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# Quasilinear Problems on Non-Reflexive Orlicz-Sobolev Spaces

**by**

**Lucas da Silva**

**Campina Grande - PB**
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**Advised by**

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Thesis presented to the Associate Graduate Program in Mathematics UFPB/UFCG as partial fulfillment of the requirements for the degree of Doctor of Mathematics.

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# ABSTRACT

The goal of this paper is to study the existence of solutions for some classes of elliptic PDEs involving the $\Phi$-Laplacian operator, $\Delta_\Phi$. Firstly, in order to generalize the results obtained in the paper [10], we present the study of two quasilinear Schrödinger equations with potential vanishing at infinity and the $\mathcal{N}$-function $\tilde{\Phi}$(Complementary of function $\Phi$) may not satisfy the $\Delta_2$-condition. Here we present new compact embeddings in $\mathbb{R}^N$ that are commonly known as Hardy-Type inequalities. These inequalities, associated with a Mountain Pass Theorem without the Palais-Smale condition for Gateaux-differentiable energy functionals (Ghoussoub-Preiss Mountain Pass Theorem), yield solutions for the classes of problems initially studied. It is worth noting that in one of the classes, we assume that the nonlinearity of the problem is a non-local type with a Stein-Weiss convolution term. The regularity of the solutions was obtained using the regularity results due to Lieberman [24].

In a second part of this thesis, we study the existence of solutions for two classes of quasilinear systems driven by the operators $\Delta_{\Phi_1}$ ($\Phi_1$-Laplacian) and $\Delta_{\Phi_2}$ ($\Phi_2$-Laplacian) where the $\mathcal{N}$-functions $\Phi_1$ and $\Phi_2$ or $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ may not satisfy the $\Delta_2$-condition. In the first class, we relax the $\Delta_2$-condition of the functions $\Phi_i(i = 1, 2)$ and present a definition for the well-known Ambrosetti-Rabinowitz condition for nonlinearity. In this class we base the results on a Rabinowitz saddle point theorem without the Palais-Smale condition for differentiable Fréchet functionals combining with properties of the weak topology$^*$. In the second class, we relax the $\Delta_2$-conditions of the $\mathcal{N}$-functions $\tilde{\Phi}_i(i = 1, 2)$ and assume that the nonlinearity has supercritical growth. Here, we use a link theorem without the Palais-Smale condition for locally Lipschitz functionals and combine it with a concentration-compactness lemma for non-reflexive Orlicz-Sobolev space to guarantee the existence of solutions for this class.

**Keywords:** $\mathcal{N}$-functions, Orlicz-Sobolev space, Variational methods, locally Lipschitz functionals, $\Delta_2$-condition, Schrödinger equation.

# RESUMO

O objetivo dessa tese é estudar a existência de solução de algumas classes de EDPs elípticas envolvendo o operador $\Phi$-Laplaciano, $\Delta_\Phi$. Num primeiro momento, com o intuito de generalizar os resultados obtidos no paper [10], apresentamos o estudo de duas equações quasilineares Schrödinger com potenciais que podem se anular no infinito e a $\mathcal{N}$-função $\tilde{\Phi}$(Complementar da função $\Phi$) pode não satisfazer a condição $\Delta_2$. Aqui, apresentamos novas imersões compactas no $\mathbb{R}^N$ que comumente são conhecidas como desigualdades do Tipo Hardy, essas desigualdades, associadas a um Teorema do Passo da Montanha sem a condição de Palais-Smale para funcionais energia Gateaux-differentiable (Teorema do Passo da Montanha de Ghoussoub-Preiss) produzem uma solução para as classes de problemas inicialmente estudadas. Vale ressaltar que em uma das classes assumimos que a não linearidade do problema é tipo não local com termo de convolução de Stein-Weiss. A regularidade das soluções foram obtidas utilizando-se dos resultados de regularidade devido a Lieberman [24].

Num segundo momento dessa tese, passamos a estudar a existência de soluções para duas classes de sistemas quasilineares dirigidos pelos operadores $\Delta_{\Phi_1}$ ($\Phi_1$-Laplacian) e $\Delta_{\Phi_2}$ ($\Phi_2$-Laplacian) onde as $\mathcal{N}$-funções $\Phi_1$ e $\Phi_2$ ou $\tilde{\Phi}_1$ e $\tilde{\Phi}_2$ podem não satisfazer a condição $\Delta_2$. Na primeira classe, relaxamos a $\Delta_2$-condition das funções $\Phi_i (i = 1, 2)$ e apresentamos uma definição para a conhecida condição de Ambrosetti-Rabinowitz para a não linearidade. Nessa classe baseamos os resultados em um teorema do ponto de sela de Rabinowitz sem a condição de Palais-Smale para funcionais Fréchet diferenciáveis combinando com propriedades da topologia fraca$^*$. Na segunda classe, relaxamos as condições $\Delta_2$ das $\mathcal{N}$-funções $\tilde{\Phi}_i (i = 1, 2)$ e assumiremos que a não-linearidade tem crescimento super-crítico. Aqui, usamos um teorema de link sem a condição de Palais-Smale para funcionais localmente de Lipschitz e combinamos com um lema de concentração-compacidade para espaço de Orlicz-Sobolev não reflexivo para garantir a existência de soluções para essa classe.

**Palavras-chave:** $\mathcal{N}$-funções, espaço de Orlicz-Sobolev, métodos variacionais, funcionais localmente Lipschitz, condição $\Delta_2$, equações de Schrödinger.

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# Contents

# Chapter 1

# Introduction

In the study of partial differential equations, we encounter problems such as

$$-\Delta_p u = f(x,u) \quad \text{in} \quad \Omega \tag{1.1}$$

where $f(x,\cdot)$ is a continuous function in $\mathbb{R}$ for each $x \in \Omega$ (where $\Omega \subset \mathbb{R}^N$ is a domain) and $p > 1$. The above operator frequently appears in physical models, for example in Newtonian and non-Newtonian fluids (see [42, 43] and references therein). There are several techniques to address problem (1.1), one of which is the well-known variational method. This method associates the problem with a functional, commonly called the energy functional in the literature, which in this case would be a functional $I : X \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\Omega} F(x,u)dx \tag{1.2}$$

where $F(t) = \int_0^t f(s)ds$ and $X$ is a Banach space. Under suitable conditions on the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ the functional (1.2) belongs to the class $C^1(X,\mathbb{R})$. Consequently, the equation (1.1) can be approached by finding critical points of the functional $I$, i.e., functions $u \in X$ such that $I'(u) = 0$, where $I'$ is the Fréchet derivative of the functional $I$. Some topological properties of the energy space $X$, such as separability and reflexivity, are also crucial when studying the critical points of the energy functional. The works of Schwartz [36], Palais and Smale [68], Ambrosetti and Rabinowitz [2], and Benci and Rabinowitz [71] are clear examples of how to obtain critical points for a functional belongs to the class $C^1$ and highlight the importance of reflexivity in obtaining critical points for the functional $I$.

In order to generalize the study of equations of the type (1.1), several authors have recently been working on quasilinear equations of the form

$$-\Delta_\Phi u = f(x, u) \quad \text{in} \quad \Omega \tag{1.3}$$

where $f(x, \cdot)$ is a continuous function on $\mathbb{R}$ for each $x \in \Omega$ (where $\Omega \subset \mathbb{R}^N$ is a domain) and $\Delta_\Phi u = div(\phi(|\nabla u|)\nabla u)$ in which $\phi : (0, \infty) \to (0, \infty)$ is a function $C^1$ so that the function $\Phi : \mathbb{R} \to [0, \infty)$ of the type

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds, \quad t \in \mathbb{R},$$

is an $\mathcal{N}$-function (See Definition 2.1). The operator described above is associated with many applications in physics, such as nonlinear elasticity, plasticity, generalized Newtonian fluids, non-Newtonian fluids, and plasma physics. The reader can find more details about this subject in [66], [47], [57], and their references. Similarly to what was described for equation (1.1), associated with the problem (1.3) we have the energy functional $I : W^{1,\Phi}(\Omega) \to \mathbb{R}$ given by

$$I(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx - \int_\Omega F(x, u)dx, \tag{1.4}$$

where $F(t) = \int_0^t f(s)ds$. It is easily seen in the literature that this functional belongs to $C^1$ when the so-called $\Delta_2$-condition (See definition 2.4) is assumed on $\Phi$ and $\tilde{\Phi}$ (Complementary function of $\Phi$). Furthermore, this ensures that the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ and $D^{1,\Phi}(\Omega)$ are reflexive Banach spaces (See for instance Chapter 2). The papers Bonanno, Bisci and Radulescu [20, 21], Cerny [67], Clément, Garcia-Huidobro and Manásevich [63], Donaldson [69], Fuchs and Li [45], Fuchs and Osmolovski [47], Fukagai, Ito and Narukawa [59], Gossez [34], Le and Schmitt [72], Mihailescu and Radulescu [50, 51], Mihailescu and Repovs [52], Mihailescu, Radulescu and Repovs [53], Mustonen and Tienari [74], Alves e.t.al [9], Orlicz [75] and their references, are clear examples of works where the so-called $\Delta_2$-condition is assumed on $\Phi$ and $\tilde{\Phi}$ (Complementary function of $\Phi$), which ensures that the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ and $D^{1,\Phi}(\Omega)$ are reflexive Banach spaces.

In recent years, problem (1.3) without the $\Delta_2$-condition on the function $\tilde{\Phi}$ have been studied. This type of problem presents many difficulties when applying variational methods. For example, it is observed in the literature that the energy functional associated with the problem might not belong to $C^1$, so classical minimax type results cannot be used here. To overcome this difficulty, some recent articles suggest the use of the minimax theory developed by Szulkin [7]. Notable examples include [11]. In that paper, Alves and Carvalho study a class of problems

$$\begin{cases} -\Delta_\Phi u + V(x)\phi(u)u = f(u), & \text{in } \mathbb{R}^N \\ u \in W^{1,\Phi}(\mathbb{R}^N) \text{ with } N \geq 2 \end{cases}$$

when $V$ is $\mathbb{Z}^N$-periodic and $f$ is a continuous function satisfying some technical conditions. The $\Delta_2$-condition of $\mathcal{N}$-function $\tilde{\Phi}$ has not been required.

Another work in which the $\Delta_2$-condition of $\mathcal{N}$-function $\tilde{\Phi}$ can be relaxed is [16]. Silva, Carvalho, Silva and Gonçalves study a class of problem

$$\begin{cases} -\Delta_\Phi u = g(x,u), & \text{in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary.

It is worth mentioning that in the works above we cannot rely on standard analysis because the Orlicz-Sobolev spaces associated with the mentioned problems might not be reflexive. This is a challenge to apply when applying variational methods. To overcome these obstacles, we consider the weak* topology to recover some compactness required in variational methods. Based on these new challenges, we dedicate the first part of this thesis to studying two problems where the $\Delta_2$-condition of the $\mathcal{N}$-function $\tilde{\Phi}$ is removed.

Specifically, in Chapter 3 we present a joint paper with Professor Marco Souto [37]. The main goal of this chapter is to prove the existence of solutions of the following class of quasilinear equations:

$$(P_1) \qquad \begin{cases} -\Delta_\Phi u + V(x)\phi(|u|)u = K(x)f(u), \text{ in } \mathbb{R}^N \\ u \in D^{1,\Phi}(\mathbb{R}^N), \; u \geq 0, \text{ in } \mathbb{R}^N \end{cases}$$

for $N \geq 2$ and assuming that $V, K : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions with $V$, $K$ being nonnegative functions and $f$ having a quasicritical growth. The motivation for studying equation $(P_1)$ initially arises from the problem proposed by

Ambrosetti, Felli and Malchiodi [4]. In that paper, the authors studied the problem

$$
\left\{
\begin{array}{l}
-\Delta u + V(x)u = K(x)u^p, \text{ in } \mathbb{R}^N \\
u \in D^{1,\Phi}(\mathbb{R}^N), \ u \geq 0, \text{ in } \mathbb{R}^N
\end{array}
\right. \tag{1.5}
$$

for $N \geq 2$ e $2 < p < \frac{N+2}{N-2}$. Furthermore, they assumed that $V$ , $K$ satisfying the following assumptions:

$V$, $K : \mathbb{R}^N \to \mathbb{R}$ are smooth functions and there exist $\tau, \xi, \tau_1, \tau_2, \tau_3 > 0$ such that

$$
\frac{a_1}{1+|x|^\tau} \leq V(x) \leq \tau_2 \text{ and } 0 < K(x) \leq \frac{\tau_3}{1+|x|^\xi}, \ \forall x \in \mathbb{R}^N \tag{$VK$}
$$

and $\tau$, $\xi$ verifying

$$
\frac{N+2}{N-2} - \frac{4\xi}{\tau(N-2)} < p, \text{ if } 0 < \xi < \tau \text{ or } 1 < p, \text{ when } \xi \geq \tau.
$$

The condition $(VK)$ is interesting, because in Opic and Kufner [?] was showed that it can be used to prove that the space E given by

$$
E = \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\}
$$

endowed with the norm

$$
\|u\|_E = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx
$$

is compactly embedded into the weighted Lebesgue space

$$
L_K^{p+1} = \left\{ u : \mathbb{R} \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^N} K(x)|u|^{p+1} dx < \infty \right\}.
$$

In [3], Ambrosetti and Wang have considered also the condition $(VK)$, however the inequality on $V$ is assumed only outside of a ball centered at origin.

Inspired by [4] and [3], Alves and Marco Souto in [10] generalized the problem (1.5) for a general class of nonlinearity. Moreover, the authors assumed that the functions $V, K : \mathbb{R}^N \to \mathbb{R}$ are continuous and satisfy:

$(K_0')$ $V > 0$, $K \in L^\infty(\mathbb{R}^N)$ and $K$ is positive almost everywhere.

(I) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borelian sets such that $\sup_n |A_n| < +\infty$, then

$$
\lim_{r \to +\infty} \int_{A_n \cap B_r^c(0)} K(x) dx = 0, \quad \text{uniformly in } n \in \mathbb{N}. \tag{$K_1'$}
$$

(II) One of the below conditions occurs:

$$\frac{K}{V} \in L^{\infty}(\mathbb{R}^N) \tag{$K_2'$}$$

or there is $p \in (2, 2^*)$ such that

$$\frac{K(x)}{[V(x)]^{\frac{2^*-p}{2^*-2}}} \to 0 \;\text{ as }\; |x| \to +\infty. \tag{$K_3'$}$$

These conditions generalize the condition $(VK)$. Related to the function $f$, the authors assumed the following conditions:

$(f^1)$ $\displaystyle \lim_{t\to 0} \frac{f(t)}{t} = 0$ if $(K_2')$ holds or $\displaystyle \limsup_{t\to 0^+} \frac{|f(t)|}{t^{p-1}} < \infty$ if $(K_3')$ holds ,

$(f^2)$ $f$ has a quasicritical growth, that is

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{2^*-1}} = 0$$

$(f^3)$ $s^{1-m}f(s)$ is an increasing function in $(0, +\infty)$ and $F(t) = \int_0^t f(s)ds$ is is superquadratic at infinity, that is,

$$\lim_{|t|\to+\infty} \frac{F(t)}{|t|^2} = +\infty.$$

Motivated by the above references, more precisely by papers [10], [11] and [16], we begin to study the problem $(P_1)$ where the $\Delta_2$-condition of the $\mathcal{N}$-function $\tilde{\Phi}$ is removed. In this sense, we suppose that $\Phi : \mathbb{R} \to [0, \infty)$ is an $\mathcal{N}$-function of the type

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds, \;\; t \in \mathbb{R} \tag{1.6}$$

for a function $\phi \in C^1((0, \infty), (0, \infty))$ satisfying

$$t \mapsto t\phi(t) \;\text{ is increasing for }\; t > 0. \tag{$\phi_1$}$$

$$\lim_{t\to 0^+} t\phi(t) = 0 \quad\text{and}\quad \lim_{t\to+\infty} t\phi(t) = +\infty. \tag{$\phi_2$}$$

$$1 \le \ell = \inf_{t>0} \frac{\phi(t)t^2}{\Phi(t)} \le \sup_{t>0} \frac{\phi(t)t^2}{\Phi(t)} = m < N, \;\; m < \ell^* = \frac{\ell N}{N-\ell}, \;\; m \ne 1, \tag{$\phi_3$}$$

$$t \mapsto \frac{\phi(t)}{t^{m-2}} \text{ is nonincreasing on } (0, \infty). \tag{$\phi_4$}$$

In Example 2.2.4, we show that the function

$$\Phi_\alpha(t) = |t|\ln(|t|^\alpha + 1) \;\text{ for } 0 < \alpha < \frac{N}{N-1} - 1 \tag{1.7}$$

is an $\mathcal{N}$-function satisfying conditions $(\phi_1) - (\phi_4)$ and $1 = \ell = \inf_{t>0} \frac{\phi_\alpha(t)t^2}{\Phi_\alpha(t)}$ and $m < \ell^*$ where

$$\phi_\alpha(t) = \frac{\ln(t^\alpha + 1)}{t} + \alpha\frac{t^{\alpha-1}}{t^\alpha + 1}, \quad \text{for } t > 0.$$

As can be seen in Lemma 2.19, the $\mathcal{N}$-function (1.7) is an example of where function $\tilde{\Phi}_\alpha$ (Complementary function of $\Phi_\alpha$) does not satisfy the $\Delta_2$-condition. Therefore, by Lemma 2.26, the Orlicz-Sobolev space associated with $\mathcal{N}$-function (1.7) may not be reflexive, consequently, the $(P_1)$ may be associated with a non-reflexive Orlicz-Sobolev space. In order to generalize the results in the paper [10] to a class of $\mathcal{N}$-functions of type (1.7) that satisfy conditions $(\phi_1) - (\phi_4)$, Silva and Marco Souto in [37] introduced the following assumptions regarding the potential $V$ and the coefficient $K$:

$(K_0)$ $V > 0$, $K \in L^\infty(\mathbb{R}^N)$ and $K$ is positive almost everywhere.

