phi_L'(t)t}{\phi_L(t)} = \frac{4(t^2 - L - 1)t^2}{(t^2 - L - 1)^2 + 4L + 3}. \tag{3.25}$$

We are going to study the behavior of the function $f(t) = 4(t^2 - L - 1)t^2$ on $[\sqrt{L}, \sqrt{L+1}]$. For this, let us note that

$$f'(t) = 16t \left(t - \sqrt{\frac{L+1}{2}} \right) \left(t + \sqrt{\frac{L+1}{2}} \right).$$

Let us now analyze the infimum and supremum of the function (3.25) on $[\sqrt{L}, \sqrt{L+1}]$ in cases.

Case 1: When $L \leq 1$, the critical point $\sqrt{\frac{L+1}{2}}$ of f is a minimum point in $[\sqrt{L}, \sqrt{L+1}]$, and so,

$$-(L+1)^2 = f\left(\sqrt{\frac{L+1}{2}}\right) \leq f(t) \leq f\left(\sqrt{L+1}\right) = 0 \quad \forall t \in [\sqrt{L}, \sqrt{L+1}].$$

Gathering (3.22) and (3.25),

$$1 + \frac{-(L+1)^2}{(t^2-L-1)^2+4L+3} \leq \frac{(\phi_L(t)t)'}{\phi_L(t)} \leq 1 \quad \text{for all} \quad t \in [\sqrt{L}, \sqrt{L+1}],$$

and from

$$4L+3 \leq (t^2-L-1)^2+4L+3 \leq 4L+4 \quad \text{for any} \quad t \in [\sqrt{L}, \sqrt{L+1}], \tag{3.26}$$

we arrive at

$$\frac{2L-L^2+2}{4L+4} \leq \frac{(\phi_L(t)t)'}{\phi_L(t)} \leq 1 \quad \text{for all} \quad t \in [\sqrt{L}, \sqrt{L+1}]. \tag{3.27}$$

Case 2: When $L > 1$, the function f is increasing on $[\sqrt{L}, \sqrt{L+1}]$ and in this case

$$-4L = f\left(\sqrt{L}\right) \leq f(t) \leq f\left(\sqrt{L+1}\right) = 0 \quad \forall t \in [\sqrt{L}, \sqrt{L+1}],$$

and hence, from (3.22), (3.25) and (3.26),

$$\frac{3}{4L+4} \leq \frac{(\phi_L(t)t)'}{\phi_L(t)} \leq 1 \quad \text{for all} \quad t \in [\sqrt{L}, \sqrt{L+1}]. \tag{3.28}$$

Finally, the lemma statement follows with a simple verification of what was described to obtain estimates (3.23), (3.24), (3.27) and (3.28). ∎

3.2.2 Existence of solution

We exhibit in this subsection the proof of the main theorem of this chapter involving the curvature equation and a function of the form $a(t)V'$, where V is a double-well potential and a is asymptotically away from zero at infinity. To this end, we will consider the class

$$\Gamma_L(\alpha) = \left\{ q \in W^{1,\Phi_L}_{\text{loc}}(\mathbb{R}) : \lim_{t\to+\infty} q(t) = \alpha, \ q \text{ is odd and } 0 \leq q(t) \text{ for } t \geq 0 \right\}$$

and the action functional

$$I_L(q) = \int_{\mathbb{R}} \left(\Phi_L(|q'|) + a(t)V(q)\right) dt.$$

Our main goal is to look for minima of I_L on $\Gamma_L(\alpha)$. More precisely, we show that the set

$$K_L(\alpha) = \{q \in \Gamma_L(\alpha) : I_L(q) = c_L(\alpha)\}$$

is not empty, where

$$c_L(\alpha) = \inf_{q \in \Gamma_L(\alpha)} I_L(q).$$

First of all, we would like to emphasize that the potential $V \in C^2(\mathbb{R}, \mathbb{R})$ satisfies conditions (V_1)-(V_2) and (V_7). From item (i) of (V_7) it is possible to find $\gamma_1, \gamma_2, \gamma_3 \in (0, +\infty)$ and $\rho \in (0, \frac{\alpha}{2})$ such that

$$|V'(t)| \leq \gamma_3 |t - \alpha|, \ \forall t \in (\alpha - \rho, \alpha + \rho) \tag{3.29}$$

and

$$\gamma_1 |t - \alpha|^2 \leq V(t) \leq \gamma_2 |t - \alpha|^2, \ \forall t \in (\alpha - \rho, \alpha + \rho). \tag{3.30}$$

Then, by Lemma 3.5-(c) and (3.30),

$$2\gamma_1 \Phi_L(|t - \alpha|) \leq V(t) \leq \frac{2\gamma_2}{y_L} \Phi_L(|t - \alpha|), \ \forall t \in (\alpha - \rho, \alpha + \rho)$$

and by Lemma 3.5-(b) and (3.29),

$$|V'(t)| \leq \frac{\gamma_3}{y_L} \phi_L(|\alpha - t|)|\alpha - t| \ \forall t \in (\alpha - \rho, \alpha + \rho).$$

Therefore, the assumption (V_7)-(i) implies that V also satisfies the conditions (V_3) and (V_8), which allows us to use the arguments contained in Section 3.1. Thus, for each fixed $L > 0$ we can now proceed analogously to the proof Lemmas 3.2, 3.3 and 3.4 to find $q_\alpha \in K_L(\alpha)$ such that q_α is odd, $0 < q_\alpha(t) < \alpha$ for any $t > 0$,

$$\lim_{t \to -\infty} q_\alpha(t) = -\alpha, \lim_{t \to +\infty} q_\alpha(t) = \alpha \ \text{ and } \ \lim_{|t| \to +\infty} q'_\alpha(t) = 0.$$

Having done the study for the modified problem $(AP)_L$, we will now work to find a solution to the equation of mean curvature (3.1). The next lemma is crucial to guarantee that if $q_\alpha \in K_L(\alpha)$, then q_α is a heteroclinic solution from $-\alpha$ to α of (3.1) whenever $\alpha > 0$ is small enough. The reader will see that the condition (V_7)-(ii) is crucial in our approach.

Lemma 3.8 *For each $L > 0$, there exists $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0)$ we have that*

$$\|q_\alpha\|_{C^1[-1+r, 1+r]} < \sqrt{L},$$

for all $r \in \mathbb{R}$ and $q_\alpha \in K_L(\alpha)$.

Proof. Assume, by contradiction, that the lemma is not true. Then there exist $(r_n) \subset \mathbb{R}$ and $(\alpha_n) \subset (0, +\infty)$ with $\alpha_n \to 0$ as $n \to +\infty$ and

$$\|q_{\alpha_n}\|_{C^1[-1+r_n, 1+r_n]} \geq \sqrt{L}, \quad \forall n \in \mathbb{N}. \tag{3.31}$$

For each $n \in \mathbb{N}$, let us consider the function

$$\tilde{q}_n(t) = q_{\alpha_n}(t + r_n) \quad \text{for} \quad t \in \mathbb{R},$$

which is a weak solution of the equation

$$-\left(\varphi_L\left(|q'|^2\right) q'\right)' + a(t + r_n)V'(q) = 0.$$

So, setting

$$B_n(t) = -a(t + r_n)V'(\tilde{q}_n(t)) \text{ for } t \in \mathbb{R},$$

by (V_7)-(ii) there exists $C > 0$ independent of n such that

$$|B_n(t)| \leq C\|a\|_{L^\infty(\mathbb{R})} \ \text{ for all } \ t \in \mathbb{R} \ \text{ and } \ n \in \mathbb{N},$$

because $|q_{\alpha_n}(t)| \leq \alpha_n$ for all $n \in \mathbb{N}$ and $\alpha_n \to 0$. The elliptic regularity theory found in [67, Theorem 1.7] implies that $\tilde{q}_n$ is in $C^{1,\beta_0}_{\text{loc}}(\mathbb{R})$ for some $\beta_0 \in (0, 1)$ with

$$\|\tilde{q}_n\|_{C^{1,\beta_0}_{\text{loc}}(\mathbb{R})} \leq R, \quad \forall\, n \in \mathbb{N},$$

for some positive constant R independent of n. Invoking ArzelÃ -Ascoli Theorem, there exists $q \in C^1([-1, 1])$ and a subsequence of $(\tilde{q}_n)$, still denoted by $(\tilde{q}_n)$, such that

$$\tilde{q}_n \to q \quad \text{in} \quad C^1([-1, 1]).$$

But since $\|q_{\alpha_n}\|_\infty \to 0$ as $n \to +\infty$, we must have $q = 0$, and so,

$$\|\tilde{q}_n\|_{C^1([-1,1])} < \sqrt{L} \quad \forall\, n \geq n_0,$$

for some $n_0 \in \mathbb{N}$, that is,

$$\|q_{\alpha_n}\|_{C^1([-1+r_n, 1+r_n])} < \sqrt{L} \quad \forall\, n \geq n_0,$$

which contradicts (3.31), and the proof is completed. ■

Finally, we are ready to prove the best result of this chapter, which is an immediate consequence of Lemma 3.8.

Theorem 3.3 *Assume that a belongs to Class 11, $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2) and (V_7). Then, for each $L > 0$ there exists $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$ equation (3.1) possesses a heteroclinic solution q_α from $-\alpha$ to α satisfying:*

(a) $q_\alpha(t) = -q_\alpha(-t)$ *for all* $t \in \mathbb{R}$,

(b) $0 < q_\alpha(t) < \alpha$ *for all* $t > 0$,

(c) $|q'_\alpha(t)| < \sqrt{L}$ *for any* $t \in \mathbb{R}$.

Proof. From Lemma 3.8, for each $L > 0$, there exists $\alpha_0 = \alpha_0(L) > 0$ such that for $\alpha \in (0, \alpha_0)$ we have that $|q'_\alpha(t)| < \sqrt{L}$ for all $t \in \mathbb{R}$ and $q_\alpha \in K_L(\alpha)$, and the proof is completed. ■

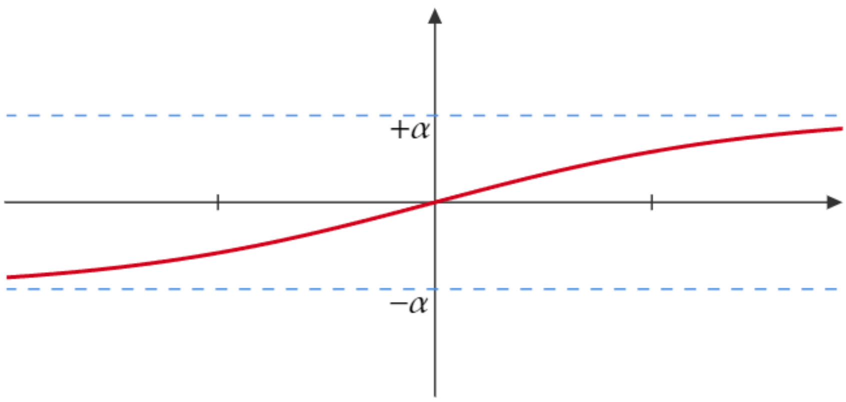

Figure 3.2: Geometric illustration of the heteroclinic solution q_α

3.3 Some remarks on the autonomous case

To conclude this chapter, we will deal with a special case where $a(t)$ belongs to Class 1 listed in the introduction, that is, $a(t) = b$ for all $t \in \mathbb{R}$ where $b > 0$. In this case, we can establish more information about the heteroclinic solutions of problem (3.1), such as the uniqueness of minimal solution. An important point that we would like to point out is that for the constant case, the same argument as in Lemma 1.7 guarantees the following result.

Lemma 3.9 *When a belongs to Class 1, any function $q_\alpha \in K_L(\alpha)$ is increasing on $\mathbb{R}$.*

3.3.1 The Cauchy problem

In what follows, we will consider $a(t) = b > 0$ for all $t \in \mathbb{R}$ and the Cauchy problem

$$\begin{cases} \left(\dfrac{q'(t)}{\sqrt{1 + q'(t)^2}} \right)' = bV'(q(t)), \quad t \in \mathbb{R}, \\ q(0) = r_1, \\ q'(0) = r_2, \end{cases} \tag{CP}$$

where $r_1, r_2 \in \mathbb{R}$. In order to study problem (CP), for each $L > 0$ we will consider the following Cauchy problem

$$\begin{cases} \left(\varphi_L \left(|q'(t)|^2\right) q'(t)\right)' = bV'(q(t)), \quad t \in \mathbb{R}, \\ q(0) = r_1, \\ q'(0) = r_2. \end{cases} \tag{$CP)_L$}$$

A simple adaptation of Theorem 1.1 allows us to prove the proposition below. However, for the reader's convenience we write some words of the proof.

Proposition 3.1 *For each $L > 0$, assume that there exists a solution $q \in C^{1,\beta}_{loc}(\mathbb{R})$, for some $\beta \in (0,1)$, for the Cauchy problem $(CP)_L$ such that there exists $r > 0$ satisfying*

(a) $q'(t) \geq 0$ for any $t \in (-r, r)$,

(b) $q \in L^\infty(\mathbb{R})$.

Then, q is unique in $(-r, r)$.

Proof. Suppose that q_1 and q_2 are two solutions of $(CP)_L$ in $C^{1,\beta}_{\text{loc}}(\mathbb{R})$ and set the functions

$$w(t) = \varphi_L(|{q_1}'(t)|^2){q_1}'(t) - \varphi_L(|{q_2}'(t)|^2){q_2}'(t), \quad t \in \mathbb{R}$$

and

$$\psi(t) = bV'(q_1(t)) - bV'(q_2(t)), \quad t \in \mathbb{R}.$$

A direct computation gives

$$w'(t) = \psi(t) \quad \text{and} \quad w(0) = 0,$$

which yields

$$w(t) = \int_0^t w'(s)ds = \int_0^t \psi(s)ds \quad \text{for} \quad t \in \mathbb{R}.$$

Consequently,

$$|w(t)| \leq t \max_{s\in[0,t]} |\psi(s)|, \quad t > 0. \tag{3.32}$$

Now, combining the fact that $V \in C^2(\mathbb{R}, \mathbb{R})$ together with item (b), there is a constant $K > 0$ such that

$$|\psi(t)| = |bV'(q_1(t)) - bV'(q_2(t))| \leq Kb|q_1(t) - q_2(t)| \quad \forall t \in \mathbb{R},$$

Hence, using the equality $q_1(0) = q_2(0) = r_1$,

$$|\psi(t)| \leq K \int_0^t |q_1{}'(s) - q_2{}'(s)| ds, \quad \forall t > 0. \tag{3.33}$$

On the other hand, given $t \in (0, r)$, the item (a) ensures that $q_2{}'(t), q_1{}'(t) \geq 0$. Thus, assuming without loss of generality that $q_1{}'(t) \leq q_2{}'(t)$, the item (e) of Lemma 3.5 guarantees that

$$\varphi_L(|q_2{}'(t)|^2)q_2{}'(t) - \varphi_L(|q_1{}'(t)|^2)q_1{}'(t) = \int_{q_1{}'(t)}^{q_2{}'(t)} \left(\varphi_L(s^2)s\right)' ds \geq 2x_L(q_2{}'(t) - q_1{}'(t)). \tag{3.34}$$

Thereby, by definition of w and (3.34),

$$|q_1{}'(t) - q_2{}'(t)| \leq \frac{1}{2x_L}|w(t)| \quad \forall t \in (0, r). \tag{3.35}$$

Gathering (3.33) and (3.35), we get

$$|\psi(t)| \leq \frac{K}{2x_L} \int_0^t |w(s)| ds \quad \forall t \in (0, r)$$

that combines with (3.32) to provide

$$|w(t)| \leq \frac{K}{2x_L} t \int_0^t |w(s)| ds \quad \forall t \in (0, r).$$

Fixing $A = \frac{K}{2x_L}$ and $\chi(t) = \frac{w(t)}{t}$ for $t \in (0, r)$, we arrive at

$$|\chi(t)| \leq A \int_0^t |w(s)| ds = A \int_0^\epsilon |w(s)| ds + A \int_\epsilon^t s|\chi(s)| ds,$$

for any $0 < \epsilon < t < r$. Now, from Gronwall's inequality found in [78, Theorem 1.2.2],

$$|\chi(t)| \leq \left(A \int_0^\epsilon |w(s)| ds \right) e^{A \int_\epsilon^t s ds} \quad \forall t \in (0, r).$$

Taking $\epsilon \to 0$ we find $w(t) = 0$ for each $t \in (0, r)$, and so,

$$\varphi_L(|q_1{}'(t)|^2)q_1{}'(t) = \varphi_L(|q_2{}'(t)|^2)q_2{}'(t).$$

Now, since $\varphi_L(t^2)t$ is increasing on $(0, +\infty)$ (see for instant the item (e) of the Lemma 3.5) and $q_1(0) = q_2(0)$ we conclude that $q_1 = q_2$ in $(0, r)$. The same argument works for $t \in (-r, 0)$, and the proof is completed. ■

As a byproduct of the last proposition we have the following result.

Theorem 3.4 *Assume that there exists a solution* $q \in C^{1,\beta}_{loc}(\mathbb{R})$, *for some* $\beta \in (0,1)$, *for the Cauchy problem* (CP) *such that there is* $r > 0$ *satisfying*

(a) $q'(t) \geq 0$ *for any* $t \in (-r, r)$,

(b) $q \in W^{1,\infty}(\mathbb{R})$.

Then, q *is unique in* $(-r, r)$.

Proof. Let us assume that q_1 and q_2 are solutions of (CP) in $C^{1,\beta}_{\text{loc}}(\mathbb{R})$ verifying the items (a) and (b). From (b), there exists $L > 0$ such that

$$|q_1'(t)|, |q_2'(t)| \leq \sqrt{L} \text{ for all } t \in \mathbb{R},$$

from where it follows that q_1 and q_2 are solutions of Cauchy problem $(CP)_L$. Therefore, invoking Proposition 3.1, $q_1 = q_2$ in $(-r, r)$, and the proof is complete. ■

3.3.2 Uniqueness of the minimal solution

Our objective in this subsection is to establish the uniqueness (up to translations) of the heteroclinic solution q from $-\alpha$ to α of equation (3.1) when $a(t)$ is a positive constant on $\mathbb{R}$. For this, we prove the following comparing result involving elements of $K_L(\alpha)$.

Lemma 3.10 *(Comparison Lemma) For each* $L > 0$, *if* $q_1, q_2 \in K_L(\alpha)$, *then*

$$q_2(t) \leq q_1(t) \quad \textit{or} \quad q_1(t) \leq q_2(t) \quad \forall t \geq 0.$$

Proof. If $q_1 = q_2$ then there is nothing to do. Now, if $q_1 \neq q_2$, then there exists $t_0 > 0$ such that $q_1(t_0) \neq q_2(t_0)$, and so, we can assume without loss of generality that $q_1(t_0) > q_2(t_0)$. By continuity, there exists $\epsilon > 0$ such that

$$q_1(t) > q_2(t), \quad \forall t \in (-\epsilon + t_0, t_0 + \epsilon). \tag{3.36}$$

In what follows, we define the functions

$$\eta(t) = \begin{cases} \min\{q_1(t), q_2(t)\}, & \text{if} \quad t \geq 0 \\ -\eta(-t), & \text{if} \quad t < 0 \end{cases} \quad \text{and} \quad \zeta(t) = \begin{cases} \max\{q_1(t), q_2(t)\}, & \text{if} \quad t \geq 0 \\ -\zeta(-t), & \text{if} \quad t < 0, \end{cases}$$

that belong to $\Gamma_L(\alpha)$, and so, a direct computation gives $\zeta, \eta \in K_L(\alpha)$. Consequently, ζ and η are solutions in $C^{1,\beta}_{\text{loc}}(\mathbb{R})$, for some $\beta \in (0,1)$, of the modified problem $(AP)_L$. Now, we claim that

$$q_1(t) \geq q_2(t) \text{ for all } t \geq t_0.$$

Indeed, suppose by contradiction that there is $t_1 > t_0$ such that $q_2(t_1) > q_1(t_1)$ and fix $r > 0$ satisfying $t_0 + r > t_1$. So, it is easily seen that the functions

$$\tilde{\zeta}(t) = \zeta(t + t_0) \quad \text{and} \quad \tilde{q}_1(t) = q_1(t + t_0)$$

satisfy the items (a)-(b) of Proposition 3.1 and are solutions for the Cauchy problem $(CP)_L$ on $[0, r)$ with $r_1 = \tilde{q}_1(0)$ and $r_2 = \tilde{q}_1'(0)$, because from (3.36) we infer that $\zeta = q_1$ on $(t_0 - \epsilon, t_0 + \epsilon)$. Invoking Proposition 3.1, $\tilde{\zeta} = \tilde{q}_1$ on $[0, r)$. In particular,

$$q_1(t) = \zeta(t), \quad \forall t \in (t_0, t_0 + r),$$

and hence $q_1(t_1) = \zeta(t_1) = q_2(t_1)$, which is impossible. Therefore, $q_1(t) \geq q_2(t)$ for any $t \geq t_0$. To complete the proof, suppose by contradiction that there is $t_2 \in (0, t_0)$ such that $q_1(t_2) < q_2(t_2)$. Similarly, taking $s \in \mathbb{R}$ such that $t_2 > t_0 + s$, it can be shown that $\zeta = q_1$ on $(t_0 + s, t_0)$. Then, in particular, $q_1(t_2) = \zeta(t_2) = q_2(t_2)$, a contradiction, and the lemma follows. ■

From now on, given $q_\alpha \in K_b(\alpha)$ and $\tau \in \mathbb{R}$, we will denote

$$q_\alpha^\tau(t) = q_\alpha(t + \tau) \text{ for } t \in \mathbb{R}.$$

So, $q_\alpha^\tau \in K_L(\alpha)$ for every $\tau \in \mathbb{R}$. Having this in mind, we have the following result.

Lemma 3.11 *For each $L > 0$, the set $K_L(\alpha)$ admits a unique element, modulo time translation.*

Proof. Firstly, let us consider $q_1, q_2 \in K_L(\alpha)$. So, by the comparison lemma,

$$q_1(t) \leq q_2(t) \quad \text{or} \quad q_2(t) \leq q_1(t), \quad \forall t \geq 0.$$

Without loss of generality we may assume that

$$q_1(t) \leq q_2(t), \quad \forall t \geq 0. \tag{3.37}$$

Our aim now is to prove that

$$q_2(t) = q_1^\tau(t) \text{ for all } t \in \mathbb{R},$$

for some $\tau \in \mathbb{R}$. To see this, let us note that from (V_7)-(i) there is $\lambda \in (0, \alpha)$ such that $V''(t) > 0$ for each $t \in (\alpha - \lambda, \alpha + \lambda)$, and so,

$$V' \text{ is increasing on } (\alpha - \lambda, \alpha + \lambda). \tag{3.38}$$

Now, since q_2 is a heteroclinic solution from $-\alpha$ to α, then we can take $t_1 > 0$ such that $q_2(t_1) = \alpha - \lambda$. Hence, as q_1 is increasing on $\mathbb{R}$ (see Lemma 3.9), there is $\tau \geq 0$ such that $q_1(t_1 + \tau) = \alpha - \lambda$. Therefore, $q_2(t_1) = q_1^\tau(t_1)$ and

$$q_1^\tau(t), q_2(t) \in (\alpha - \lambda, \alpha + \lambda) \quad \forall t \geq t_1. \tag{3.39}$$

Setting the functions

$$\psi_1(t) = \begin{cases} (q_1^\tau - q_2)^+(t), & \text{if } \ t \geq t_1 \\ 0, & \text{if } \ t < t_1 \end{cases} \quad \text{and} \quad \psi_2(t) = \begin{cases} (q_2 - q_1^\tau)^+(t), & \text{if } \ t \geq t_1 \\ 0, & \text{if } \ t < t_1, \end{cases}$$

a simple computation yields that $\psi_1, \psi_2 \in H^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} \left(\varphi_L(|q_1^{\tau\prime}|^2) q_1^{\tau\prime} \psi_1' - \varphi_L(|q_2{}'|^2) q_2{}' \psi_1' \right) dt = b \int_{\mathbb{R}} \left(V'(q_2) - V'(q_1^\tau) \right) \psi_1 dt.$$

In this way, putting

$$P_1 = \{ t \in \mathbb{R} : q_1^\tau(t) \geq q_2(t) \},$$

from (3.38) and (3.39) we get

$$\int_{P_1 \cap (t_1, +\infty)} \left(\varphi_L(|q_1^{\tau\prime}|^2) q_1^{\tau\prime} - \varphi_L(|q_2{}'|^2) q_2{}' \right) \left(q_1^{\tau\prime} - q_2{}' \right) dt \leq 0.$$

From Lemma A.8-(c),

$$0 < \left(\varphi_L(|s|^2) s - \varphi_L(|r|^2) r \right) (s - r) \text{ for all } s, r \in \mathbb{R} \text{ with } s \neq r.$$

This fact implies $q_1^{\tau\prime} = q_2{}'$ on $P_1 \cap (t_1, +\infty)$. Similarly,

$$\int_{P_2 \cap (t_1, +\infty)} \left(\varphi_L(|q_2'|^2) q_2{}' - \varphi_L(|q_1^{\tau\prime}|^2) q_1^{\tau\prime} \right) \left(q_2{}' - q_1^{\tau\prime} \right) dt \leq 0,$$

where

$$P_2 = \{ t \in \mathbb{R} : q_1^\tau(t) \leq q_2(t) \},$$

and so $q_1^{\tau\prime} = q_2{}'$ on $P_2 \cap (t_1, +\infty)$. Since $P_1 \cup P_2 = (t_1, +\infty)$ and $q_2(t_1) = q_1^\tau(t_1)$ we infer that

$$q_2(t) = q_1^\tau(t) \ \text{ for any } \ t \in [t_1, +\infty).$$

Finally, the Proposition 3.1 ensures that $q_2 = q_1^\tau$ on $(0, 2t_1)$, which finishes the proof. ∎

With the above results we may now establish a result about the existence, uniqueness and qualitative properties of solutions for the problem (3.1) in the constant case.