(I) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borelian sets such that $\sup_n |A_n| < +\infty$, then

$$\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N}. \tag{$K_1$}$$

(II) One of the below conditions occurs:

$$\frac{K}{V} \in L^\infty(\mathbb{R}^N) \tag{$K_2$}$$

or there are $a_1, a_2 \in (m, \ell^*)$ and a $\mathcal{N}$-function $A(t) = \int_0^{|t|} sa(s)ds$ verifying the following properties:

$$a_1 \le \frac{a(t)t^2}{A(t)} \le a_2 \tag{1.8}$$

and

$$\frac{K(x)}{H(x)} \to 0 \;\text{ as }\; |x| \to +\infty \tag{$K_3$}$$

with $H : \mathbb{R}^N \mapsto \mathbb{R}$ given by $H(x) = \min_{s>0}\left\{V(x)\frac{\Phi(s)}{A(s)} + \frac{\Phi_*(s)}{A(s)}\right\}$ where $\Phi_*$ is the conjugate function of $\Phi$ (see the definition of $\Phi_*$ in Chapter 2 of tesis).

This hypotheses above leads us to define that $(V, K) \in \mathcal{K}_1$ if conditions $(K_0)$, $(K_1)$, and $(K_2)$ are satisfied. Conversely, when conditions $(K_0)$, $(K_1)$, and $(K_3)$ are met, we denote $(V, K) \in \mathcal{K}_2$. In the appendix B, you can see examples of functions $V$ and $K$ belonging to the classes $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively.

To study the main results of this chapter, we will divide the study of problem $(P_1)$ into two conditions: $(V, K) \in \mathcal{K}_1$ and $(V, K) \in \mathcal{K}_2$.

In the case $(V, K) \in \mathcal{K}_1$. In our first main result, by means of some conditions imposed on $\Phi$ and $f$, we will show that the problem $(P_1)$ has a $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ positive ground state solution. More specifically, we assume that $\Phi$ satisfies conditions $(\phi_1) - (\phi_4)$ and $\Phi \in \mathcal{C}_m$, i.e, there is a constant $C > 0$ satisfying

$$(\mathcal{C}_m) \qquad \Phi(t) \geq C|t|^m, \quad \text{for all} \;\; t \in \mathbb{R}.$$

Furthermore, $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions

$$(f_1) \qquad \lim_{t\to 0} \frac{f(t)}{t\phi(t)} = 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{f(t)}{t\phi_*(t)} = 0,$$

where $\phi_*(t)t$ is such that the Sobolev conjugate function $\Phi_*$ of $\Phi$ is its primitive, that is, $\Phi_*(t) = \displaystyle\int_0^{|t|} \phi_*(s)sds$.

$(f_2)$ $s^{1-m}f(s)$ is an increasing function in $(0, +\infty)$.

$(f_3)$ $F(t) = \int_0^t f(s)ds$ is $m$-superlinear at infinity, that is,

$$\lim_{|t|\to+\infty} \frac{F(t)}{|t|^m} = +\infty.$$

Under these conditions, our first main result can be stated as follows.

**Theorem 1.1** *Assume that $(V, K) \in \mathcal{K}_1$ and $\Phi \in \mathcal{C}_m$. Suppose that $(\phi_1) - (\phi_4)$ and $(f_1) - (f_3)$ hold. Then problem $(P_1)$ possesses a nonnegative solutions that are locally bounded.*

To study the regularity of the solutions provided by Theorem 1.1, we add the following assumptions:

$(\phi_5)$ There are $0 < \delta < 1$, $C_1, C_2 > 0$ and $1 < \beta \leq \ell^*$ such that

$$C_1 t^{\beta-1} \leq t\phi(t) \leq C_2 t^{\beta-1} \;\; \text{for} \;\; t \in [0, \delta].$$

$(\phi_6)$ There are constants $\delta_0 > 0$ and $\delta_1 > 0$ such that

$$\delta_0 \leq \frac{(\phi(t)t)'}{\phi(t)} \leq \delta_1 \;\; \text{for } t > 0.$$

We are in position to state the following regularity result:

**Theorem 1.2** *Suppose that $\Phi$ satisfies $(\phi_5)-(\phi_6)$. Under the assumptions of Theorem 1.1, the problem $(P)$ possesses a $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ positive ground state solution.*

For the next main result, we consider the problem $(P_1)$ without condition $\mathcal{C}_m$. The removal of this condition on the $\mathcal{N}$-function $\Phi$, forced us to present a more restricted growth condition that $(f_1)$. This constraint under the nonlinearity $f$ will be necessary to show that the nonnegative solutions of $(P_1)$, are positive. In this way, we will consider $B : \mathbb{R} \to [0, \infty)$ being a $\mathcal{N}$-function given by $B(t) = \int_0^{|t|} b(s)sds$, where $b : (0, \infty) \to (0, \infty)$ is a function satisfying the following conditions:

$(B_1)$ $t \mapsto tb(t)$ is increasing for $t > 0$,

$(B_2)$ $\lim\limits_{t\to 0^+} tb(t) = 0$ and $\lim\limits_{t\to +\infty} tb(t) = +\infty$,

$(B_3)$ There exist $b_1 \in [m, \ell^*]$ such that

$$b_1 = \inf_{t>0} \frac{b(t)t^2}{B(t)} \quad \text{and} \quad \ell^* \geq \sup_{t>0} \frac{b(t)t^2}{B(t)}, \quad \forall t > 0.$$

For this case, we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditions $(f_2)$ and $(f_3)$. Moreover, we will consider the following growth condition

$$(f_4) \qquad \lim_{t\to 0} \frac{f(t)}{t\phi(t)} = 0 \quad \text{and} \quad \limsup_{t\to\infty} \frac{|f(t)|}{tb(t)} = 0.$$

Our second main result can be written in the following form.

**Theorem 1.3** *Assume that $(V, K) \in \mathcal{K}_1$. Suppose that $(\phi_1) - (\phi_4)$, $(f_2)$, $(f_3)$ and $(f_4)$ hold. Then problem $(P_1)$ possesses a nonnegative solutions that are locally bounded. If $\Phi$ also satisfies $(\phi_5)$ and $(\phi_6)$, the solutions for the problem $(P_1)$ are $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ positive ground state solution.*

To study this second class of problem where $(V, K) \in \mathcal{K}_2$, we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $(f_2)$ and $(f_3)$. Moreover, we will consider the following condition

$$(f_5) \qquad \limsup_{t\to 0} \frac{|f(t)|}{ta(t)} < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{t\phi_*(t)} = 0.$$

Our first main result of this section can be stated as follows.

**Theorem 1.4** *Assume that $(V, K) \in \mathcal{K}_2$ and $\Phi \in \mathcal{C}_m$. Suppose that $(\phi_1) - (\phi_4)$, $(f_2)$, $(f_3)$ and $(f_5)$ hold. Then problem $(P_1)$ possesses a nonnegative solutions that are locally bounded. If $\Phi$ also satisfies $(\phi_5)$ and $(\phi_6)$, the solutions for the problem $(P_1)$ are $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ positive ground state solution.*

In a second moment, as in Theorem 1.3, we relax the condition $\Phi \in \mathcal{C}_m$ and present a more restricted growth condition that $(f_5)$. More precisely, $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$(f_6) \qquad \limsup_{t\to 0} \frac{|f(t)|}{ta(t)} < \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{tb(t)} = 0.$$

Furthermore, we will assume that $f$ satisfies $(f_2)$ and $(f_3)$.

Our second main result of this subsection can be written in the following form.

**Theorem 1.5** *Assume that $(V, K) \in \mathcal{K}_2$. Suppose that $(\phi_1) - (\phi_4)$, $(f_2)$, $(f_3)$ and $(f_6)$ hold. Then problem $(P_1)$ possesses a nonnegative solutions that are locally bounded. If $\Phi$ also satisfies $(\phi_5)$ and $(\phi_6)$, the solutions for the problem $(P_1)$ are $C^{1,\alpha}_{loc}(\mathbb{R}^N)$ positive ground state solution.*

It can be observed that Theorems 1.2 and 1.4 generalizes and strengthens the Theorem 1.1 presented by Alves and Marco Souto in [10].

Now, continuing the study of the existence of positive solutions for a class of quasilinear Schrödinger equations with a potential vanishing at infinity in non-reflexive Orlicz-Sobolev spaces, in Chapter 4, we present a joint paper with Professor Marco Souto [38]. We aim to extend the ideas presented in Chapter 3 by modifying the structure of the right-hand side of equation $(P_1)$ with an Stein-Weiss convolution term. More specifically, we propose to study equations of the type:

$$(P_2) \qquad \begin{cases} -\Delta_\Phi u + V(x)\phi(|u|)u = \dfrac{1}{|x|^\alpha}\left(\displaystyle\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^\lambda |y|^\alpha}dy\right) K(x)f(u(x)),\ x \in \mathbb{R}^N \\ u \in D^{1,\Phi}(\mathbb{R}^N) \end{cases}$$

where $\alpha \geq 0$, $N \geq 2$, $\lambda > 0$, $V, K \in C(\mathbb{R}^N, [0,\infty))$ are nonnegative functions that may vanish to infinity, the function $f \in C(\mathbb{R}, \mathbb{R})$ is quasicritical and $F(t) = \int_0^t f(s)ds$. To address the above problem, we suppose that $\phi : (0,\infty) \to (0,\infty)$ is a $C^1$ function satisfying the conditions $(\phi_1) - (\phi_4)$ defined above, furthermore we will assume that $0 \leq \alpha < \lambda$ and $\lambda + 2\alpha \in (0, N) \cap (0, 2N - \frac{2N}{m})$.

When $\alpha = 0$, due to the presence of the Choquard type nonlinearity, the problem $(P_2)$ is known as a Choquard equation. In that case, to show the existence of solution using variational methods, a tool of the main tool to deal with such type of equations is Hardy-Littlewood-Sobolev inequality [18]. Several works use this approach, we can mention [17,25,65]

It is clear that there is a physical interpretation for Choquard type of equations, we refer to [73] and survey of such type of equations.

Recently, in [13], Alves, Rădulescu and Tavares studied the equation $(P_2)$ with $V = K = 1$ and $\alpha = 0$ using different assumptions on the $\mathcal{N}$-function $\Phi$. In this work, the authors aimed to show that the variational methods could be applied to establish the existence of solutions assuming that the $\mathcal{N}$-function $\Phi$ satisfies the conditions $(\phi_1)$–$(\phi_3)$ with $\ell > 1$. One of the main difficulties was to prove that the energy functional associated with equation $(P_2)$ is differentiable. However, good conditions involving the function $f$ made it possible to show the differentiability of the energy functional and consequently allowed to guarantee the existence of a solution through the mountain pass theorem. It is also worth mentioning that in the same work, Alves, Rădulescu and Tavares extended the result to the case where $K = 1$ and $V$ is one of the following potentials: periodic function, asymptotic periodic function, coercive or Bartsch-Wang-like potential.

Before we present the first results of this chapter, we will write a fundamental tool for studying problems with anisotropic Stein-Weiss convolution term that is the Stein-Weiss inequality [19], that is the extension of the Hardy-Littlewood-Sobolev inequality.

**Proposition 1.1** *[Stein-Weiss inequality] Set $t, r > 1$, $\lambda \in (0, N)$ $\sigma + \beta \geq 0$ and $\sigma + \beta + \lambda \leq N$. If $1/t + 1/r + (\lambda + \sigma + \beta)/N = 2$ and $1 - 1/t - \lambda/N < \alpha/N < 1 - 1/t$. Then there exists a constant $C_0 = C(t, r, \sigma, \beta, N, \lambda)$ such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g_1(x) g_2(y)}{|x|^\sigma |x-y|^\lambda |y|^\beta} dx dy \right| \leq C_0 \|g_1\|_{L^r(\mathbb{R}^N)} \|g_2\|_{L^t(\mathbb{R}^N)}. \tag{1.9}$$

*for all $g_1 \in L^r(\mathbb{R}^N)$ and $g_2 \in L^t(\mathbb{R}^N)$, where $C_0$ is independent of $g_1$, $g_2$. For $\sigma = \beta = 0$, it is reduced to the Hartree type (also called the Choquard type) nonlinearity, which is driven by the classical Hardy-Littlewood-Sobolev inequality (See [18]).*

Inspired by [10] and [37], to proceed with the study outlined in this chapter, it will be necessary to present new assumptions regarding the potential $V$ and the coefficient $K$, so that the study carried out in the previous chapter can serve as a guide to determine the existence of a solution for the equation $(P_2)$. In this sense, we consider the constant $\theta = \dfrac{2N}{2N - 2\alpha - \lambda} > 0$ and we will assume that $V$ and $K$ satisfy:

$(Q_0)$ $V > 0$, $Q \in L^\infty(\mathbb{R}^N)$ and $K$ is positive almost everywhere.

(I) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borelian sets such that $\sup_n |A_n| < +\infty$, then

$$\lim_{r \to +\infty} \int_{A_n \cap B_r^c(0)} K(x)^\theta dx = 0, \quad \text{uniformly in } n \in \mathbb{N}. \tag{$Q_1$}$$

(II) One of the below conditions occurs:

$$\frac{K^\theta}{V} \in L^\infty(\mathbb{R}^N) \tag{$Q_2$}$$

or there are $b_1, b_2 \in (m, \ell^*)$ and an $\mathcal{N}$-function $B(t) = \int_0^{|t|} b(\tau)\tau d\tau$ verifying the following properties:

$$(B_1) \qquad t \longmapsto tb(t) \ \text{ is increasing for } \ t > 0.$$

$$(B_2) \qquad \lim_{t\to 0^+} tb(t) = 0 \quad \text{and} \quad \lim_{t\to +\infty} tb(t) = +\infty.$$

$$(B_3) \qquad b_1 \leq \frac{b(t)t^2}{B(t)} \leq b_2, \ \text{ for all } \ t > 0$$

$$(B_4) \qquad \text{The function } B(|t|^{1/\theta}) \text{ is convex in } \mathbb{R}$$

and

$$\frac{K(x)^\theta}{H(x)} \longrightarrow 0 \ \text{ as } \ |x| \to +\infty \tag{$Q_3$}$$

where $H(x) = \min_{\tau>0}\left\{V(x)\dfrac{\Phi(\tau)}{B(\tau)} + \dfrac{\Phi_*(\tau)}{B(\tau)}\right\}$ and $\Phi_*$ is the conjugate function of $\Phi$ (see Section 2).

This hypotheses above leads us to define that $(V, K) \in \mathcal{Q}_1$ if conditions $(Q_0)$, $(Q_1)$, and $(Q_2)$ are satisfied. Conversely, when conditions $(Q_0)$, $(Q_1)$, and $(Q_3)$ are met, we denote $(V, K) \in \mathcal{Q}_2$. In the appendix B, you can see examples of functions $V$ and $K$ belonging to the classes $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively.

To study the main results of this chapter, we will divide the study of problem $(P_2)$ into two conditions: $(V, K) \in \mathcal{Q}_1$ and $(V, K) \in \mathcal{Q}_2$.

Note that the constant $\theta = \dfrac{2N}{2N - 2\alpha - \lambda}$ satisfies

$$1 - \frac{1}{\theta} - \frac{\lambda}{N} < \frac{\theta}{N} < 1 - \frac{1}{\theta} \quad \text{and} \quad \frac{2}{\theta} - \frac{\lambda + 2\alpha}{N} = 2.$$

These inequalities will be fundamental for us to apply the Proposition 1.1.

Inspired by the Chapter 3 and by papers [37] and [11], we need to assume certain conditions on $f$.

We will consider $A : \mathbb{R} \to [0,+\infty)$ and $Z : \mathbb{R} \to [0,+\infty)$ $\mathcal{N}$-functions given by $A(w) = \int_0^{|w|} ta(t)dt$ and $Z(w) = \int_0^{|w|} tz(t)dt$ where $a : (0,+\infty) \to (0,+\infty)$ and $z : (0,+\infty) \to (0,+\infty)$ are functions satisfying the following conditions:

$(A_1)$ $\quad t \mapsto ta(t)$ is increasing for $t > 0$ and $t \mapsto tz(t)$ is increasing for $t > 0$.

$(A_2)$ $\quad \lim_{t\to 0^+} ta(t) = 0, \quad \lim_{t\to+\infty} ta(t) = +\infty$ and $\lim_{t\to 0^+} tz(t) = 0, \quad \lim_{t\to+\infty} tz(t) = +\infty.$

$(A_3)$ There exist $a_1, a_2, z_1, z_2 \in [m, \ell^*]$ with $a_1 \le a_2 \le z_1 \le z_2$ such that

$$a_1 \le \frac{a(t)t^2}{A(t)} \le a_2, \quad \forall t > 0. \tag{1.10}$$

and

$$z_1 = \inf_{t>0} \frac{z(t)t^2}{Z(t)} \quad \text{and} \quad z_2 \ge \sup_{t>0} \frac{z(t)t^2}{Z(t)}. \tag{1.11}$$

$(A_4)$ The functions $A(|t|^{1/\theta})$ and $Z(|t|^{1/\theta})$ are convex in $\mathbb{R}$.

We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions:

$(f_1')$ $$\limsup_{t\to 0} \frac{f(t)}{\big(a(|t|)|t|^{2-\theta}\big)^{1/\theta}} = 0 \text{ and } \lim_{t\to+\infty} \frac{f(t)}{\big(z(|t|)|t|^{2-\theta}\big)^{1/\theta}} = 0$$

$(f_2')$ $$t^{1-m/2} f(t) \text{ is nondecreasing on } (0,+\infty)$$

$(f_3')$ $$f(t) \ge 0 \text{ for } t \ge 0 \text{ and } f(t) = 0 \text{ for } t \le 0$$

$(f_4')$ $$\lim_{|t|\to\infty} \frac{F(t)}{|t|^{\frac{m}{2}}} = +\infty.$$

Assuming the conditions above, our first main result can be stated as follows.

**Theorem 1.6** *Assume that $\Phi$ satisfies $(\phi_1) - (\phi_4)$, $0 \le \alpha < \lambda$ and $\lambda + 2\alpha \in (0, N) \cap (0, 2N - \frac{2N}{m})$. Suppose that $(V, K) \in \mathcal{Q}_1$, $(A_1) - (A_4)$ and $(f_1)$, $(f_2)$, $(f_3)$, $(f_4)$ hold. Then, problem $(P_2)$ possesses a nonnegative ground state solution. If $2\alpha + \lambda < 2\ell$, then the nonnegative solutions are locally bounded.*

To study the regularity of the solutions provided by Theorem 1.6, we add the assumptions $(\phi_5)$ and $(\phi_6)$.