Theorem 3.5 *Assume $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2), (V_7) and that a belongs to Class 1. For each $L > 0$ there exists $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0)$ equation (3.1) possesses a unique heteroclinic solution q_α from $-\alpha$ to α in $K_L(\alpha) \cap C^{1,\beta}_{loc}(\mathbb{R})$, for some $\beta \in (0, 1)$, satisfying*

(a) q_α is odd,

(b) $0 < q_\alpha(t) < \alpha$ for all $t > 0$,

(c) q_α is increasing on $\mathbb{R}$,

(d) $|q'_\alpha(t)| < \sqrt{L}$ for any $t \in \mathbb{R}$.

Proof. From Theorem 3.3, given $L > 0$ there is $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$ there exists $q_\alpha \in K_L(\alpha) \cap C^{1,\beta}_{\text{loc}}(\mathbb{R})$, for some $\beta \in (0, 1)$. Moreover, q_α verifies the items (a)-(d). In order to show the uniqueness let $q \in K_L(\alpha)$. So, by Lemma 3.11 there is $\tau \in \mathbb{R}$ such that $q = q^\tau_\alpha$. Since q is odd, $0 = q(0) = q_\alpha(\tau)$, from where it follows that $\tau = 0$, and the proof is complete. ■

3.4 Final remarks

In this subsection, we present some additional observations and comments on the results discussed in this chapter. We would like to start by pointing out that in the study carried out in Section 3.1, it was not necessary for $\phi(0)$ to be well defined, which leads to the conclusion that the classic case $\phi(t) = t^{p-2}$ with $p \in (1, 2)$ fits that scenario. Moreover, we would also like to emphasize that the condition (V_8) was not necessary to prove the existence of heteroclinic solution for (3.2), however it together with (ϕ_3) are used to obtain more information about the behavior of the heteroclinic solution.

Reexamining what was done in Proposition 3.1, we can see that Theorem 1.5, which involves a quasilinear Cauchy problem, can be refined when ϕ satisfies the following condition

$(\tilde{\phi}_1)$ $\phi > 0$ on $(0, +\infty)$ and there exists $c > 0$ such that $(\phi(t)t)' \geq c$ for all $t > 0$.

Specifically, we have the following result:

Theorem 3.6 *Assume $a \in L^\infty(\mathbb{R})$, $(\tilde{\phi}_1)$ and that there is a solution $q \in C^{1,\gamma}_{loc}(\mathbb{R})$, for some $\gamma \in (0,1)$, for the following quasilinear Cauchy problem*

$$\begin{cases} (\phi(|q'(t)|)q'(t))' = a(t)V'(q(t)) \quad t \in \mathbb{R}, \\ q(0) = q_0, \\ q'(0) = q_0', \end{cases}$$

such that there exists $r > 0$ satisfying

(a) $q'(t) \geq 0$ for any $t \in (-r, r)$,

(b) $q \in L^\infty(\mathbb{R})$.

Then, q is unique in $(-r, r)$.

As a direct consequence of the above result, Theorem 3.4, which involves a Cauchy problem for the prescribed mean curvature operator, can be replaced by introducing the factor $a(t)$, as follows:

Theorem 3.7 *Assume $a \in L^\infty(\mathbb{R})$ that there is a solution $q \in C^{1,\gamma}_{loc}(\mathbb{R})$, for some $\gamma \in (0,1)$, for the Cauchy problem*

$$\begin{cases} \left(\dfrac{q'(t)}{\sqrt{1+q'(t)^2}}\right)' = a(t)V'(q(t)), \quad t \in \mathbb{R}, \\ q(0) = q_0, \\ q'(0) = q_0', \end{cases} \tag{CP}$$

such that there is $r > 0$ satisfying

(a) $q'(t) \geq 0$ for any $t \in (-r, r)$,

(b) $q \in W^{1,\infty}(\mathbb{R})$.

Then, q is unique in $(-r, r)$.

Finally, we would like to point out that in Subsection 3.3 we show that the heteroclinic problem

$$-\left(\frac{q'}{\sqrt{1+(q')^2}}\right)' + bV'(q) = 0 \text{ in } \mathbb{R}, \; q(0) = 0, \; \lim_{t \to \pm\infty} q(t) = \pm\alpha, \tag{3.40}$$

has a unique minimal heteroclinic solution whenever the global minimum α of V satisfies $\alpha \in (0, \alpha_0)$ for some $\alpha_0 > 0$. An interesting question is whether uniqueness holds for heteroclinic solutions that are not necessarily minimal. This problem was answered positively by Alves, Isneri and Montecchiari in [18], where the authors proved that under certain conditions in potential V, there exists $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$ the problem (3.40) has a unique twice differentiable solution q in $C^{1,\gamma}_{\text{loc}}(\mathbb{R})$ for some $\gamma \in (0,1)$ satisfying the following exponential decay estimates

$$0 < \alpha - q(t) \leq \theta_1 e^{-\theta_2 t} \quad \text{and} \quad 0 < q'(t) \leq \beta_1 e^{-\beta_2 t} \quad \text{for all } t \geq 0$$

and

$$0 < \alpha + q(t) \leq \theta_3 e^{\theta_4 t} \quad \text{and} \quad 0 < q'(t) \leq \beta_3 e^{\beta_4 t} \quad \text{for all } t \leq 0,$$

for some real numbers $\theta_i, \beta_i > 0$. Moreover, when $V(t) = (t^2 - \alpha^2)^2$ then q satisfies

$$\alpha \tanh\left(\alpha\sqrt{2b}t\right) \leq q(t) \leq \alpha \tanh\left(\frac{\alpha\sqrt{2b}t}{\kappa}\right) \quad \text{for} \quad t \geq 0,$$

where κ is a positive constant that depends on $\|q'\|_{L^\infty(\mathbb{R})}$.

CHAPTER 4

HETEROCLINIC SOLUTIONS FOR PRESCRIBED MEAN CURVATURE EQUATIONS IN $\mathbb{R}^2$

The purpose of this chapter consists in using variational methods to establish the existence of heteroclinic solutions for some classes of prescribed mean curvature equations of the type

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+A(\epsilon x,y)V'(u)=0 \quad \text{in} \quad \mathbb{R}^2, \tag{4.1}$$

where $\epsilon > 0$ and V is a double-well potential with minima at $t = \alpha$ and $t = \beta$ with $\alpha < \beta$. Here, we consider some class of functions $A(x, y)$ that are oscillatory in the variable y and satisfy different geometric conditions such as periodicity in all variables or asymptotically periodic at infinity, for more details, see classes A, B, C and D listed in the introduction. The idea here is to reduce the study of (4.1) to a equation of the form

$$-\Delta_\Phi u + A(\epsilon x, y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{4.2}$$

and for that, it was necessary to truncate the following function involved in the prescribed mean curvature operator

$$\phi(t) = \frac{1}{\sqrt{1+t^2}}$$

and develop new estimates. In this chapter we also intend to analyze the qualitative properties of heteroclinic solutions, as well as the regularity of these solutions. A part of our arguments was inspired by papers due to Rabinowitz [79] and Alves [13].

4.1 Existence of heteroclinic solution for quasilinear equations

The goal of this section is to establish the existence of a solution for (4.2) that is heteroclinic in both x taking into account the case where A assumes different geometric conditions. The proof of the existence of solution is given by a minimization argument. To formulate the minimization problem of this section, let us first consider the infinite strip $\Omega = \mathbb{R}\times(0,1)$ of $\mathbb{R}^2$ and for each $j \in \mathbb{Z}$ we define the functional $a_j : W^{1,\Phi}_{\text{loc}}(\Omega) \to \mathbb{R}\cup\{+\infty\}$ by

$$a_j(w) = \iint_{\Omega_j} \mathcal{L}(w)dxdy, \quad w \in W^{1,\Phi}_{\text{loc}}(\Omega),$$

where $\Omega_j = (j, j+1) \times (0,1)$ and

$$\mathcal{L}(w) = \Phi(|\nabla w|) + A(\epsilon x, y)V(w).$$

Under this notation, we also define the energy functional $I : W^{1,\Phi}_{\text{loc}}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$I(w) = \sum_{j\in\mathbb{Z}} a_j(w), \quad w \in W^{1,\Phi}_{\text{loc}}(\Omega).$$

In what follows, for each $k \in \mathbb{Z}$ and $w \in W^{1,\Phi}_{\text{loc}}(\Omega)$ we consider function $\tau_k w$ given by

$$\tau_k w(x,y) = w(x+k, y) \ \text{ for all } \ (x,y) \in \Omega.$$

Clearly, $\tau_0 w \equiv w$ on Ω. Hereafter, let us identify $\tau_k w|_{\Omega_0}$ with $\tau_k w$ itself. Now, for the purposes of this section, we will designates by $\Gamma_\Phi(\alpha,\beta)$ the class of admissible functions given by

$$\Gamma_\Phi(\alpha,\beta) = \Big\{w \in W^{1,\Phi}_{\text{loc}}(\Omega) : \tau_k w \to \alpha \text{ in } L^\Phi(\Omega_0) \text{ as } k \to -\infty \text{ and } \tau_k w \to \beta \text{ in } L^\Phi(\Omega_0) \text{ as } k \to +\infty\Big\}. \tag{4.3}$$

We would like to point out that $\tau_k w$ goes to α in $L^\Phi(\Omega_0)$ as k goes to $-\infty$ if, and only if,

$$\iint_{\Omega_k} \Phi(|w-\alpha|)dxdy \to 0 \text{ as } k \to -\infty.$$

Here the fact that Φ satisfies Δ_2-condition applies an important rule in the proof of the last limit. Analogously, $\tau_k w$ goes to β in $L^\Phi(\Omega_0)$ as k goes to $+\infty$ if, and only if,

$$\iint_{\Omega_k} \Phi(|w-\beta|)dxdy \to 0 \text{ as } k \to +\infty.$$

On the other hand, it is easy to check that the class $\Gamma_\Phi(\alpha,\beta)$ is not empty, because the function $\varphi_{\alpha,\beta} : \Omega \to \mathbb{R}$ defined by

$$\varphi_{\alpha,\beta}(x,y) = \begin{cases} \beta, & \text{if} \quad \beta \le x \quad \text{and} \quad y \in (0,1), \\ x, & \text{if} \quad \alpha \le x \le \beta \quad \text{and} \quad y \in (0,1), \\ \alpha, & \text{if} \quad x \le \alpha \quad \text{and} \quad y \in (0,1) \end{cases} \tag{4.4}$$

belongs to $\Gamma_\Phi(\alpha,\beta)$. By the properties of Φ, A and V,

$$a_j(w) \ge 0 \text{ for all } j \in \mathbb{Z} \text{ and } w \in \Gamma_\Phi(\alpha,\beta),$$

and hence, I is bounded from below on $\Gamma_\Phi(\alpha,\beta)$. Furthermore, it is easy to see that the function given in (4.4) has finite energy, that is, $I(\varphi_{\alpha,\beta}) < +\infty$, and so,

$$c_\Phi(\alpha,\beta) = \inf_{w \in \Gamma_\Phi(\alpha,\beta)} I(w)$$

is well defined. Here it is worth mentioning that we will see throughout this section that critical points of the functional I on the class $\Gamma_\Phi(\alpha,\beta)$ are heteroclinic solution from α to β for the equation (4.2).

4.1.1 The case periodic

In this subsection, we intend to investigate the existence of a heteroclinic solution from α to β for (4.2) with $\epsilon = 1$ by assuming that A belongs to Class A and, unless indicated, the potential V satisfies the assumptions $(\tilde{V}_1)$-$(\tilde{V}_3)$. With the preliminaries contained at the beginning of this section we may state and prove our first result that will be useful in the next lemma.

Lemma 4.1 *If $w \in \Gamma_\Phi(\alpha,\beta)$, then for all $k \in \mathbb{Z}$ we have that $\tau_k w \in \Gamma_\Phi(\alpha,\beta)$ and $I(\tau_k w) = I(w)$.*

Proof. Initially, it is easy to see that $\tau_k w \in \Gamma_\Phi(\alpha,\beta)$ for any $k \in \mathbb{Z}$ and $w \in \Gamma_\Phi(\alpha,\beta)$. On the other hand, for each $j \in \mathbb{Z}$, a simple change variable combined with the periodicity of

A in the variable x leads to

$$\begin{aligned} a_j(\tau_k w) &= \iint_{\Omega_j} (\Phi(|\nabla \tau_k w|) + A(x,y)V(\tau_k w))\, dxdy \\ &= \iint_{\Omega_j} (\Phi(|\nabla w(x+k,y)|) + A(x+k,y)V(w(x+k,y)))\, dxdy \\ &= \iint_{\Omega_{j+k}} (\Phi(|\nabla w|) + A(x,y)V(w))\, dxdy = a_{j+k}(w), \end{aligned}$$

from where it follows that

$$I(\tau_k w) = \sum_{j\in\mathbb{Z}} a_j(\tau_k w) = \sum_{j\in\mathbb{Z}} a_{j+k}(w) = \sum_{j\in\mathbb{Z}} a_j(w) = I(w),$$

and the proof is completed. ■

Now we employ the Lemma 4.1 to prove that the energy functional I reaches the minimum energy in some function of $\Gamma_\Phi(\alpha,\beta)$.

Proposition 4.1 *There exists $u \in \Gamma_\Phi(\alpha,\beta)$ such that $I(u) = c_\Phi(\alpha,\beta)$ and*

$$\alpha \le u(x,y) \le \beta \text{ almost everywhere in } \Omega.$$

Proof. Let (u_n) be a minimizing sequence for I on $\Gamma_\Phi(\alpha,\beta)$, that is, $I(u_n) \to c_\Phi(\alpha,\beta)$ as $n \to +\infty$. Thus, there is a constant $M > 0$ verifying

$$I(u_n) \le M \quad \text{for all } n \in \mathbb{N}. \tag{4.5}$$

We claim that we may assume without loss of generality that the sequence u_n satisfies

$$\alpha \le u_n(x,y) \le \beta \text{ for all } (x,y) \in \Omega \text{ and } n \in \mathbb{N}.$$

Indeed, just consider

$$\tilde{u}_n(x,y) = \max\{\alpha, \min\{u_n(x,y),\beta\}\}, \quad (x,y) \in \Omega,$$

instead of u_n. Moreover, we can also assume that for each $n \in \mathbb{N}$,

$$\iint_{\Omega_0} \Phi(|u_n - \alpha|)dxdy > \delta \text{ and } \iint_{\Omega_{k-1}} \Phi(|u_n - \alpha|)dxdy \le \delta \text{ for } k \le 0, \tag{4.6}$$

for some $\delta > 0$ such that

$$\delta < \Phi(\beta - \alpha). \tag{4.7}$$

Indeed, note first that for each $n \in \mathbb{N}$ fixed,

$$\tau_k u_n \to \beta \text{ in } L^\Phi(\Omega_0) \text{ as } k \to +\infty,$$

and so, since $\alpha \neq \beta$ and $\Phi \in \Delta_2$, there are $\delta > 0$ and a subsequence of $(\tau_k u_n)_{k\geq 0}$, still denoted $(\tau_k u_n)$, such that

$$\iint_{\Omega_0} \Phi(|\tau_k u_n - \alpha|)dxdy > \delta, \quad \forall k > 0.$$

Note that, without loss of generality, we may assume that δ satisfies (4.7). On the other hand, using again the fact that $\Phi \in \Delta_2$, $\tau_k u_n$ goes to α in $L^\Phi(\Omega_0)$ as k goes to $-\infty$ implies

$$\iint_{\Omega_0} \Phi(|\tau_k u_n - \alpha|)dxdy \to 0 \text{ as } k \to -\infty,$$

and so, there exists an integer $\overline{k}_n < 0$ such that

$$\iint_{\Omega_0} \Phi(|\tau_k u_n - \alpha|)dxdy \leq \delta, \quad \forall k \leq \overline{k}_n.$$

From this, it is possible to find the bigger integer $k_n \in \mathbb{Z}$ such that

$$\iint_{\Omega_{k-1}} \Phi(|u_n - \alpha|)dxdy \leq \delta \text{ for all } k \leq k_n \text{ and } \iint_{\Omega_{k_n}} \Phi(|u_n - \alpha|)dxdy > \delta,$$

that is,

$$\iint_{\Omega_{j-1}} \Phi(|\tau_{k_n} u_n - \alpha|)dxdy \leq \delta \text{ for all } j \leq 0 \text{ and } \iint_{\Omega_0} \Phi(|\tau_{k_n} u_n - \alpha|)dxdy > \delta.$$

Now, we can apply Lemma 4.1 to consider $\tau_{k_n} u_n$ in the place of u_n.

Now, since $\alpha \leq u_n \leq \beta$ in Ω, it is straightforward to check that (u_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega)$. Thereby, in view of Lemma A.4, $W^{1,\Phi}(K)$ is reflexive Banach spaces whenever K is relatively compact in Ω, and so, by a classical diagonal argument, there are a subsequence of (u_n), still denoted by (u_n), and $u \in W^{1,\Phi}_{\text{loc}}(\Omega)$ satisfying

$$u_n \rightharpoonup u \ \text{ in } \ W^{1,\Phi}_{\text{loc}}(\Omega) \text{ as } n \to +\infty, \tag{4.8}$$

$$u_n \to u \ \text{ in } \ L^{\Phi}_{\text{loc}}(\Omega) \text{ as } n \to +\infty \tag{4.9}$$

and

$$u_n(x,y) \to u(x,y) \ \text{ a.e. in } \ \Omega \text{ as } n \to +\infty. \tag{4.10}$$

As a consequence of (4.10),

$$\alpha \leq u(x,y) \leq \beta \text{ almost everywhere in } \Omega. \tag{4.11}$$

Moreover, from (4.5), we have the inequality below

$$\int_D \int_{-j}^{j} \mathcal{L}(u_n)dxdy \leq M \quad \forall n, j \in \mathbb{N},$$

which combines with weak lower semicontinuity of I to give

$$\int_D \int_{-j}^{j} \mathcal{L}(u)dxdy \leq M, \quad \forall j \in \mathbb{N}.$$

Therefore, since $j \in \mathbb{N}$ is arbitrary, we conclude that $I(u) \leq M$. With the aid of the previous preliminaries, our goal is to ensure that u belongs to $\Gamma_\Phi(\alpha, \beta)$. Towards that end, we will show that

$$\tau_k u \to \alpha \text{ in } L^\Phi(\Omega_0) \text{ as } k \to -\infty. \tag{4.12}$$

To show (4.12), let us consider the sequence $(\tau_k u)_{k\leq 0}$ with $k \in \mathbb{Z}$. Due to Lemma 4.1 and the estimate (4.11), it is simple to prove that $(\tau_k u)_{k\leq 0}$ is bounded in $W^{1,\Phi}(\Omega_0)$. Consequently, for some subsequence, there exists $u^* \in W^{1,\Phi}(\Omega_0)$ such that

$$\tau_k u \rightharpoonup u^* \text{ in } W^{1,\Phi}(\Omega_0) \ \text{ as } \ k \to -\infty, \tag{4.13}$$

$$\tau_k u \to u^* \ \text{ in } \ L^\Phi(\Omega_0) \text{ as } k \to -\infty \tag{4.14}$$

and

$$\alpha \leq u^*(x,y) \leq \beta \ \text{ almost everywhere on } \ \Omega_0. \tag{4.15}$$

Now, since $I(u) \leq M$, the definition of I ensures that

$$a_k(u) \to 0 \text{ as } |k| \to +\infty.$$

This together with the periodicity of A yields that

$$a_0(\tau_k u) \to 0 \text{ as } |k| \to +\infty. \tag{4.16}$$

Now, the fact that a_0 is weakly lower semicontinuous on $W^{1,\Phi}(\Omega_0)$ and $a_0 \geq 0$ together (4.13) and (4.16) guarantee that $a_0(u^*) = 0$. Thereby, (4.15) together with the assumptions on functions A and V ensures that $u^* = \alpha$ or $u^* = \beta$ a.e. in Ω_0. On the other hand, it follows from (4.6) and (4.9) that

$$\iint_{\Omega_0} \Phi(|u-\alpha|)dxdy \geq \delta \text{ and } \iint_{\Omega_0} \Phi(|\tau_k u-\alpha|)dxdy \leq \delta \text{ for } k<0.$$

Consequently, taking the limit as $k \to -\infty$ in the inequality above and employing (4.14), we arrive at

$$\iint_{\Omega_0} \Phi(|u^*-\alpha|)dxdy \leq \delta.$$

From (4.7), one has $u^* = \alpha$ a.e. in Ω_0, showing that the limit (4.12) is valid. Now we claim that

$$\tau_k u \to \beta \text{ in } L^{\Phi}(\Omega_0) \text{ as } k \to +\infty. \tag{4.17}$$

Indeed, considering the sequence $(\tau_k u)_{k>0}$ with $k \in \mathbb{N}$, there exist $u^{**} \in W^{1,\Phi}(\Omega_0)$ and a subsequence of $(\tau_k u)$, still denoted $(\tau_k u)$, such that

$$\tau_k u \rightharpoonup u^{**} \text{ in } W^{1,\Phi}(\Omega_0) \text{ as } k \to +\infty, \tag{4.18}$$

$$\tau_k u \to u^{**} \text{ in } L^{\Phi}(\Omega_0) \text{ as } k \to +\infty, \tag{4.19}$$

$$\tau_k u \to u^{**} \text{ in } L^{1}(\Omega_0) \text{ as } k \to +\infty \tag{4.20}$$

and

$$\tau_k u(x,y) \to u^{**}(x,y) \text{ a.e in } \Omega_0 \text{ as } k \to +\infty. \tag{4.21}$$

Arguing as above, we will get that $u^{**} = \alpha$ or $u^{**} = \beta$ a.e in Ω_0. The claim (4.17) follows if we prove that $u^{**} = \beta$ a.e in Ω_0, and to do that, we will split the proof into two steps. So, seeking for a contradiction we assume that $u^{**} = \alpha$ a.e. in Ω_0.

Step 1: There are $\epsilon_0 > 0$ and $n_1 \in \mathbb{N}$ such that

$$a_{-1}(u_n) + a_0(u_n) = \int_0^1 \int_{-1}^1 (\Phi(|\nabla u_n|) + A(x,y)V(u_n))\,dxdy \geq \epsilon_0, \quad \forall n \geq n_1. \tag{4.22}$$

Indeed, if this does not hold, then there is a subsequence (u_{n_i}) of (u_n) such that

$$\int_0^1 \int_{-1}^1 (\Phi(|\nabla u_{n_i}|) + A(x,y)V(u_{n_i}))\,dxdy \to 0.$$

Consequently, there is $v \in W^{1,\Phi}((-1,1)\times(0,1))$ such that

$$u_{n_i} \rightharpoonup v \quad \text{in} \quad W^{1,\Phi}((-1,1)\times(0,1)), \quad u_{n_i} \to v \quad \text{in} \quad L^{\Phi}((-1,1)\times(0,1))$$

and

$$v = \alpha \quad \text{or} \quad v = \beta \quad \text{a.e. in} \quad (-1,1)\times(0,1). \tag{4.23}$$

Making a simple analysis of the estimates contained in (4.6), we infer that (4.23) is impossible, which ends this step.

To proceed to the next step, let us fix $\tilde{\epsilon} \in (0, \epsilon_0/2)$ and $n_0 \geq n_1$ such that

$$I(u_n) \leq c_{\Phi}(\alpha,\beta) + \frac{\tilde{\epsilon}}{2} \;\; \forall n \geq n_0, \tag{4.24}$$

where ϵ_0 and n_1 were given in Step 1.

Step 2: There are $k \in \mathbb{N}$ and $n \geq n_0$ large enough satisfying

$$a_0\left(x(\tau_k u_n - \alpha) + \alpha\right) \leq \frac{\tilde{\epsilon}}{2}. \tag{4.25}$$

In order to show estimate (4.25), we will separately analyze the terms of the functional a_0 that will be divided into four parts as follows:

Part 1: There exists $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ there is $n(k) \geq n_0$ verifying

$$\iint_{\Omega_0} A(x,y)V(\tau_k u_n)dxdy \leq \frac{\tilde{\epsilon}}{24 \cdot 4^m} \quad \forall n \geq n(k), \tag{4.26}$$

$$\iint_{\Omega_0} A(x,y)V\left(x(\tau_k u_n - \alpha) + \alpha\right)dxdy \leq \frac{\tilde{\epsilon}}{4} \quad \forall n \geq n(k) \tag{4.27}$$

and

$$\iint_{\Omega_0} \Phi(|\tau_k u_n - \alpha|)dxdy \leq \frac{\tilde{\epsilon}}{12 \cdot 4^m} \quad \forall n \geq n(k). \tag{4.28}$$

In fact, let us initially note that, since $V \in C^1$ and $\tau_k u(x,y) \in [\alpha, \beta]$ for any $(x,y) \in \Omega_0$, the Mean Value Theorem together with $(\tilde{V}_2)$ gives us

$$V(\tau_k u),\ V(x(\tau_k u - \alpha) + \alpha) \leq R|\tau_k u - \alpha| \quad \forall (x,y) \in \Omega_0,$$

for some $R > 0$. Consequently,

$$\iint_{\Omega_0} A(x,y)V(x(\tau_k u - \alpha) + \alpha)dxdy \leq R \sup_{\Omega_0} A(x,y) \iint_{\Omega_0} |\tau_k u - \alpha| dxdy$$

and

$$\iint_{\Omega_0} A(x,y)V(\tau_k u)dxdy \leq R \sup_{\Omega_0} A(x,y) \iint_{\Omega_0} |\tau_k u - \alpha| dxdy.$$

Now, as we are assuming that $u^{**} = \alpha$ a.e in Ω_0, it follows from (4.20) that there is $k_0 \in \mathbb{N}$ such that

$$\iint_{\Omega_0} A(x,y)V(x(\tau_k u - \alpha) + \alpha)dxdy \leq \frac{\tilde{\epsilon}}{8} \quad \forall k \geq k_0 \tag{4.29}$$

and

$$\iint_{\Omega_0} A(x,y)V(\tau_k u)dxdy \leq \frac{\tilde{\epsilon}}{48 \cdot 4^m} \quad \forall k \geq k_0, \tag{4.30}$$

where m was given in (ϕ_2). Furthermore, from (4.19), increasing k_0 if necessary, one gets

$$\iint_{\Omega_0} \Phi(|\tau_k u - \alpha|)dxdy \leq \frac{\tilde{\epsilon}}{24 \cdot 8^m} \quad \forall k \geq k_0. \tag{4.31}$$

On the other hand, for each $k \in \mathbb{N}$ fixed, Lebesgue Dominated Convergence Theorem yields

$$\left| \iint_{\Omega_0} A(x,y)\left(V(x(\tau_k u_n - \alpha) + \alpha) - V(x(\tau_k u - \alpha) + \alpha)\right)dxdy \right| \to 0 \text{ as } n \to +\infty$$

and

$$\left|\iint_{\Omega_0} A(x,y)\left(V(\tau_k u_n) - V(\tau_k u)\right) dxdy\right| \to 0 \text{ as } n \to +\infty.$$

Moreover, as $\Phi \in \Delta_2$, for each $k \in \mathbb{N}$ we can use the limit (4.9) to find

$$\iint_{\Omega_0} \Phi(|\tau_k u_n - \tau_k u|)dxdy \to 0 \text{ as } n \to +\infty.$$

With everything, for every $k \geq k_0$ there exists $n(k) \geq n_0$ satisfying

$$\iint_{\Omega_0} A(x,y)V(x(\tau_k u_n - \alpha) + \alpha)dxdy \leq \iint_{\Omega_0} A(x,y)V(x(\tau_k u - \alpha) + \alpha)dxdy + \frac{\tilde{\epsilon}}{8} \quad \forall n \geq n(k), \tag{4.32}$$

$$\iint_{\Omega_0} A(x,y)V(\tau_k u_n)dxdy \leq \iint_{\Omega_0} A(x,y)V(\tau_k u)dxdy + \frac{\tilde{\epsilon}}{48 \cdot 4^m} \quad \forall n \geq n(k), \tag{4.33}$$

and

$$\iint_{\Omega_0} \Phi(|\tau_k u_n - \tau_k u|)dxdy \leq \frac{\tilde{\epsilon}}{24 \cdot 8^m} \quad \forall n \geq n(k). \tag{4.34}$$

Finally, analyzing all the estimates, a direct computation of (4.29)-(4.30) and (4.32)-(4.33) load to (4.26) and (4.27). To see the inequality (4.28), note that from Lemma A.8-(a) one gets

$$\Phi(|\tau_k u_n - \alpha|) \leq 2^m \left(\Phi(|\tau_k u_n - \tau_k u|) + \Phi(|\tau_k u - \alpha|)\right).$$

Now, (4.28) follows from (4.31) and (4.34), finishing the first part.

Part 2: There are $k \geq k_0$ and $n \geq n(k)$ such that

$$a_0(\tau_k u_n) \leq \frac{\tilde{\epsilon}}{12 \cdot 4^m}. \tag{4.35}$$

If the estimate above does not occur, for each $k \geq k_0$ there is $j(k) \geq n(k)$ satisfying

$$a_0(\tau_k u_j) > \frac{\tilde{\epsilon}}{12 \cdot 4^m} \quad \forall j \geq j(k).$$

Then, by definition of a_0,

$$\iint_{\Omega_0} \Phi(|\nabla(\tau_k u_j)|)dxdy \geq \frac{\tilde{\epsilon}}{12 \cdot 4^m} - \iint_{\Omega_0} A(x,y)V(\tau_k u_j)dxdy, \quad \forall j \geq j(k),$$

which combines with (4.26) to give

$$\iint_{\Omega_0} \Phi(|\nabla(\tau_k u_j)|)dxdy \geq \frac{\tilde{\epsilon}}{24 \cdot 4^m}, \quad \forall j \geq j(k).$$

Now let be $p \in \mathbb{N}$ such that

$$(p+1)\frac{\tilde{\epsilon}}{24 \cdot 4^m} > M,$$

where M was given in (4.5). Fixing $i \in \mathbb{N}$ such that $i > \max\{j(k) : k_0 \leq k \leq k_0 + p\}$ we have

$$I(u_i) \geq \sum_{t=k_0}^{k_0+p} a_t(u_i) = \sum_{t=k_0}^{k_0+p} a_0(\tau_t u_i) \geq \sum_{t=k_0}^{k_0+p} \left(\iint_{\Omega_0} \Phi(|\nabla(\tau_t u_i)|) dxdy \right) \geq (p+1) \frac{\tilde{\epsilon}}{24 \cdot 4^m} > M,$$

which contradicts (4.5), showing (4.35).

Part 3: For k and n as in Part 2, one has

$$\iint_{\Omega_0} \Phi(|\partial_x(\tau_k u_n)|) dxdy, \iint_{\Omega_0} \Phi(|\partial_y(\tau_k u_n)|) dxdy \leq \frac{\tilde{\epsilon}}{12 \cdot 4^m}. \tag{4.36}$$

Indeed, just notice that the inequality at (4.35) together with the facts that Φ is increasing on $[0, +\infty)$ and

$$|\partial_x(\tau_k u_n)|, |\partial_y(\tau_k u_n)| \leq |\nabla(\tau_k u_n)|,$$

leads to estimate (4.36).

Part 4: For k and n as in Part 2, one has

$$\iint_{\Omega_0} \Phi(|\nabla(x(\tau_k u_n - \alpha) + \alpha)|) dxdy \leq \frac{\tilde{\epsilon}}{4}. \tag{4.37}$$

To show the estimate (4.37), we first observe that

$$\partial_x \left(x(\tau_k u_n - \alpha) + \alpha \right) = \tau_k u_n - \alpha + x \partial_x(\tau_k u_n)$$

and

$$\partial_y \left(x(\tau_k u_n - \alpha) + \alpha \right) = x \partial_y \left(\tau_k u_n \right).$$

Therefore, from Lemma A.8-(a),

$$\Phi(|\nabla(x(\tau_k u_n - \alpha) + \alpha)|) \leq 4^m \left(\Phi(|\partial_x(\tau_k u_n)|) + \Phi(|\tau_k u_n - \alpha|) + \Phi(|\partial_y(\tau_k u_n)|) \right) \text{ on } \Omega_0,$$

and the Part 4 follows from Parts 1 and 3.

Finally, the estimate (4.25) contained in Step 2 is immediately verified from (4.27) and (4.37). We are now ready to use Steps 1 and 2 to complete the proof of Claim (4.17). To this end, fix k and n as in Step 2 and define the following function

$$U_n(x, y) = \begin{cases} \alpha, & \text{if } \; x \leq k \quad \text{and} \quad y \in (0, 1), \\ (u_n(x, y) - \alpha)(x - k) + \alpha, & \text{if } \; k \leq x \leq k + 1 \quad \text{and} \quad y \in (0, 1), \\ u_n(x, y), & \text{if } \; x > k + 1 \quad \text{and} \quad y \in (0, 1). \end{cases}$$

So, it is clear that $U_n \in \Gamma_\Phi(\alpha,\beta)$ and

$$a_k(U_n) = a_0(x(\tau_k u_n - \alpha) + \alpha).$$

Hence,

$$c_\Phi(\alpha,\beta) \le I(U_n) = a_k(U_n) + \sum_{j=k+1}^{+\infty} a_j(u_n) \le a_k(U_n) + I(u_n) - a_0(u_n) - a_{-1}(u_n),$$

that is,

$$c_\Phi(\alpha,\beta) \le a_0(x(\tau_k u_n - \alpha) + \alpha) + I(u_n) - a_0(u_n) - a_{-1}(u_n).$$

Invoking estimates (4.22), (4.24) and (4.25), one gets

$$c_\Phi(\alpha,\beta) \le \frac{\tilde{\epsilon}}{2} + c_\Phi(\alpha,\beta) + \frac{\tilde{\epsilon}}{2} - \epsilon_0 = \tilde{\epsilon} + c_\Phi(\alpha,\beta) - \epsilon_0 < c_\Phi(\alpha,\beta) - \frac{\epsilon_0}{2},$$

which is absurd. Therefore, (4.17) occurs and $u \in \Gamma_\Phi(\alpha,\beta)$. To conclude the proof, it remains to show that $I(u) = c_\Phi(\alpha,\beta)$. For this purpose, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{-j}^{j} a_k(u_n) \le c_\Phi(\alpha,\beta) + \epsilon \quad \forall n \ge n_0 \text{ and } \forall j \in \mathbb{N}.$$

Letting $n \to +\infty$ and after $j \to +\infty$, we find

$$I(u) \le c_\Phi(\alpha,\beta) + \epsilon.$$

Since ϵ is arbitrary, we derive that $I(u) = c_\Phi(\alpha,\beta)$, and the proof is completed. ■

In order to find a periodic solution $u(x,y)$ in the variable y for the equation (4.2), we will consider the following class

$$K_\Phi(\alpha,\beta) = \left\{u \in \Gamma_\Phi(\alpha,\beta) : I(u) = c_\Phi(\alpha,\beta), u(x,0) = u(x,1) \text{ in } \mathbb{R}, \alpha \le u \le \beta \text{ a.e. on } \Omega\right\}.$$

Next, we are going to show that $K_\Phi(\alpha,\beta)$ is not empty.

Lemma 4.2 *It holds that $K_\Phi(\alpha,\beta) \neq \emptyset$.*

Proof. Initially, for each $w \in \Gamma_\Phi(\alpha,\beta)$ we set

$$I_1(w) = \iint_{\mathbb{R}\times(0,\frac{1}{2})} \mathcal{L}(w)\,dxdy \quad \text{and} \quad I_2(w) = \iint_{\mathbb{R}\times(\frac{1}{2},1)} \mathcal{L}(w)\,dxdy.$$

Now, choosing $u \in \Gamma_\Phi(\alpha,\beta)$ as in Proposition 4.1, we can write

$$I(u) = I_1(u) + I_2(u) = c_\Phi(\alpha,\beta). \tag{4.38}$$

Suppose for a moment that $I_1(u) \leq I_2(u)$ holds. Then considering the function

$$v(x,y) = \begin{cases} u(x,y), & \text{if} \quad x \in \mathbb{R} \quad \text{and} \quad 0 \leq y \leq \frac{1}{2}, \\ u(x,1-y), & \text{if} \quad x \in \mathbb{R} \quad \text{and} \quad \frac{1}{2} \leq y \leq 1, \end{cases}$$

it is clear that $v \in \Gamma_\Phi(\alpha,\beta)$ and thanks to the assumptions $(\tilde{A}_2)$-$(\tilde{A}_3)$ a straightforward computation gives

$$I_2(v) = I_1(v) = I_1(u). \tag{4.39}$$

According to (4.38) and (4.39),

$$c_\Phi(\alpha,\beta) \leq I(v) = I_1(v) + I_2(v) \leq I(u) = c_\Phi(\alpha,\beta),$$

from where it follows that $I(v) = c_\Phi(\alpha,\beta)$ with $v(x,0) = v(x,1)$ for every $x \in \mathbb{R}$ and $v(x,y) \in [\alpha,\beta]$ a.e. in Ω. On the other hand, if $I_2(u) \leq I_1(u)$ occurs then in this case we define the function

$$\tilde{v}(x,y) = \begin{cases} u(x,1-y), & \text{if} \quad x \in \mathbb{R} \quad \text{and} \quad 0 \leq y \leq \frac{1}{2} \\ u(x,y), & \text{if} \quad x \in \mathbb{R} \quad \text{and} \quad \frac{1}{2} \leq y \leq 1. \end{cases}$$

Consequently, using the same ideas discussed just above, we obtain that $\tilde{v} \in \Gamma_\Phi(\alpha,\beta)$ with $I_1(\tilde{v}) = I_2(\tilde{v}) = I_2(u)$, from where it follows that $I(\tilde{v}) = c_\Phi(\alpha,\beta)$, $\tilde{v}(x,0) = \tilde{v}(x,1)$ for any $x \in \mathbb{R}$ and $\alpha \leq \tilde{v} \leq \beta$ a.e in Ω, which completes the proof. ■

We would like to emphasize here that the functions of $K_\Phi(\alpha,\beta)$ can be extended periodically in the variable y on $\mathbb{R}^2$ with period 1. For this reason, it will be convenient to assume that the elements of $K_\Phi(\alpha,\beta)$ are extended to the whole real plane.

Lemma 4.3 *If $u \in K_\Phi(\alpha,\beta)$, then u is a weak solution of (4.2) with $\epsilon = 1$. Moreover, u is a heteroclinic solution from α to β which belongs to $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$.*

Proof. Initially, let be $u \in K_\Phi(\alpha,\beta)$ and $\psi \in C_0^\infty(\mathbb{R}^2)$. An direct computation shows that

$$\iint_\Omega (\phi(|\nabla u|)\nabla u \nabla \psi + A(x,y)V'(u)\psi)\, dxdy = 0. \tag{4.40}$$

Indeed, taking $v = u + t\psi$ we obtain for x large enough, let us say, $|x| \geq R$ for some $R > 0$, that

$$\iint_{\Omega\setminus((-p,p)\times(0,1))} \mathcal{L}(v)dxdy = \iint_{\Omega\setminus((-p,p)\times(0,1))} \mathcal{L}(u)dxdy, \ \forall p \geq R.$$

Thus,

$$\frac{\partial I}{\partial \psi}(u) = \lim_{t\to 0} \frac{I(u+t\psi)-I(u)}{t}$$
$$= \lim_{t\to 0} \iint_{(-p,p)\times(0,1)} \left(\frac{\Phi(|\nabla(u+t\psi)|) - \Phi(|\nabla u|) + A(x,y)(V(u+t\psi)-V(u))}{t} \right) dxdy,$$

from which it follows that

$$\frac{\partial I}{\partial \psi}(u) = \iint_{(-p,p)\times(0,1)} (\phi(|\nabla u|)\nabla u \nabla \psi + A(x,y)V'(u)\psi)\, dxdy,$$

and therefore, by the arbitrariness of p,

$$\frac{\partial I}{\partial \psi}(u) = \iint_{\Omega} (\phi(|\nabla u|)\nabla u \nabla \psi + A(x,y)V'(u)\psi)\, dxdy.$$

As $v \in \Gamma_\Phi(\alpha,\beta)$, $I(u) \le I(v)$, and so, a standard argument ensures that $\frac{\partial I}{\partial \psi}(u) = 0$, which implies (4.40). The equality (4.40) allows us to use the same arguments found in the proof Theorem 2.2 to prove that

$$\iint_{\mathbb{R}^2} (\phi(|\nabla u|)\nabla u \nabla \psi + A(x,y)V'(u)\psi)\, dxdy = 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}^2),$$

from where it follows that u is a weak solution for (4.2) with $\epsilon = 1$. The assumption (ϕ_2) permits to apply a well known regularity result developed by Lieberman [67, Theorem 1.7] to conclude that $u \in C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$. Moreover, similar to the proof of Theorem 2.2, we also have that u is a heteroclinic solution from α to β, that is,

$$u(x,y) \to \alpha \text{ as } x \to -\infty \text{ and } u(x,y) \to \beta \text{ as } x \to +\infty, \text{ uniformly in } y \in \mathbb{R},$$

and the lemma follows. ■

Now, we will show our last lemma in this subsection, which ends the study of the equation (4.2) in the case where A is periodic in all variables.

Lemma 4.4 *Assume* (ϕ_3) *and* $(\tilde{V}_6)$. *Then, if* $u \in K_\Phi(\alpha,\beta)$ *we have that*

$$\alpha < u(x,y) < \beta \text{ for all } (x,y) \in \mathbb{R}^2.$$

Proof. Let $u \in K_\Phi(\alpha,\beta)$ and observe that $\alpha \le u \le \beta$ on $\mathbb{R}^2$. In what follows, we will show that $u(x,y) < \beta$ for any $(x,y) \in \mathbb{R}^2$. Indeed, assume for the sake of contradiction that there exists $(x_0,y_0) \in \mathbb{R}^2$ such that $u(x_0,y_0) = \beta$. Therefore, by the geometry of u, we can consider a compact set $\mathcal{O}$ contained in $\mathbb{R}^2$ such that there exists $(x_1, y_1) \in \mathcal{O}$ with $u(x_1, y_1) < \beta$. Having that in mind, setting the function $\tilde{\phi} : (0, +\infty) \to (0, +\infty)$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & \text{if} \quad t \in (0, R], \\ \dfrac{\phi(R)t^{s-2}}{R^{s-2}}, & \text{if} \quad t \in (R, +\infty), \end{cases}$$

where $R > \max\{\|\nabla u\|_{L^\infty(\mathcal{O})}, \eta\}$ and the constants s and η were given in (ϕ_3), a direct computation implies that there are positive real numbers γ_1 and γ_2, which dependent on η, s, R, c_1 and c_2, such that

$$\tilde{\phi}(t)t \leq \gamma_1 t^{s-1} \quad \text{and} \quad \tilde{\phi}(t)t^2 \geq \gamma_2 t^s \quad \text{for all} \quad t \geq 0.$$

Using the function $\tilde{\phi}$, let us define the vector measurable function $G : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G(z, t, p) = \frac{\tilde{\phi}(|p|)p}{\gamma_2},$$

which satisfies

$$|G(z, t, p)| \leq \frac{\gamma_1}{\gamma_2}|p|^{s-1} \quad \text{and} \quad pG(z, t, p) \geq |p|^s \quad \text{for all} \quad (z, t, p) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2.$$

Furthermore, we will also consider the scalar measurable function $B : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$B(z, t, p) = \frac{A(z)V'(\beta - t)}{\gamma_2}.$$

Now, combining (ϕ_3) with $(\tilde{V}_6)$, it is possible to ensure that for each $M > 0$ there exists $C_M > 0$ satisfying

$$|B(z, t, p)| \leq C_M |t|^{s-1} \quad \text{for all} \quad (z, t, p) \in \mathbb{R}^2 \times (-M, M) \times \mathbb{R}^2.$$

All these information are necessary to guarantee that G and B fulfill the structure required in the Harnack type inequality found in Trudinger [91, Theorem 1.1]. So, setting $v(z) = \beta - u(z)$ for $z \in \mathbb{R}^2$, we infer that v is a weak solution of the quasilinear equation

$$div\ G(z, v, \nabla v) + B(z, v, \nabla v) = 0 \quad \text{in} \quad \mathcal{O}.$$

Employing [91, Theorem 1.1], we deduce that $v = 0$ on $\mathcal{O}$, that is, $u = \beta$ on $\mathcal{O}$, which contradicts the fact that $(x_1, y_1) \in \mathcal{O}$ with $u(x_1, y_1) < \beta$. Likewise, we can apply a similar argument to show that $u(x, y) > \alpha$ for any $(x, y) \in \mathbb{R}^2$, and hence the proof is completed.
■

Finally, the following theorem is an immediate consequence of Proposition 4.1 and Lemmas 4.3 and 4.4.

Theorem 4.1 *Assume* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$, $\epsilon = 1$ *and that* A *belongs to Class A. Then equation* (4.2) *has a heteroclinic solution from* α *to* β *in* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that*

(a) $u(x,y) = u(x,y+1)$ *for any* $(x,y) \in \mathbb{R}^2$.

(b) $\alpha \leq u(x,y) \leq \beta$ *for all* $(x,y) \in \mathbb{R}^2$.

Moreover, taking into account the assumptions (ϕ_3) *and* $(\tilde{V}_6)$ *then the inequalities in (b) are strict.*

4.1.2 The case asymptotic at infinity to a periodic function

In this subsection we will study the existence of a heteroclinic solution for (4.2) with $\epsilon = 1$ and A belongs to Class B, that is, A is asymptotic at infinity to a periodic function A_p. Moreover, unless otherwise indicated, we will consider here the conditions (ϕ_1)-(ϕ_2) on ϕ and $(\tilde{V}_1)$-$(\tilde{V}_3)$ on V. The fact that we are assuming that the function A is only assumed to be asymptotically periodic with respect to x brings a lot of difficulties and some arguments explored in the periodic case do not work anymore.

In this section, let us consider the functional $I_p : W^{1,\Phi}_{\text{loc}}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$I_p(w) = \sum_{j \in \mathbb{Z}} a_{p,j}(w), \quad w \in W^{1,\Phi}_{\text{loc}}(\Omega),$$

where

$$a_{p,j}(w) = \iint_{\Omega_j} (\Phi(|\nabla w|) + A_p(x,y)V(w))\, dxdy.$$

Moreover, we use $c_{p,\Phi}(\alpha,\beta)$ to denote the real number given by

$$c_{p,\Phi}(\alpha,\beta) = \inf_{w \in \Gamma_\Phi(\alpha,\beta)} I_p(w).$$

From Subsection 4.1.1, we know that there is $w_0 \in \Gamma_\Phi(\alpha,\beta)$ such that $I_p(w_0) = c_{p,\Phi}(\alpha,\beta)$, and so,

$$c_\Phi(\alpha,\beta) \leq I(w_0) < I_p(w_0) = c_{p,\Phi}(\alpha,\beta). \tag{4.41}$$

The inequality (4.41) establishes an important relation between $c_\Phi(\alpha,\beta)$ and $c_{p,\Phi}(\alpha,\beta)$, which will be useful to achieve the objective of this subsection. With these information, we are ready to prove the main result of this subsection.