We are in position to state the following regularity result:

**Theorem 1.7** *Supoose that $\Phi$ satisfies $(\phi_5)-(\phi_6)$. Under the assumptions of Theorem 1.6, if $K \in L^1(\mathbb{R}^N)$, $\alpha = 0$ and $\lambda < 2\ell$, then the problem $(P_2)$ possesses a $C^{1,\gamma}_{loc}(\mathbb{R}^N)$ positive ground state solution.*

To study this second class of problem where $(V, K) \in \mathcal{Q}_2$, we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $(f_2)$, $(f_3)$ and $(f_4)$. Furthermore, for this case, we replace the condition $(f_1)$ with the following condition:

$$(f_5) \qquad \limsup_{t\to 0} \frac{f(t)}{\left(\frac{1}{\theta}b(|t|)|t|^{2-\theta}\right)^{1/\theta}} < \infty \quad \text{and} \quad \lim_{t\to+\infty} \frac{f(t)}{\left(\frac{1}{\theta}\phi_*(|t|)|t|^{2-\theta}\right)^{1/\theta}} = 0.$$

where $\phi_*(t)t$ is such that the Sobolev conjugate function $\Phi_*$ of $\Phi$ is its primitive, that is, $\Phi_*(t) = \displaystyle\int_0^{|t|} \phi_*(s)sds.$

Our first main result of this subsection can be stated as follows. Under these conditions, the next result of the existence of a nonnegative solution has the following statement:

**Theorem 1.8** *Assume that $\Phi$ satisfies $(\phi_1)-(\phi_4)$, $0 \le \alpha < \lambda$ and $\lambda + 2\alpha \in (0, N) \cap (0, 2N - \frac{2N}{m})$. Suppose that $(V, K) \in \mathcal{Q}_2$, $(B_1)-(B_4)$ and $(f_2)$, $(f_3)$, $(f_4)$, $(f_5)$ hold. If $\Phi_*(|t|^{1/\theta})$ is convex in $\mathbb{R}$, then the problem $(P_2)$ possesses a nonnegative ground state solution.*

In a second part of this thesis, we study the existence of solutions for two classes of quasilinear systems of the type:

$$(S) \qquad \begin{cases} -\Delta_{\Phi_1} u = F_u(x, u, v) + \lambda R_u(x, u, v) & \text{in } \Omega \\ -\Delta_{\Phi_2} v = -F_v(x, u, v) - \lambda R_v(x, u, v) & \text{in } \Omega \\ \quad u = v = 0 \quad \text{on } \partial\Omega \end{cases}$$

where $\Delta_{\Phi_i} u = div(\phi_i(|\nabla u|)\nabla u)$, $i = 1, 2$. This type of system has been explored using variational methods techniques by several authors. For example, in [23], Ding and Figueiredo consider the noncooperative system $(S)$ with $\phi_1(t) = 1$, $\phi_2(t) = 1$, $\lambda = 1$ allowing that the function $F(x, u, v)$ can assume a supercritical and subcritical growth on $v$ and $u$ respectively. They established the existence of infinitely many solutions to $(S)$ provided the nonlinear terms $F$ and $R$ are even in $(u, v)$. Already in [41], Clapp, Ding and Hernández showed that multiple existence of solutions to the noncooperative system $(S)$ with some supercritical growth can be established without the symmetry assumption. Motivated by some results found in [41] and [23], Alves and Monari in [12] studied the existence of nontrivial solutions for $(S)$ when $\phi_1(t) = |t|^{p-2}$, $\phi_2(t) = |t|^{q-2}$ $(p, q > 1)$ with $p$ and $q$ are different from 2, $\lambda = 1$ and $F(x, u, v)$ has a supercritical growth on variable $v$ and has a critical growth at infinity on variable $u$ of the type $|u|^{p^*}$ with $p^* = pN/(N - p)$, the critical exponent of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. The main difficulty in this case is the lack of compactness of the functional energy associated to system. To overcome this difficulty, they carefully estimate and prove through the concentration-compactness principle due to Lions [65] the existence of a Palais-Smale sequence that has a strongly convergent subsequence.

In a brief bibliographical research, we can mention some contributions devoted to the study of system where $\Phi_1$ and $\Phi_2$ are less trivial functions, as can be seen in [33,40]. We would like to highlight the paper [40], Wang et al. considered the following quasilinear elliptic system in Orlicz-Sobolev spaces:

$$\begin{cases} -\Delta_{\Phi_1} u = R_u(x, u, v) \;\text{ in } \Omega \\ -\Delta_{\Phi_2} v = R_v(x, u, v) \;\text{ in } \Omega \\ \;u = v = 0 \;\text{ on } \partial\Omega \end{cases} \tag{1.12}$$

where $\Omega$ is a bounded domain in $R^N (N \geq 2)$ with smooth boundary $\partial\Omega$. In that paper when $R$ satisfies some appropriate conditions including $(\Phi_1, \Phi_2)$-superlinear and subcritical growth conditions at infinity as well as symmetric condition, by using the mountain pass theorem and the symmetric mountain pass theorem, they obtained that system (1.12) has a nontrivial weak solution and infinitely many weak solutions, respectively. Some of the results obtained extend and improve those corresponding results in Carvalho et al [49]. In [33], Huentutripay-Manásevich studied an eigenvalue problem to the following system:

$$\begin{cases} -\Delta_{\Phi_1} u = \lambda R_u(x, u, v) \;\text{ in } \Omega \\ -\Delta_{\Phi_2} v = \lambda R_v(x, u, v) \;\text{ in } \Omega \\ \;u = v = 0 \;\text{ on } \partial\Omega \end{cases}$$

where the functions $(\Phi_i)_*$, $i = 1, 2$, exists (See Lemma 2.15). Furthermore, the function $R$ has the form

$$R(x, s_1, s_2) = A_1(x, s_1) + b(x)\Gamma_1(s_1)\Gamma_2(s_2) + A_2(x, s_2),$$

with $b \in L^{\infty}(\Omega)$ and the functions $\Gamma_1$, $\Gamma_2$ are given by

$$\Gamma_i(t) = \int_0^t \gamma_i(s) s ds$$

and $\gamma_i$ an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, $i = 1, 2$. Additionally, $A_i$, $i = 1, 2$ are given by

$$A_i(x,t) = \int_0^t a_i(x,s) ds$$

where $a_1, a_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ are Caratheodory functions that satisfy the following growth conditions:

$$|a_i(x,t)| \leq \alpha_i(x) + \tilde{C}_i \tilde{P}_i^{-1} P_i(C_i t)$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$. Here, for $i = 1, 2$, $C_i$, $\tilde{C}_i$ are positive constants, $P_i$ is an $\mathcal{N}$-function with $P_i \prec\prec (\Phi_i)_*$ (See Definition 2.6) and $\alpha_i \in L_{\tilde{P}_i}(\Omega)$, with $\alpha_i(x) \geq 0$, for a.e. $x \in \Omega$. It is obvious that in [33], the Orlicz-Sobolev spaces need not be reflexive.

In a brief bibliographical research, one can find several other works involving the system $(S)$ where the functions $\Phi_1$ and $\Phi_2$ satisfy the $\Delta_2$-condition and works in which this condition is relaxed are rare. Given this, in Chapter 5, we present the work [39], which is a joint collaboration with Professor Marco Souto. In these chapter we will show the existence of solutions for the system $(S)$ where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and $F, R : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous function verifying some conditions which will be mentioned later. Initially, we will assume that the functions $\phi_i (i = 1, 2) \in C^1(0, +\infty)$ are two functions which satisfy:

$(\phi'_{i,1})$ $\quad t \mapsto t\phi_i(t)$ are stricly increasing and $t \mapsto t^2\phi_i(t)$ is convex in $(0, \infty)$;

$(\phi'_{i,2})$ $$t\phi_i(t) \to 0 \text{ as } t \to 0 \text{ and } t\phi_i(t) \to +\infty \text{ as } t \to +\infty;$$

$(\phi'_{i,3})$ $$1 < \ell_i \leq \frac{t^2\phi_i(t)}{\Phi_i(t)}, \quad \text{where } \Phi_i(t) = \int_0^{|t|} s\phi_i(s) ds, \ t \in \mathbb{R};$$

$(\phi'_{i,4})$ $$\liminf_{t \to +\infty} \frac{\Phi_i(t)}{t^{q_i}} > 0, \text{ for some } q_i > N;$$

$(\phi'_{i,5})$ $$\left| 1 - \frac{\Phi_1(t)}{t^2\phi_1(t)} \left( 1 + \frac{t\phi'_1(t)}{\phi_1(t)} \right) \right| \leq 1, \quad \forall t > 0.$$

Let $d$ twice the diameter of $\Omega$, then we will assume that there exists $\delta \geq 0$ such that

$$(\phi'_{i,6}) \qquad \frac{t^2}{d^2} \leq \Phi_1(t/d), \quad \forall |t| \geq \delta$$

Regarding the above conditions, in order to find a solution to the systems $(S)$, we will assume that we consider $F = 0$, $\lambda = 1$ and that the function $R$ satisfying the following conditions:

$(R'_1)$ $\qquad R \in C^1(\overline{\Omega} \times \mathbb{R}^2)$ and $R_v(x, u, 0) \neq 0$ for all $(x, u) \in \Omega \times \mathbb{R}$

$(R'_2)$ $\qquad R(x, u, 0) \leq \frac{1}{2}\Phi_1(u/d) + \frac{1}{2d^2}|u|^2,$ for all $(x, u) \in \Omega \times \mathbb{R}$

$(R'_3)$ $R(x, 0, v) \geq -\frac{1}{2}\Phi_2(v/d) - Mv,$ for all $(x, v) \in \Omega \times \mathbb{R},$ for some constant $M > 0$.

$(R'_4)$ There are $\nu > 0$, $\mu > 1$ and $0 < \beta < 1$ such that

$(i)$ $\qquad \frac{1}{\mu}h(u)R_u(x, u, v)u + \frac{1}{\nu}R_v(x, u, v)v - R(x, u, v) \geq 0, \quad \forall (x, u, v) \in \Omega \times \mathbb{R}^2$

and

$(ii)$ $\qquad \beta R(x, u, v) - \frac{1}{\mu}h(u)R_u(x, u, v)u \geq 0, \quad \forall (x, u, v) \in \Omega \times \mathbb{R}^2$

where $h(u) = \dfrac{\Phi_1(u)}{u^2\phi_1(u)}$.

We note that $R(u, v) = \Phi_1(u)^\sigma \Phi_2(v)^\theta + v^+$ satisfies $(R'_1) - (R'_4)$ for add $\theta, \sigma > 1$, where $v^+(x) := \max\{0, v(x)\}$. Furthermore, the functions $\Phi_1(t) = (e^{t^2} - 1)/2$ and $\Phi_2(t) = |t|^p/p$ (or $\Phi_2(t) = (e^{t^2} - 1)/2$) with $p > N$ satisfying $(\phi'_{i,1}) - (\phi'_{i,6})$. These functions are examples of $\mathcal{N}$-functions whose the complementary functions $\Phi_1$ and $\Phi_2$ do not satisfy the $\Delta_2$-condition, consequently $W_0^{1,\Phi_{\alpha_1}}(\Omega) \times W_0^{1,\Phi_{\alpha_2}}(\Omega)$ is nonreflexive.

Before we state the main result of this chapter, let us recall that $(u, v) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ is a weak solution of $(S)$ if

$$\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla \varphi_1 dx - \int_\Omega \phi_2(|\nabla u|)\nabla u \nabla \varphi_2 dx = \int_\Omega R_u(x, u, v)\varphi_1 dx + \int_\Omega R_v(x, u, v)\varphi_2 dx,$$

for all $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$.

The main result of this chapter is the following.

**Theorem 1.9** *Assume that* $(\phi'_{i,1}) - (\phi'_{i,6})$ *and* $(R'_1) - (R'_4)$ *hold. If* $F = 0$ *and* $\lambda = 1$*, then, the system* $(S)$ *possesses a nontrivial solution.*

In proving this theorem, some difficulties arise when relaxing the $\Delta_2$-condition of the functions $\Phi_1$ and $\Phi_2$. The first of them arises from the fact that the energy functional $J : W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \to \mathbb{R}$ associated with the system $(S)$ given by

$$J(u, v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega R(x, u, v)dx.$$

no belongs to $C^1(W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega), \mathbb{R})$. Have this in mind, we have decide to work in the space $W_0^1 E^{\Phi_i}(\Omega)$, because it is topologically more rich than $W_0^{1,\Phi_i}(\Omega)$, for example, it is possible to prove that the energy functional $J$ is $C^1(W_0^1 E^{\Phi_1}(\Omega) \times W_0^1 E^{\Phi_2}(\Omega), \mathbb{R})$. Even knowing that the result contained in Proposition 3.7 in [8] presents inconsistency when dropping the $\Delta_2$-condition, we add a "Ambrosetti-Rabinowitz" condition under the function $R$ and we refine part of the technique presented by Alves et al., so that, together with the saddle-point theorem of Rabinowitz without Palais-Smale condition, we can prove the existence of a Palais-Smale bounded sequence. Finally, due to the possible lack of reflexivity of the spaces $W_0^{1,\Phi_i}(\Omega)(i = 1, 2)$, we will utilize properties of the weak* topology of these spaces to guarantee the existence of nontrivial solutions for the system $(S)$, thereby proving Theorem 1.9.

Continuing the study of systems in non-reflexive Orlicz-Sobolev spaces, in Chapter 5, we investigate the existence of solutions for the system $(S)$ where $\lambda > 0$ is a parameter, $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, and $\phi_i(i = 1, 2) : (0, \infty) \to (0, \infty)$ are two functions which satisfy:

$(\phi_{1,i})$ $\qquad \phi_i \in C^1(0, +\infty)$ and $t \mapsto t\phi_i(t)$ are stricly increasing;

$(\phi_{2,i})$ $\qquad t\phi_i(t) \to 0$ as $t \to 0$ and $t\phi_i(t) \to +\infty$ as $t \to +\infty$;

$(\phi_{3,i})$ $1 \leq \ell_i = \inf_{t>0} \dfrac{t^2\phi_i(t)}{\Phi_i(t)} \leq \sup_{t>0} \dfrac{t^2\phi_i(t)}{\Phi_i(t)} = m_i < N$, where $\Phi_i(t) = \int_0^{|t|} s\phi_i(s)ds$ and $\ell_i < m_i < \ell_i^*$.

With regard to the function $F$, we will assume that $F(x, u, v) = \Phi_{1*}(u) + G(v)$ where $\Phi_{1*}$ denotes the Sobolev conjugate function of $\Phi_1$ and that $G$ is a function satisfying the following conditions:

$(G_1)$ There are $C > 0$, $G \in C^1(\mathbb{R},\mathbb{R})$, $a_1, a_2 \in (1,\infty)$ and a $\mathcal{N}$-function $A(t) = \int_0^{|t|} sa(s)ds$ satisfying

$$(i) \qquad m_2 < a_1 \leq \frac{a(t)t^2}{A(t)} \leq a_2, \quad \forall t > 0$$

and

$$(ii) \qquad |g(s)| \leq a_1 C a(|s|)|s|, \quad \text{for all } s \in \mathbb{R}$$

where $g(s) = G'(s)$. If $a_2 \geq \ell_2^*$, we add that

$$(iii) \qquad (g(t) - g(s))(t - s) \geq Ca(|t - s|)|t - s|^2, \quad \text{for all } t, s \in \mathbb{R}.$$

$(G_2)$ There exists $\nu \in (0, \ell_1)$ such that

$$0 \leq \nu G(s) \leq sg(s), \quad \text{for all } s \in \mathbb{R}.$$

Furthermore, we will assume that the function $R$ satisfies the following conditions:
$(R_1)$ $R \in C^1(\overline{\Omega} \times \mathbb{R}^2)$, $R_u(x,0,0) = 0$, $R_v(x,0,0) = 0$, $R(x,u,v) \geq 0$ and $R_u(x,u,v)u \geq 0$, for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R}^2$.
$(R_2)$ There are $\mathcal{N}$-functions $B(t) = \int_0^{|t|} sb(s)ds$, $P(t) = \int_0^{|t|} sp(s)ds$, $Q(t) = \int_0^{|t|} sq(s)ds$ and $Z(t) = \int_0^{|t|} sz(s)ds$ satisfying

$$(i) \qquad m_1 < p_1 \leq \frac{p(t)t^2}{P(t)} \leq p_2 < \ell_1^*$$

$$(ii) \qquad m_1 < b_1 \leq \frac{b(t)t^2}{B(t)} \leq b_2 < \ell_1^*$$

$$(iii) \qquad m_2 < q_1 \leq \frac{q(t)t^2}{Q(t)} \leq q_2 < \ell_2^*$$

$$(iv) \qquad m_2 < z_1 \leq \frac{z(t)t^2}{Z(t)} \leq z_2 < \ell_2^*,$$

with $\max\{b_2, q_2\} < \min\{\ell_1^*, \ell_2^*\}$ such that

$$(1.13) \quad |R_u(x,u,v)| \leq C(p(|u|)u + q(|v|)v) \text{ and } |R_v(x,u,v)| \leq C(b(|u|)u + z(|v|)v),$$

for all $(x,u,v) \in \Omega \times \mathbb{R}^2$ and for some constant $C > 0$.

$(R_3)$ There exists $\mu \in (m_1, \ell_1^*)$ such that

$$\frac{1}{\mu}R_u(x,u,v) + \frac{1}{\nu}R_v(x,u,v) - R(x,u,v) \geq 0, \quad \text{for all } x \in \Omega \text{ and } (u,v) \in \mathbb{R}^2,$$

where $\nu$ is given by condition $(G_2)$.

$(R_4)$ There exists $s \in (m_1, \max\{p_2, b_2\}]$, a nonempty open subset $\Omega_0 \subset \Omega$ and a constant $\omega > 0$ such that

$$R(x,u,v) \geq \omega |u|^s \quad \text{for all } x \in \Omega_0 \text{ and } (u,v) \in \mathbb{R}^2.$$

Let us there are examples of functions that satisfy the conditions listed above. Consider $\alpha_1, \alpha_2 \in (0, \frac{N}{N-1} - 1)$ such that $\alpha_1 \leq \alpha_2$. We would like to point out that $\Phi_1(t) = |t|\ln(|t|^{\alpha_1} + 1)$ and $\Phi_2(t) = |t|\ln(|t|^{\alpha_2} + 1)$ satisfying $(\phi_{1,i}) - (\phi_{3,i})$ with $\ell_1 = \ell_2 = 1$ and $m_1 = 1 + \alpha_1$, $m_2 = 1 + \alpha_2$ respectively. These functions are examples of $\mathcal{N}$-functions whose the complementary functions $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ do not satisfy the $\Delta_2$-condition, consequently $W_0^{1,\Phi_{\alpha_1}}(\Omega) \times W_0^{1,\Phi_{\alpha_2}}(\Omega)$ is nonreflexive.

Before stating the main result of this chapter, we would like to remember that $(u,v) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ is a weak solution of $(S)$ if

$$\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla w_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla w_2 dx = \int_\Omega H_u(x,u,v) w_1 dx + \int_\Omega H_v(x,u,v) w_2 dx,$$

for all $(w_1, w_2) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ where $H(x,u,v) = F(x,u,v) + \lambda R(x,u,v)$.

The main result of this chapter is the following.

**Theorem 1.10** *If $(\phi_{1,i}) - (\phi_{3,i})$, $(i = 1,2)$, $(G_1) - (G_2)$, $(R_1) - (R_4)$ hold, then there exists $\lambda_0 > 0$ such that $(S)$ possesses a nontrivial solution for all $\lambda > \lambda_0$.*

This theorem was inspired by the results presented in [12]. However, the first difficulty in studying this case arises from the lack of differentiability of the energy functional $J_\lambda : W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \to \mathbb{R}$ associated with the system $(S)$ given by

$$J_\lambda(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega F(x,u,v)dx - \lambda \int_\Omega R(x,u,v)dx.$$

To get around this difficulty we will use the critical point theory for locally lipschitz fuctionals, here, in particular we apply a version of the linking theorem without Palais-Smale condition for locally lipschitz fuctionals (The version that will be applied in Chapter 5 we took care to enunciate in Appendix A). A second difficulty of studying this case is the lack of compactness of the energy functional $J_\lambda$. To overcome this difficulty, we adapted some arguments presented in the works of Alves and Soares in [12] and from Fukagai et al, in [58]. Here, we carefully estimate and prove through the second concentration-compactness lemma of P. L. Lions for nonreflexive Orlicz-Sobolev space that there exists a constant $\lambda_0 > 0$ such that the sistem has a nontrivial solution for any $\lambda > \lambda_0$.

In Appendix A, we present some minimax results involving critical point theory for locally Lipschitz functionals. These results are utilized in Chapter 5.

This thesis concludes with Appendix B, where we provide detailed examples of functions that satisfy the conditions $(V, K) \in \mathcal{K}_1$, $(V, K) \in \mathcal{K}_2$, $(V, Q) \in \mathcal{Q}_1$ and $(V, Q) \in \mathcal{Q}_1$.

# Chapter 2

# Orlicz and Orlicz-Sobolev spaces: A review

In this chapter, we will make a brief study of the main properties involving Orlicz and Orlicz-Sobolev spaces. It is important to emphasize that the information present in this chapter constitutes only the minimum language necessary for the study present in this thesis. We suggest to interested readers the references [1,54,55,58,59] for a more complete study on the subject. In Portuguese, we suggest the thesis [32].

## 2.1 $\mathcal{N}$-function

In this section we recall some properties of $\mathcal{N}$-function.

**Definition 2.1** *We will say that $\Phi : \mathbb{R} \to [0, +\infty)$ is a **$\mathcal{N}$-function** if*

*(i)* $\Phi$ *is convex and continuous;*

*(ii)* $\Phi(t) = 0 \;\Leftrightarrow\; t = 0$;

*(iii)* $\Phi$ *is even;*

*(iv)* $\displaystyle\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ *and* $\displaystyle\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty$.

**Example 2.1.1** *Below, we list some classic examples of $\mathcal{N}$-functions.*

*(i)* $\Phi_1(t) = \frac{1}{p}|t|^p$, *where* $p \in (1, \infty)$ *and* $t \in \mathbb{R}$;

*(ii)* $\Phi_2(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q$, *where* $p, q \in (1, \infty)$ *and* $t \in \mathbb{R}$;

*(iii)* $\Phi_3(t) = (1+t)^\alpha - 1$, *where* $\alpha > 1$ *and* $t \in \mathbb{R}$;

*(iv)* $\Phi_4(t) = |t|^p \ln(1+t)$, *where* $p \in (1, \infty)$ *and* $t \in \mathbb{R}$;

*(v)* $\Phi_5(t) = e^{t^2} - 1$ *for* $t \in \mathbb{R}$;

*(vi)* $\Phi_6(t) = e^{|t|} - |t| - 1$ *for* $t \in \mathbb{R}$.

In the following, we list a result that characterizes the $\mathcal{N}$-functions.

**Lemma 2.1** *Let* $\Phi : \mathbb{R} \to [0, +\infty)$ *be a function. Then* $\Phi$ *is a* $\mathcal{N}$*-function if and only if*

$$\Phi(t) = \int_0^{|t|} \varphi(s)ds, \quad t \in \mathbb{R} \tag{2.1}$$

*where* $\varphi : [0, +\infty) \mapsto [0, +\infty)$ *is a function satisfying*

*(i)* $\varphi$ *is right-continuous and non-decreasing in* $(0, \infty)$;

*(ii)* $\varphi(t) = 0$ *if and only if* $t = 0$;

*(iii)* $\lim_{t \to +\infty} \varphi(t) = \infty$;

*(iv)* $\varphi(t) > 0$, *for* $t > 0$.

For each $\mathcal{N}$-function, we can define a special class of functions called **complementary functions**.

**Definition 2.2** *(Complementary Function of* $\Phi$*) Let* $\Phi$ *be a* $\mathcal{N}$*-function. The* ***complementary function*** *of* $\Phi$*, denoted by* $\tilde{\Phi}$*, is the function given by*

$$\tilde{\Phi}(t) = \sup_{s \geq 0}\{st - \Phi(s)\}, \quad \text{for } \ t \geq 0.$$

**Example 2.1.2** *The* $\mathcal{N}$*-function* $\Phi_1(t) = \frac{1}{p}|t|^p$ *with* $p \in (1, \infty)$ *and* $t \in \mathbb{R}$*, has as a complementary function*

$$\tilde{\Phi}_1(t) = \frac{1}{q}|t|^q, \ t \in \mathbb{R}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

It is clear in the example 2.1.2 that the definition of complementary function generalizes the concept of conjugate function to Lebesgue spaces. The above definitions allow us to do the following lemma:

**Lemma 2.2** *If* $\Phi$ *is a* $\mathcal{N}$*-function, then* $\tilde{\Phi}$ *is also a* $\mathcal{N}$*-function.*

Next, we list some properties involving $\mathcal{N}$-functions.

**Lemma 2.3** *The $\mathcal{N}$-function $\Phi$ satisfies:*

1. $\Phi(\alpha t) \leq \alpha\Phi(t)$, $\alpha \in [0,1]$ *and* $t \geq 0$;

2. $\Phi(\beta t) \geq \beta\Phi(t)$, $\beta > 1$ *and* $t \geq 0$;

3. *(Young's inequality) Given* $s, t \in \mathbb{R}$, *then*

$$st \leq \Phi(s) + \tilde{\Phi}(t),$$

*equality holds if, and only if,* $s = \tilde{\varphi}(t)$ *or* $t = \varphi(s)$, *where* $\varphi$ *and* $\tilde{\varphi}$ *satisfy*

$$\Phi(s) = \int_0^s \varphi(r)dr \quad \text{and} \quad \tilde{\Phi}(t) = \int_0^t \tilde{\varphi}(r)dr;$$

4. *If* $\Phi$ *is of the form* (2.1) *with* $\varphi$ *continuous and increasing, then*

$$\tilde{\Phi}(t) = \int_0^{|t|} \varphi^{-1}(s)ds;$$

4. $\tilde{\Phi}(\varphi(t)) \leq \Phi(2t)$, *for* $\geq 0$.

5. $\tilde{\Phi}(\frac{\Phi(t)}{t}) \leq \Phi(t)$, *for* $t > 0$.

## 2.2 Orlicz Space

In this section, we aim to present Orlicz spaces. For more details, we suggest the reader the references cited at the beginning of this chapter.

To continue this brief review of Orlicz spaces, from now on, unless otherwise indicated, we will always assume that $\Omega$ is an open set of $\mathbb{R}^N$, with $N \geq 1$, and that $\Phi$ is an $\mathcal{N}$-function. In these configurations, we will present the definition of Orlicz Spaces.

In what follows, fixed an open set $\Omega \subset \mathbb{R}^N$.

**Definition 2.3** *Let $\Phi$ be a $\mathcal{N}$-function. We define the* ***Orlicz space*** *associated with $\Phi$ as*

$$L^\Phi(\Omega) = \left\{ u \in L^1_{loc}(\Omega) : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right\}.$$

The convexity of the $\mathcal{N}$-function $\Phi$ guarantees that $L^\Phi(\Omega)$ is a vector space. Furthermore, the space $L^\Phi(\Omega)$ is a Banach space equipped with the **Luxemburg norm** given by

$$\|u\|_\Phi = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

**Example 2.2.1** *Considere a $\mathcal{N}$-function $\Phi_1(t) = \frac{1}{p}|t|^p$ with $p \in (1, \infty)$ and $t \in \mathbb{R}$. Then,*

$$L^{\Phi}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) : \int_{\Omega} \left| \frac{u}{\lambda} \right|^p dx < +\infty \text{ for some } \lambda > 0 \right\} = L^p(\Omega).$$

*As a consequence of this equality, we can conclude that Orlicz spaces are generalizations of Lebesgue spaces.*

Throughout this thesis, the separability and reflexivity of Orlicz's spaces are sometimes questioned. The next definition plays a key role in these properties.

**Definition 2.4** *Let $\Phi$ be a $\mathcal{N}$-function. We say that a $\mathcal{N}$-function $\Phi$ verifies the* **$\Delta_2$-*condition*** *($\Phi \in (\Delta_2)$), if there are constants $K > 0$, $t_0 \geq 0$ such that*

$$\Phi(2t) \leq K\Phi(t), \quad \forall t \geq t_0.$$

**Remark 2.1** *$|\Omega| = \infty$, $\Phi \in \Delta_2$ with $t_0 = 0$.*

**Example 2.2.2** *The $\mathcal{N}$-functions $\Phi_1$, $\Phi_2$ and $\Phi_4$ given in example 2.1.1 are examples of $\mathcal{N}$-functions that check the $\Delta_2$-condition. Already the $\mathcal{N}$-function $\Phi_3$ defined in example 2.1.1 also satisfies the $\Delta_2$-condition whenever $\alpha \in \left(1, \frac{N}{N-2}\right)$. The $\mathcal{N}$-functions $\Phi_5$ and $\Phi_6$ are examples of $\mathcal{N}$-functions that do not satisfy the $\Delta_2$-condition.*

**Lemma 2.4** *Let $\Phi$ be a $\mathcal{N}$-function given by*

$$\Phi(t) = \int_0^{|t|} \varphi(s)ds.$$

*Then $\Phi \in (\Delta_2)$ if and only if there are $\alpha > 0$ and $t_0 > 0$ such that*

$$\frac{t\varphi(t)}{\Phi(t)} \leq \alpha, \quad t \geq t_0.$$

**Lemma 2.5** *(Young's Integral Inequality) Given $u \in L^{\Phi}(\Omega)$ and $v \in L^{\tilde{\Phi}}(\Omega)$, then*

$$uv \in L^1(\Omega) \quad \text{and} \quad \int_{\Omega} uv dx \leq \int_{\Omega} \Phi(u)dx + \int_{\Omega} \tilde{\Phi}(v)dx.$$

**Lemma 2.6** *(Young's inequality) Given $u \in L^{\Phi}(\Omega)$ and $v \in L^{\tilde{\Phi}}(\Omega)$, then*

$$\int_{\Omega} uv dx \leq \|u\|_{\Phi} + \|v\|_{\tilde{\Phi}}.$$

**Definition 2.5** *Let $\Phi$ be a $\mathcal{N}$-function. If $|\Omega| < \infty$, the space $E^{\Phi}(\Omega)$ denotes the closing of $L^{\infty}(\Omega)$ in $L^{\Phi}(\Omega)$ with respect to the norm $\|\cdot\|_{\Phi}$. When $|\Omega| = \infty$, the space $E^{\Phi}(\Omega)$ denotes the closure of $B_0(\Omega)$ in $L^{\Phi}(\Omega)$ with respect to norm $\|\cdot\|_{\Phi}$, where $B_0(\Omega) = \{u \in L^{\infty}(\Omega) : supp(u) \subset\subset \Omega\}$.*

**Remark 2.2** *$E^{\Phi}(\Omega) = L^{\Phi}(\Omega)$ if and only if $\Phi$ satisfies the $\Delta_2$-condition.*

**Lemma 2.7** *Let $\Phi$ be a $\mathcal{N}$-function. Then:*

1. *$E^{\Phi}(\Omega)$ is separable;*
2. *$E^{\Phi}(\Omega) = \overline{C_0^{\infty}(\Omega)}^{\|\cdot\|_{\Phi}}$ ;*
3. *$L^{\tilde{\Phi}}(\Omega) = \big(E^{\Phi}(\Omega)\big)'$ and $L^{\Phi}(\Omega) = \big(E^{\tilde{\Phi}}(\Omega)\big)'$;*
4. *$L^{\infty}(\Omega) \cdot E^{\Phi}(\Omega) = E^{\Phi}(\Omega)$*
5. *$L^{\Phi}(\Omega)$ is separable if and only if $\Phi$ satisfies the $\Delta_2$-condition;*
6. *$L^{\Phi}(\Omega)$ is reflexive if and only if $\Phi$ and $\tilde{\Phi}$ satisfy the $\Delta_2$-condition ;*
7. *(Riesz Representation Theorem) Let $F \in \big(E^{\Phi}(\Omega)\big)'$, then there is a unique $v \in L^{\tilde{\Phi}}(\Omega)$ such that*
$$F(u) = \int_{\Omega} uv dx, \quad u \in E^{\Phi}(\Omega).$$

**Lemma 2.8** *If $\Phi$ is $\mathcal{N}$-function and $(\int_{\Omega} \Phi(|u_n|)dx)$ is a bounded sequence, then $(u_n)$ is a bounded sequence in $L^{\Phi}(\Omega)$. When $\Phi \in (\Delta_2)$, the equivalence is valid.*

**Lemma 2.9** *Let $\Phi$ satisfying the $\Delta_2$-condition. Then, $u_n \to u$ in $L^{\Phi}(\Omega)$ if and only if $\int_{\Omega} \Phi(|u_n - u|)dx \to 0$.*

The following result will be useful in applications involving Lebesgue's Theorem in the context of Orlicz Spaces.

**Lemma 2.10** *Let $\Phi$ a $\mathcal{N}$-function and $(u_n)$ a sequence in $E^{\Phi}(\Omega)$ with $u_n \to u$ in $E^{\Phi}(\Omega)$. Then, there is $H \in E^{\Phi}(\Omega)$ and a subsequence $(u_{n_j})$ such that*

*(i)* *$|u_{n_j}(x)| \leq H(x)$ a.e. in $\Omega$;*

*(ii)* *$u_{n_j}(x) \to u(x)$ a.e. in $\Omega$ and all $j \in \mathbb{N}$.*

The next results is a classic Brezis-Lieb lemma for reflexive Orlicz spaces.

**Lemma 2.11** *Let $\Phi$ be a $\mathcal{N}$-function such that $\Phi$, $\tilde{\Phi} \in (\Delta_2)$ and $(u_n) \subset L^{\Phi}(\Omega)$ is bounded. Suppose that $u_n \to u$ a.e. in $\Omega$. Then $u \in L^{\Phi}(\Omega)$ and $u_n \rightharpoonup u$.*

The following lemma is an immediate consequence of the Banach-Alaoglu-Bourbaki theorem [27], and is crucial when the space $L^{\Phi}(\Omega)$ can be nonreflexive.

**Lemma 2.12** *Assume that $\Phi$ is a $\mathcal{N}$-function . If $(u_n) \subset L^{\Phi}(\Omega)$ is a bounded sequence, then there exists a subsequence of $(u_n)$, which we will still denote by $(u_n)$, and $u \in L^{\Phi}(\Omega)$ such that*

$$u_n \overset{*}{\rightharpoonup} u \quad in \;\; L^{\Phi}(\Omega)$$

*or equivalently,*

$$\int_{\mathbb{R}^N} u_n v dx \to \int_{\mathbb{R}^N} uv dx, \;\; \forall v \in E^{\tilde{\Phi}}(\Omega).$$

Next, we present the first embedding result.

**Lemma 2.13** *Let $\Phi$ be a $\mathcal{N}$-function. Suppose $|\Omega| < \infty$, then*

$$L^{\Phi}(\Omega) \underset{cont}{\hookrightarrow} L^{1}(\Omega)$$

For the next embedding result, we will make the following definition:

**Definition 2.6** *Let $\Phi_1$ and $\Phi_2$ $\mathcal{N}$-functions. We say that $\Phi_2$* ***grows strictly slower*** *than $\Phi_1$, if for any $k > 0$*

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(kt)} = 0.$$

*In this case, we use the notation $\Phi_2 \prec\prec \Phi_1$.*

**Lemma 2.14** *Let $\Phi_1$ and $\Phi_2$ $\mathcal{N}$-functions such that $\Phi_2 \prec\prec \Phi_1$. Suppose that $|\Omega| < \infty$, then*

$$L^{\Phi_1}(\Omega) \underset{cont}{\hookrightarrow} L^{\Phi_2}(\Omega).$$

Now, we will define another class important set of $\mathcal{N}$-functions called critical growth functions.

**Lemma 2.15** *Let $\Phi$ the $\mathcal{N}$-function satisfying*

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}} ds < \infty \quad and \quad \int_1^\infty \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}} ds = \infty. \tag{2.2}$$

*Then, the function $\Phi_*^{-1} : [0,\infty) \to [0,\infty)$ given by*

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}} ds,$$

*is bijective and its inverse $\Phi_*$, extended in $\mathbb{R}$ so that $\Phi_*$ is an even function, is a $\mathcal{N}$-function.*

The $\mathcal{N}$-function $\Phi_*$ is called **critical growth function**. The motivation for this denomination is seen in the example below.

**Example 2.2.3** *The $\mathcal{N}$-function $\Phi_1(t) = \frac{1}{p}|t|^p$, with $p \in [1, N)$, has the critical growth function*

$$\Phi_*(t) = \frac{1}{p^*}|t|^{p^*}, \quad t \in \mathbb{R},$$

*where $p^* = \frac{N-p}{pN}$.*

Now, we list results that play a crucial role in this work. To do this, let us consider $\phi : (0, \infty) \to (0, \infty)$ a continuous function satisfying:

$(\phi_1)$ $t \mapsto t\phi(t)$ is stricly increasing.