Proposition 4.2 *There is $u \in \Gamma_\Phi(\alpha,\beta)$ such that $I(u) = c_\Phi(\alpha,\beta)$ satisfying*

$$\alpha \leq u(x,y) \leq \beta \text{ almost everywhere in } \Omega.$$

Proof. First of all, note that there exists a minimizing sequence $(u_n) \subset \Gamma_\Phi(\alpha,\beta)$ for I satisfying

$$\alpha \leq u_n(x,y) \leq \beta \ \ \forall n \in \mathbb{N} \text{ and } (x,y) \in \Omega.$$

Moreover, there are $u \in W^{1,\Phi}_{\text{loc}}(\Omega)$ and a subsequence of (u_n), still denoted by (u_n), such that

$$u_n \rightharpoonup u \ \text{ in } \ W^{1,\Phi}_{\text{loc}}(\Omega), \tag{4.42}$$

$$u_n \to u \ \text{ in } \ L^{\Phi}_{\text{loc}}(\Omega) \tag{4.43}$$

and

$$u_n(x,y) \to u(x,y) \text{ a.e. in } \Omega. \tag{4.44}$$

From (4.42)-(4.44),

$$I(u) \leq c_\Phi(\alpha,\beta) \tag{4.45}$$

and

$$\alpha \leq u(x,y) \leq \beta \text{ a.e. in } \quad \Omega.$$

Now our goal is to show that $u \in \Gamma_\Phi(\alpha,\beta)$. To achieve this goal, similar to the proof of Proposition 4.1, we have that $(\tau_k u)_{k>0}$ is a bounded sequence in $W^{1,\Phi}(\Omega_0)$. Thereby, for some subsequence of $(\tau_k u)$, still denoted by itself, there is $u^* \in W^{1,\Phi}(\Omega_0)$ such that

$$\tau_k u \rightharpoonup u^* \text{ in } W^{1,\Phi}(\Omega_0) \ \text{ as } \ k \to +\infty,$$

$$\tau_k u \to u^* \ \text{ in } \ L^{\Phi}(\Omega_0) \text{ as } k \to +\infty$$

and

$$\tau_k u(x,y) \to u^*(x,y) \text{ a.e on } \Omega_0 \text{ as } k \to +\infty.$$

We claim that $u^* = \alpha$ or $u^* = \beta$ a.e. in Ω_0. Indeed, since $I(u) \leq c_\Phi(\alpha,\beta)$, we infer that $a_k(u)$ goes to 0 as k goes to $+\infty$, and so, by change of variable,

$$\iint_{\Omega_0} (\Phi(|\nabla \tau_k u|) + A(x+k,y)V(\tau_k u))\, dxdy \to 0 \text{ as } k \to +\infty.$$

Consequently, by (A_1),

$$\iint_{\Omega_0} (\Phi(|\nabla \tau_k u|) + A_0 V(\tau_k u))\, dxdy \to 0 \text{ as } k \to +\infty,$$

from where it follows that

$$\iint_{\Omega_0} (\Phi(|\nabla u^*|) + A_0 V(u^*))\, dxdy = 0.$$

By the assumptions on Φ and V, we derive that $u^* = \alpha$ or $u^* = \beta$ a.e. in Ω_0. Next, we claim that

$$u^* = \beta \text{ a.e in } \Omega_0. \tag{4.46}$$

To establish the claim above, let us assume by contradiction that $u^* = \alpha$ a.e. in Ω_0. So, as a consequence, we will prove that given $\delta \in (0, \Phi(\beta - \alpha))$ there are $(u_{n_i}) \subset (u_n)$, $(k_i) \subset \mathbb{N}$ and $i_* \in \mathbb{N}$ such that

$$i_* < k_i \text{ for all } i \in \mathbb{N}, k_i \to +\infty \text{ and } n_i \to +\infty \text{ as } i \to +\infty, \tag{4.47}$$

$$\iint_{\Omega_0} \Phi(|\tau_j u_{n_i} - \alpha|)dxdy < \delta \text{ and } \iint_{\Omega_0} \Phi(|\tau_{k_i} u_{n_i} - \alpha|)dxdy \geq \delta \ \forall j \in [i_*, k_i - 1] \cap \mathbb{N}. \tag{4.48}$$

Indeed, since $\tau_k u$ goes to α in $L^\Phi(\Omega_0)$ as k goes to $+\infty$, given $\delta \in (0, \Phi(\beta - \alpha))$, there is $i_* = i_*(\delta) \in \mathbb{N}$ satisfying

$$\iint_{\Omega_0} \Phi(|\tau_k u - \alpha|)dxdy < \frac{\delta}{2^{m+1}}, \quad \forall k \geq i_*. \tag{4.49}$$

In particular,

$$\iint_{\Omega_0} \Phi(|\tau_{i_*} u - \alpha|)dxdy < \frac{\delta}{2^{m+1}}. \tag{4.50}$$

Gathering (4.43) and (4.50) with the fact that $\Phi \in \Delta_2$ (see for a moment the Lemma A.8-(a)), we find $n_1 \in \mathbb{N}$ satisfying

$$\iint_{\Omega_0} \Phi(|\tau_{i_*} u_{n_1} - \alpha|)dxdy < \delta.$$

Thereby, since $u_{n_1} \in \Gamma_\Phi(\alpha, \beta)$, we may fix $k_1 \geq i_* + 1$ as the first natural number such that

$$\iint_{\Omega_0} \Phi(|\tau_j u_{n_1} - \alpha|)dxdy < \delta \text{ and } \iint_{\Omega_0} \Phi(|\tau_{k_1} u_{n_1} - \alpha|)dxdy \geq \delta, \quad \forall j \in [i_*, k_1 - 1] \cap \mathbb{N}.$$

On the other hand, according to (4.49),

$$\iint_{\Omega_0} \Phi(|\tau_{i_*} u - \alpha|)dxdy, \ \iint_{\Omega_0} \Phi(|\tau_{i_*+1} u - \alpha|)dxdy < \frac{\delta}{2^{m+1}}.$$

Hence, in the same manner we can see that there is $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and

$$\iint_{\Omega_0} \Phi(|\tau_{i_*} u_{n_2} - \alpha|)dxdy, \ \iint_{\Omega_0} \Phi(|\tau_{i_*+1} u_{n_2} - \alpha|)dxdy < \delta.$$

Using the fact that $u_{n_2} \in \Gamma_\Phi(\alpha,\beta)$, we can find $k_2 \geq i_* + 2$ as the first natural number satisfying

$$\iint_{\Omega_0} \Phi(|\tau_j u_{n_2} - \alpha|)dxdy < \delta \text{ and } \iint_{\Omega_0} \Phi(|\tau_{k_2} u_{n_2} - \alpha|)dxdy \geq \delta, \quad \forall j \in [i_*, k_2 - 1] \cap \mathbb{N}.$$

Repeating the above argument, there are sequences $(u_{n_i}) \subset (u_n)$ and $(k_i) \subset \mathbb{N}$ such that $k_i \geq i_* + i$ satisfying (4.47) and (4.48). So, for some subsequence, there is $w \in W^{1,\Phi}_{\text{loc}}(\Omega)$ such that

$$\tau_{k_i} u_{n_i} \rightharpoonup w \text{ in } W^{1,\Phi}_{\text{loc}}(\Omega) \quad \text{as} \quad i \to +\infty, \tag{4.51}$$

$$\tau_{k_i} u_{n_i} \to w \quad \text{in} \quad L^{\Phi}_{\text{loc}}(\Omega) \text{ as } i \to +\infty, \tag{4.52}$$

$$\tau_{k_i} u_{n_i}(x,y) \to w(x,y) \text{ a.e. in } \Omega \text{ as } i \to +\infty \tag{4.53}$$

and

$$\alpha \leq w(x,y) \leq \beta \text{ a.e. in } \Omega. \tag{4.54}$$

Now, setting the functional

$$a^i_j(v) = \iint_{\Omega_j} (\Phi(|\nabla v|) + A(x + k_i, y)V(v))\, dxdy, \quad v \in W^{1,\Phi}_{\text{loc}}(\Omega), \ i \in \mathbb{N}, \ j \in \mathbb{Z},$$

a simple change of variables gives us

$$a^i_j(\tau_{k_i} u_{n_i}) = a_{j+k_i}(u_{n_i}),$$

and so,

$$\sum_{j\in\mathbb{Z}} a^i_j(\tau_{k_i} u_{n_i}) = \sum_{j\in\mathbb{Z}} a_j(u_{n_i}) = I(u_{n_i}) \quad \forall i \in \mathbb{N}. \tag{4.55}$$

From (4.55), one has

$$a_{p,0}(\tau_k w) \to 0 \quad \text{as} \quad |k| \to +\infty. \tag{4.56}$$

To see this, it suffices to show that $I_p(w) \leq c_\Phi(\alpha,\beta)$. Indeed, combining the fact that $A(x + k_i, y)$ goes to $A_p(x,y)$ as i goes to $+\infty$ with (4.53), one gets

$$A(x + k_i, y)V(\tau_{k_i} u_{n_i}(x,y)) \to A_p(x,y)V(w(x,y)) \text{ a.e. in } \Omega.$$

Therefore, the Fatou's Lemma and (4.51) provide

$$\sum_{-j}^{j} a_{p,j}(w) \leq \liminf_{i\to+\infty} \sum_{-j}^{j} a^i_j(\tau_{k_i} u_{n_i}) \quad \forall j \in \mathbb{N}.$$

As j is arbitrary, (4.55) guarantees that

$$I_p(w) \leq \liminf_{i\to+\infty} I(u_{n_i}) = c_\Phi(\alpha,\beta), \tag{4.57}$$

and (4.56) is proved. Thereby, passing to a subsequence if necessary, a direct computation shows that

$$\tau_k w \to \alpha \text{ or } \beta \text{ in } L^\Phi(\Omega_0) \text{ as } k \to -\infty$$

and

$$\tau_k w \to \alpha \text{ or } \beta \text{ in } L^\Phi(\Omega_0) \text{ as } k \to +\infty.$$

Our goal now is to ensure that $w \in \Gamma_\Phi(\alpha,\beta)$.

Claim 1: $\tau_k w \to \alpha$ in $L^\Phi(\Omega_0)$ as $k \to -\infty$.

Indeed, note first that for each $j \in \mathbb{N}$, there is $i_0 = i_0(j) \in \mathbb{N}$ such that

$$k_i - 1 \geq k_i - j \geq i_* \text{ for all } i \geq i_0,$$

where $i_* \in \mathbb{N}$ was given in (4.47). According to (4.48),

$$\iint_{\Omega_0} \Phi(|\tau_{k_i-j}u_{n_i} - \alpha|)dxdy < \delta \quad \forall j \in \mathbb{N},$$

that is,

$$\iint_{\Omega_{-j}} \Phi(|\tau_{k_i}u_{n_i} - \alpha|)dxdy < \delta \quad \forall j \in \mathbb{N}.$$

Invoking (4.52), we can increase i if necessary to obtain

$$\iint_{\Omega_0} \Phi(|\tau_{-j}w - \alpha|)dxdy \leq \delta \ \forall j \in \mathbb{N}.$$

Our claim is proved by noting that $\delta \in (0, \Phi(\beta-\alpha))$.

Claim 2: $\tau_k w \to \beta$ in $L^\Phi(\Omega_0)$ as $k \to +\infty$.

Assume by contradiction that $\tau_k w \to \alpha$ in $L^\Phi(\Omega_0)$ as $k \to +\infty$. Let us break down the proof of Claim 2 into two steps.

Step 1: There are $\epsilon_0 > 0$ and $i_0 \in \mathbb{N}$ such that

$$\int_{k_i-1}^{k_i+1}\int_0^1 (\Phi(|\nabla u_{n_i}|) + V(u_{n_i}))\,dxdy \geq \frac{\epsilon_0}{\tilde{A}}, \quad \forall i \geq i_0, \tag{4.58}$$

where $\tilde{A} = \min\{1, A_0\}$.

Indeed, if this does not occur, then there is a subsequence $(\tau_{k_{i_j}}u_{n_{i_j}})$ of $(\tau_{k_i}u_{n_i})$ such that

$$\int_{-1}^1\int_0^1 \left(\Phi(|\nabla\tau_{k_{i_j}}u_{n_{i_j}}|) + V(\tau_{k_{i_j}}u_{n_i})\right) dxdy \to 0 \text{ as } j \to +\infty.$$

Recalling that (u_n) is bounded in $W^{1,\Phi}_{\text{loc}}(\Omega)$, then going to a subsequence if necessary, there exists $v \in W^{1,\Phi}((-1,1)\times(0,1))$ such that

$$\tau_{k_{i_j}} u_{n_{i_j}} \rightharpoonup v \text{ in } W^{1,\Phi}((-1,1)\times(0,1)) \text{ and } \tau_{k_{i_j}} u_{n_{i_j}} \to v \text{ in } L^{\Phi}((-1,1)\times(0,1)). \quad (4.59)$$

From the assumptions on Φ and V, we have $v = \alpha$ or $v = \beta$ a.e. in $(-1,1)\times(0,1)$. On the other hand, from (4.48),

$$\iint_{\Omega_{-1}} \Phi(|\tau_{k_i} u_{n_i} - \alpha|)dxdy < \delta \text{ and } \iint_{\Omega_0} \Phi(|\tau_{k_i} u_{n_i} - \alpha|)dxdy \geq \delta \quad \forall i \in \mathbb{N}. \quad (4.60)$$

Finally, taking the limit of $k_i \to +\infty$ in (4.60) and using the limit (4.59) we find a contradiction, finishing the proof of Step 1.

In what follows, fixing $\epsilon \in (0, \epsilon_0/2)$ and increasing i_0 if necessary, we obtain

$$I(u_{n_i}) \leq c_\Phi(\alpha,\beta) + \frac{\epsilon}{4}, \quad \forall i \geq i_0. \quad (4.61)$$

Step 2: There exist $j \in \mathbb{N}$ and $i \geq i_0$ large enough satisfying

$$a_{p,j}\left((\tau_{k_i} u_{n_i} - \alpha)(x-j) + \alpha\right) \leq \frac{\epsilon}{2}. \quad (4.62)$$

The proof of Step 2 follows as in the proof of Proposition 4.1, and so, it will be omitted. In the sequel, let us consider $j \in \mathbb{N}$ and $i \geq i_0$ as in Step 2. Setting the function

$$U_{j,i}(x,y) = \begin{cases} \alpha, & \text{if } \; x \leq j \quad \text{and} \quad y \in (0,1), \\ (\tau_{k_i} u_{n_i}(x,y) - \alpha)(x-j) + \alpha, & \text{if } \; j \leq x \leq j+1 \quad \text{and} \quad y \in (0,1), \\ \tau_{k_i} u_{n_i}(x,y), & \text{if } \; x > j+1 \quad \text{and} \quad y \in (0,1), \end{cases}$$

it is simple to check that $U_{j,i} \in \Gamma_\Phi(\alpha,\beta)$ and

$$c_{p,\Phi}(\alpha,\beta) \leq I_p(U_{j,i}) = a_{p,j}(U_{j,i}) + \sum_{t=j+1}^{+\infty} a_{p,t}(\tau_{k_i} u_{n_i}) = a_{p,j}(U_{j,i}) + \sum_{t=j+1+k_i}^{+\infty} a_{p,t}(u_{n_i}). \quad (4.63)$$

We claim that increasing i_0 if necessary, one gets

$$\sum_{t=j+1+k_i}^{+\infty} a_{p,t}(u_{n_i}) \leq \sum_{t=j+1+k_i}^{+\infty} a_t(u_{n_i}) + \frac{\epsilon}{4}. \quad (4.64)$$

Indeed, since the function A belongs to Class B, we infer that there is $R > 0$ such that

$$A_p(x,y) - A(x,y) \leq \frac{A_0 \epsilon}{4C} \quad \forall |x| \geq R \text{ and } \forall y \in (0,1),$$

where $C > 0$ is a constant such that $I(u_n) \leq C$ for all $n \in \mathbb{N}$. Consequently,

$$\int_R^{+\infty} \int_0^1 (A_p(x,y) - A(x,y))\, V(u_{n_i}) dxdy \leq \frac{A_0 \epsilon}{4C} \int_R^{+\infty} \int_0^1 V(u_{n_i}) dxdy \leq \frac{\epsilon}{4},$$

and therefore, increasing i if necessary the last inequality is sufficient to justify (4.64). In view of (4.63) and (4.64), one has

$$c_{p,\Phi}(\alpha, \beta) \leq a_{p,j}(U_{j,i}) + \sum_{t=j+1+k_i}^{+\infty} a_t(u_{n_i}) + \frac{\epsilon}{4}. \tag{4.65}$$

On the other hand, according to Step 1,

$$\sum_{t=j+1+k_i}^{+\infty} a_t(u_{n_i}) + \epsilon_0 \leq \sum_{t=j+1+k_i}^{+\infty} a_t(u_{n_i}) + \tilde{A} \int_{k_i-1}^{k_i+1} \int_0^1 (\Phi(|\nabla u_{n_i}|) + V(u_{n_i}))\, dxdy \leq I(u_{n_i}),$$

which together with (4.65) yields that

$$c_{p,\Phi}(\alpha, \beta) \leq a_{p,j}(U_{j,i}) + I(u_{n_i}) - \epsilon_0 + \frac{\epsilon}{4}.$$

This together with (4.61) leads to

$$c_{p,\Phi}(\alpha, \beta) \leq a_{p,j}(U_{j,i}) + c_{\Phi}(\alpha, \beta) + \frac{\epsilon}{2} - \epsilon_0.$$

Recalling that $\epsilon \in (0, \epsilon_0/2)$ and using (4.62), we arrive at

$$c_{p,\Phi}(\alpha, \beta) \leq c_{\Phi}(\alpha, \beta) + \epsilon - \epsilon_0 < c_{\Phi}(\alpha, \beta) - \frac{\epsilon_0}{2},$$

contradicting (4.41). This proves the Claim 2.

Finally, by virtue of Claims 1 and 2, we infer that $w \in \Gamma_{\Phi}(\alpha, \beta)$. Furthermore, from (4.57) we also have

$$c_{p,\Phi}(\alpha, \beta) \leq I_p(w) \leq c_{\Phi}(\alpha, \beta),$$

obtaining a new contradiction, and the our claim (4.46) is proved. As a byproduct,

$$\tau_k u \to \beta \ \text{ in } \ L^{\Phi}(\Omega_0) \text{ as } k \to +\infty. \tag{4.66}$$

A similar argument works to prove that

$$\tau_k u \to \alpha \ \text{ in } \ L^{\Phi}(\Omega_0) \text{ as } k \to -\infty. \tag{4.67}$$

Combining (4.66) and (4.67) with (4.45) we get precisely the assertion of the proposition.

■

Considering here $K_{\Phi}(\alpha, \beta)$ as described in Subsection 4.1.1, the same argument explored in that subsection guarantees that $K_{\Phi}(\alpha, \beta)$ is a non-empty set and allows us to write the following result.

Theorem 4.2 *Assume* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$, $\epsilon = 1$ *and that* A *belongs to Class B. Then equation* (4.2) *has a heteroclinic solution from* α *to* β *in* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that*

(a) $u(x,y) = u(x, y+1)$ *for any* $(x,y) \in \mathbb{R}^2$.

(b) $\alpha \leq u(x,y) \leq \beta$ *for all* $(x,y) \in \mathbb{R}^2$.

Moreover, taking into account the assumptions (ϕ_3) *and* $(\tilde{V}_6)$ *then the inequalities in (b) are strict.*

4.1.3 The case of Rabinowitz's condition

The main objective of this subsection is to establish a heteroclinic solution through the variational method for equation (4.2) in the case where the function A belongs to Class C, in which it was listed in the introduction. In [17], Alves called this class of *Rabinowitz's condition*, because an assumption like that has been introduced by Rabinowitz [83, Theorem 4.33] to build up a variational framework to study the existence of solution for a partial differential equation of the type

$$-\epsilon^2 \Delta u + A(x)u = f(u) \ \text{ in } \ \mathbb{R}^N,$$

where $\epsilon > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth and $A : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$0 < \inf_{x \in \mathbb{R}^N} A(x) < \liminf_{|x| \to +\infty} A(x).$$

Now we will mainly focus on some preliminary results that are crucial in our approach. As a beginning, let us denote by $I_\epsilon, I_\infty : W^{1,\Phi}_{\text{loc}}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ the following functionals

$$I_\epsilon(v) = \sum_{j \in \mathbb{Z}} a_{\epsilon,j}(v) \ \text{ and } \ I_\infty(v) = \sum_{j \in \mathbb{Z}} a_{\infty,j}(v),$$

where

$$a_{\epsilon,j}(v) = \iint_{\Omega_j} (\Phi(|\nabla v|) + A(\epsilon x, y)V(v))\, dxdy$$

and

$$a_{\infty,j}(v) = \iint_{\Omega_j} (\Phi(|\nabla v|) + A_\infty V(v))\, dxdy.$$

Moreover, we indicate by $c_{\epsilon,\Phi}(\alpha,\beta)$ and $c_{\infty,\Phi}(\alpha,\beta)$ the real numbers

$$c_{\epsilon,\Phi}(\alpha,\beta)=\inf_{v\in\Gamma_\Phi(\alpha,\beta)} I_\epsilon(v) \ \text{ and } \ c_{\infty,\Phi}(\alpha,\beta)=\inf_{v\in\Gamma_\Phi(\alpha,\beta)} I_\infty(v).$$

Here we would like to emphasize that throughout this subsection the potential V satisfies the conditions $(\tilde{V}_1)$-$(\tilde{V}_3)$. The next lemma establishes an important relation between the real numbers $c_{\epsilon,\Phi}(\alpha,\beta)$ and $c_{\infty,\Phi}(\alpha,\beta)$, which will play an essential in our approach.

Lemma 4.5 *According to the notation above,*

$$\limsup_{\epsilon\to 0^+} c_{\epsilon,\Phi}(\alpha,\beta)<c_{\infty,\Phi}(\alpha,\beta).$$

Proof. The proof is similar to that discussed in the proof of [17, Lemma 4.1] and its proof is omitted. ■

We are now ready to prove the following result.

Proposition 4.3 *There exists $\epsilon_0>0$ such that for each $\epsilon\in(0,\epsilon_0)$ there is $u_\epsilon\in\Gamma_\Phi(\alpha,\beta)$ satisfying $I_\epsilon(u_\epsilon)=c_{\epsilon,\Phi}(\alpha,\beta)$ and $\alpha\le u_\epsilon(x,y)\le\beta$ almost everywhere $(x,y)\in\Omega$.*

Proof. The idea here is to use a variant of the proof of Proposition 4.2 to establish the proposition. First of all, thanks to Lemma 4.5 we may fix $\epsilon_0>0$ small enough verifying

$$c_{\epsilon,\Phi}(\alpha,\beta)<c_{\infty,\Phi}(\alpha,\beta) \ \ \forall\epsilon\in(0,\epsilon_0). \tag{4.68}$$

Now, arguing as in Subsection 4.1.1, for each $\epsilon\in(0,\epsilon_0)$ there exist a minimizing sequence $(u_n)\subset\Gamma_\Phi(\alpha,\beta)$ for I_ϵ and $u_\epsilon\in W^{1,\Phi}_{\text{loc}}(\Omega)$ such that

$$\alpha\le u_n(x,y)\le\beta \ \ \forall(x,y)\in\Omega \ \text{ and } \ \forall n\in\mathbb{N},$$

$$u_n\rightharpoonup u_\epsilon \text{ in } W^{1,\Phi}_{\text{loc}}(\Omega),$$

$$u_n\to u_\epsilon \text{ in } L^{\Phi}_{\text{loc}}(\Omega),$$

$$u_n(x,y)\to u_\epsilon(x,y) \ \text{ a.e in } \Omega,$$

$$\alpha\le u_\epsilon(x,y)\le\beta \ \text{ a.e in } \Omega$$

and

$$I_\epsilon(u_\epsilon)\le c_{\epsilon,\Phi}(\alpha,\beta). \tag{4.69}$$

By a similar argument to the one used in the proof of Proposition 4.2, there are $u_\epsilon^* \in W^{1,\Phi}(\Omega_0)$ and a subsequence of $(\tau_k u_\epsilon)$, still denoted by itself, such that

$$\tau_k u_\epsilon \rightharpoonup u_\epsilon^* \ \text{ in } \ W^{1,\Phi}(\Omega_0) \ \text{ as } \ k \to +\infty,$$

$$\tau_k u_\epsilon \to u_\epsilon^* \ \text{ in } \ L^{\Phi}(\Omega_0) \ \text{ as } \ k \to +\infty$$

and

$$\tau_k u_\epsilon(x,y) \to u_\epsilon^*(x,y) \ \text{ a.e in } \ \Omega_0 \text{ as } k \to +\infty,$$

where $u_\epsilon^* = \alpha$ or $u_\epsilon^* = \beta$ a.e. in Ω_0. As in the proofs of Propositions 4.1 and 4.2, we want to show that $u_\epsilon \in \Gamma_\Phi(\alpha,\beta)$. Toward that end, we show that $u_\epsilon^* = \beta$ a.e. in Ω_0. The argument is similar to that developed in Proposition 4.2, but we present the proof in detail for the reader's convenience. Indeed, arguing by contradiction, assume that $u_\epsilon^* = \alpha$ a.e. in Ω_0. Thus, given $\delta \in (0, \Phi(\beta-\alpha))$ there exist $i_* \in \mathbb{N}$, a sequence $(k_i) \subset \mathbb{N}$ and a subsequence (u_{n_i}) of (u_n) such that $i_* < k_i$ for all $i \in \mathbb{N}$, $k_i \to +\infty$ and $n_i \to +\infty$ as $i \to +\infty$ and

$$\iint_{\Omega_0} \Phi(|\tau_j u_{n_i} - \alpha|)dxdy < \delta \text{ and } \iint_{\Omega_0} \Phi(|\tau_{k_i} u_{n_i} - \alpha|)dxdy \geq \delta \ \forall j \in [i_*, k_i - 1] \cap \mathbb{N}.$$

Consequently, considering the sequence $(\tau_{k_i} u_{n_i})$, for some subsequence, there exists $w_\epsilon \in W^{1,\Phi}_{\text{loc}}(\Omega)$ satisfying

$$\tau_{k_i} u_{n_i} \rightharpoonup w_\epsilon \text{ in } W^{1,\Phi}_{\text{loc}}(\Omega) \ \text{ as } \ i \to +\infty,$$

$$\tau_{k_i} u_{n_i} \to w_\epsilon \ \text{ in } \ L^{\Phi}_{\text{loc}}(\Omega) \text{ as } i \to +\infty$$

and

$$\alpha \leq w_\epsilon(x,y) \leq \beta \text{ a.e. in } \Omega.$$

Setting the functional

$$a^i_{\epsilon,j}(v) = \iint_{\Omega_j} \left(\Phi(|\nabla v|) + A(\epsilon x + \epsilon k_i, y)V(v)\right) dxdy, \quad v \in W^{1,\Phi}_{\text{loc}}(\Omega), \ i \in \mathbb{N} \text{ and } j \in \mathbb{Z},$$

it is easy to check that

$$\sum_{j\in\mathbb{Z}} a^i_{\epsilon,j}(\tau_{k_i} u_{n_i}) = \sum_{j\in\mathbb{Z}} a_{\epsilon,j}(u_{n_i}) = I_\epsilon(u_{n_i}) \ \ \forall i \in \mathbb{N}.$$

This fact together with the limit below

$$\liminf_{i\to+\infty} A(\epsilon x + \epsilon k_i, y) = A_\infty$$

implies that

$$I_\infty(w_\epsilon) \leq c_{\epsilon,\Phi}(\alpha,\beta), \tag{4.70}$$

and so, $a_{\infty,j}(w_\epsilon)$ goes to 0 as j goes to $\pm\infty$. So, by passing to a subsequence if necessary, it is easy to see that

$$\tau_k w_\epsilon \to \alpha \text{ or } \beta \text{ in } L^\Phi(\Omega_0) \text{ as } k \to \pm\infty.$$