$(\phi_2)$ $t\phi(t) \to 0$ as $t \to 0$ and $t\phi(t) \to +\infty$ as $t \to +\infty$

$(\phi_3)$ $1 \leq \ell = \inf\limits_{t>0} \dfrac{t^2\phi(t)}{\Phi(t)} \leq \sup\limits_{t>0} \dfrac{t^2\phi(t)}{\Phi(t)} = m < N$, where $\Phi(t) = \displaystyle\int_0^{|t|} s\phi(s)ds$.

Extend the function $t \mapsto \phi(s)$ to $\mathbb{R}$ as an odd function and define the function $\Phi$ by

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds, \quad t \in \mathbb{R}. \tag{2.3}$$

It is clear that $\Phi$ is a $\mathcal{N}$-function. In fact, define

$$\varphi(t) = \begin{cases} t\phi(t), & t > 0 \\ 0 \ , & t = 0 \end{cases}.$$

It is clear that $\varphi(t) > 0$ for $t > 0$, and by $(\phi_2)$ we conclude that $\varphi$ is continuous in $[0, \infty)$ and $\lim\limits_{t\to\infty} \varphi(t) = \infty$. Furthermore, from the condition $(\phi_1)$ it follows that $\varphi$ is non-decreasing in $(0, \infty)$. Therefore, by Lemma 2.1, we conclude that $\Phi$ is a $\mathcal{N}$-function.

**Remark 2.3** *By the hypothesis $(\phi_3)$, it follows that $\Phi \in (\Delta_2)$. This implication is a consequence of Lemma 2.4.*

The next three lemmas involve the $\mathcal{N}$-function $\Phi$ defined in (2.3), its conjugate function $\tilde{\Phi}$ and critical growth function $\Phi_*$.

**Lemma 2.16** *Consider $\Phi$ a $\mathcal{N}$-function of the form* (2.3) *and satisfying* $(\phi_1)$, $(\phi_2)$ *and* $(\phi_3)$. *Define*

$$\xi_0(t) = \min\{t^\ell, t^m\} \quad \textit{and} \quad \xi_1(t) = \max\{t^\ell, t^m\}, \quad \forall t \geq 0.$$

*Then,*

$$\xi_0(t)\Phi(\rho) \leq \Phi(\rho t) \leq \xi_1(t)\Phi(\rho), \quad \forall \rho, t \geq 0$$

*and*

$$\xi_0(\|u\|_\Phi) \leq \int_\Omega \Phi(u)dx \leq \xi_1(\|u\|_\Phi), \quad \forall u \in L^\Phi(\Omega).$$

**Lemma 2.17** *Consider $\Phi$ a $\mathcal{N}$-function of the form* (2.3) *and satisfying* $(\phi_1)$, $(\phi_2)$ *and* $(\phi_3)$ *with* $\ell > 1$. *Define*

$$\xi_2(t) = \min\{t^{\frac{\ell}{\ell-1}}, t^{\frac{m}{m-1}}\} \quad \textit{and} \quad \xi_3(t) = \max\{t^{\frac{\ell}{\ell-1}}, t^{\frac{m}{m-1}}\}, \quad \forall t \geq 0.$$

*Then,*

$$\frac{m}{m-1} \leq \frac{t\tilde{\Phi}'(t)}{\tilde{\Phi}(t)} \leq \frac{\ell}{\ell-1} \tag{2.4}$$

$$\xi_2(t)\tilde{\Phi}(\rho) \leq \tilde{\Phi}(\rho t) \leq \xi_3(t)\tilde{\Phi}(\rho), \quad \forall \rho, t \geq 0$$

*and*

$$\xi_2(\|u\|_{\tilde{\Phi}}) \leq \int_\Omega \tilde{\Phi}(u)dx \leq \xi_3(\|u\|_{\tilde{Phi}}), \quad \forall u \in L^{\tilde{\Phi}}(\Omega).$$

**Remark 2.4** *The inequality* (2.4) *guarantees that $\tilde{\Phi} \in (\Delta_2)$, just apply Lemma 2.4. Therefore, $L^\Phi(\Omega)$ is reflexive.*

**Lemma 2.18** *If $\Phi$ is an $\mathcal{N}$-function of the form* (2.3) *satisfying* $(\phi_1)$, $(\phi_2)$ *and* $(\phi_3)$ *with $\ell = 1$ , then*

$$\tilde{\Phi}(\rho t) \leq t^{\frac{m}{m-1}}\tilde{\Phi}(\rho), \quad \textit{for all} \quad \rho > 0 \quad \textit{and} \quad 0 \leq t < 1.$$

**Lemma 2.19** *If $\Phi$ is an $\mathcal{N}$-function of the form* (2.3) *satisfying* $(\phi_1)$, $(\phi_2)$ *and* $(\phi_3)$ *with $\ell = 1$, then $\tilde{\Phi}$ does not verify the $\Delta_2$-condition.*

**Proof.** Suppose by contradiction that $\tilde{\Phi} \in (\Delta_2)$, then there is $\beta \geq 1$ such that $\dfrac{\tilde{\Phi}'(t)t}{\Phi(t)} \leq \beta$, for all $t > 0$. We know that $\tilde{\tilde{\Phi}} = \Phi$ and $\tilde{\tilde{\Phi}}(t) = \sup\{st - \tilde{\Phi}(t) : s \in \mathbb{R}\}$. An easy computation shows that

$$\tilde{\Phi}(\Phi'(s)) = \Phi'(s)s - \Phi(s), \quad \forall s \geq 0. \tag{2.5}$$

Deriving this expression with respect to $s$, we get

$$\tilde{\Phi}'(\Phi'(s))\Phi''(s) = \Phi''(s)s.$$

As $\Phi''(s) > 0$, we have

$$\tilde{\Phi}'(\Phi'(s)) = s, \quad \forall s > 0. \tag{2.6}$$

Since $\dfrac{\tilde{\Phi}'(t)t}{\tilde{\Phi}(t)} \leq \beta$, for all $t \geq 0$, making $t = \Phi'(s)$, we are left with

$$\frac{\tilde{\Phi}'(\Phi'(s))\Phi'(s)}{\tilde{\Phi}(\Phi'(s))} \leq \beta, \quad \forall s > 0, \tag{2.7}$$

in other words, by (2.6) and (2.7), it follows

$$s\Phi'(s) \leq \beta\tilde{\Phi}(\Phi'(s)), \quad \forall s > 0. \tag{2.8}$$

According to (2.5) and (2.8)

$$s\Phi'(s) \leq \beta(\Phi'(s)s - \Phi(s))$$

and therefore

$$\frac{\beta}{\beta - 1} \leq \frac{\Phi'(s)s}{\Phi(s)}, \quad \forall s > 0.$$

Since $(\phi_3)$ occurs when $\ell = 1$, we can conclude that $\frac{\beta}{\beta-1} = 1$, that is, $\beta = \beta - 1$. An contradiction. Therefore, $\tilde{\Phi} \notin (\Delta_2)$. ∎

**Example 2.2.4** *It can be observed that the $\mathcal{N}$-functions $\Phi_1$, $\Phi_2$, $\Phi_3$ and $\Phi_4$ satisfy the condition $(\phi_3)$ for $\ell > 1$. Although it might appear that there is no valid example of an $\mathcal{N}$-function satisfying conditions $(\phi_1) - (\phi_4)$ with $\ell = 1$, we introduce the function*

$$\Phi_\alpha(t) = |t| \ln(|t|^\alpha + 1) \ \ \textit{for } 0 < \alpha < \frac{N}{N-1} - 1 \tag{2.9}$$

*as an example of an $\mathcal{N}$-function that satisfies $(\phi_1) - (\phi_4)$ for the case $\ell = 1$.*

In fact, consider $\phi_\alpha : (0, +\infty) \to (0, +\infty)$ a continuous function defined by

$$\phi_\alpha(t) = \frac{\ln(t^\alpha + 1)}{t} + \alpha\frac{t^{\alpha-1}}{t^\alpha + 1}, \quad \text{for } \ t > 0.$$

It is easy to check that

$$\Phi_\alpha(t) = \int_0^{|t|} s\phi_\alpha(s)ds = |t| \ln(|t|^\alpha + 1), \quad \forall t \in \mathbb{R}.$$

Now, we will prove that $\Phi_\alpha$ is an $\mathcal{N}$-function. Firstly, notice that

$$t\phi_\alpha(t) = \ln(t^\alpha + 1) + \alpha \frac{t^\alpha}{t^\alpha + 1},$$

so

$$\lim_{t\to 0^+} t\phi_\alpha(t) = \lim_{t\to 0^+} (\ln(t^\alpha + 1) + \alpha \frac{t^\alpha}{t^\alpha + 1}) = 0$$

and

$$\lim_{t\to +\infty} t\phi_\alpha(t) = \lim_{t\to +\infty} (\ln(t^\alpha + 1) + \alpha \frac{t^\alpha}{t^\alpha + 1}) \geq \lim_{t\to +\infty} (\ln(t^\alpha + 1)) = +\infty.$$

It remains to show that the function $t \mapsto t\phi_\alpha(t)$ is increasing, for all $t > 0$. Indeed,

$$\begin{aligned}
(t\phi_\alpha(t))' &= \alpha \frac{t^{\alpha-1}}{t^\alpha + 1} + \alpha^2 \frac{t^{\alpha-1}(t^\alpha + 1)}{(t^\alpha + 1)^2} + \alpha^2 \frac{t^\alpha t^{\alpha-1}}{(t^\alpha + 1)^2} \\
&= \alpha \frac{t^{\alpha-1}}{t^\alpha + 1}(1 + \alpha - \alpha \frac{t^\alpha}{t^\alpha + 1}), \quad \forall t > 0.
\end{aligned}$$

Since $\dfrac{t^\alpha}{t^\alpha + 1} < 1$, for all $t > 0$, we conclude that $(t\phi_\alpha(t))' > 0$, for all $t > 0$. Having done this study, it follows from Lemma 2.1 that $\Phi_\alpha$ is a $\mathcal{N}$-function.

Now, see that

$$\begin{aligned}
\left(\frac{t^2\phi_\alpha(t)}{\Phi_\alpha(t)}\right)' &= \frac{\alpha^2 t^{\alpha-1}(t^\alpha + 1)\ln(t^\alpha + 1) - \alpha^2 t^\alpha (t^{\alpha-1}\ln(t^\alpha + 1) + t^{\alpha-1})}{[(t^\alpha + 1)\ln(t^\alpha + 1)]^2} \\
&= \frac{\alpha^2 t^{\alpha-1}}{[(t^\alpha + 1)\ln(t^\alpha + 1)]^2}[(t^\alpha + 1)\ln(t^\alpha + 1) - t^\alpha \ln(t^\alpha + 1) - t^\alpha] \\
&= \frac{\alpha^2 t^{\alpha-1}}{[(t^\alpha + 1)\ln(t^\alpha + 1)]^2}[\ln(t^\alpha + 1) - t^\alpha].
\end{aligned}$$

Since $\ln(t^\alpha + 1) - t^\alpha < 0$ for all $t > 0$, it follows that $\left(\dfrac{t^2\phi_\alpha(t)}{\Phi_\alpha(t)}\right)' < 0$, for all $t > 0$. Thus the function $t \mapsto \dfrac{t^2\phi_\alpha(t)}{\Phi_\alpha(t)}$ is decreasing in $(0, +\infty)$. Clearly $\dfrac{t^2\phi_\alpha(t)}{\Phi_\alpha(t)} \geq 1$, for all $t > 0$, since

$$\frac{t^2\phi_\alpha(t)}{\Phi_\alpha(t)} = \frac{t\ln(t^\alpha + 1)}{t\ln(t^\alpha + 1)} + \alpha \frac{t^{\alpha+1}}{t(t^\alpha + 1)\ln(t^\alpha + 1)} = 1 + \frac{\alpha t^\alpha}{(t^\alpha + 1)\ln(t^\alpha + 1)}. \tag{2.10}$$

Let us now see that if there is $\ell \geq 1$ such that $\dfrac{t^2\phi_\alpha(t)}{\Phi_\alpha(t)} \geq \ell$, then $\ell = 1$. In effect, suppose that $\ell > 1$. By L'Hôpital's rule, we have

(2.11)

$$\lim_{t\to +\infty} \frac{\alpha t^\alpha}{(t^\alpha + 1)\ln(t^\alpha + 1)} = \lim_{t\to +\infty} \frac{\alpha^2 t^{\alpha-1}}{\alpha t^{\alpha-1}\ln(t^\alpha + 1) + \alpha t^{\alpha-1}} = \lim_{t\to +\infty} \frac{\alpha}{\ln(t^\alpha + 1) + 1} = 0.$$

Therefore, by (2.10) and (2.11)

$$1 = \lim_{t \to +\infty} \frac{t^2 \phi_\alpha(t)}{\Phi_\alpha(t)} \geq \ell.$$

Which is absurd. So, we conclude that $\ell = 1$.

Still by (2.10) and L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{t \to 0^+} \frac{t^2 \phi_\alpha(t)}{\Phi_\alpha(t)} &= 1 + \lim_{t \to 0^+} \frac{\alpha t^\alpha}{(t^\alpha + 1)\ln(t^\alpha + 1)} \\
&= 1 + \lim_{t \to 0^+} \frac{\alpha^2 t^{\alpha - 1}}{\alpha t^{\alpha-1}\ln(t^\alpha + 1) + \alpha t^{\alpha - 1}} \\
&= 1 + \lim_{t \to 0^+} \frac{\alpha}{\ln(t^\alpha + 1) + 1} \\
&= 1 + \alpha.
\end{aligned}$$

Therefore,

$$1 = \ell \leq \frac{t^2 \phi_\alpha(t)}{\Phi_\alpha(t)} \leq 1 + \alpha, \quad \forall t > 0.$$

At this point, just choose $\alpha > 0$, small enough, so that $1 + \alpha < l^*$. And so, we will have the desired example.

**Remark 2.5** *The Lemma 2.15 guarantees that $\Phi_*$ is a $\mathcal{N}$-function. Hence, by Lemma 2.1, it follows that*

$$\Phi_*(t) = \int_0^{|t|} s\phi_*(s)ds,$$

*where $\phi_* : [0, \infty) \to [0, \infty)$ satisfies*

1. *$\phi_*(0) = 0$, $s\phi_*(s) > 0$ for $s > 0$, $\lim_{s \to \infty} s\phi_*(s) = \infty$;*
2. *$\phi_*$ is continuous and non-decreasing.*

**Lemma 2.20** *Consider $\Phi$ a $\mathcal{N}$-function of the form (2.3) and satisfying $(\phi_1)$, $(\phi_2)$ and $(\phi_3)$. Define*

$$\xi_4(t) = \min\{t^{\ell^*}, t^{m^*}\} \;\; \text{and} \;\; \xi_5(t) = \max\{t^{\ell^*}, t^{m^*}\}, \;\; t \geq 0.$$

*Then,*

$$\ell^* \leq \frac{\Phi_*'(t) t^2}{\Phi_*(t)} \leq m^*, \quad \forall t > 0; \tag{2.12}$$

$$\xi_2(t)\Phi_*(\rho) \leq \Phi_*(\rho t) \leq \xi_3(t)\Phi_*(\rho), \quad \forall \rho, t > 0 \tag{2.13}$$

*and*

$$\xi_2(\|u\|_{\Phi_*}) \leq \int_\Omega \Phi_*(u)dx \leq \xi_3(\|u\|_{\Phi_*}), \quad \forall u \in L^{\Phi_*}(\Omega).$$

**Remark 2.6** *The inequality (2.12) guarantees that* $\Phi$ *and* $\tilde{\Phi}$ *satisfy the* $\Delta_2$*-condition. Therefore,* $L^{\Phi_*}(\Omega)$ *is separable and reflexive.*

## 2.3 Orlicz-Sobolev Space

In this section, we study Orlicz-Sobolev spaces. We present some basic properties as well as embeddings of Orlicz-Sobolev spaces into Orlicz spaces.

**Definition 2.7** *For a* $\mathcal{N}$*-function* $\Phi$*, we define the* ***Orlicz-Sobolev space*** $W^{1,\Phi}(\Omega)$ *as*

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^{\Phi}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega), i = 1, ..., N \right\}.$$

**Definition 2.8** *For a* $\mathcal{N}$*-function* $\Phi$*, we define the* ***Orlicz-Sobolev space*** $W^1E^{\Phi}(\Omega)$ *as*

$$W^1E^{\Phi}(\Omega) = \left\{ u \in E^{\Phi}(\Omega) : \frac{\partial u}{\partial x_i} \in E^{\Phi}(\Omega), i = 1, ..., N \right\}.$$

*The spaces* $W^{1,\Phi}(\Omega)$ *and* $W^1E^{\Phi}(\Omega)$ *are Banach spaces equipped with the norm*

$$\|u\|_{1,\Phi} = \|\nabla u\|_{\Phi} + \|u\|_{\Phi}. \tag{2.14}$$

**Remark 2.7** *(i)* $W^1E^{\Phi}(\Omega) \subseteq \big(E^{\Phi}(\Omega)\big)^{N+1}$*, in addition,* $W^1E^{\Phi}(\Omega)$ *is closed in the topology of the norm of* $\big(E^{\Phi}(\Omega)\big)^{N+1}$*;*

*(ii)* $W^{1,\Phi}(\Omega) \subseteq \big(L^{\Phi}(\Omega)\big)^{N+1}$*, in addition,* $W^{1,\Phi}(\Omega)$ *is closed in the norm topology of* $\big(L^{\Phi}(\Omega)\big)^{N+1}$*;*

*(iii)* $W^{1,\Phi}(\Omega)$ *is closed in the weak* topology of* $\big(L^{\Phi}(\Omega)\big)^{N+1}$*;*

*(iv)* $W^1E^{\Phi}(\Omega)$ *is separable.*

*(v) For each* $F \in \big(W^1E^{\Phi}(\Omega)\big)'$ *there are* $v_0, v_1, \cdots, v_N \in L^{\tilde{\Phi}}(\Omega)$ *such that*

$$F(u) = \int_{\Omega} u v_0 dx + \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u}{\partial x_i} v_i dx, \quad u \in W^1E^{\Phi}(\Omega);$$

*(vi)* $W^1E^{\Phi}(\Omega) = \overline{C^{\infty}(\Omega) \cap W^1E^{\Phi}(\Omega)}^{\|\cdot\|_{1,\Phi}}$*;*

*(vii) If* $\Omega$ *has the segment property*[1]*, then* $W^1E^{\Phi}(\Omega) = \overline{C^{\infty}(\overline{\Omega})}^{\|\cdot\|_{1,\Phi}}$*.*

**Theorem 2.1** *Let* $\Phi$ *be a* $\mathcal{N}$*-function satisfying the* $\Delta_2$*-condition. Then:*

[1]Segment property, we understand the domains $\Omega$ has the segment property if for every $x \in \partial\Omega$ there exists an open set $U_x$, and a nonzero vector $y_x$, such that $x \in U_x$, and if $z \in \overline{\Omega} \cap U_x$, then $z + t y_x \in \Omega$, for $0 < t < 1$.

*(i)* $W^{1,\Phi}(\Omega) = W^1E^\Phi(\Omega)$*;*

*(ii)* $W^{1,\Phi}(\Omega)$ *is separable;*

*(iii)* $W^{1,\Phi}(\Omega)$ *is reflexive if and only if* $\tilde{\Phi}$.

The main embedding result involving this function class can be found in [70, Theorem 3.2].

**Theorem 2.2** *Let* $\Omega \subset \mathbb{R}^N$ *open and admissible*[2]*. If* $\Phi$ *is a* $\mathcal{N}$*-function verifying* (2.2)*, then*

$$W^{1,\Phi}(\Omega) \underset{cont}{\hookrightarrow} L^{\Phi_*}(\Omega).$$

*Furthermore, if* $|\Omega| < \infty$ *and* $\Psi$ *is a* $\mathcal{N}$*-function such that* $\Psi \prec\prec \Phi_*$*, then*

$$W^{1,\Phi}(\Omega) \underset{comp}{\hookrightarrow} L^{\Psi}(\Omega).$$

**Theorem 2.3** *Let* $\Omega \subset \mathbb{R}^N$ *be open and admissible. If* $\Phi$ *is a* $\mathcal{N}$*-function which does not check* (2.2)*, then we have*

$$W^{1,\Phi}(\Omega) \underset{cont}{\hookrightarrow} C(\Omega) = C(\Omega) \cap L^\infty(\Omega).$$

The following result is a version of the Lemma 2.12 for the space $W^{1,\Phi}(\Omega)$.

**Lemma 2.21** *Assume that* $\Phi$ *is an N-function. If* $(u_n) \subset W^{1,\Phi}(\Omega)$ *is a bounded sequence, then there exists a subsequence of* $(u_n)$*, which we will still denote by* $(u_n)$*, and* $u \in W^{1,\Phi}(\Omega)$ *such that*

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\Phi(\Omega) \quad \text{and} \quad \frac{\partial u_n}{\partial x_i} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial x_i} \quad \text{in } L^\Phi(\Omega) \tag{2.15}$$

*or equivalently,*

$$\int_\Omega u_n v dx \longrightarrow \int_\Omega uv dx, \quad \forall v \in E^{\tilde{\Phi}_*}(\Omega)$$

*and*

$$\int_\Omega \frac{\partial u_n}{\partial x_i} w dx \longrightarrow \int_\Omega \frac{\partial u}{\partial x_i} w dx, \quad \forall w \in E^{\tilde{\Phi}}(\Omega).$$

From now on, we denote the limit (2.15) by $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\Phi}(\Omega)$. As an immediate consequence of the last lemma, we have the following corollary.

[2] By admissible, we understand the domains in which embedding occur $W^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$

**Corollary 2.1** *If $(u_n) \subset W^{1,\Phi}(\Omega)$ is a bounded sequence with $u_n \to u$ in $L^{\Phi}_{loc}(\Omega)$, then $u \in W^{1,\Phi}(\Omega)$.*

**Lemma 2.22** *Suppose $(u_n) \subset W^{1,\Phi}(\Omega)$ is a bounded sequence in $W^{1,\Phi}(\Omega)$, then there is $u \in W^{1,\Phi}(\Omega)$ such that $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\Phi}(\Omega)$ and*

$$\int_{\Omega} \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_{\Omega} \Phi(|\nabla u_n|)dx.$$

**Proof.** Since $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\Phi}(\Omega)$, thus

$$\int_{\Omega} \frac{\partial u_n}{\partial x_i}\varphi dx \to \int_{\Omega} \frac{\partial u}{\partial x_i}\varphi dx, \quad \forall \varphi \in L^{\infty}(\Omega),$$

i.e.

$$\frac{\partial u_n}{\partial x_i} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial x_i} \quad \text{in} \quad L^1(\Omega),$$

because $(L^1(\Omega))^* = L^{\infty}(\Omega)$. Thus we can apply [30, Theorem 2.1] to get

$$\int_{\Omega} \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_{\Omega} \Phi(|\nabla u_n|)dx,$$

which completes the proof. ■

## 2.4 The spaces $W^1_0E^{\Phi}(\Omega)$ and $W^{1,\Phi}_0(\Omega)$

In this section, we shell consider $\Phi$ a $\mathcal{N}$-function and $\Omega \subset \mathbb{R}^N$ an open set. We define the Banach space $W^1_0E^{\Phi}(\Omega)$ as being the closure of $C^{\infty}_0(\Omega)$ in $W^{1,\Phi}(\Omega)$ with respect to the norm (2.14). The Banach space $W^{1,\Phi}_0(\Omega)$ is defined as the weak* closure of $C^{\infty}_0(\Omega)$ in $W^{1,\Phi}(\Omega)$.

**Lemma 2.23** *(i) $W^1_0E^{\Phi}(\Omega)$ is separable;*

*(ii) $W^{1,\Phi}_0(\Omega)$ is the Kernel of the dash operator.*

*(iii) (Poincaré-type inequality [See [35]]) If $d = 2diam(\Omega) < \infty$, then*

$$\int_{\Omega} \Phi(|u|/d)dx \leq \int_{\Omega} \Phi(|\nabla u|)dx, \quad \forall u \in W^{1,\Phi}_0(\Omega)$$

**Lemma 2.24** *Let $\Phi$ a $\mathcal{N}$-function satisfying the $\Delta_2$-condition. Then:*

*(i) $W^{1,\Phi}_0(\Omega) = W^1_0E^{\Phi}(\Omega)$;*

*(ii) $W^{1,\Phi}_0(\Omega)$ is separable;*

*(iii)* $W_0^{1,\Phi}(\Omega)$ *is reflexive if and only if* $\tilde{\Phi}$.

**Theorem 2.4** *Let* $\Omega \subset \mathbb{R}^N$ *open limited and admissible. If* $\Phi$ *is a* $\mathcal{N}$*-function verifying* (2.2)*, then*

$$W_0^{1,\Phi}(\Omega) \underset{comp}{\hookrightarrow} L^{\Phi}(\Omega).$$

## 2.5 The space $D^{1,\Phi}(\mathbb{R}^N)$

Considering $\Phi$ a $\mathcal{N}$-function verifying the $\Delta_2$-condition, the space $D^{1,\Phi}(\mathbb{R}^N)$ is defined to be the complement of the space $C_0^{\infty}(\mathbb{R}^N)$ with respect to the standard

$$|u|_{D^{1,\Phi}(\mathbb{R}^N)} = \|u\|_{\Phi_*} + \|\nabla u\|_{\Phi}. \tag{2.16}$$

It is immediate to verify that

$$D^{1,\Phi}(\mathbb{R}^N) \underset{\text{cont}}{\hookrightarrow} L^{\Phi_*}(\mathbb{R}^N).$$

**Lemma 2.25** *There exists* $S_N > 0$*, such that*

$$\|u\|_{\Phi_*} \leq S_N \|\nabla u\|_{\Phi}, \quad u \in D^{1,\Phi}(\mathbb{R}^N). \tag{2.17}$$

By lemma above, it follows that the norm $|u|_{D^{1,\Phi}(\mathbb{R}^N)}$ is equivalent to the norm $\|\nabla u\|_{\Phi}$. For this reason, in this thesis we will assume the norm of $D^{1,\Phi}(\mathbb{R}^N)$ as being the norm $\|\nabla u\|_{\Phi}$. Being $L^{\Phi}(\mathbb{R}^N)$ and $L^{\Phi_*}(\mathbb{R}^N)$ Banach spaces, we conclude that $D^{1,\Phi}(\mathbb{R}^N)$ is Banach.

**Remark 2.8** *Let* $\Omega \subset \mathbb{R}^N$ *open bounded. Then,* $D^{1,\Phi}(\Omega) = W_0^{1,\Phi}(\Omega)$.

**Lemma 2.26** *(i)* $D^{1,\Phi}(\mathbb{R}^N) \subseteq L^{\Phi_*}(\mathbb{R}^N) \times (L^{\Phi}(\mathbb{R}^N))^N$*, furthermore,* $D^{1,\Phi}(\mathbb{R}^N)$ *is closed in the topology of the norm of* $L^{\Phi_*}(\mathbb{R}^N) \times (L^{\Phi}(\mathbb{R}^N))^N$*;*

*(ii)* $D^{1,\Phi}(\mathbb{R}^N)$ *is separable;*

*(iii)* $D^{1,\Phi}(\mathbb{R}^N)$ *is reflexive if and only if* $\Phi$*,* $\tilde{\Phi}$*,* $\Phi_*, \tilde{\Phi}_* \in (\Delta_2)$.

The following result is a version of the Lemma 2.12 for the space $D^{1,\Phi}(\mathbb{R}^N)$.

**Lemma 2.27** *Assume that* $\Phi$ *is a N-function verificando the* $\Delta_2$*-condition. If* $(u_n) \subset D^{1,\Phi}(\mathbb{R}^N)$ *is a bounded sequence, then there exists a subsequence of* $(u_n)$*, which we will still denote by* $(u_n)$*, and* $u \in D^{1,\Phi}(\mathbb{R}^N)$ *such that*

$$u_n \overset{*}{\rightharpoonup} u \quad in \ L^{\Phi_*}(\mathbb{R}^N) \quad and \quad \frac{\partial u_n}{\partial x_i} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial x_i} \quad in \ L^{\Phi}(\mathbb{R}^N) \tag{2.18}$$

*or equivalently,*

$$\int_{\mathbb{R}^N} u_n v dx \to \int_{\mathbb{R}^N} uv dx, \ \ \forall v \in E^{\tilde{\Phi}_*}(\mathbb{R}^N)$$

*and*

$$\int_{\mathbb{R}^N} \frac{\partial u_n}{\partial x_i} w dx \to \int_{\mathbb{R}^N} \frac{\partial u}{\partial x_i} w dx, \ \ \forall w \in E^{\tilde{\Phi}}(\mathbb{R}^N).$$

From now on, we denote the limit (2.18) by $u_n \overset{*}{\rightharpoonup} u$ in $D^{1,\Phi}(\mathbb{R}^N)$. As an immediate consequence of the last lemma, we have the following corollary.

**Corollary 2.2** *If $(u_n) \subset D^{1,\Phi}(\mathbb{R}^N)$ is a bounded sequence with $u_n \to u$ in $L^{\Phi}_{loc}(\mathbb{R}^N)$, then $u \in D^{1,\Phi}(\mathbb{R}^N)$.*

**Lemma 2.28** *Suppose $(u_n) \subset D^{1,\Phi}(\mathbb{R}^N)$ is a bounded sequence in $D^{1,\Phi}(\mathbb{R}^N)$, then there is $u \in E$ such that $u_n \overset{*}{\rightharpoonup} u$ in $D^{1,\Phi}(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx.$$

**Proof.** Consider $\varphi \in L^\infty(\mathbb{R}^N)$ arbitrary. For every $R > 1$, define the function

$$\omega_R(t) = \begin{cases} 1\,, & \text{if } \ x \in B_R(0) \\ 0\,, & \text{if } x \in B_R^c(0) \end{cases}.$$

It is clear that $\omega_R \in E^{\tilde{\Phi}}(\mathbb{R}^N)$, because $\omega_R \in L^\infty(\mathbb{R}^N)$ and $supp(\omega_R) \subset\subset \mathbb{R}^N$. As a consequence, we have $\varphi\omega_R \in E^{\tilde{\Phi}}(\mathbb{R}^N)$. We know that $L^\Phi(\mathbb{R}^N) \underset{\text{cont}}{\hookrightarrow} L^1_{loc}(\mathbb{R}^N)$, such as $u_n, u \in D^{1,\Phi}(\mathbb{R}^N)$, then

$$\frac{\partial u_n}{\partial x_i}, \ \frac{\partial u}{\partial x_i} \in L^1_{loc}(\mathbb{R}^N), \ \ \forall i = 1, 2, \cdots, N$$

and hence

$$\frac{\partial u_n}{\partial x_i}\omega_R, \ \frac{\partial u}{\partial x_i}\omega_R \in L^1_{loc}(\mathbb{R}^N), \ \ \forall i = 1, 2, \cdots, N.$$

By Lemma 2.27, $u_n \overset{*}{\rightharpoonup} u$ in $D^{1,\Phi}(\mathbb{R}^N)$, thus

$$\int_{\mathbb{R}^N} \left(\frac{\partial u_n}{\partial x_i}\omega_R\right) \varphi dx = \int_{\mathbb{R}^N} \frac{\partial u_n}{\partial x_i}(\omega_R\varphi)dx \to \int_{\mathbb{R}^N} \frac{\partial u}{\partial x_i}(\omega_R\varphi)dx = \int_{\mathbb{R}^N} \left(\frac{\partial u}{\partial x_i}\omega_R\right) \varphi dx.$$

By the arbitrariness of $\varphi$ in $L^\infty(\mathbb{R}^N)$,

$$\frac{\partial u_n}{\partial x_i}\omega_R \overset{*}{\rightharpoonup} \frac{\partial u}{\partial x_i}\omega_R, \ \ \text{in} \ \ L^1(\mathbb{R}^N)$$

because $(L^1(\mathbb{R}^N))^* = L^\infty(\mathbb{R}^N)$. Therefore, applying [30, Theorem 2.1], we can conclude that

$$\int_{B_R(0)} \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|\omega_R)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx.$$

Passing the limit at $R \to +\infty$, we get

$$\int_{\mathbb{R}^N} \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx,$$

which completes the proof. ∎

# Chapter 3

# Existence of positive solution for a class of quasilinear Schrödinger equations with potential vanishing at infinity on nonreflexive Orlicz-Sobolev spaces

Our main goal in this chapter is to prove the sequence of Theorem 1.1, 1.2, 1.3, 1.4 and 1.5 that correspond to the existence of positive solutions to the problem quasilinear of type

$$(P_1) \qquad \begin{cases} -\Delta_\Phi u + V(x)\phi(|u|)u = K(x)f(u), \text{ in } \mathbb{R}^N \\ u \in D^{1,\Phi}(\mathbb{R}^N), \ u \geq 0, \text{ in } \mathbb{R}^N \end{cases}$$

where $N \geq 2$, $V, K : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions with $V$, $K$ being nonnegative functions and $f$ having a quasicritical growth and $\tilde{\Phi} \notin (\Delta_2)$. In this sense, we divide this chapter into 3 sections, where in Section 3.1 we present preliminary results involving the energy functional associated with the problem $(P_1)$, where the conditions $(\phi_1) - (\phi_4)$ (mentioned in the introduction) are satisfied. In the following Sections 3.2 and 3.3 we study the cases in which $(V, K) \in \mathcal{K}_1$ or $(V, K) \in \mathcal{K}_2$.

## 3.1 Preliminary results

Since the potential V may vanish at infinity, we cannot study equation $(P_1)$ on the Sobolev space $D^{1,\Phi}(\mathbb{R}^N)$ by variational methods. As in [10], we work in the space

$$E = \left\{ u \in D^{1,\Phi}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx < +\infty \right\} \tag{3.1}$$

with norm

$$\|u\|_E = \|u\|_{D^{1,\Phi}(\mathbb{R}^N)} + \|u\|_{V,\Phi},$$

where

$$\|u\|_{V,\Phi} = \inf \left\{ \alpha > 0 : \int_{\mathbb{R}^N} V(x)\Phi\left(|u|/\alpha\right) dx \leq 1 \right\}$$

is the norm of Banach Space

$$L_V^{\Phi}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx < +\infty \right\}.$$

It is immediate that $E$ is continuously embedded in the spaces $D^{1,\Phi}(\mathbb{R}^N)$ and $L_V^{\Phi}(\mathbb{R}^N)$.

Now let us list some properties involving the space $E$.