The same ideas explored in the proof of Claim 1 of Proposition 4.2 ensures that

$$\tau_k w_\epsilon \to \alpha \text{ in } L^\Phi(\Omega_0) \text{ as } k \to -\infty.$$

Next, we are going to prove that

$$\tau_k w_\epsilon \to \beta \text{ in } L^\Phi(\Omega_0) \text{ as } k \to +\infty. \tag{4.71}$$

Assume for contradiction that (4.71) is not true. Arguing as in the proof of Proposition 1.2, it follows that there are $\tilde{\epsilon}_0 > 0$, $j \in \mathbb{N}$ and $i \in \mathbb{N}$ large enough such that for some fixed $\tilde{\epsilon} \in (0, \tilde{\epsilon}_0/2)$ one has

$$\int_{k_i-1}^{k_i+1}\int_0^1 (\Phi(|\nabla u_{n_i}|) + V(u_{n_i}))\,dxdy \geq \frac{\tilde{\epsilon}_0}{\tilde{A}}, \tag{4.72}$$

$$I_\epsilon(u_{n_i}) \leq c_{\epsilon,\Phi}(\alpha,\beta) + \frac{\tilde{\epsilon}}{4} \tag{4.73}$$

and

$$a_{\infty,j}\left((\tau_{k_i}u_{n_i} - \alpha)(x-j) + \alpha\right) \leq \frac{\tilde{\epsilon}}{2}. \tag{4.74}$$

Using the function $U_{j,i} \in \Gamma_\Phi(\alpha,\beta)$ given by

$$U_{j,i}(x,y) = \begin{cases} \alpha, & \text{if} \quad x \leq j \quad \text{and} \quad y \in (0,1), \\ (\tau_{k_i}u_{n_i}(x,y) - \alpha)(x-j) + \alpha, & \text{if} \quad j \leq x \leq j+1 \quad \text{and} \quad y \in (0,1), \\ \tau_{k_i}u_{n_i}(x,y), & \text{if} \quad x > j+1 \quad \text{and} \quad y \in (0,1), \end{cases}$$

we derive that

$$c_{\infty,\Phi}(\alpha,\beta) \leq I_\infty(U_{j,i}) = a_{\infty,j}(U_{j,i}) + \sum_{t=j+1+k_i}^{+\infty} a_{\infty,t}(u_{n_i}). \tag{4.75}$$

Now, since the function A belongs to Class C, increasing i if necessary, an easy computation shows that

$$\sum_{t=j+1+k_i}^{+\infty} a_{\infty,t}(u_{n_i}) \leq \sum_{t=j+1+k_i}^{+\infty} a_{\epsilon,t}(u_{n_i}) + \frac{\tilde{\epsilon}}{4}. \tag{4.76}$$

Thus, from (4.72)-(4.76),

$$c_{\infty,\Phi}(\alpha,\beta) \leq a_{\infty,j}(U_{j,i}) + I_\epsilon(u_{n_i}) - \epsilon_0 + \frac{\tilde{\epsilon}}{4} \leq c_{\epsilon,\Phi}(\alpha,\beta) - \frac{\tilde{\epsilon}_0}{2},$$

contrary to (4.68). Therefore, $w_\epsilon \in \Gamma_\Phi(\alpha,\beta)$ and (4.70) leads to

$$c_{\infty,\Phi}(\alpha,\beta) \leq I_\infty(w_\epsilon) \leq c_{\epsilon,\Phi}(\alpha,\beta),$$

which again contradicts (4.68). Consequently, we conclude from the study carried out here that $\tau_k u_\epsilon \to \beta$ in $L^\Phi(\Omega_0)$ as $k \to +\infty$. By a similar argument, we can conclude that $u_\epsilon \in \Gamma_\Phi(\alpha,\beta)$ for $\epsilon \in (0,\epsilon_0)$. Moreover, by (4.69), we must have $I_\epsilon(w_\epsilon) = c_{\epsilon,\Phi}(\alpha,\beta)$, finishing the proof. ■

Finally, we can now prove our main result of this subsection.

Theorem 4.3 *Assume* (ϕ_1)*-*(ϕ_2)*,* $(\tilde{V}_1)$*-*$(\tilde{V}_3)$ *and that* A *belongs to Class C. Then there is a constant* $\epsilon_0 > 0$ *such that for each* $\epsilon \in (0,\epsilon_0)$ *equation* (4.2) *has a heteroclinic solution from* α *to* β *in* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that*

(a) $u(x,y) = u(x,y+1)$ *for any* $(x,y) \in \mathbb{R}^2$*.*

(b) $\alpha \leq u(x,y) \leq \beta$ *for all* $(x,y) \in \mathbb{R}^2$*.*

Moreover, assuming (ϕ_3) *and* $(\tilde{V}_6)$ *we have that the inequalities in (b) are strict.*

Proof. Initially, we will consider the following set

$$K_{\epsilon,\Phi}(\alpha,\beta) = \{u \in \Gamma_\Phi(\alpha,\beta) : I_\epsilon(u) = c_{\epsilon,\Phi}(\alpha,\beta), u(x,0) = u(x,1) \text{ in } \mathbb{R}, \alpha \leq u \leq \beta \text{ a.e. on } \Omega\}, \tag{4.77}$$

which consists of minimum points of I_ϵ on $\Gamma_\Phi(\alpha,\beta)$ that are seen as functions defined on $\mathbb{R}^2$ being 1-periodic on the variable y. Next, from Proposition 4.3 we can proceed analogously to the proof of Lemma 4.2 for show that $K_{\epsilon,\Phi}(\alpha,\beta)$ is non empty whenever $\epsilon \in (0,\epsilon_0)$. Finally, we point out that the Theorem 4.3 follows following the same steps of Subsection 4.1.1 and the details are left to the reader. ■

4.1.4 The case asymptotically away from zero at infinity

We exhibit in this subsection a heteroclinic solution for (4.2) when A is asymptotically away from zero at infinity, that is, in the case where A belongs to Class D. Specifically, let's assume that A satisfies $(\tilde{A}_2)$-$(\tilde{A}_3)$ and that A is a continuous non-negative function, even in x, $A \in L^\infty(\mathbb{R}^2)$ and there exists $K > 0$ such that

$$\inf_{|x|\geq K,\ y\in[0,1]} A(x,y) > 0.$$

To build a framework for this heteroclinic problem and avoid some bothersome technicalities, we will always assume here that the potential V satisfies the conditions $(\tilde{V}_1)$-$(\tilde{V}_3)$, (V_2) and $(\tilde{V}_7)$. Furthermore, we will consider assumptions (ϕ_1)-(ϕ_2) on ϕ and that $\epsilon = 1$. Next we consider the following class of admissible functions

$$\Gamma_\Phi^o(\beta) = \{v \in \Gamma_\Phi(-\beta,\beta) : v(x,y) = -v(-x,y) \text{ a.e. in } \Omega \text{ and } 0 \leq v(x,y) \leq \beta \text{ for a.e } x \geq 0\}$$

and the real number

$$c_\Phi^o(\beta) = \inf_{v\in\Gamma_\Phi^o(\beta)} I(v),$$

where $\Gamma_\Phi(-\beta,\beta)$ is given as in (4.3). Now it is important to point out that $\Gamma_\Phi^o(\beta)$ is not empty, because the function $\varphi_{-\beta,\beta}$ defined as in (4.4) belongs to $\Gamma_\Phi^o(\beta)$ with $I(\varphi_{-\beta,\beta}) < +\infty$. Having said that, we will now explore the conditions (V_2) and $(\tilde{V}_7)$ to show that the following class

$$K_\Phi^o(\beta) = \{v \in \Gamma_\Phi^o(\beta) : I(v) = c_\Phi^o(\beta) \text{ and } v(x,0) = v(x,1) \text{ in } \mathbb{R}\} \tag{4.78}$$

is not empty. Hereafter, we will assume that the functions of $K_\Phi^o(\beta)$ are periodically extended in $\mathbb{R}^2$ on the variable y. Therefore, $K_\Phi^o(\beta)$ is constituted by (minimal) heteroclinic type solutions of (4.2) with $\epsilon = 1$ that are 1-periodic in y and odd in x.

Lemma 4.6 *It holds that $K_\Phi^o(\beta) \neq \emptyset$.*

Proof. By some standard computations, one easily verifies that there exists a minimizing sequence $(u_n) \subset \Gamma_\Phi^o(\beta)$ for I such that

$$0 \leq u_n(x,y) \leq \beta \ \ \forall n \in \mathbb{N} \ \text{ and } \ x \geq 0.$$

Besides that, there exist $u \in W_{\text{loc}}^{1,\Phi}(\Omega)$ and a subsequence of (u_n), still denoted by (u_n), satisfying

$$u_n \rightharpoonup u \ \text{ in } \ W_{\text{loc}}^{1,\Phi}(\Omega), \tag{4.79}$$

$$u_n \to u \ \text{ in } \ L_{\text{loc}}^{\Phi}(\Omega) \tag{4.80}$$

and

$$u_n(x,y) \to u(x,y) \text{ a.e. on } \Omega. \tag{4.81}$$

We conclude from (4.81) that $u(x,y) = -u(-x,y)$ almost everywhere $(x,y) \in \Omega$ and $0 \leq u(x,y) \leq \beta$ for almost every $x \geq 0$, and finally by (4.79)-(4.80) it is easy to check that

$$I(u) \leq c^o_\Phi(\beta). \tag{4.82}$$

Now we claim that $u \in \Gamma^o_\Phi(\beta)$. To establish our claim, we assume for the sake of contradiction that $\tau_k u$ not goes to β as k goes to $+\infty$ in $L^\Phi(\Omega_0)$. Thereby, since $\Phi \in \Delta_2$ there are $\epsilon > 0$ and a subsequence (k_i) of natural numbers with $k_i \to +\infty$ such that

$$\iint_{\Omega_0} \Phi\left(|\tau_{k_i} u - \beta|\right) dxdy \geq \epsilon \quad \forall i \in \mathbb{N}. \tag{4.83}$$

On the other hand, $(\tilde{V}_1)$-$(\tilde{V}_3)$ and $(\tilde{V}_7)$ yield

$$\tilde{\mu}\Phi(|t-\beta|) \leq V(t) \ \ \forall t \in [0,\beta],$$

for some $\tilde{\mu} > 0$. Consequently,

$$I(u) \geq \sum_{i\in\mathbb{N}} \left(\iint_{\Omega_{k_i}} A(x,y)V(u)dxdy \right) \geq \tilde{\mu} \sum_{i\in\mathbb{N}} \left(\iint_{\Omega_{k_i}} A(x,y)\Phi\left(|u-\beta|\right) dxdy \right),$$

that is,

$$I(u) \geq \tilde{\mu} \sum_{i\in\mathbb{N}} \left(\iint_{\Omega_0} A(x+k_i,y)\Phi\left(|\tau_{k_i} u - \beta|\right) dxdy \right).$$

Now, fixing $i_0 \in \mathbb{N}$ such that $|x+k_i| \geq K$ for any $x \in [0,1]$ and $i \geq i_0$, the fact that A belongs to Class D leads to

$$I(u) \geq \tilde{\mu} a_0 \sum_{i\geq i_0} \left(\iint_{\Omega_0} \Phi\left(|\tau_{k_i} u - \beta|\right) dxdy \right),$$

where

$$a_0 = \inf_{|x|\geq K,\ y\in[0,1]} A(x,y) > 0.$$

Hence, by (4.83),

$$I(u) \geq a_0 \tilde{\mu} \sum_{i\geq i_0} \epsilon = +\infty,$$

contrary to (4.82). For this reason,

$$\tau_k u \to \beta \text{ in } L^\Phi(\Omega_0) \text{ as } k \to +\infty,$$

and therefore, since u is odd in x we conclude that $u \in \Gamma_\Phi^o(\beta)$. This fact combined with (4.82) produces that $I(u) = c_\Phi^o(\beta)$. Now assumptions $(\tilde{A}_2)$-$(\tilde{A}_3)$ allow us to proceed as in the proof of Lemma 4.2 to find a function $v \in \Gamma_\Phi^o(\beta)$ dependent on u such that $v \in K_\Phi^o(\beta)$, and the proof is over. ■

We now finish this subsection by proving the following theorem as follows.

Theorem 4.4 *Assume* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$, $(\tilde{V}_7)$ *and* (V_2) *with* $\alpha = -\beta$, $\epsilon = 1$ *and that* A *belongs to Class D. Then equation* (4.2) *possesses a heteroclinic solution* u *from* $-\beta$ *to* β *in* $C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that*

(a) $u(x,y) = -u(-x,y)$ *for any* $(x,y) \in \mathbb{R}^2$.

(b) $u(x,y) = u(x,y+1)$ *for all* $(x,y) \in \mathbb{R}^2$.

(c) $0 \leq u(x,y) \leq \beta$ *for any* $x > 0$ *and* $y \in \mathbb{R}$.

Moreover, if (ϕ_3) *and* $(\tilde{V}_6)$ *occur then the inequalities in (c) are strict.*

Proof. Our proof follows the method developed in Chapter 2 and we will do it in detail for the reader's convenience. To begin with, thanks to Lemma 4.6 we can take $u \in K_\Phi^o(\beta)$. Here we will first show that

$$\iint_{\Omega_0} \left(\phi\left(|\nabla u|\right) \nabla u \nabla \psi + A(x,y)V'(u)\psi\right) dxdy \geq 0 \ \text{ for all } \ \psi \in C_0^\infty(\mathbb{R}^2), \tag{4.84}$$

which will guarantee that

$$\iint_{\Omega_0} \left(\phi\left(|\nabla u|\right) \nabla u \nabla \psi + A(x,y)V'(u)\psi\right) dxdy = 0 \ \text{ for all } \ \psi \in C_0^\infty(\mathbb{R}^2),$$

implying that u is a weak solution of (4.2). In what follows, for each $\psi \in C_0^\infty(\mathbb{R}^2)$ we will use the fact that

$$\psi(x,y) = \psi_o(x,y) + \psi_e(x,y),$$

where

$$\psi_e(x,y) = \frac{\psi(x,y) + \psi(-x,y)}{2} \quad \text{and} \quad \psi_o(x,y) = \frac{\psi(x,y) - \psi(-x,y)}{2}.$$

In addition to these functions, let us consider for $t > 0$ the function

$$\tilde{\varphi}_t(x,y) = \max\left\{-\beta, \min\{\beta, \varphi_t(x,y)\}\right\}, \quad (x,y) \in \Omega,$$

where

$$\varphi_t(x,y) = \begin{cases} u(x,y) + t\psi_o(x,y), & \text{if} \quad x \geq 0 \text{ and } u(x,y) + t\psi_o(x,y) \geq 0, \\ -u(x,y) - t\psi_o(x,y), & \text{if} \quad x \geq 0 \text{ and } u(x,y) + t\psi_o(x,y) \leq 0, \\ -\varphi_t(-x,y), & \text{if} \quad x < 0. \end{cases}$$

Now, a direct computation shows that $\tilde{\varphi}_t \in \Gamma^o_\Phi(\beta)$. Then (V_2) together with $\tilde{\varphi}_t$ yields that

$$I(u + t\psi_o) = I(\varphi_t) \geq I(\tilde{\varphi}_t) \geq c^o_\Phi(\beta) = I(u). \tag{4.85}$$

On the other hand, by Lemma A.8-(b),

$$\begin{aligned} I(u+t\psi) - I(u+t\psi_o) \geq t \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} &\phi(|\nabla(u+t\psi_o)|)\nabla u \nabla \psi_e dxdy \\ &+ t^2 \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} \phi(|\nabla(u+t\psi_o)|)\nabla \psi_o \nabla \psi_e dxdy \\ &+ \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} A(x,y)\left(V(u+t\psi) - V(u+t\psi_o)\right) dxdy. \end{aligned} \tag{4.86}$$

Since the functions $\phi(|\nabla(u + t\psi_o)|)\nabla u \nabla \psi_e$ and $\phi(|\nabla(u + t\psi_o)|)\nabla \psi_o \nabla \psi_e$ are odd in the variable x, then it is easily seen that

$$\sum_{j\in\mathbb{Z}} \iint_{\Omega_j} \phi(|\nabla(u+t\psi_o)|)\nabla u \nabla \psi_e dxdy = \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} \phi(|\nabla(u+t\psi_o)|)\nabla \psi_o \nabla \psi_e dxdy = 0, \tag{4.87}$$

and so, from (4.85)-(4.87),

$$I(u+t\psi) - I(u) \geq \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} A(x,y)\left(V(u+t\psi) - V(u+t\psi_o)\right) dxdy.$$

Consequently, as $A(x,y)V'(u)\psi_e$ is odd in the variable x, one gets

$$\begin{aligned} \iint_\Omega (\phi(|\nabla u|)\nabla u \nabla \psi + A(x,y)V'(u)\psi) dxdy &= \lim_{t\to 0^+} \frac{I(u+t\psi) - I(u)}{t} \\ &\geq \lim_{t\to 0^+} \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} A(x,y) \frac{V(u+t\psi) - V(u+t\psi_o)}{t} dxdy \\ &\geq \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} A(x,y)V'(u)(\psi - \psi_o) dxdy = \sum_{j\in\mathbb{Z}} \iint_{\Omega_j} A(x,y)V'(u)\psi_e dxdy = 0, \end{aligned} \tag{4.88}$$

showing that the inequality (4.84) occurs for every $\psi \in C^\infty_0(\mathbb{R}^2)$. Finally, slightly varying the same ideas discussed in the Subsection 4.1.1, we can conclude the proof of Theorem 4.4. ■

4.2 Heteroclinic solution of the prescribed curvature equation

Throughout this section, we adapt for our problem the approach explored in Chapter 3 to find solutions that are periodic in the variable y and heteroclinic in x from α to β to the prescribed mean curvature equation (4.1). Since the ideas are so close to those of Chapter 3, the presentation will be brief.

4.2.1 Auxiliary results

In the following, we consider for each $L > 0$ the quasilinear equation

$$-\Delta_{\Phi_L} u + A(\epsilon x, y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{$(E)_L$}$$

where $\Phi_L : \mathbb{R} \to [0, +\infty)$ is an N-function of the form

$$\Phi_L(t) = \int_0^{|t|} \phi_L(s)s ds,$$

where $\phi_L(t) = \varphi_L(t^2)$ and φ_L is defined by

$$\varphi_L(t) = \begin{cases} \dfrac{1}{\sqrt{1+t}}, & \text{if} \quad t \in [0, L], \\ x_L(t - L - 1)^2 + y_L, & \text{if} \quad t \in [L, L+1], \\ y_L, & \text{if} \quad t \in [L+1, +\infty), \end{cases}$$

with

$$x_L = \frac{\sqrt{1+L}}{4(1+L)^2} \quad \text{and} \quad y_L = (4L+3)x_L.$$

We point out that the main purpose of this section is to use the arguments of Sect. 4.1 to investigate the existence of a heteroclinic solution $u_{\alpha,\beta}$ from α to β for $(E)_L$ that satisfies

$$\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}, \tag{4.89}$$

because this inequality implies that $u_{\alpha,\beta}$ is a heteroclinic solution from α to β for (4.1). Here, we will prove that the inequality (4.89) holds when $\max\{|\alpha|, |\beta|\}$ is small enough. In order to do that, a control involving the roots α and β of V is necessary, and at this point the condition $(\tilde{V}_4)$ applies an important rule in our argument.

The next result is about functions ϕ_L and Φ_L, which makes it clear that Φ_L is an N-function.

Lemma 4.7 *For each $L > 0$, the functions ϕ_L and Φ_L have the following properties:*

(a) ϕ_L *is* C^1.

(b) $y_L \leq \phi_L(t) \leq 1$ *for all* $t \geq 0$.

(c) $\dfrac{y_L}{2}t^2 \leq \Phi_L(t) \leq \dfrac{1}{2}t^2$ *for any* $t \in \mathbb{R}$.

(d) Φ_L *is a convex function.*

(e) $(\phi_L(t)t)' > 0$ *for all* $t > 0$.

Proof. The argument follow the same lines as the proof of Lemma 3.5. ■

We would like to point out that our focus now is on examining if the N-function Φ_L is in the settings of Sect. 4.1, that is, if ϕ_L satisfies conditions (ϕ_1)-(ϕ_3). Indeed, it is clear that by Lemma 4.7-(e) ϕ_L checks (ϕ_1), and by Lemma 4.7-(b), it checks (ϕ_3) with $q = 2$. Moreover, with direct computations one can get that there are real numbers $m_L, l_L > 1$ such that $l_L \leq m_L$ and

$$l_L - 1 \leq \frac{(\phi_L(t)t)'}{\phi_L(t)} \leq m_L - 1 \quad \text{for any} \quad t \geq 0,$$

from which it follows that ϕ_L verifies (ϕ_2). For more details see Lemma 3.7. As a direct consequence the N-functions Φ_L and $\tilde{\Phi}_L$ satisfy Δ_2-condition, where $\tilde{\Phi}_L$ is the complementary function associated with Φ_L, which ensures that the space L^{Φ_L} is reflexive (see for instance Appendix A). Actually, the study made in Lemma 3.6 shows that the space L^{Φ_L} is exactly L^2 space and the norm of L^{Φ_L} is equivalent to the norm of L^2.

4.2.2 Existence of heteroclinic solution

Assuming for a moment that function A belongs to Class A or B, $\epsilon = 1$, and that the potential V satisfies $(\tilde{V}_1)$-$(\tilde{V}_4)$, the same arguments from Subsections 4.1.1 and 4.1.2 guarantee that there exist a periodic function $u_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}$ on the variable y such that $u_{\alpha,\beta} \in K_{\Phi_L}(\alpha, \beta)$, where

$$K_{\Phi_L}(\alpha,\beta) = \{w \in \Gamma_{\Phi_L}(\alpha, \beta) \,|\, I(w) = c_{\Phi_L}(\alpha, \beta), w(x, 0) = w(x, 1) \text{ in } \mathbb{R}, \alpha \leq w \leq \beta \text{ a.e. on } \Omega\}.$$

Moreover, $u_{\alpha,\beta}$ is a weak solution of equation $(E)_L$ with $\epsilon = 1$ in $C^{1,\gamma}_{\text{loc}}(\mathbb{R}^2)$ for some $\gamma \in (0, 1)$ that is heteroclinic in x from α to β. The next lemma is crucial to guarantee the existence of a heteroclinic solution for (4.1).

Lemma 4.8 *Given $L > 0$ there exists $\delta > 0$ such that for each pair (α, β) of real numbers with $\alpha < \beta$ and $\max\{|\alpha|, |\beta|\} \in (0, \delta)$ we have that*

$$\|u_{\alpha,\beta}\|_{C^1(B_1(z))} < \sqrt{L}$$

for all $z \in \mathbb{R}^2$ and $u_{\alpha,\beta} \in K_{\Phi_L}(\alpha, \beta)$, where $B_1(z)$ denotes the ball in $\mathbb{R}^2$ of center z and radius 1.

Proof. If the lemma does not hold, there are $(r_n, s_n) \subset \mathbb{R}^2$ and $(\alpha_n, \beta_n) \subset \mathbb{R}^2$ such that $\max\{|\alpha_n|, |\beta_n|\}$ goes to 0 as n goes to $+\infty$ and for some $u_{\alpha_n,\beta_n} \in K_{\Phi_L}(\alpha_n, \beta_n)$ one has

$$\|u_{\alpha_n,\beta_n}\|_{C^1(B_1(r_n,s_n))} \geq \sqrt{L}, \quad \forall n \in \mathbb{N}. \tag{4.90}$$

Now, we note that for each $n \in \mathbb{N}$ the function $\tilde{u}_n$ defined by

$$\tilde{u}_n(x, y) = u_{\alpha_n,\beta_n}(x + r_n, y + s_n) \text{ for } (x, y) \in \mathbb{R}^2$$

is a weak solution of the quasilinear equation

$$-\Delta_{\Phi_L} u + B_n(x, y) = 0 \text{ in } \mathbb{R}^2,$$

where

$$B_n(x, y) = A(x + r_n, y + s_n)V'(u_{\alpha_n,\beta_n}(x, y)).$$

Furthermore, it is easy to see that $(\tilde{V}_4)$ ensures the existence of a positive number $M > 0$, which is independent of n, such that

$$|B_n(x, y)| \leq M\|A\|_{L^\infty(\mathbb{R}^2)} \;\; \forall (x, y) \in \mathbb{R}^2 \text{ and } \forall n \in \mathbb{N}.$$

Therefore, the elliptic regularity theory found in [67, Theorem 1.7] implies that $\tilde{u}_n \in C^{1,\gamma_0}_{\text{loc}}(\mathbb{R}^2)$, for some $\gamma_0 \in (0, 1)$, and that there is a positive constant R independent of n verifying

$$\|\tilde{u}_n\|_{C^{1,\gamma_0}_{\text{loc}}(\mathbb{R}^2)} \leq R \;\; \forall\, n \in \mathbb{N}.$$

The above estimate allows us to use Arzelà-Ascoli Theorem to find $u \in C^1(B_1(0))$ and a subsequence of $(\tilde{u}_n)$, still denoted by $(\tilde{u}_n)$, such that

$$\tilde{u}_n \to u \quad \text{in} \quad C^1(B_1(0)).$$

Now since $\|u_{\alpha_n,\beta_n}\|_{L^\infty(\mathbb{R}^2)} \to 0$ as $\max\{|\alpha_n|,|\beta_n|\} \to 0$, we obtain that $u = 0$ on $B_1(0)$, and so,

$$\|\tilde{u}_n\|_{C^1(B_1(0))} < \sqrt{L} \ \ \forall\, n \geq n_0,$$

for some $n_0 \in \mathbb{N}$. Therefore,

$$\|u_{\alpha_n,\beta_n}\|_{C^1(B_1(r_n,s_n))} < \sqrt{L} \ \ \forall\, n \geq n_0,$$

which contradicts (4.90), and the proof is completed. ■

We are finally ready to prove one of our best results from this chapter involving prescribed mean curvature equation (4.1).