**Lemma 3.1** *$(E, \|\cdot\|_E)$ is a Banach space.*

**Proof.** Let $(u_n) \subset E$ be a Cauchy sequence. So $(u_n)$ is a Cauchy sequence in $(D^{1,\Phi}(\mathbb{R}^N), \|\cdot\|)$, that is, there is $u \in D^{1,\Phi}(\mathbb{R}^N)$ satisfying

$$u_n \to u \quad \text{in} \quad D^{1,\Phi}(\mathbb{R}^N). \tag{3.2}$$

As the embedding $D^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^{\Phi_*}(\mathbb{R}^N)$ is continuous, then $u_n \to u$ in $L^{\Phi_*}(\mathbb{R}^N)$, thus, there is a subsequence $(u_{n_j}) \subset (u_n)$, verifying

$$u_{n_j}(x) \to u(x) \quad \text{in} \ \ \mathbb{R}^N. \tag{3.3}$$

Since $(u_n)$ is a Cauchy sequence in $E$ and the embedding $E \hookrightarrow L_V^{\Phi}(\mathbb{R}^N)$ is continuous, then $(u_n)$ is a Cauchy sequence in $L_V^{\Phi}(\mathbb{R}^N)$. Thus, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\|u_n - u_m\|_{V,\Phi} < \varepsilon, \quad \forall n, m \geq n_0.$$

Particularly,

$$\|u_n - u_{n_j}\|_{V,\Phi} < \varepsilon, \quad \forall n, j \geq n_0.$$

Thus,

$$\int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u_n(x) - u_{n_j}(x)|}{\varepsilon}\right) dx \leq 1, \quad \forall n, j \geq n_0. \tag{3.4}$$

From Fatou's Lemma and by (3.3)-(3.4),

$$\int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u_n(x) - u(x)|}{\varepsilon}\right) dx \leq 1, \quad \forall n \geq n_0.$$

Therefore,

$$\|u_n - u\|_{V,\Phi} < \varepsilon, \quad \forall n \geq n_0,$$

from where it follows that

$$u_n \to u \quad \text{in} \quad L^\Phi_V(\mathbb{R}^N). \tag{3.5}$$

Gathering (3.2) and (3.5) we get

$$u_n \to u \quad \text{in} \quad E.$$

To finish the proof, let us show that $u \in E$. As $u_n \to u$ in $L^\Phi_V(\mathbb{R}^N)$, then $(\|u_{n_j}\|_{V,\Phi})$ is bounded, that is, there is $K > 0$ so that $\|u_{n_j}\|_{V,\Phi} \leq K$, for all $j \in \mathbb{N}$. So,

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x)\Phi\left(|u_{n_j}(x)|\right) dx &\leq \frac{\|u_{n_j}\|_{V,\Phi}}{K} \int_{\mathbb{R}^N} V(x)\Phi\left(K\frac{|u_{n_j}(x)|}{\|u_{n_j}\|_{V,\Phi}}\right) dx \\
&\leq C_K \frac{\|u_{n_j}\|_{V,\Phi}}{K} \int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u_{n_j}(x)|}{\|u_{n_j}\|_{V,\Phi}}\right) dx \\
&\leq C_K,
\end{aligned}$$

for every $j \in \mathbb{N}$, where $C_K > 0$ is a constant that depends only on $K$. By Fatou's Lemma, we obtain

$$\int_{\mathbb{R}^N} V(x)\Phi\left(|u(x)|\right) dx \leq C_K,$$

proving that $u \in E$. Thus we conclude that $E$ is a Banach space. ■

**Lemma 3.2** *$E = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_E}$ and $E$ is compactly embedded in $L^\Phi_{loc}(\mathbb{R}^N)$.*

**Proof.** It follows the same ideas as Theorem 8.21, which can be found in [1]. For that reason, we will omit its proof. ■

**Lemma 3.3** *Suppose $(u_n) \subset E$ is a bounded sequence, then there is $u \in E$ such that*

$$u_n \overset{*}{\rightharpoonup} u \text{ in } D^{1,\Phi}(\mathbb{R}^N), \quad u_n(x) \to u(x) \ \text{a.e. in } \mathbb{R}^N$$

*and*

$$\int_\Omega \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)dx,$$

**Proof.** Since $(u_n)$ is a bounded sequence in $E$, then $(u_n)$ is a bounded sequence in $D^{1,\Phi}(\mathbb{R}^N)$ and by the Lemma 2.28, there is $u \in D^{1,\Phi}(\mathbb{R}^N)$ such that $u_n \overset{*}{\rightharpoonup} u$ in $D^{1,\Phi}(\mathbb{R}^N)$ and

$$\int_\Omega \Phi(|\nabla u|)dx \leq \liminf_{n\to\infty} \int_\Omega \Phi(|\nabla u_n|)dx.$$

Let us show that $u \in E$, because by Lemma 3.2 and Corollary 2.2, we can conclude that less than one subsequence

$$u_n(x) \to u(x) \ \ a.e. \text{ in } \mathbb{R}^N.$$

By Fatou's Lemma

$$\int_{\mathbb{R}^N} V(x)\Phi(|u(x)|)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)\Phi(|u_n(x)|)dx.$$

Since $(u_n)$ is bounded in $E$, then $(u_n)$ is bounded in $L^\Phi_V(\mathbb{R}^N)$. As $\Phi \in (\Delta_2)$, there is $C > 0$ such that

$$\int_{\mathbb{R}^N} V(x)\Phi(|u_n(x)|)dx \leq C, \quad \forall n \in \mathbb{N}$$

Therefore,

$$\int_{\mathbb{R}^N} V(x)\Phi(|u(x)|)dx < +\infty,$$

showing that $u \in E$, and the proof is complete. ■

Now, we consider the functional $Q : E \to \mathbb{R}$ which is given by

$$Q(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx. \tag{3.6}$$

It is well known in the literature that $Q \in C^1(E,\mathbb{R})$ when $\Phi$ and $\tilde{\Phi}$ satisfy the condition $(\Delta_2)$ and this occurs when the condition $(\phi_3)$ is satisfied with $\ell > 1$ and $m < \infty$. When $\ell = 1$, we know that $\tilde{\Phi} \notin (\Delta_2)$ and therefore cannot guarantee the differentiability of functional $Q$. However, we will show that the functional $Q$ is continuous and Gateaux-differentiable with derivative $Q' : E \to E^*$ defined by

$$Q'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)\phi(|u|)uv dx, \quad \forall u, v \in E$$

is continuous from the norm topology of $E$ to the weak$^*$-topology of $E^*$.

The lemma below illustrates the computation of the Gateaux derivative of the functional $Q$, and this result can be found in [11, Lemma 4.1].

**Lemma 3.4** *The functional $Q$ is Gateaux differentiable, that is, $Q'(u)v$ exists for all $u, v \in E$ with*

$$Q'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)\phi(|u|)uv dx.$$

**Proof.** For each $v \in E$ and $t \in [-1,1] \setminus \{0\}$,

$$\Phi(|\nabla u + t\nabla v|) - \Phi(|\nabla v|) = t\phi(|\nabla u + st\nabla v|)(\nabla u + st\nabla v)\nabla v,$$

for some $s \in (0,1)$. Consequently,

$$\left| \frac{\Phi(|\nabla u + t\nabla v|) - \Phi(|\nabla v|)}{t} \right| \le \phi(|\nabla u + st\nabla v|)|\nabla u + st\nabla v||\nabla v|. \tag{3.7}$$

Since $\Phi$ satisfies the $\Delta_2$-condition, by Young's inequality, there is $C > 0$ such that

$$\phi(|\nabla u + st\nabla v|)|\nabla u + st\nabla v||\nabla v| \le C\big(\Phi(|\nabla u|) + \Phi(|\nabla v|)\big) \quad \in L^1(\mathbb{R}^N). \tag{3.8}$$

A similar argument works to show that

$$V(x)\phi(|u + stv|)|u + stv||v| \le C\big(V(x)\Phi(|u|) + V(x)\Phi(|v|)\big) \quad \in L^1(\mathbb{R}^N). \tag{3.9}$$

Now, by using Lebesgue dominated convergence theorem, we derive that

$$\lim_{t \to 0} \frac{Q(u + tv) - Q(u)}{t} = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)\phi(|u|)uv dx,$$

for all $u, v \in E$. ■

**Lemma 3.5** *Let $\Phi$ an $\mathcal{N}$-function of the form (1.6) satisfying $(\phi_1)$, $(\phi_2)$ and $(\phi_3)$. If $u_n \to u$ in $L_V^\Phi(\mathbb{R}^N)$, then there exists $H_1 \in L_V^\Phi(\mathbb{R}^N)$ and a subsequence $(u_{n_j})$ such that*

*i)* $|u_{n_j}(x)| \leq H_1(x)$ *for every* $x \in \mathbb{R}^N$ *and every* $j \in \mathbb{N}$

*ii)* $u_{n_j}(x) \to u(x)$ *a.e. in* $\mathbb{R}^N$ *and every* $j \in \mathbb{N}$.

**Proof.** Since that

$$\begin{aligned}\|V(x)\Phi(2|u_n-u|)\|_1 &= \int_\Omega V(x)\Phi\left(2\|u_n-u\|_{V,\Phi}\frac{|u_n-u|}{\|u_n-u\|_{V,\Phi}}\right)dx\\ &\leq 2\|u_n-u\|_{V,\Phi}\int_\Omega V(x)\Phi\left(\frac{|u_n-u|}{\|u_n-u\|_{V,\Phi}}\right)dx\\ &= 2\|u_n-u\|_{V,\Phi},\end{aligned}$$

where $2\|u_n-u\|_{V,\Phi}\leq 1$. Thus,

$$\|V(x)\Phi(2|u_n-u|)\|_1 \to 0 \text{ as } n\to\infty.$$

Therefore, there is $H \in L^1(\mathbb{R}^N)$ and a subsequence $(u_{n_j})$ such that

*i)* $V(x)\Phi(2|u_{n_j}(x)-u(x)|) \leq H(x)$, for all $x\in\mathbb{R}^N$ and all $j\in\mathbb{N}$

*ii)* $u_{n_j}(x) \to u(x)$, *a.e.* in $\mathbb{R}^N$.

From item *ii)*, we have

$$|u_{n_j}(x)| \leq |u(x)| + \frac{1}{2}\Phi^{-1}\left(\frac{H(x)}{V(x)}\right).$$

Clearly $H_1 \in L_V^\Phi(\mathbb{R}^N)$ where

$$H_1(x) = |u(x)| + \frac{1}{2}\Phi^{-1}\left(\frac{H(x)}{V(x)}\right),$$

and the lemma follows. ■

As an immediate consequence of the Lemma 3.5, we have the following result.

**Lemma 3.6** *The functional $Q : E \to \mathbb{R}$ is continuous in the norm topology.*

**Lemma 3.7** *The Gateaux derivative $Q' : E \to E^*$ is continuous from the norm topology of $E$ to the weak$^*$ topology of $E^*$.*

**Proof.** By the Proposition 3.2 of [27] is sufficient prove that, any sequence $(u_n) \subset E$ such that $u_n \to u$ in $E$, implies

$$\langle Q'(u_n), v\rangle \to \langle Q'(u), v\rangle, \quad \forall v \in E.$$

Consider $(u_n) \subset E$ such that $u_n \to u$ in $E$, then

$$|\nabla u_n| \to |\nabla u| \quad \text{in} \quad L^{\Phi}(\mathbb{R}^N) \quad \text{and} \quad u_n \to u \;\; \text{in} \;\; L^{\Phi}_V(\mathbb{R}^N).$$

By Lemma 3.5, there are $H_1 \in L^{\Phi}_V(\mathbb{R}^N)$, $H_2 \in L^{\Phi}_V(\mathbb{R}^N)$ and a subsequence $(u_{n_j}) \subset (u_n)$ such that

*i)* $|u_{n_j}(x)| \leq H_1(x)$ and $|\nabla u_{n_j}(x)| \leq H_2(x)$ for $x \in \mathbb{R}^N$ and $j \in \mathbb{N}$

*ii)* $u_{n_j}(x) \to u(x)$ and $|\nabla u_{n_j}(x)| \to |\nabla u(x)|$ *a.e.* in $\mathbb{R}^N$ and $j \in \mathbb{N}$.

Set $v \in E$ arbitrary. By the continuity of the function $\phi$, it follows that

$$\phi(|\nabla u_{n_j}(x)|)\nabla u_{n_j}(x)\nabla v(x) \to \phi(|\nabla u(x)|)\nabla u(x)\nabla v(x), \; a.e. \text{ in } \mathbb{R}^N.$$

Also, by $(\phi_1)$ the function $\phi(t)t$ is increasing for every $t > 0$, thus

$$\phi(|\nabla u_{n_j}(x)|)|\nabla u_{n_j}(x)||\nabla v(x)| \leq \phi(|H(x)|)|H(x)|\nabla v(x)|.$$

Hence by Lebesgue dominated convergence theorem

$$\int_{\mathbb{R}^N} \phi(|\nabla u_{n_j}|)\nabla u_{n_j}\nabla v dx \to \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx,$$

with this, we ensure that

$$\int_{\mathbb{R}^N} \phi(|\nabla u_n|)\nabla u_n \nabla v dx \to \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx.$$

Similarly, we have

$$\int_{\mathbb{R}^N} \phi(|u_n|)u_n v dx \to \int_{\mathbb{R}^N} \phi(|u|)u v dx.$$

Therefore,

$$\langle Q'(u_n), v\rangle \to \langle Q'(u), v\rangle.$$

By the arbitrariness of $v \in E$, we conclude the results. ■

With these preliminary results established, we can now present the main outcomes that will be developed throughout this chapter. To achieve this, we will divide the study of problem $(P_1)$ into two conditions: $(V, K) \in \mathcal{K}_1$ and $(V, K) \in \mathcal{K}_2$.

## 3.2 Existence of a solution in the case $(V, K) \in \mathcal{K}_1$

Initially, we list some results that will be true if the conditions $(f_1)$ or $(f_4)$ (mentioned in the introduction) is hold. Note that the condition $(f_4)$ implies that $\lim_{t\to+\infty} \dfrac{f(t)}{\phi_*(t)t} = 0$. Then, by the conditions $(f_1)$ or $(f_4)$, given $\varepsilon > 0$ there exists $\delta_0 > 0$, $\delta_1 > 0$ and $C_\varepsilon > 0$ such that

$$K(x)|f(t)| \leq \varepsilon C_1\big(V(x)t\phi(t) + t\phi_*(t)\big) + C_\varepsilon K(x)t\phi_*(t)\chi_{[\delta_0,\delta_1]}(t), \tag{3.10}$$

for every $t \geq 0$ and $x \in \mathbb{R}^N$, where $C_1 = \max\left\{\|K\|_\infty, \left\|\frac{K}{V}\right\|_\infty\right\}$. This inequality yields that the functional $\mathcal{F} : E \to \mathbb{R}$ given by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} K(x)F(u)dx \tag{3.11}$$

is well defined and belongs to $C^1(E, \mathbb{R})$ with derivative

$$\mathcal{F}'(u)v = \int_{\mathbb{R}^N} K(x)f(u)vdx, \quad \forall u, v \in E.$$

From the results presented in the previous section, we can conclude that the energy functional $J : E \to \mathbb{R}$ associated with the problem $(P_1)$, which is given by

$$J(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - \int_{\mathbb{R}^N} K(x)F(u)dx$$

is a continuous and Gateaux-differentiable functional such that $J' : E \to E^*$ given by

$$J'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u\nabla vdx + \int_{\mathbb{R}^N} V(x)\phi(|u|)uvdx - \int_{\mathbb{R}^N} K(x)f(u)vdx$$

is continuous from the norm topology of $E$ to the weak*-topology of $E^*$. (By Corollary A.1, we have $J$ is locally Lipschitz functional)

Once that we intend to find nonnegative solutions for the problem $(P_1)$, we will assume that

$$f(s) = 0, \quad \forall s \in (-\infty, 0]. \tag{3.12}$$

Since $J$ is locally Lipschitz functional (See Definition in Appendix A) and $Q$, given in (3.6), is convex, then this allows us to present a definition of a critical point for $J$. In this sense, we will say that $u \in E$ is a critical point for the functional $J$ if

$$Q(v) - Q(u) \geq \int_{\mathbb{R}^N} K(x)f(u)(v - u)dx, \quad \forall v \in E. \tag{3.13}$$

Our next lemma establishes that a critical point $u$ in the sense (3.13) is a weak solution for $(P_1)$.

**Proposition 3.1** *If $u \in E$ is a critical point of $J$ in $E$, then $u$ is a weak solution to $(P_1)$.*

**Proof.** See [11, Lemma 4.1]. ■

Now, let us check that $J$ also satisfies the mountain pass geometry.

**Lemma 3.8** *There are $\rho, \eta > 0$ such that $J(u) \geq \eta$ if $\|u\|_E = \rho$.*

**Proof.** Consider $0 < \varepsilon < \frac{1}{2C_1}$ with $C_1 = \left\|\frac{K}{V}\right\|_\infty$. By (3.10), there is $C_\varepsilon > 0$, such that

$$K(x)|F(t)| \leq \frac{1}{2}V(x)\Phi(t) + C_\varepsilon \Phi_*(t), \quad \forall t \geq 0 \ \text{ and } \ x \in \mathbb{R}^N. \tag{3.14}$$

Thus,

$$\begin{aligned} J(u) &\geq \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - C_\varepsilon \int_{\mathbb{R}^N} \Phi_*(|u|)dx \\ &\geq C_2\big(\xi_0(\|\nabla u\|_\Phi) + \xi_0(\|u\|_{V,\Phi})\big) - C_2\xi_3(\|u\|_{\Phi_*}), \end{aligned}$$

for some $C_2 > 0$, where $\xi_0(t) = \min\{t^\ell, t^m\}$ and $\xi_3(t) = \max\{t^{\ell^*}, t^{m^*}\}$. Choose $\rho > 0$ such that

$$\|u\|_E = \|u\|_{D^{1,\Phi}(\mathbb{R}^N)} + \|u\|_{V,\Phi} = \rho < 1.$$

As $E$ is continuously embedded in $L^{\Phi_*}(\mathbb{R}^N)$, we get $\|u\|_{\Phi^*} \leq 1$. Furthermore,

$$J(u) \geq C_2\big(\|\nabla u\|_\Phi^m + \|u\|_{V,\Phi}^m\big) - C_2\|u\|_{\Phi_*}^{\ell^*}.$$

Using classical inequality

$$(x+y)^\alpha \leq 2^{\alpha-1}(x^\alpha + y^\alpha), \quad x, y \geq 0 \quad \text{with} \quad \alpha > 1,$$

we concluded that

$$J(u) \geq C_3\|u\|_E^m - C_3\|u\|_E^{\ell^*},$$

for some positive constant $C_3$. Since $0 < m < \ell^*$, there is $\eta > 0$ such that

$$J(u) \geq \eta \quad \text{for all} \quad \|u\|_E = \rho.$$

This finishes the proof. ■

**Lemma 3.9** *There is $e \in E$ with $\|e\|_E > \rho$ and $J(e) < 0$.*

**Proof.** Consider $\psi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ and $C_1 \in \mathbb{R}$ such that

$$C_1 > \xi_1(\|\psi\|_{D^{1,\Phi}(\mathbb{R}^N)}) + \xi_1(\|\psi\|_{V,\Phi}). \tag{3.15}$$

By $(f_3)$, there exists $C_2 > 0$ satisfying

$$F(t) \geq C_1|t|^m - C_2, \quad \forall t \in \mathbb{R}.$$

Thus

$$K(x)F(t) \geq C_1 K(x)|t|^m - C_2 K(x), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N.$$

That said, considering $t > 0$, we have

$$J(t\psi) \leq \int_{\mathbb{R}^N} \Phi(t|\nabla\psi|)dx + \int_{\mathbb{R}^N} V(x)\Phi(t|\psi|)dx - C_1 t^m \int_{\mathbb{R}^N} K(x)|\psi|^m dx + C_3|supp(\psi)|,$$

before that, it follows from the 2.16 that

$$J(t\psi) \leq \xi_1(t)\big(\xi_1(\|\psi\|_{D^{1,\Phi}(\mathbb{R}^N)}) + \xi_1(\|\psi\|_{V,\Phi})\big) - C_1 t^m \int_{\mathbb{R}^N} K(x)|\psi|^m dx + C_3|supp(\psi)|,$$

therefore, for $t > 1$,

$$J(t\psi) \leq t^m\big(\xi_1(\|\psi\|_{D^{1,\Phi}(\mathbb{R}^N)}) + \xi_1(\|\psi\|_{V,\Phi})\big) - C_1 t^m \int_{\mathbb{R}^N} K(x)|\psi|^m dx + C_3|supp(\psi)|. \tag{3.16}$$

By (3.15) and (3.16), we can conclude that

$$J(t\psi) \to -\infty \quad \text{as} \quad t \to +\infty.$$

The last limit guarantees the existence of $t > 0$ large enough that the result is verified with $e = t\psi$. ■

In what follows, let us denote by $c > 0$ the mountain pass level associated with $J$, that is,

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0 \ \text{ and } \ \gamma(1) = e\}.$$

Associated with $c$, we have a Cerami sequence $(u_n) \subset E$, that is,

$$J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|J'(u_n)\|_* \to 0. \tag{3.17}$$

The above sequence is obtained from the Corollary A.1 in Appendix A.