Theorem 4.5 *Assume* $(\tilde{V}_1)$-$(\tilde{V}_4)$, $\epsilon = 1$ *and that A belongs to Class A or B. Given $L > 0$ there exists $\delta > 0$ such that if* $\max\{|\alpha|,|\beta|\} \in (0,\delta)$ *then equation* (4.1) *possesses a heteroclinic solution $u_{\alpha,\beta}$ from α to β in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, satisfying*

(a) $u_{\alpha,\beta}$ *is 1-periodic on y.*

(b) $\alpha \leq u_{\alpha,\beta}(x,y) \leq \beta$ *for any* $(x,y) \in \mathbb{R}^2$.

(c) $\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$.

Moreover, if $V \in C^2(\mathbb{R},\mathbb{R})$ then the inequalities in (b) are strict.

Proof. To begin with, we claim that given $L > 0$ there exists $\delta > 0$ such that if $\max\{|\alpha|,|\beta|\} \in (0,\delta)$ we have

$$\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L} \ \text{ for all } \ u_{\alpha,\beta} \in K_{\Phi_L}(\alpha,\beta). \tag{4.91}$$

Indeed, for each $(x,y) \in \mathbb{R}^2$ we can choose $z \in \mathbb{R}^2$ verifying $(x,y) \in B_1(z)$. Thanks to Lemma 4.8, there is $\delta = \delta(L) > 0$ such that for each pair (α,β) of real numbers with $\alpha < \beta$ and $\max\{|\alpha|,|\beta|\} \in (0,\delta)$ one has

$$\|u_{\alpha,\beta}\|_{C^1(B_1(z))} < \sqrt{L}$$

whenever $u_{\alpha,\beta} \in K_{\Phi_L}(\alpha,\beta)$. Now, from the arbitrariness of (x,y) it is easy to see that our claim is established. Therefore, the estimate (4.91) ensures $u_{\alpha,\beta}$ is a heteroclinic solution of (4.1). To complete the proof, the fact that $V \in C^2(\mathbb{R},\mathbb{R})$ combined with the assumptions $(\tilde{V}_2)$-$(\tilde{V}_3)$ yields that there are $\lambda, d_1, d_2 > 0$ such that

$$|V'(t)| \leq d_1|t-\alpha|, \quad \forall t \in [\alpha-\lambda, \alpha+\lambda]$$

and

$$|V'(t)| \leq d_2|t-\beta|, \quad \forall t \in [\beta-\lambda, \beta+\lambda].$$

Thus, by Lemma 4.7-(b),

$$|V'(t)| \leq \frac{d_1}{y_L}\phi_L(|t-\alpha|)|t-\alpha|, \quad \forall t \in [\alpha-\lambda, \alpha+\lambda]$$

and

$$|V'(t)| \leq \frac{d_2}{y_L}\phi_L(|t-\beta|)|t-\beta|, \quad \forall t \in [\beta-\lambda, \beta+\lambda].$$

Consequently, V satisfies $(\tilde{V}_6)$ with ϕ_L, and so, proceeding as in the proof of Lemma 4.4, we see that $u_{\alpha,\beta}$ verifies

$$\alpha < u_{\alpha,\beta}(x,y) < \beta \text{ for all } (x,y) \in \mathbb{R}^2,$$

which is the desired conclusion. ■

Now, let us assume that A belongs to Class C and that V satisfies $(\tilde{V}_1)$-$(\tilde{V}_4)$. Considering the set $K_{\epsilon,\Phi_L}(\alpha,\beta)$ as in (4.77), then we can argue similarly to the proof of Theorem 4.3 to obtain that $K_{\epsilon,\Phi_L}(\alpha,\beta) \neq \emptyset$ whenever $\epsilon > 0$ is small. With everything, proceeding analogously as in the proof of Lemma 4.8 we get the following result.

Lemma 4.9 *There exists $\epsilon_0 > 0$ such that for $\epsilon \in (0,\epsilon_0)$ and $L > 0$ there is $\delta > 0$ such that for each pair (α,β) of real numbers with $\alpha < \beta$ and $\max\{|\alpha|,|\beta|\} \in (0,\delta)$ we have for all $z \in \mathbb{R}^2$ and $u_{\epsilon,\alpha,\beta} \in K_{\epsilon,\Phi_L}(\alpha,\beta)$ that*

$$\|u_{\epsilon,\alpha,\beta}\|_{C^1(B_1(z))} < \sqrt{L}.$$

We now present the following result.

Theorem 4.6 *Assume $(\tilde{V}_1)$-$(\tilde{V}_4)$ and that A belongs to Class C. There is $\epsilon_0 > 0$ such that for each $\epsilon \in (0,\epsilon_0)$ and $L > 0$ there exists $\delta > 0$ such that if $\max\{|\alpha|,|\beta|\} \in (0,\delta)$ then equation (4.1) possesses a heteroclinic solution $u_{\alpha,\beta}$ from α to β in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, verifying*

(a) $u_{\alpha,\beta}$ is 1-periodic on y.

(b) $\alpha \leq u_{\alpha,\beta}(x,y) \leq \beta$ for any $(x,y) \in \mathbb{R}^2$.

(c) $\|\nabla u_{\alpha,\beta}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$.

Moreover, if $V \in C^2(\mathbb{R}, \mathbb{R})$ occurs then the inequalities in (b) are strict.

Proof. The proof can be done via a comparison argument like that of the proof Theorem 4.5 and we omit it here. Details are left to the reader. ■

For the final exhibition of these ideas, we will see below that requiring more of V we can relax the conditions on A to guarantee the existence of a heteroclinic solution for (4.1). Let's assume that the function A belongs to Class D, $\alpha = -\beta$, $\epsilon = 1$ and that $V \in C^2(\mathbb{R}, \mathbb{R})$ and satisfies conditions $(\tilde{V}_2)$-$(\tilde{V}_5)$ and (V_2). We want to point out that condition $(\tilde{V}_5)$ implies that the potential V satisfies $(\tilde{V}_7)$ with Φ_L. In fact, note that by $(\tilde{V}_5)$ there are $\rho > 0$ and $\theta \in (0, \frac{\beta}{2})$ such that

$$\rho|t-\beta|^2 \leq V(t), \ \forall t \in (\beta - \theta, \beta + \theta), \tag{4.92}$$

from which it follows by Lemma 4.7-(c) and (4.92),

$$2\rho\Phi_L(|t-\beta|) \leq V(t), \ \forall t \in (\beta - \theta, \beta + \theta).$$

Consequently, the argument of Subsection 4.1.4 shows that for each $L > 0$ the set $K^o_{\Phi_L}(\beta)$ is not empty, where $K^o_{\Phi_L}(\beta)$ is given as in (4.78). We would like to remind here that each element of $K^o_{\Phi}(\beta)$ can be seen as a function on $\mathbb{R}^2$ being periodic in the variable y. Moreover, if $u_\beta \in K^o_{\Phi_L}(\beta)$, then u_β is a weak solution for $(E)_L$ with $\epsilon = 1$ in $C^{1,\gamma}_{\text{loc}}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$, odd and heteroclinic from $-\beta$ to β in x satisfying

$$0 \leq u_\beta(x,y) \leq \beta \text{ for all } (x,y) \in \mathbb{R}_+ \times \mathbb{R}.$$

Now, the following result is a similar version of Lemma 4.8 and is proved in an essentially identical fashion, which will play an analogous role to that developed in Theorem 4.5 in the present setting.

Lemma 4.10 *Given $L > 0$ there is $\delta > 0$ such that for each $\beta \in (0, \delta)$ we have that*

$$\|u_\beta\|_{C^1(B_1(z))} < \sqrt{L}$$

for all $z \in \mathbb{R}^2$ and $u_\beta \in K^o_{\Phi_L}(\beta)$.

Finally, to conclude this subsection, we prove the following theorem using the framework discussed above.

Theorem 4.7 *Assume $V \in C^2(\mathbb{R},\mathbb{R})$, $(\tilde{V}_2)$-$(\tilde{V}_5)$, (V_2), $\alpha = -\beta$, $\epsilon = 1$ and that A belongs to Class D. Then for each $L > 0$ there exists $\delta > 0$ such that if $\beta \in (0,\delta)$ then equation (4.1) possesses a heteroclinic solution u_β from $-\beta$ to β in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, verifying*

(a) $u_\beta(x,y) = -u_\beta(-x,y)$ *for any* $(x,y) \in \mathbb{R}^2$.

(b) $u_\beta(x,y) = u_\beta(x,y+1)$ *for all* $(x,y) \in \mathbb{R}^2$.

(c) $0 < u_\beta(x,y) < \beta$ *for* $x > 0$.

(d) $\|\nabla u_\beta\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}$.

Proof. The proof is established using Lemma 4.10 and arguing as in the proof of Theorem 4.5. Detailed verification is left to the reader. ■

4.3 Final remarks

In this final section, we highlight some observations that complement the study of the previous sections. First, we would like to point out that in the study of Section 4.1 the conditions (ϕ_1) and (ϕ_2) on ϕ are enough to show the existence of heteroclinic solution from α to β for (4.1), while assumption (ϕ_3) together with $(\tilde{V}_6)$ are used to get more information about the behavior of the heteroclinic solution, because it permits to apply a Harnack type inequality found in Trudinger [91]. Secondly, Theorems 4.1, 4.2, 4.3 and 4.4 hold for all pair of real numbers (α,β) with $\alpha < \beta$ and cover the cases $\Phi(t) = |t|^p$ for $p \in (1,2)$. Moreover, in Theorems 4.1-4.3 we can consider a variety of potentials as the prototypes

$$V(t) = (t-\alpha)^2(t-\beta)^2$$

and

$$V(t) = \beta + \beta\cos\left(\frac{t\pi}{\beta}\right)$$

when $\alpha = -\beta$, while in Theorem 4.4 the potential V must have a strong interaction with the N-function Φ, see for example

$$V(t) = \Phi(|(t-\alpha)(t-\beta)|).$$

Applying the same argument as in Chapter 2, we can find a heteroclinic solution $u : \mathbb{R}^2 \to \mathbb{R}$ from α to β being 2-periodic in y and satisfying $\alpha \le u \le \beta$ on $\mathbb{R}^2$ for the quasilinear equation (4.2). Moreover, interchanging the roles of α and β in Section 4.1 we obtain a variational framework to show that there exists a solution $v : \mathbb{R}^2 \to \mathbb{R}$ for (4.2) that is 1 or 2-periodic in the variable y, $\alpha \le v \le \beta$ on $\mathbb{R}^2$ and heteroclinic from β to α in x, that is,

$$v(x, y) \to \beta \ \text{ as } \ x \to -\infty \ \text{ and } \ v(x, y) \to \alpha \ \text{ as } \ x \to +\infty \ \text{ uniformly in } \ y \in \mathbb{R}.$$

Likewise, switching x and y produces solutions that are heteroclinic from α to β or from β to α in the variable y and 1 or 2-periodic in x. We can also find solutions in the same settings for the prescribed mean curvature equation (4.1) when $\max\{|\alpha|, |\beta|\}$ is small.

A more general consideration of the ideas presented here would be to contemplate higher order problems. The reader is invited to see that the study in this chapter can be applied to elliptic problems on a cylindrical domain in $\mathbb{R}^N$, with $N \ge 2$, of the form $\Omega = \mathbb{R} \times D$, where D is a bounded open set in $\mathbb{R}^{N-1}$ such that $\partial D \in C^1$. Specifically, adapting Classes A, B, C and D to the case of functions A defined in Ω we can write the following results

Theorem 4.8 *Assume* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$, $\epsilon = 1$ *and that* A *belongs to Class A or B excluding assumptions* $(\tilde{A}_2)$ *and* $(\tilde{A}_3)$. *Then the quasilinear elliptic problem*

$$\begin{cases} -\Delta_\Phi u + A(\epsilon x, y)V'(u) = 0 \ \ in \ \ \Omega \\ \dfrac{\partial u}{\partial \eta}(x, y) = 0 \ \ on \ \ \partial\Omega \end{cases} \tag{P_1}$$

has a heteroclinic solution from α *to* β *in* $C^{1,\gamma}_{loc}(\Omega)$ *for some* $\gamma \in (0, 1)$ *such that*

$$\alpha \le u(x, y) \le \beta \ \ for\ all \ \ (x, y) \in \Omega.$$

Moreover, taking into account the assumptions (ϕ_3) *and* $(\tilde{V}_6)$ *then the above inequalities are strict.*

Theorem 4.9 *Assume* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$ *and that* A *belongs to Class C excluding assumptions* $(\tilde{A}_2)$ *and* $(\tilde{A}_3)$. *Then there is a constant* $\epsilon_0 > 0$ *such that for each* $\epsilon \in (0, \epsilon_0)$ *problem* (P_1) *has a heteroclinic solution from* α *to* β *in* $C^{1,\gamma}_{loc}(\Omega)$ *for some* $\gamma \in (0, 1)$ *such that*

$$\alpha \le u(x, y) \le \beta \ \ for\ all \ \ (x, y) \in \Omega.$$

Moreover, assuming (ϕ_3) *and* $(\tilde{V}_6)$ *we have that the above inequalities are strict.*

Theorem 4.10 *Assume* (ϕ_1)-(ϕ_2), $(\tilde{V}_1)$-$(\tilde{V}_3)$, $(\tilde{V}_7)$, (V_2), $\alpha = -\beta$, $\epsilon = 1$ *and that* A *belongs to Class D excluding assumptions* $(\tilde{A}_2)$ *and* $(\tilde{A}_3)$. *Then problem* (P_1) *possesses a heteroclinic solution* u *from* $-\beta$ *to* β *in* $C^{1,\gamma}_{loc}(\Omega)$ *for some* $\gamma \in (0,1)$ *such that*

(a) $u(x,y) = -u(-x,y)$ *for any* $(x,y) \in \Omega$.

(b) $0 \leq u(x,y) \leq \beta$ *for any* $x > 0$ *and* $y \in D$.

Moreover, if (ϕ_3) *and* $(\tilde{V}_6)$ *occur then the inequalities in (b) are strict.*

Now let us list some results where the existence of a heteroclinic solution for a prescribed mean curvature equation is addressed in Ω.

Theorem 4.11 *Assume* $(\tilde{V}_1)$-$(\tilde{V}_4)$, $\epsilon = 1$ *and that* A *belongs to Class A or B excluding assumptions* $(\tilde{A}_2)$ *and* $(\tilde{A}_3)$. *Given* $L > 0$ *there exists* $\delta > 0$ *such that if* $\max\{|\alpha|, |\beta|\} \in (0,\delta)$ *then problem*

$$\begin{cases} -div\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + A(\epsilon x, y)V'(u) = 0 \quad in \quad \Omega \\ \dfrac{\partial u}{\partial \eta}(x,y) = 0 \quad on \quad \partial\Omega \end{cases} \tag{P_2}$$

possesses a heteroclinic solution u *from* α *to* β *in* $C^{1,\gamma}_{loc}(\Omega)$ *for some* $\gamma \in (0,1)$ *satisfying*

(a) $\alpha \leq u(x,y) \leq \beta$ *for any* $(x,y) \in \Omega$.

(b) $\|\nabla u\|_{L^\infty(\Omega)} \leq \sqrt{L}$.

Moreover, if $V \in C^2(\mathbb{R},\mathbb{R})$ *then the inequalities in (b) are strict.*

Theorem 4.12 *Assume* $(\tilde{V}_1)$-$(\tilde{V}_4)$ *and that* A *belongs to Class C excluding assumptions* $(\tilde{A}_2)$ *and* $(\tilde{A}_3)$. *There is* $\epsilon_0 > 0$ *such that for each* $\epsilon \in (0,\epsilon_0)$ *and* $L > 0$ *there exists* $\delta > 0$ *such that if* $\max\{|\alpha|, |\beta|\} \in (0,\delta)$ *then problem* (P_2) *possesses a heteroclinic solution* u *from* α *to* β *in* $C^{1,\gamma}_{loc}(\Omega)$ *for some* $\gamma \in (0,1)$ *verifying*

(a) $\alpha \leq u(x,y) \leq \beta$ *for any* $(x,y) \in \Omega$.

(b) $\|\nabla u\|_{L^\infty(\Omega)} \leq \sqrt{L}$.

Moreover, if $V \in C^2(\mathbb{R},\mathbb{R})$ *occurs then the inequalities in (b) are strict.*

Theorem 4.13 *Assume* $V \in C^2(\mathbb{R},\mathbb{R})$, $(\tilde{V}_2)$-$(\tilde{V}_5)$, *(*V_2*)*, $\alpha = -\beta$, $\epsilon = 1$ *and that* A *belongs to Class D excluding assumptions* $(\tilde{A}_2)$ *and* $(\tilde{A}_3)$. *Then for each* $L > 0$ *there exists* $\delta > 0$ *such that if* $\beta \in (0,\delta)$ *then problem* (P_2) *possesses a heteroclinic solution* u *from* $-\beta$ *to* β *in* $C^{1,\gamma}_{loc}(\Omega)$ *for some* $\gamma \in (0,1)$ *verifying*

(a) $u(x,y) = -u(-x,y)$ *for any* $(x,y) \in \Omega$.

(b) $0 < u(x,y) < \beta$ *for* $x > 0$.

(c) $\|\nabla u\|_{L^\infty(\Omega)} \leq \sqrt{L}$.

CHAPTER 5

SADDLE SOLUTIONS FOR PRESCRIBED MEAN CURVATURE EQUATIONS IN $\mathbb{R}^2$

In this last chapter of this thesis, we will combine some of the arguments introduced in the previous chapters to show that the prescribed mean curvature equation given by

$$-div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)+A(x,y)V'(u)=0 \quad \text{in} \quad \mathbb{R}^2 \tag{5.1}$$

has a saddle solution whenever the distance between the roots of the double well symmetric potential V is small. Throughout the chapter, the oscillatory factor $A(x,y)$ satisfies precisely the following assumptions:

(A_1) A is a continuous function and $A(x,y)>0$ for each $(x,y)\in\mathbb{R}^2$,

(A_2) $A(x,y)=A(-x,y)=A(x,-y)$ for all $(x,y)\in\mathbb{R}^2$,

(A_3) $A(x,y)=A(x+1,y)=A(x,y+1)$ for any $(x,y)\in\mathbb{R}^2$,

(A_4) $A(x,y)=A(y,x)$ for all $(x,y)\in\mathbb{R}^2$,

while V is a double well potential with absolute minima at $t=\pm\alpha$ satisfying conditions (V_1), (V_2) and (V_7), which were introduced in the Introduction. An important prototype of this scenario is the following Ginzburg-Landau type potential

$$V(t)=(t^2-\alpha^2)^2.$$

Another prototype is the following Sine-Gordon type potential given by

$$V(t) = \alpha + \alpha \cos\left(\frac{t\pi}{\alpha}\right).$$

In the particular case, when $A(x, y)$ is a positive constant, we get an infinite number of geometrically distinct saddle-type solutions for (5.1). All these solutions are characterized by the fact that, along different directions parallel to the end lines, they are uniformly asymptotic to $\pm\alpha$.

5.1 Existence of saddle solutions for quasilinear equations

We will show in this subsection that the main results about saddle-type solutions in Chapters 1 and 2 are extended to a larger class of N-functions than the class that was presented there. We will start with a brief review of what has been done here in this thesis for saddle solutions. To recapitulate, for example, in Chapter 2, conditions (ϕ_1), (ϕ_2) and (ϕ_3) guarantee the existence of a heteroclinic solution u from $-\alpha$ to α being 1-periodic in the variable y and odd in x for the following quasilinear elliptic equation

$$-\Delta_\Phi u + A(x, y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{5.2}$$

while condition (ϕ_4) came into play to employ only the exponential decay estimates for $u \pm \alpha$. Specifically, (ϕ_4) was assumed to obtain the inequality

$$\phi(|\zeta'(x)|) \leq \phi(\omega_2 \zeta(x)) \ \text{ for all } \ x \in \mathbb{R}, \tag{5.3}$$

which involves the real function $\zeta : \mathbb{R} \to \mathbb{R}$ given by

$$\zeta(x) = \delta_\alpha \frac{\cosh\left(a\left(x - \frac{j - \frac{1}{4} + L}{2}\right)\right)}{\cosh\left(a\frac{j - \frac{1}{4} - L}{2}\right)}, \tag{5.4}$$

where ω_2, L and j are chosen properly and the constant a is small enough. Inequality (5.3) allowed us to use direct calculations to get the desired exponential-type estimates. To check the details, see Lemmas 2.11, 2.12 and 2.13. This study of the asymptotic behavior at infinity of heteroclinic solutions plays a fundamental role in the search of saddle-type solutions for (5.2) in the approach adopted in Chapters 1 and 2.

Our goal now is to replace (ϕ_4) with another condition on ϕ that includes a larger number of examples for ϕ, including the monotonically non-decreasing functions. Precisely we assume the following assumption:

$(\tilde{\phi}_4)$ There exist $\kappa_1, \kappa_2 > 0$ such that the inequality

$$\phi(|\zeta'(t)|) \leq \kappa_1 \phi(\kappa_2 \zeta(t)) \text{ for all } t \in \mathbb{R}$$

occurs whenever $a > 0$ is small enough in (5.4).

The reader is invited to verify that the exponential decay estimates involving the heteroclinic solutions in Chapters 1 and 2 still occur replacing (ϕ_4) by $(\tilde{\phi}_4)$. Now, we would like to point out that we have already seen that functions ϕ satisfying (ϕ_4) also verify $(\tilde{\phi}_4)$. However, the reverse is not true, as there are functions satisfying $(\tilde{\phi}_4)$ but not fulfill (ϕ_4), which makes $(\tilde{\phi}_4)$ more general than (ϕ_4). An explicit example is given in the next section.

Under our current assumptions on ϕ, we provide below a result that generalizes Theorem 2.4.

Theorem 5.1 *Assume (ϕ_1)-(ϕ_3), $(\tilde{\phi}_4)$, $V \in C^1(\mathbb{R}, \mathbb{R})$, (V_1)-(V_4) and (A_1)-(A_4). Then, there is $v \in C^{1,\gamma}_{loc}(\mathbb{R}^2)$ for some $\gamma \in (0,1)$ such that v is a weak solution of (5.2) that verifies the following:*

(a) $0 < v(x,y) < \alpha$ *on the fist quadrant in* $\mathbb{R}^2$,

(b) $v(x,y) = -v(-x,y) = -v(x,-y)$ *for all* $(x,y) \in \mathbb{R}^2$,

(c) $v(x,y) = v(y,x)$ *for any* $(x,y) \in \mathbb{R}^2$,

(d) $v(x,y) \to \alpha$ *as* $x \to \pm\infty$ *and* $y \to \pm\infty$,

(e) $v(x,y) \to -\alpha$ *as* $x \to \mp\infty$ *and* $y \to \pm\infty$.

The items (d) and (e) of the theorem above tells us that along directions parallel to the axes, v is uniformly asymptotic to stationary solutions $\pm\alpha$. Moreover, when $A(x,y)$ is a positive constant, we can demand more conditions on the geometry of the graph of potential V to obtain the existence of infinitely many geometrically distinct saddle-type solutions of equation (5.2). Finally, motivated by Theorem 5.1, we can follow the steps of Chapter 1 to obtain the following result.

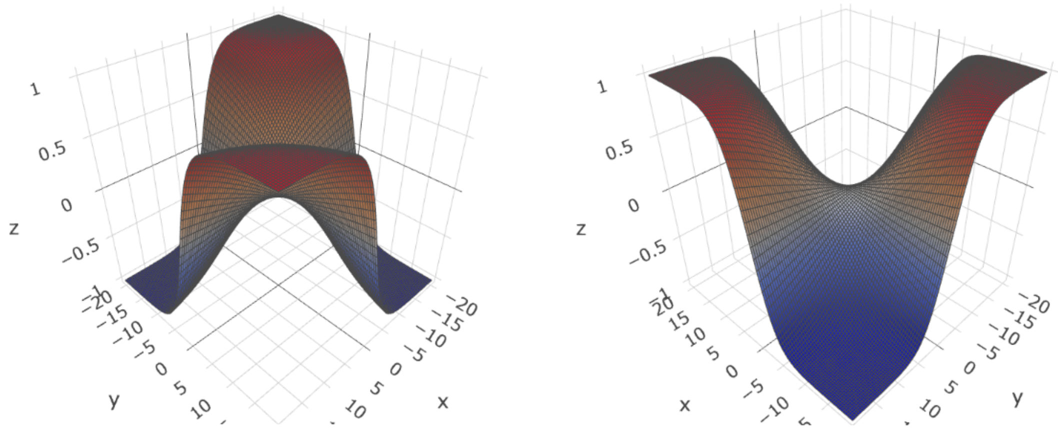

Figure 5.1: Geometric illustration of the saddle solution with asymptotic behavior.

Theorem 5.2 *Assume* (ϕ_1)*-*(ϕ_3)*,* $(\tilde{\phi}_4)$*,* (V_1)*-*(V_6) *and that* $A(x,y)$ *is a positive constant. Then, For each* $j \geq 2$ *there exists* $v_j \in C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that* v_j *is a weak solution of* (5.2) *satisfying*

(a) $0 < \tilde{v}_j(\rho,\theta) < \alpha$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *and* $\rho > 0$*,*

(b) $\tilde{v}_j(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_j(\rho, \frac{\pi}{2} - \theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$*,*

(c) $\tilde{v}_j(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_j(\rho,\theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$*,*

(d) $\tilde{v}_j(\rho,\theta) \to (-\alpha)^{k+1}$ *as* $\rho \to +\infty$ *whenever* $\theta \in \left(\frac{\pi}{2} + k\frac{\pi}{j}, \frac{\pi}{2} + (k+1)\frac{\pi}{j}\right)$ *for* $k = 0, \ldots, 2j-1$*,*

where $\tilde{v}_j(\rho,\theta) = v_j(\rho\cos(\theta), \rho\sin(\theta))$*.*

In other words, the saddle-type solution $\tilde{v}_j$ is antisymmetric with respect to the half-line $\theta = \frac{\pi}{2}$, $\frac{2\pi}{j}$-periodic in the angle variable and has L^∞-norm less than or equal to α with the asymptotic behavior at infinity described in item (d). In true, the conditions on V in the theorem above can be refined, that is, conditions (V_5) and (V_6) can be omitted. To see this, combine many of the arguments from Chapter 2 and apply the idea of partitioning $\mathbb{R}^2$ into $2j$, with $j \geq 2$, disjoint triangular sets when constructing saddle solutions in Chapter 1, specifically in Theorem 1.4. Hence, we may write the following result which improves Theorem 5.2 and the verification details are left to the reader.

Theorem 5.3 *Assume* (ϕ_1)*-*(ϕ_3)*,* $(\tilde{\phi}_4)$*,* (V_1)*-*(V_4) *and that* $A(x,y)$ *is a positive constant. Then, For each* $j \geq 2$ *there exists* $v_j \in C^{1,\gamma}_{loc}(\mathbb{R}^2)$ *for some* $\gamma \in (0,1)$ *such that* v_j *is a weak solution of* (5.2) *satisfying*

(a) $0 < \tilde{v}_j(\rho,\theta) < \alpha$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *and* $\rho > 0$*,*

(b) $\tilde{v}_j(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_j(\rho, \frac{\pi}{2} - \theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$*,*

(c) $\tilde{v}_j(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_j(\rho, \theta)$ *for all* $(\rho,\theta) \in [0,+\infty) \times \mathbb{R}$*,*

(d) $\tilde{v}_j(\rho,\theta) \to (-\alpha)^{k+1}$ *as* $\rho \to +\infty$ *whenever* $\theta \in \left(\frac{\pi}{2} + k\frac{\pi}{j}, \frac{\pi}{2} + (k+1)\frac{\pi}{j}\right)$ *for* $k = 0, \ldots, 2j-1$*,*

where $\tilde{v}_j(\rho,\theta) = v_j(\rho\cos(\theta), \rho\sin(\theta))$*.*

5.2 Saddle solution of the prescribed mean curvature equation

Our main objective in this subsection is to prove the existence of a saddle solution to the prescribed mean curvature equation (5.1) whenever the global minima of the potential V are close enough. To this aim we first study an auxiliary problem of the form (5.2) proving the existence of a saddle solution in this scenario. The idea here is similar to those presented in Chapters 3 and 4, so the exposition will be brief. For our purposes, for each $L > 0$ we will truncate the prescribed mean curvature operator as follows

$$\varphi_L(t) = \begin{cases} \dfrac{1}{\sqrt{1+t}}, & \text{if} \quad t \in [0,L], \\ x_L(t-L-1)^2 + y_L, & \text{if} \quad t \in [L, L+1], \\ y_L, & \text{if} \quad t \in [L+1, +\infty), \end{cases}$$

where the numbers x_L and y_L are expressed by

$$x_L = \frac{\sqrt{1+L}}{4(1+L)^2} \quad \text{and} \quad y_L = (4L+3)x_L.$$

As a consequence, for each $L > 0$ we get the following quasilinear equation

$$-\Delta_{\Phi_L} u + A(x,y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{5.5}$$

where $\Phi_L : \mathbb{R} \to [0,+\infty)$ is an N-function of the form

$$\Phi_L(t) = \int_0^{|t|} \phi_L(s)sds \quad \text{with} \quad \phi_L(t) = \varphi_L(t^2).$$

Due to the study carried out in the previous chapters, it is well known that for each $L > 0$ the function ϕ_L checks conditions (ϕ_1)-(ϕ_3). However, a direct check shows that ϕ_L is non-increasing on $(0, +\infty)$ and therefore ϕ_L does not satisfy (ϕ_4). But, each ϕ_L satisfies $(\tilde{\phi}_4)$, as the following lemma says.

Lemma 5.1 *For each $L > 0$, the function ϕ_L satisfies $(\tilde{\phi}_4)$.*

Proof. Indeed, according to Lemma 4.7-(b) we have that

$$y_L \leq \phi_L(t) \leq 1 \quad \text{for all} \quad t \geq 0.$$

In particular,

$$y_L \leq \phi_L(|\zeta'(t)|), \phi_L(\zeta(t)) \leq 1 \quad \text{for all} \quad t \in \mathbb{R},$$

and therefore,

$$\phi_L(|\zeta'(t)|) \leq \frac{1}{y_L}\phi_L(\zeta(t)) \quad \text{for all} \quad t \in \mathbb{R}.$$

Finally, since there is no restriction for the constant $a > 0$ in (5.4), the lemma follows. ∎

Remark 5.1 *By the argument of the previous lemma, we can conclude that any function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying*

$$c_1 \leq \phi(t) \leq c_2 \quad \textit{for all} \quad t \geq 0$$

for some constants $c_1, c_2 > 0$ satisfies condition $(\tilde{\phi}_4)$.

To find saddle solutions for (5.1), let's analyze the existence of these solutions for the auxiliary problem (5.5). To begin with, we will assume that the potential V satisfies (V_1)-(V_2) and (V_7). Consequently, V also satisfies the conditions (V_3) with Φ_L and (V_4) with ϕ_L. Indeed, by $(V_7) - (i)$ there are $\rho_1, \rho_2, d_1, d_2, d_3 > 0$ such that

$$|V'(t)| \leq d_1|t - \alpha| \quad \text{for all} \quad t \in [\alpha - \rho_1, \alpha + \rho_1]$$

and

$$d_2|t - \alpha|^2 \leq V(t) \leq d_3|t - \alpha|^2 \quad \text{for all} \quad t \in (\alpha - \rho_2, \alpha + \rho_2),$$

from which it follows by Lemma 4.7 that

$$|V'(t)| \leq \frac{d_1}{y_L}\phi_L(|t - \alpha|)|t - \alpha| \quad \text{for all} \quad t \in [\alpha - \rho_1, \alpha + \rho_1]$$

and

$$2d_2\Phi_L(|t-\alpha|) \le V(t) \le \frac{2d_3}{y_L}\Phi_L(|t-\alpha|) \text{ for all } t \in (\alpha-\rho_2, \alpha+\rho_2).$$

Therefore, we may use Theorem 5.1 to obtain for each $L > 0$ a saddle solution $v_{\alpha,L} : \mathbb{R}^2 \to \mathbb{R}$ to equation (5.5), that is, a weak solution $v_{\alpha,L} \in C^{1,\gamma}_{\rm loc}(\mathbb{R}^2)$ of (5.5) such that it has the same sign as xy, odd in both the variables x and y, symmetric with respect to the diagonals $y = \pm x$ and presenting the asymptotic behavior described in items (d) and (e) of Theorem 5.1. We are now going to use this information together with condition $(V_7) - (ii)$ to prove an estimate involving the functions $v_{\alpha,L}$.

Lemma 5.2 *Given $L > 0$ there is $\delta > 0$ such that for each $\alpha \in (0, \delta)$ we have that*

$$\|v_{\alpha,L}\|_{C^1(B_1(z))} < \sqrt{L} \tag{5.6}$$

for any open ball $B_1(z)$ in $\mathbb{R}^2$ of radius 1.

Proof. To prove this lemma, we will argue by contradiction. So, suppose that there are sequences $(z_n) \subset \mathbb{R}^2$ and $(\alpha_n) \subset (0, +\infty)$ such that $\alpha_n \to 0$ as $n \to +\infty$ and

$$\|v_{\alpha_n,L}\|_{C^1(B_1(z_n))} \ge \sqrt{L} \text{ for all } n \in \mathbb{N}. \tag{5.7}$$

For our purposes, we will study the regularity of some specific solutions to the following elliptic equation

$$-\Delta_{\Phi_L} u + B_n(x, y) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{5.8}$$

where the scalar measurable function $B_n : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$B_n(x, y) = A(x + z_{n,1}, y + z_{n,2})V'(v_{\alpha_n,L}(x, y))$$

with $z_n = (z_{n,1}, z_{n,2})$, which by $(V_7) - (ii)$ B_n satisfies the following estimate

$$|B_n(x, y)| \le M\|A\|_{L^\infty(\mathbb{R}^2)} \quad \forall (x, y) \in \mathbb{R}^2 \text{ and } \forall n \in \mathbb{N},$$

for some positive number $M > 0$ independent of n. A weak solution to (5.8) of particular interest is

$$u_n(x, y) = v_{\alpha_n,L}(x + z_{n,1}, y + z_{n,2}) \text{ for } (x, y) \in \mathbb{R}^2.$$

Now, applying elliptic regularity estimates on u_n developed by Lieberman in [67, Theorem 1.7] one has $u_n \in C^{1,\gamma_0}_{\rm loc}(\mathbb{R}^2)$, for some $\gamma_0 \in (0, 1)$, and

$$\|u_n\|_{C^{1,\gamma_0}_{\rm loc}(\mathbb{R}^2)} \le R \text{ for all } n \in \mathbb{N},$$

for some positive constant R independent of n. Consequently, from the above estimate, via Arzelà-Ascoli's theorem, we conclude that there exists $u \in C^1(B_1(0))$ and a subsequence of (u_n), still denoted by (u_n), such that

$$u_n \to u \quad \text{in} \quad C^1(B_1(0)). \tag{5.9}$$

Since $\|v_{\alpha_n,L}\|_{L^\infty(\mathbb{R}^2)}$ goes to 0 as $\alpha_n \to 0$,

$$\|u_n\|_{L^\infty(\mathbb{R}^2)} \to 0 \quad \text{as} \quad n \to +\infty,$$

and thus from the convergence (5.9) we naturally get that $u = 0$ on $B_1(0)$. Therefore, there is $n_0 \in \mathbb{N}$ such that

$$\|u_n\|_{C^1(B_1(0))} < \sqrt{L} \quad \text{for all} \quad n \geq n_0,$$

which results from the definition of u_n that

$$\|v_{\alpha_n,L}\|_{C^1(B_1(z_n))} < \sqrt{L} \quad \text{for all} \quad n \geq n_0,$$

which contradicts (5.7). The proof of the lemma is complete. ■

The estimate (5.6) will be used in a crucial way in the theorem bellow for studying the existence of saddle solutions for prescribed mean curvature equation (5.1) whenever the roots $\pm\alpha$ of V are close to enough.

Theorem 5.4 *Assume $V \in C^2(\mathbb{R},\mathbb{R})$, (V_1)-(V_2), (V_7), and (A_1)-(A_4). Given $L > 0$ there exists $\delta > 0$ such that if $\alpha \in (0,\delta)$ then the prescribed mean curvature equation (5.1) possesses a weak solution $v_{\alpha,L}$ in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0,1)$, satisfying the following properties:*

(a) $0 < v_{\alpha,L}(x,y) < \alpha$ *on the fist quadrant in* $\mathbb{R}^2$,

(b) $v_{\alpha,L}(x,y) = -v_{\alpha,L}(-x,y) = -v_{\alpha,L}(x,-y)$ *for all* $(x,y) \in \mathbb{R}^2$,

(c) $v_{\alpha,L}(x,y) = v_{\alpha,L}(y,x)$ *for any* $(x,y) \in \mathbb{R}^2$,

(d) $v_{\alpha,L}(x,y) \to \alpha$ *as* $x \to \pm\infty$ *and* $y \to \pm\infty$,

(e) $v_{\alpha,L}(x,y) \to -\alpha$ *as* $x \to \mp\infty$ *and* $y \to \pm\infty$,

(f) $\|\nabla v_{\alpha,L}\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{L}$.

Proof. We claim that for each $L > 0$ there exists $\delta > 0$ such that if $\alpha \in (0, \delta)$ one has

$$\|\nabla v_{\alpha,L}\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}. \tag{5.10}$$

In fact, given any $(x, y) \in \mathbb{R}^2$ we can choose a point $z \in \mathbb{R}^2$ such that $(x, y) \in B_1(z)$. By virtue of Lemma 5.2, there exists $\delta > 0$, which depends on L, such that for each $\alpha \in (0, \delta)$ one gets

$$\|v_{\alpha,L}\|_{C^1(B_1(z))} < \sqrt{L}.$$

In particular,

$$\|\nabla v_{\alpha,L}\|_{L^\infty(B_1(z))} \le \sqrt{L}.$$

Therefore, the claim (5.10) is valid from the arbitrariness of $(x, y) \in \mathbb{R}^2$, and thus, the estimate (5.10) guarantees that $v_{\alpha,L}$ is a saddle solution to the equation (5.1) satisfying items (a) to (f), thanks to the study developed in the auxiliary problem (5.5). ■

In the case where $A(x, y)$ is a positive constant, we can also obtain the existence of infinitely many saddle-type solutions for the prescribed mean curvature equation (5.1), where such solutions may be named as "pizza solutions" due to the geometry of their graphs. Following the strategy developed to prove Theorem 5.4, we can show the following multiplicity result of saddle-type solutions to (5.1).

Theorem 5.5 *Assume $V \in C^2(\mathbb{R}, \mathbb{R})$, (V_1)-(V_2), (V_7), and that $A(x, y)$ is a positive constant. Then, given $L > 0$ there is $\delta > 0$ such that if $\alpha \in (0, \delta)$ then for each $j \ge 2$ the prescribed mean curvature equation (5.1) possesses a weak solution $v_{\alpha,L,j}$ in $C^{1,\gamma}_{loc}(\mathbb{R}^2)$, for some $\gamma \in (0, 1)$, satisfying*

(a) $0 < \tilde{v}_{\alpha,L,j}(\rho, \theta) < \alpha$ *for any* $\theta \in [\frac{\pi}{2} - \frac{\pi}{2j}, \frac{\pi}{2})$ *and* $\rho > 0$,

(b) $\tilde{v}_{\alpha,L,j}(\rho, \frac{\pi}{2} + \theta) = -\tilde{v}_{\alpha,L,j}(\rho, \frac{\pi}{2} - \theta)$ *for all* $(\rho, \theta) \in [0, +\infty) \times \mathbb{R}$,

(c) $\tilde{v}_{\alpha,L,j}(\rho, \theta + \frac{\pi}{j}) = -\tilde{v}_{\alpha,L,j}(\rho, \theta)$ *for all* $(\rho, \theta) \in [0, +\infty) \times \mathbb{R}$,

(d) $\tilde{v}_{\alpha,L,j}(\rho, \theta) \to (-\alpha)^{k+1}$ *as* $\rho \to +\infty$ *whenever* $\theta \in \left(\frac{\pi}{2} + k\frac{\pi}{j}, \frac{\pi}{2} + (k+1)\frac{\pi}{j}\right)$ *for* $k = 0, \ldots, 2j - 1$,

(e) $\|\nabla v_{\alpha,L,j}\|_{L^\infty(\mathbb{R}^2)} \le \sqrt{L}$,

where $\tilde{v}_{\alpha,L,j}(\rho, \theta) = v_{\alpha,L,j}(\rho\cos(\theta), \rho\sin(\theta))$.

5.3 Final remarks

As far as we know, Theorems 5.4 and 5.5 are the first results in the literature on saddle-type solutions for some stationary Allen-Cahn-type equations involving the prescribed mean curvature operator in the whole plane. Transition-type solutions to equations involving the prescribed mean curvature operator is an extremely fascinating field of mathematics and there are still many open questions one can work on. For example, a possible extension to Theorems 5.4 and 5.5 would be to study the existence of a saddle solution for (5.1) without requiring that the distance between the absolute minima $\pm\alpha$ of V be small. We believe that a natural approach to solve such a problem would be to look for minima of an action functional on a convex subset of the space of functions of bounded variation $BV_{\rm loc}(\mathbb{R})$.

We would like to end this last section by stating that, although we improved on the results of Chapters 1 and 2 in Section 5.1 on saddle solutions to quasilinear elliptic equations of the form

$$-\Delta_\Phi u + A(x,y)V'(u) = 0 \quad \text{in} \quad \mathbb{R}^2 \tag{5.11}$$

by imposing condition $(\tilde{\phi}_4)$ instead of (ϕ_4) on ϕ, the case $\phi(t) = t^{p-2}$ with $p \in (1,2)$ still remains an open problem, since ϕ does not satisfy $(\tilde{\phi}_4)$ because in this case we have by direct calculation that

$$\frac{\phi(|\zeta'(t)|)}{\phi(\kappa_2\zeta(t))} \to +\infty \quad \text{as} \quad |t| \to \frac{j - \frac{1}{4} + L}{2}$$

for any positive constant κ_2. Therefore, the problem of obtaining exponential decay type estimates to find saddle solutions in the case where (5.11) involves the p-Laplacian operator with $1 < p < 2$ is potentially difficult and interesting.

APPENDIX A

ORLICZ AND ORLICZ-SOBOLEV SPACES

In this appendix we will highlight some basic properties about the Orlicz and Orlicz-Sobolev spaces that will be useful throughout the text. We would like to emphasize here that we will give the minimum on the topic to better contextualize the reader about some points of the text, because it is not our intention to make an exposition in all the details. For a quite comprehensive account of this topic, the interested reader might start by referring to [1, 64, 85] and the bibliography therein.

A.1 A brief overview on Orlicz spaces

In our brief review of Orlicz and Orlicz-Sobolev spaces, we will begin by presenting the following definition:

Definition A.1 *A function $\Phi : \mathbb{R} \to [0, +\infty)$ is said to be an **N-function** if it satisfies the following properties*

(a) Φ *is continuous, convex and even,*

(b) $\Phi(t) = 0$ *if and only if* $t = 0$,

(c) $\displaystyle\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ *and* $\displaystyle\lim_{t\to +\infty} \frac{\Phi(t)}{t} = +\infty.$

*Moreover, we say that an N-function Φ verifies the Δ_2-**condition** ($\Phi \in \Delta_2$ for short), if there are constants $K > 0$ and $t_0 \geq 0$ such that*

$$\Phi(2t) \leq K\Phi(t) \text{ for all } t \geq t_0. \tag{Δ_2}$$

Before proceeding with this theory, we would like to list below some examples of N-functions that satisfy (Δ_2) with $t_0 = 0$:

(1) $\Phi_1(t) = \dfrac{|t|^p}{p}$, $1 < p < +\infty$.

(2) $\Phi_2(t) = \dfrac{|t|^p}{p} + \dfrac{|t|^q}{q}$ for $1 < p < q < +\infty$.

(3) $\Phi_3(t) = (1+t^2)^\gamma - 1$ with $\gamma > 1$.

(4) $\Phi_4(t) = \int_0^t s^{1-\gamma}(\sinh^{-1} s)^\beta ds$ with $0 \leq \gamma < 1$ and $\beta > 0$.

(5) $\Phi_5(t) = |t|^p \ln(1 + |t|)$, where $p \in (1, +\infty)$.

(6) $\Phi_6(t) = (\sqrt{1+t^2} - 1)^\gamma$ for $\gamma > 1$.

(7) $\Phi_7(t) = (1 + |t|)\ln(1 + |t|) - |t|$.

In addition, we also list below two N-functions that do not satisfy the Δ_2-condition:

(8) $\Phi_8(t) = \dfrac{e^{t^2} - 1}{2}$.

(9) $\Phi_9(t) = e^{|t|} - |t| - 1$.

It is reasonable that, through the examples above, the reader may hastily conclude that an N-function that satisfies (Δ_2) behaves in a powerlike way at infinity and at the origin, in other words, Φ does not increase more rapidly than exponential functions. In true, if Φ satisfies (Δ_2) then there are $a, b > 0$ such that $\Phi(t) \leq a|t|^b$ for all $t \geq t_0$. This fact can be found in [64].

To continue this brief review of Orlicz spaces, from now on, unless otherwise indicated, we will always assume that $\mathcal{O}$ is an open set of $\mathbb{R}^N$, with $N \geq 1$, and that Φ is an N-function. In these configurations, we will present the definition of Orlicz Spaces.

Definition A.2 *The following class functions*

$$L^\Phi(\mathcal{O}) = \left\{ u \in L^1_{loc}(\mathcal{O}) \mid \int_{\mathcal{O}} \Phi\left(\frac{|u|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right\}$$

is called **Orlicz space associated to Φ over $\mathcal{O}$**. *When there is no confusion of notation, it is simply called* **Orlicz space**.

The vector space $L^{\Phi}(\mathcal{O})$ was introduced and explored by W. Orlicz [75] in the year 1932, where he also introduced the additional condition on Φ the so-called Δ_2-condition in (Δ_2). In 1936, Orlicz investigated $L^{\Phi}(\mathcal{O})$ in the absence of the Δ_2-condition in [76].

We can introduce several norms on L^{Φ}, but one of special interest is the Minkowski functional

$$\|u\|_{L^{\Phi}(\mathcal{O})} = \inf\left\{\lambda > 0 : \int_{\mathcal{O}} \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\}, \quad u \in L^{\Phi}(\mathcal{O}),$$

which was introduced by W. A. J. Luxemburg in his thesis [68] in 1955 and is therefore called **Luxemburg norm associated to Φ over $\mathcal{O}$**. The reader can verify that $L^{\Phi}(\mathcal{O})$ endowed with the Luxemburg norm associated to Φ over $\mathcal{O}$ carries the structure of a Banach space.

The spaces L^{Φ} have very rich topological structure and are a very elegant generalization of ordinary Lebesgue's spaces.

Example A.1 *If $\Phi(t) = \dfrac{|t|^p}{p}$ with $p \in (1, +\infty)$, then*

$$L^{\Phi}(\mathcal{O}) = L^p(\mathcal{O}) \quad \text{and} \quad \|u\|_{L^{\Phi}(\mathcal{O})} = p^{-\frac{1}{p}}\|u\|_{L^p(\mathcal{O})},$$

where $\|\cdot\|_{L^p(\mathcal{O})}$ denotes the usual norm of $L^p(\mathcal{O})$.

The spaces $L^{\Phi}(\mathcal{O})$ are more general than $L^p(\mathcal{O})$ spaces and may have peculiar properties that do not occur in Lebesgue's spaces. However, in some cases there are relationships between them, such as continuous embedding

$$L^{\Phi}(\mathcal{O}) \hookrightarrow L^1(\mathcal{O}), \tag{A.1}$$

whenever Lebesgue measure on $\mathbb{R}^N$ of $\mathcal{O}$ is finite (in short, $|\mathcal{O}| < +\infty$). In other words, this embedding is equivalent to the simple containment $L^{\Phi}(\mathcal{O}) \subset L^1(\mathcal{O})$ in which some topological properties are preserved such as notions of convergences.

Normally, the investigation of Orlicz spaces is divided into two classes: in the presence of $\Phi \in \Delta_2$ and in the absence of $\Phi \in \Delta_2$. Without the Δ_2-condition, the study of $L^{\Phi}(\mathcal{O})$ becomes more delicate and Orlicz was the first to investigate this case in his famous work [76]. We would like to emphasize that here we will limit ourselves to addressing only the case where the Δ_2-condition is assumed, but for the reader interested in the absence of this assumption we recommend starting with [1, 76]. Let us now see some properties about Orlicz spaces when $\Phi \in \Delta_2$ is assumed, for example,

$$L^{\Phi}(\mathcal{O}) = \left\{u \in L^1_{\text{loc}}(\mathcal{O}) : \int_{\mathcal{O}} \Phi(|u|)dx < +\infty\right\},$$

$$u_n \to u \ \text{ in } \ L^{\Phi}(\mathcal{O}) \quad \Leftrightarrow \quad \int_{\mathcal{O}} \Phi(|u_n - u|)dx \to 0$$

and

$$\int_{\mathcal{O}} \Phi\left(\frac{|u|}{\|u\|_{L^{\Phi}(\mathcal{O})}}\right) dx = 1,$$

while in the absence of $\Phi \in \Delta_2$ these facts are generally not valid. This slightly shows that $L^{\Phi}(\mathcal{O})$ with the Δ_2-condition has its topological structure modified.

We will now present a very relevant concept in the theory of Orlicz spaces.

Definition A.3 *Given an N-function Φ, the function defined by*

$$\tilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\} \quad \textit{for} \quad s \geq 0$$

is called **complementary function of** Φ.

It is convenient to extend the definition domain of the function $\tilde{\Phi}$ for $\mathbb{R}$ by putting the even condition. It turns out that $\tilde{\Phi}$ is also an N-function and that the functions Φ and $\tilde{\Phi}$ are complementary each other. Now, to illustrate this phenomenon in particular cases, a classic example follows below.

Example A.2 *If $p \in (1, +\infty)$ and $\Phi(t) = \dfrac{|t|^p}{p}$, then*

$$\tilde{\Phi}(t) = \frac{|t|^q}{q}, \quad \textit{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

In this case, q is known as the **conjugate exponent** *of p.*

An important property that relates Φ and $\tilde{\Phi}$ is a Young type inequality which states that

$$st \leq \Phi(t) + \tilde{\Phi}(s) \quad \forall s, t \geq 0.$$

It can be used to prove the following Hölder type inequality

$$\int_{\mathcal{O}} |uv| dx \leq 2\|u\|_{L^{\Phi}(\mathcal{O})} \|v\|_{L^{\tilde{\Phi}}(\mathcal{O})} \ \text{ for all } \ u \in L^{\Phi}(\mathcal{O}) \ \text{ and } \ v \in L^{\tilde{\Phi}}(\mathcal{O}).$$

We will see in the next result that the Δ_2-condition applies an important rule in the development of Orlicz spaces.

Lemma A.1 *The space $L^{\Phi}(\mathcal{O})$ is reflexive if, and only if, Φ, $\tilde{\Phi} \in \Delta_2$.*

Proof. See for instance [85, Ch. IV, Theorem 10]. ■

The structure of Orlicz spaces also allows us to naturally generalize the ideas of Sobolev spaces $W^{1,p}(\mathcal{O})$, as we will see in the next lines.

Definition A.4 *Let be Φ an N-function. The vector space*

$$W^{1,\Phi}(\mathcal{O}) = \left\{ u \in L^{\Phi}(\mathcal{O}) \mid \frac{\partial u}{\partial x_i} = u_{x_i} \in L^{\Phi}(\mathcal{O}),\ i = 1, ..., N \right\}$$

*is called **Orlicz-Sobolev space associated with** Φ **over** $\mathcal{O}$ or simply* **Orlicz-Sobolev space** *whenever there is no confusion of notation.*

The Orlicz-Sobolev space $W^{1,\Phi}(\mathcal{O})$ equipped with the norm

$$\|u\|_{W^{1,\Phi}(\mathcal{O})} = \|\nabla u\|_{L^{\Phi}(\mathcal{O})} + \|u\|_{L^{\Phi}(\mathcal{O})}, \quad u \in W^{1,\Phi}(\mathcal{O}),$$

where $\nabla u = (u_{x_1}, ..., u_{x_N})$, is a Banach space. Moreover, under the Δ_2-condition, $W^{1,\Phi}(\mathcal{O})$ is a reflexive space.

A.2 Auxiliary results

In this section, we present and develop some preliminary results that will be frequently applied throughout this thesis. To begin with, the reader can verify by means of direct calculations that the N-functions cited at the beginning of this appendix have the form

$$\Phi(t) = \int_0^{|t|} \phi(s) s ds, \quad t \in \mathbb{R}. \tag{A.2}$$

To better understand this class of N-functions, we list in the next lines some results about N-functions of type (A.2) that verify the hypotheses (ϕ_1) and (ϕ_2).

Lemma A.2 *Let Φ be an N-function of the form (A.2) satisfying (ϕ_1)-(ϕ_2). Setting*

$$\xi_0(t) = \min\{t^l, t^m\} \quad \text{and} \quad \xi_1(t) = \max\{t^l, t^m\} \quad \text{for} \quad t \geq 0,$$

then Φ satisfies

$$\xi_0(t)\Phi(s) \leq \Phi(st) \leq \xi_1(t)\Phi(s) \quad \forall s, t \geq 0$$

and

$$\xi_0\left(\|u\|_{L^{\Phi}(\mathcal{O})}\right) \leq \int_{\mathcal{O}} \Phi(u) dx \leq \xi_1\left(\|u\|_{L^{\Phi}(\mathcal{O})}\right) \quad \forall u \in L^{\Phi}(\mathcal{O}).$$

Proof. We will first show that condition (ϕ_2) produces

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m \ \text{ for all } \ t > 0. \tag{A.3}$$

Indeed, by (ϕ_2) we can write

$$l\phi(t) \leq (\phi(t)t)' + \phi(t) \leq m\phi(t) \quad \text{for all} \quad t > 0,$$

from which it follows that

$$l\phi(t)t \leq \left(\phi(t)t^2\right)' \leq m\phi(t)t \quad \text{for all} \quad t > 0.$$

Consequently, integrating the last inequality we obtain the estimate (A.3). Finally, from (A.3) the proof becomes similar to that given in [49, Lemma 2.1]. The details are left to the reader. ■

Lemma A.3 *Let Φ be an N-function of the form (A.2) satisfying (ϕ_1)-(ϕ_2). Then, Φ and $\tilde{\Phi}$ satisfy the Δ_2-condition.*

Proof. See Lemma 2.7 in [49] for the proof. ■

Lemma A.4 *If Φ is an N-function of the form (A.2) satisfying (ϕ_1)-(ϕ_2), then the spaces $L^{\Phi}(\mathcal{O})$ and $W^{1,\Phi}(\mathcal{O})$ are reflexive.*

Proof. The proof follows directly from Lemmas A.1, A.2 and A.3. ■

Next, we will discuss some continuous immersions.

Lemma A.5 *Let Φ be an N-function of the form (A.2) satisfying (ϕ_1)-(ϕ_2). If $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, then*

(a) $L^{\Phi}(\mathcal{O}) \hookrightarrow L^{l}(\mathcal{O})$.

(b) $W^{1,\Phi}(\mathcal{O}) \hookrightarrow W^{1,l}(\mathcal{O})$.

(c) If I is an open interval in $\mathbb{R}$, then $W^{1,\Phi}(I) \hookrightarrow L^{\infty}(I)$.

Proof. Considering $w \in L^{\Phi}(\mathcal{O})$ with $w \neq 0$, we may write

$$\int_{\mathcal{O}} \left|\frac{w}{\|w\|_{L^{\Phi}(\mathcal{O})}}\right|^{l} dx = \int_{\left\{w/\|w\|_{L^{\Phi}(\mathcal{O})} \leq 1\right\}} \left|\frac{w}{\|w\|_{L^{\Phi}(\mathcal{O})}}\right|^{l} dx + \int_{\left\{w/\|w\|_{L^{\Phi}(\mathcal{O})} > 1\right\}} \left|\frac{w}{\|w\|_{L^{\Phi}(\mathcal{O})}}\right|^{l} dx,$$

which produces

$$\int_{\mathcal{O}} \left|\frac{w}{\|w\|_{L^{\Phi}(\mathcal{O})}}\right|^{l} dx \leq |\mathcal{O}| + \int_{\left\{w/\|w\|_{L^{\Phi}(\mathcal{O})} > 1\right\}} \left|\frac{w}{\|w\|_{L^{\Phi}(\mathcal{O})}}\right|^{l} dx. \tag{A.4}$$

Now, thanks to Lemma A.2, we deduce

$$\int_{\{w/\|w\|_{L^\Phi(\mathcal{O})}>1\}} \left|\frac{w}{\|w\|_{L^\Phi(\mathcal{O})}}\right|^l dx \leq \frac{1}{\Phi(1)} \int_{\mathcal{O}} \Phi\left(\frac{w}{\|w\|_{L^\Phi(\mathcal{O})}}\right) dx \leq \frac{1}{\Phi(1)}, \tag{A.5}$$

from which it follows by (A.4) and (A.5) that

$$\|w\|_{L^l(\mathcal{O})} \leq \Lambda_0 \|w\|_{L^\Phi(\mathcal{O})}, \quad \text{where} \quad \Lambda_0^l = |\mathcal{O}| + \frac{1}{\Phi(1)},$$

showing (a). Finally, it is easy to see that item (b) follows from (a) and (c) follows from (b) via the embedding $W^{1,l}(I) \hookrightarrow L^\infty(I)$ (see for instance [26, Corollary 9.14.]). ■

To finish this section, we will address some elementary inequalities that aim to fulfill the objective of this thesis.

Lemma A.6 *Let Φ be an N-function of the form (A.2) satisfying (ϕ_1)-(ϕ_2). Then,*

$$\tilde{\Phi}(\phi(t)t) \leq \Phi(2t) \quad \textit{for all} \quad t \geq 0.$$

Proof. See for instance [49, Lemma A.2]. ■

Lemma A.7 *Let Φ be an N-function of the form (A.2) satisfying (ϕ_4). Then, if $a, b \in \mathbb{R}$ we have that*

$$\Phi(|a|) + \Phi(|b|) \leq \Phi(|(a,b)|), \quad \textit{where} \quad |(a,b)| = \sqrt{a^2+b^2}.$$

Proof. To begin with, we define the function $\varphi : [0,+\infty) \mapsto \mathbb{R}$ by $\varphi(t) = \Phi(\sqrt{t})$. Thereby, $\varphi \in C^1([0,+\infty),\mathbb{R})$ and

$$\varphi'(t) = \frac{1}{2}\phi(\sqrt{t}), \quad \forall t \geq 0.$$

Now, fixing $s \geq 0$ and considering the application $f_s(t) : [0,+\infty) \mapsto \mathbb{R}$ given by

$$f_s(t) = \varphi(t+s) - \varphi(t) - \varphi(s),$$

it is clear that $f_s \in C^1([0,+\infty),\mathbb{R})$ and

$$f_s'(t) = \frac{1}{2}\phi(\sqrt{t+s}) - \frac{1}{2}\phi(\sqrt{t}).$$

Condition (ϕ_4) easily implies that $f_s'(t) \geq 0$ for any $t \geq 0$. Consequently, f_s is non-decreasing in $[0,+\infty)$. As $f_s(0) = 0$, we must have $f_s(t) \geq 0$ for all $t \geq 0$, that is,

$$\Phi(\sqrt{t+s}) - \Phi(\sqrt{t}) - \Phi(\sqrt{s}) \geq 0, \quad \forall s, t \geq 0.$$

Finally, given $a, b \in \mathbb{R}$ and taking $s = a^2$ and $t = b^2$ we derive

$$\Phi(\sqrt{b^2 + a^2}) \geq \Phi(\sqrt{b^2}) + \Phi(\sqrt{a^2}),$$

as asserted. ■

Lemma A.8 *Let Φ be an N-function of the type (A.3) satisfying (ϕ_1)-(ϕ_2). Then the following inequalities hold*

(a) $\Phi(|a+b|) \leq 2^{m-1}\left(\Phi(|a|) + \Phi(|b|)\right)$ *for all* $a, b \in \mathbb{R}$.

(b) $\phi(|z|)z \cdot (w - z) \leq \Phi(|w|) - \Phi(|z|)$ *for all* $w, z \in \mathbb{R}^N$ *with* $z \neq 0$ *where "·" denotes the usual inner product in* $\mathbb{R}^N$.

(c) $(\phi(|s|)s - \phi(|r|)r)(s - r) > 0$ *for all* $s, r \in \mathbb{R}$ *with* $s \neq r$.

Proof. The proof proceeds through a strong exploration of the convexity of Φ and the details are left to the reader. ■

A.3 Models for Φ

In this last section we will highlight some models for Φ in which conditions (ϕ_1)-(ϕ_4) are all verified. We will start with the classic model

$$\Phi(t) = \frac{|t|^p}{p}, \tag{A.6}$$

which is related to the celebrated p-Laplacian operator.

Proposition A.1 *The N-function given in (A.6) satisfies conditions (ϕ_1)-(ϕ_3) for all $p \in (1, +\infty)$. Moreover, (ϕ_4) is satisfied when $p \geq 2$.*

Proof. First, notice that in this case $\phi(t) = t^{p-2}$. Clearly, ϕ satisfies (ϕ_1). Furthermore, by a direct computation we see that

$$\frac{(\phi(t)t)'}{\phi(t)} = p - 1 \quad \text{for all} \quad t > 0,$$

from which follows (ϕ_2) with $l = m = p$. Finally, (ϕ_3) is satisfied with $s = p$ and $c_1 = c_2 = 1$, since

$$\phi(t)t = t^{p-1} \quad \text{for all} \quad t \geq 0,$$

and the result follows. ■

The next model for Φ that we will study is the following

$$\Phi(t) = \frac{|t|^p}{p} + \frac{|t|^q}{q}, \tag{A.7}$$

which is directly associated with the famous (p, q)-Laplacian operator.

Proposition A.2 *The N-function given in (A.7) satisfies (ϕ_1)-(ϕ_3) for all $p, q \in (1, +\infty)$ with $q > p$. Moreover, (ϕ_4) is satisfied when $p \geq 2$.*

Proof. Note first that $\phi(t) = t^{p-2} + t^{q-2}$ for $t > 0$. Immediately (ϕ_1) is satisfied. Now, by a direct calculation we have

$$\frac{(\phi(t)t)'}{\phi(t)} = \frac{(p-1)t^{p-2} + (q-1)t^{q-2}}{t^{p-2} + t^{q-2}} \quad \text{for all} \quad t > 0,$$

from which it follows that

$$p - 1 \leq \frac{(\phi(t)t)'}{\phi(t)} \leq q - 1 \quad \text{for all} \quad t > 0.$$

Therefore, ϕ satisfies (ϕ_2) by defining $l = p$ and $m = q$. To show that ϕ verifies condition (ϕ_3) it suffices to note that

$$t^{q-1} \leq \phi(t)t \leq 2t^{q-1} \quad \text{for all} \quad t \in [0, 1],$$

and choose $\eta = 1$, $s = q$, $c_1 = 1$ and $c_2 = 2$. It is clear that when $p \in [2, +\infty)$ the function ϕ is non-decreasing on $(0, +\infty)$, and the proof is complete. ■

We are going to consider the following N-function

$$\Phi(t) = (t^2 + 1)^\gamma - 1 \quad \text{for} \quad \gamma > 1. \tag{A.8}$$

Proposition A.3 *The N-function given in (A.8) satisfies (ϕ_1)-(ϕ_4) for all $\gamma > 1$.*

Proof. We observe that $\phi(t) = 2\gamma(t^2 + 1)^{\gamma-1}$. Obviously, ϕ satisfies (ϕ_1). Moreover, an easy computation shows that

$$1 \leq \frac{(\phi(t)t)'}{\phi(t)} \leq 2\gamma - 1 \quad \text{for any} \quad t > 0,$$

and so (ϕ_2) occurs for $l = 2$ and $m = 2\gamma$. Now, to show that ϕ satisfies (ϕ_3) choose $\eta > 0$ and observe that there exists $C_\eta > 0$ such that

$$1 \leq (t^2 + 1)^{\gamma-1} \leq C_\eta, \quad \forall t \in [0, \eta].$$

Consequently,

$$2\gamma t \le \phi(t)t \le 2\gamma C_\eta t, \quad \forall t \in [0,\eta],$$

and therefore take $s=2$, $c_1 = 2\gamma$ and $c_2 = 2\gamma C_\eta$. Finally, since $\phi'(t) > 0$ for every $t > 0$ it follows that (ϕ_4) is satisfied, and the proof is complete. ■

Let us also consider the logarithmic model described below

$$\Phi(t) = |t|^p \ln(1+|t|) \quad \text{for} \quad p \in (1,+\infty). \tag{A.9}$$

Proposition A.4 *The N-function Φ given in (A.9) satisfies (ϕ_1)-(ϕ_3) for all $p \in (1,+\infty)$. Condition (ϕ_4) is checked by Φ whenever $p \ge 2$.*

Proof. The N-function Φ checks (ϕ_1) because

$$\phi(t) = pt^{p-2}\ln(1+t) + \frac{t^{p-1}}{1+t} \quad \text{for} \quad t > 0.$$

Moreover, it is verified by direct calculations that

$$p-1 \le \frac{(\phi(t)t)'}{\phi(t)} \le p, \quad \forall t > 0,$$

from which condition (ϕ_2) follows. Detailed verification is left to the reader. On the other hand, to show that ϕ satisfies assumption (ϕ_3) we write

$$\phi(t)t = t^p\left(p\frac{\ln(1+t)}{t} + \frac{1}{1+t}\right),$$

and since

$$\frac{\ln(1+t)}{t} \to 1 \text{ as } t \to 0 \text{ and } 0 < \frac{\ln(1+t)}{t}, \frac{1}{1+t} \le 1 \text{ for all } t > 0$$

we conclude that for $\delta > 0$ there exists $c_\delta > 0$ such that

$$c_\delta \le p\frac{\ln(1+t)}{t} + \frac{1}{1+t} \le p+1, \quad \forall t \in (0,\delta).$$

Therefore,

$$c_\delta t^p \le \phi(t)t \le (p+1)t^p, \quad \forall t \in (0,\delta).$$

Finally, it is easy to see that for $p \ge 2$ we have $\phi'(t) > 0$ for all $t > 0$, which ends the proof. ■

Finally, let us consider the following model

$$\Phi(t) = \int_0^t s^{1-\gamma}(\sinh^{-1} s)^\beta ds, \tag{A.10}$$

where $0 \le \gamma < 1$ and $\beta > 0$.

Proposition A.5 *The N-function Φ given in (A.10) satisfies (ϕ_1)-(ϕ_3) for all $\gamma \in [0,1)$ and $\beta \in (0,+\infty)$. Moreover, if $\gamma = 0$ then Φ satisfies (ϕ_4) for each $\beta > 0$.*

Proof. Let us first note that

$$\phi(t) = \frac{\left(\sinh^{-1}(t)\right)^{\beta}}{t^{\gamma}} \quad \text{for} \quad t > 0.$$

A strong computation guarantees that ϕ verifies (ϕ_1) and (ϕ_2) with

$$1 - \gamma \le \frac{(\phi(t)t)'}{\phi(t)} \le 1 - \gamma + \beta, \quad \forall t > 0.$$

The details are left to the reader. To show that ϕ satisfies (ϕ_3) we first observe that

$$\sinh^{-1}(t) = \ln\left(t + \sqrt{t^2+1}\right). \tag{A.11}$$

Now, since $t + \sqrt{t^2+1} \le 2t + 1 \le e^{2t}$ for $t \in (0,+\infty)$, one has

$$\ln\left(t + \sqrt{t^2+1}\right) \le 2t \quad \text{for all} \quad t > 0. \tag{A.12}$$

Considering the function

$$f(t) = \frac{e^{t/2}}{t + \sqrt{t^2+1}},$$

a direct calculation shows that $f(0) = 1$ and $f'(t) < 0$ for all $t \in (0,1)$, and hence, f is decreasing on $(0,1)$. Consequently,

$$e^{t/2} \le t + \sqrt{t^2+1}, \quad \forall t \in (0,1),$$

and so,

$$\frac{t}{2} \le \ln\left(t + \sqrt{t^2+1}\right), \quad \forall t \in (0,1). \tag{A.13}$$

Therefore, combining estimates (A.11), (A.12) and (A.13), one gets

$$2^{-\beta}t^{\beta} \le \left(\sinh^{-1}(t)\right)^{\beta} \le 2^{\beta}t^{\beta}, \quad \forall t \in (0,1),$$

that is,

$$2^{-\beta}t^{\beta+1-\gamma} \le \phi(t)t \le 2^{\beta}t^{\beta+1-\gamma}, \quad \forall t \in (0,1).$$

Finally, when $\gamma = 0$ it is easy to see that $\phi'(t) > 0$ for all $t \in (0,+\infty)$, and thus (ϕ_4) is checked. This concludes the proof. ■

APPENDIX B

A NEW CLASS OF DOUBLE-WELL POTENTIALS

A class of functions that has been extensively explored in many fields of physics is the class of double-well potentials. Some models of potentials well-known in the literature are the Ginzburg-Landau potential given by

$$V(t) = \frac{1}{4}(t^2 - 1)^2,$$

and the Sine-Gordon potential that has the following configuration

$$V(t) = 1 + \cos(t\pi).$$

In this last appendix, let us consider a new class of double-well potentials and explore some of their properties.

B.1 Symmetric double-well potentials

Let's assume that Φ is an N-function satisfying conditions (ϕ_1) and (ϕ_2). Then, for each $\alpha > 0$ we define the Φ-double-well potential by

$$V(t) = \Phi(|t^2 - \alpha^2|). \tag{B.1}$$

We therefore have the following result.

Proposition B.1 *The potential given in* (B.1) *satisfies* (V_1)-(V_6).

Proof. Let V be the potential considered in (B.1). From the properties of Φ it easily follows that V satisfies (V_1). Condition (V_2) is trivially satisfied. To show that V also verifies (V_3), let us first note that for $\delta \in (0, \alpha)$ one has

$$\alpha|t-\alpha| \le |t^2-\alpha^2| \le 3\alpha|t-\alpha|, \quad \forall t \in [0, \alpha+\delta]. \tag{B.2}$$

So, since Φ is increasing on $(0, +\infty)$,

$$\Phi(\alpha|t-\alpha|) \le V(t) \le \Phi(3\alpha|t-\alpha|), \quad \forall t \in [0, \alpha+\delta].$$

Thanks to Lemma A.2, one gets

$$\xi_0(\alpha)\Phi(|t-\alpha|) \le V(t) \le \xi_1(3\alpha)\Phi(|t-\alpha|), \quad \forall t \in [0, \alpha+\delta].$$

On the other hand, we note that

$$V'(t) = -2t\phi(|t^2-\alpha^2|)(\alpha^2-t^2).$$

Consequently, combining the fact that $\phi(t)t$ is increasing on $(0, +\infty)$ with inequality (B.2), we get

$$-6\alpha t\phi\left(3\alpha|t-\alpha|\right)(\alpha-t) \le V'(t) \le -2\alpha t\phi(\alpha|t-\alpha|)(\alpha-t) \text{ for all } t \in [0, \alpha]$$

and

$$-2\alpha t\phi\left(\alpha|t-\alpha|\right)(\alpha-t) \le V'(t) \le -6\alpha t\phi(3\alpha|t-\alpha|)(\alpha-t) \text{ for all } t \in [\alpha, \alpha+\delta].$$

The last two inequalities guarantee that the potential V fulfills (V_4). To see that V verifies (V_5) note that

$$V''(t) = 4t^2f'(\alpha^2-t^2) - 2f(\alpha^2-t^2) \quad \text{for} \quad t \in (0, \alpha)$$

where $f(t) = \phi(t)t$. As Φ satisfies (ϕ_2), one has $f'(\alpha^2-t^2) \ge (l-1)\phi(\alpha^2-t^2)$, and so

$$V''(t) = 4t^2f'(\alpha^2-t^2) - 2f(\alpha^2-t^2) \ge 2\phi(\alpha^2-t^2)\left(lt^2-\alpha^2\right).$$

Thus, for each $t \in (\alpha/\sqrt{l}, \alpha)$ one gets $V''(t) > 0$, and hence, V' is increasing on $(\alpha/\sqrt{l}, \alpha)$. Finally, as $\tilde{\Phi}$ is even, we obtain that

$$\tilde{\Phi}\left(V'(t)\right) = \tilde{\Phi}\left(2t\phi(|t^2-\alpha^2|)(\alpha^2-t^2)\right)$$

and so,

$$\tilde{\Phi}\left(V'(t)\right) \leq \tilde{\Phi}\left(2\alpha\phi(|t^2-\alpha^2|)|t^2-\alpha^2|\right), \quad \forall t \in [0,\alpha].$$

Now, by Lemma A.3 we have that $\tilde{\Phi}$ satisfy the Δ_2-condition, and hence, there is $c_1 > 0$ such that

$$\tilde{\Phi}\left(V'(t)\right) \leq c_1\tilde{\Phi}\left(\phi(|t^2-\alpha^2|)|t^2-\alpha^2|\right), \quad \forall t \in [0,\alpha].$$

Thereby, from Lemma A.6,

$$\tilde{\Phi}\left(V'(t)\right) \leq c_1\Phi\left(|t^2-\alpha^2|\right), \quad \forall t \in [0,\alpha].$$

Using again the fact that $\tilde{\Phi} \in \Delta_2$ we can find $c_2 > 0$ satisfying

$$\tilde{\Phi}\left(V'(t)\right) \leq c_2\Phi\left(|t-\alpha|\right), \quad \forall t \in [0,\alpha],$$

which guarantees that V satisfies (V_6), and the proposition follows. ■

B.2 Nonsymmetric double-well potentials

To finish this appendix, let's highlight an important class of nonsymmetric double-well potentials that is given by

$$V(t) = \Phi(|(t-\alpha)(t-\beta)|), \tag{B.3}$$

where $\beta \neq \alpha$ and Φ is an N-function satisfying (ϕ_1) and (ϕ_2). The same argument from the proof of Proposition B.1 works to show the following result.

Proposition B.2 *The potential given in* (B.3) *satisfies conditions* $(\tilde{V}_1)$*-*$(\tilde{V}_3)$ *and* $(\tilde{V}_6)$*.*

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