To show that the sequences obtained in (3.17) is bounded, let us prove a Hardy Type Inequality.

**Proposition 3.2** *(Hardy Type Inequality) Suppose that $(V, K) \in \mathcal{K}_1$, then $E$ is compactly embedded in $L^Z_K(\mathbb{R}^N)$, where $Z(t) = \int_0^{|t|} sz(s)ds$ is a $\mathcal{N}$-function satisfying*

$$0 < z_1 \leq \frac{t^2 z(t)}{Z(t)} \leq z_2, \quad \forall t \geq 0, \tag{3.18}$$

*where $m < z_1 \leq z_2 < \ell^*$.*

**Remark 3.1** *The inequality (3.18) implies the following inequalities*

$$\xi_{0,Z}(t)Z(\rho) \leq Z(\rho t) \leq \xi_{1,Z}(t)Z(\rho), \quad \forall \rho, t \geq 0$$

*when*

$$\xi_{0,Z}(t) = \min\{t^{z_1}, t^{z_2}\} \quad \textit{and} \quad \xi_{1,Z}(t) = \max\{t^{z_1}, t^{z_2}\}, \quad \forall t \geq 0.$$

*Besides by Lemma 2.16 and Lemma 2.20, we have*

$$\lim_{t \to 0} \frac{Z(|t|)}{\Phi(|t|)} = 0 \quad \textit{and} \quad \lim_{t \to \infty} \frac{Z(|t|)}{\Phi_*(|t|)} = 0.$$

**Proof of Proposition 3.2:** We will assume that $(K_2)$ is true. In this case, by Remark 3.1, given $\varepsilon > 0$, there are $0 < s_0 < s_1$ and $C > 0$, such that

$$K(x)Z(|s|) \leq \varepsilon C(V(x)\Phi(|s|) + \Phi_*(|s|)) + CK(x)\chi_{[s_0,s_1]}(|s|)\Phi_*(|s|), \tag{3.19}$$

for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Thus, for $r > 0$ large enough,

$$\int_{B_r(0)^c} K(x)Z(|u|)dx \leq \varepsilon CQ(u) + C\Phi_*(s_1) \int_{A_u \cap B_r(0)^c} K(x)dx, \ \ \forall u \in E, \tag{3.20}$$

where $Q : E \to \mathbb{R}$ is the function defined in (3.6) and

$$A_u = \{x \in \mathbb{R}^N : \ s_0 \leq |u(x)| \leq s_1\}.$$

Consider $(v_n)$ a bounded sequence in $E$. To see that the operator $i : E \to L^B_K(\mathbb{R}^N)$ is compact just prove that $(v_n)$ has a convergent subsequence on $L^B_K(\mathbb{R}^N)$. By Lemma 3.3, there is $v \in E$ such that

$$v_n \xrightarrow{*} v \text{ in } D^{1,\Phi}(\mathbb{R}^N), \quad v_n(x) \to v(x) \ \ a.e. \ \text{ in } \ \mathbb{R}^N$$

or equivalently

$$w_n \overset{*}{\rightharpoonup} 0 \text{ in } D^{1,\Phi}(\mathbb{R}^N), \quad w_n(x) \to 0 \;\; a.e. \;\text{ in }\; \mathbb{R}^N,$$

where $w_n = v_n - v$. By the boundedness of $(v_n)$ in $E$ and by the fact that $\Phi$ and $\Phi_*$ satisfy the $\Delta_2$-condition, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)\Phi(|w_n|)dx \le M_1 \;\text{ and }\; \int_{\mathbb{R}^N} \Phi_*(|w_n|)dx \le M_1, \;\; \forall n \in \mathbb{N},$$

implying that $(Q(w_n))$ is bounded.

On the other hand, defining

$$A_n = \{x \in \mathbb{R}^N : \; s_0 \le |w_n(x)| \le s_1\}$$

the last inequality implies that

$$\Phi_*(s_0)|A_n| \le \int_{A_n} \Phi_*(|w_n|)dx \le M_1, \;\; \forall n \in \mathbb{N}.$$

With this, we can guarantee that $\sup\limits_{n\in\mathbb{N}} |A_n| < +\infty$. Therefore, by $(K_1)$,

$$\int_{A_n \cap B_r(0)^c} K(x)dx < \frac{\varepsilon}{\Phi_*(s_1)}, \quad \forall n \in \mathbb{N}. \tag{3.21}$$

From (3.20) and (3.21),

$$\int_{B_r(0)^c} K(x)Z(|w_n|)dx \le \varepsilon C M_1 + \Phi_*(s_1)\int_{A_n\cap B_r(0)^c} K(x)dx \le \varepsilon(CM_1+1), \quad \forall n \in \mathbb{N},$$

and hence,

$$\limsup_{n\to\infty} \int_{B_r(0)^c} K(x)Z(|w_n|)dx \le \varepsilon(CM_1+1). \tag{3.22}$$

Consider the functions $P_1 : \mathbb{R} \to \mathbb{R}$ and $P_2 : \mathbb{R} \to \mathbb{R}$ given by $P_1(t) = Z(|t|)$ and $P_2(t) = \Phi_*(|t|)$. Of course, $P_1$ and $P_2$ are continuous in addition

$$\lim_{|t|\to+\infty} \frac{P_1(t)}{P_2(t)} = 0.$$

Finally, it follows from the boundedness of $(v_n)$ in $E$ and from the fact that $\Phi_*$ satisfy the $\Delta_2$-condition the existence of a constant $C_1 > 0$, so that

$$\int_{\mathbb{R}^N} P_2(w_n)dx \le \int_{\mathbb{R}^N} \Phi_*(|w_n|)dx < C_1, \;\; \forall n \in \mathbb{N}.$$

Therefore, by compactness Lemma of Strauss [26, Theorem A.I, p. 338], we have

$$\lim_{n\to\infty}\int_{B_r(0)} P_1(w_n)dx = 0.$$

Thus,

$$\limsup_{n\to\infty}\int_{B_r(0)} K(x)Z(|w_n|)dx = 0. \tag{3.23}$$

According to (3.22) and (3.23), we get

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} K(x)Z(|w_n|)dx \leq \varepsilon(CM_1+1).$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} K(x)Z(|w_n|)dx = 0.$$

As $Z$ verifies the $\Delta_2$-condition, we have that

$$w_n \to 0 \quad \text{in} \quad L^Z_K(\mathbb{R}^N),$$

in other words

$$v_n \to v \quad \text{in} \quad L^Z_K(\mathbb{R}^N).$$

Showing the result for the case $(K_2)$.

Next lemma is an important step to prove that the Cerami sequence obtained in (3.17) is bounded.

**Lemma 3.10** *Let $(v_n)$ be a bounded sequence in $E$ such that $v_n \overset{*}{\rightharpoonup} v$ in $D^{1,\Phi}(\mathbb{R}^N)$. Suppose that $f$ satisfies $(f_1)$ or $(f_4)$, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)F(v_n)dx = \int_{\mathbb{R}^N} K(x)F(v)dx, \tag{3.24}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)f(v_n)v_ndx = \int_{\mathbb{R}^N} K(x)f(v)vdx \tag{3.25}$$

*and*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)f(v_n)\psi dx = \int_{\mathbb{R}^N} K(x)f(v)\psi dx, \quad \forall\psi\in C_0^\infty(\mathbb{R}^N). \tag{3.26}$$

**Proof.** As in (3.10), given $\varepsilon > 0$, there exists $\delta_0 > 0$, $\delta_1 > 0$ and $C_\varepsilon > 0$ such that

$$K(x)f(t) \leq \varepsilon C_1\big(V(x)t\phi(t) + t\phi_*(t)\big) + C_\varepsilon K(x)tb(t), \quad \forall t \geq 0 \text{ and } x \in \mathbb{R}^N \tag{3.27}$$

where $C_1 = \max\{\|K\|_\infty, \left\|\frac{K}{V}\right\|_\infty\}$. Hence,

$$K(x)F(t) \leq \varepsilon C_1(V(x)\Phi(t) + \Phi_*(t)) + C_\varepsilon B(t), \quad \forall t \geq 0 \;\text{ and }\; x \in \mathbb{R}^N. \tag{3.28}$$

From Proposition 3.2,

$$\int_{\mathbb{R}^N} K(x)B(v_n)dx \to \int_{\mathbb{R}^N} K(x)B(v)dx \tag{3.29}$$

then there is $r_0 > 0$, so that

$$\int_{B^c_{r_0}(0)} K(x)B(v_n)dx < \frac{\varepsilon}{C_\varepsilon}, \quad \forall n \in \mathbb{N}. \tag{3.30}$$

Moreover, as $(v_n)$ is bounded in $E$, there is a constant $M_1 > 0$ satisfying

$$\int_{\mathbb{R}^N} V(x)\Phi(|v_n|)dx \leq M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} \Phi_*(|v_n|)dx \leq M_1, \quad \forall n \in \mathbb{N}.$$

Combining the last inequalities with (3.28) and (3.30),

$$\left|\int_{B^c_{r_0}(0)} K(x)F(v_n)dx\right| \leq \varepsilon(C_1M_1 + 1), \quad \forall n \in \mathbb{N}.$$

Therefore

$$\limsup_{n\to+\infty} \int_{B^c_{r_0}(0)} K(x)F(v_n)dx \leq \varepsilon(C_1M_1 + 1). \tag{3.31}$$

On the other hand, using $(f_2)$ and the compactness Lemma of Strauss [26, Theorem A.I, p. 338], it follows that

$$\lim_{n\to+\infty} \int_{B_{r_0}(0)} K(x)F(v_n)dx = \int_{B_{r_0}(0)} K(x)F(v)dx. \tag{3.32}$$

In light of this, we can conclude that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)F(v_n)dx = \int_{\mathbb{R}^n} K(x)F(v)dx.$$

To show (3.25), consider $r_0 > 0$ given in (3.30). By (3.27),

$$\begin{aligned}\int_{B^c_{r_0}(0)} K(x)f(v_n)v_ndx \leq& \varepsilon C_1\left(m\int_{B^c_{r_0}(0)} V(x)\Phi(|v_n|)dx + m^*\int_{B^c_{r_0}(0)} \Phi_*(|v_n|)dx\right)\\ &+ C_\varepsilon z_2\int_{B^c_{r_0}(0)} K(x)Z(v_n)dx\\ \leq& \varepsilon((m^* + m)C_1M + z_2),\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore

$$\limsup_{n \to +\infty} \int_{B^c_{r_0}(0)} K(x) f(u_n) u_n dx \leq \varepsilon((m^* + m) C_1 M + z_2). \tag{3.33}$$

On the other hand, using $(f_1)$ or $(f_4)$ and the compactness Lemma of Strauss [26, Theorem A.I, p. 338], we can conclude that

$$\lim_{n \to +\infty} \int_{B_{r_0}(0)} K(x) f(u_n) u_n dx = 0. \tag{3.34}$$

Thus, the limit (3.24) is obtained from (3.33) and (3.34). Related the limit (3.26), it follows directly from the condition $(f_1)$ or $(f_4)$ together with a version of the compactness lemma of Strauss for non-autonomous problem. (This version is an immediate consequence of [26, Theorem A.I, p. 338] where $K(x)dx$ is used as the new measure)

■

Now, we can prove that the Cerami sequence $(u_n)$ obtained is bounded.

**Lemma 3.11** *Let $(u_n)$ be the Cerami sequence given in (3.17). There is a constant $M > 0$ such that $J(tu_n) \leq M$ for every $t \in [0, 1]$ and $n \in \mathbb{N}$.*

**Proof.** Let $t_n \in [0, 1]$ be such that $J(t_n u_n) = \max_{t \in [0,1]} J(tu_n)$. If $t_n = 0$ and $t_n = 1$, we are done. Thereby, we can assume $t_n \in (0, 1)$, and so, $J'(t_n u_n) u_n = 0$. From this,

$$\begin{aligned} mJ(t_n u_n) =& mJ(t_n u_n) - J'(t_n u_n)(t_n u_n) \\ =& \int_{\mathbb{R}^N} \big( m\Phi(|\nabla(t_n u_n)|) - \phi(|\nabla(t_n u_n)|) |\nabla(t_n u_n)|^2 \big) dx \\ &+ \int_{\mathbb{R}^N} V(x) \big( m\Phi(|t_n u_n|) - \phi(|t_n u_n|) |t_n u_n|^2 \big) dx + \int_{\mathbb{R}^N} K(x) \mathcal{H}(t_n u_n) dx, \end{aligned}$$

where $\mathcal{H}(s) = sf(s) - mF(s)$. By $(f_2)$ the a function $\mathcal{H}(s)$ is increasing for all $s > 0$ and decreasing for $s < 0$, and by $(\phi_4)$ the function $s \mapsto m\Phi(s) - \phi(s)s^2$ is increasing for $s > 0$. Thus,

$$mJ(tu_n) \leq mJ(t_n u_n) \leq mJ(u_n) - J'(u_n)u_n = mJ(u_n) - o_n(1), \quad \forall t \in [0, 1].$$

Since $(J(u_n))$ is bounded, there is $M > 0$ such that

$$J(tu_n) \leq M, \quad \forall t \in [0, 1] \;\text{ and }\; n \in \mathbb{N}.$$

■

**Proposition 3.3** *The Cerami sequence $(u_n)$ given in (3.17) is bounded.*

**Proof.** Suppose for contradiction that, up to a subsequence, $\|u_n\|_E \to \infty$. This way, we need to study the following situations:

*i)* $\|\nabla u_n\|_\Phi \to +\infty$ and $(\|u_n\|_{V,\Phi})$ is bounded,

*ii)* $\|u_n\|_{V,\Phi} \to \infty$ and $(\|\nabla u_n\|_\Phi)$ is bounded,

and

*iii)* $\|\nabla u_n\|_\Phi \to +\infty$ and $\|u_n\|_{V,\Phi} \to +\infty$.

In the case *iii)*, consider $w_n = \dfrac{u_n}{\|u_n\|_E}$. Since $\|w_n\|_E = 1$, by Lemma 3.3, there exists $w \in E$ such that $w_n \overset{*}{\rightharpoonup} w$ in $D^{1,\Phi}(\mathbb{R}^N)$. Now, let us show that $w = 0$. Before that, as $J(u_n) \to c$, we have $J(u_n) \geq 0$, for every $n$ large enough. Thus, there is $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u_n|)dx \geq \int_{\mathbb{R}^N} K(x)F(u_n)dx, \ \forall n \geq n_0. \tag{3.35}$$

As $\|\nabla u_n\|_\Phi \geq 1$ and $\|u_n\|_{V,\Phi} \geq 1$ for every $n \geq n_1$, we have

$$\int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx \leq \|\nabla u_n\|_\Phi^m \ \text{ and } \ \int_{\mathbb{R}^N} V(x)\Phi(|u_n|)dx \leq \|u_n\|_{V,\Phi}^m, \ \forall n \geq n_1.$$

So, by (3.35),

$$\|\nabla u_n\|_\Phi^m + \|u_n\|_{V,\Phi}^m \geq \int_{\mathbb{R}^N} K(x)F(u_n)dx, \ \ \forall n \geq \max\{n_0, n_1\}.$$

Therefore, there is a constant $C > 0$ such that

$$C(\|\nabla u_n\|_\Phi + \|u_n\|_{V,\Phi})^m \geq \int_{\mathbb{R}^N} K(x)F(u_n)dx, \ \ \forall n \geq \max\{n_0, n_1\},$$

or equivalently,

$$C(\|u_n\|_E)^m \geq \int_{\mathbb{R}^N} K(x)F(u_n)dx, \ \ \forall n \geq \max\{n_0, n_1\}.$$

Thus,

$$C \geq \int_{\mathbb{R}^N} K(x)\frac{F(u_n)}{\|u_n\|_E^m}dx \geq \int_{\mathbb{R}^N} K(x)\frac{F(u_n)}{|u_n|^m}|w_n|^m dx.$$

The condition $(f_3)$ implies that for every $\tau > 0$, there is $\xi > 0$ sufficiently large such that

$$\frac{F(s)}{|s|^m} \geq \tau, \ \ \forall |s| \geq \xi.$$

So

$$1+C \geq \int_{\Omega\cap\{|u_n|\geq\xi\}} K(x)\frac{F(u_n)}{|u_n|^m}|w_n|^m dx \geq \tau \int_{\Omega\cap\{|u_n|\geq\xi\}} K(x)|w_n|^m dx,$$

where $\Omega = \{x \in \mathbb{R}^N : \ w(x) \neq 0\}$. By Fatou's Lemma,

$$1+C \geq \tau \int_\Omega K(x)|w(x)|^m dx, \ \ \forall\, \tau > 0.$$

Therefore,

$$\int_\Omega K(x)|w(x)|^m dx = 0.$$

As $K(x) > 0$ for almost everywhere in $\mathbb{R}^N$, we have $w = 0$.

Note that for every $M > 1$, there is $n_0 \in \mathbb{N}$ such that $\dfrac{M}{\|\nabla u_n\|_\Phi} \in [0,1]$ for all $n \geq n_0$. Given this, we get

$$\begin{aligned}
J(t_n u_n) \geq& J(\frac{M}{\|\nabla u_n\|_\Phi}u_n) = J(\frac{M}{\|\nabla u_n\|_\Phi}|u_n|) = J(Mw_n) \\
\geq& \int_{\mathbb{R}^N} \Phi(M|\nabla w_n|)dx + \int_{\mathbb{R}^N} V(x)\Phi(M|w_n|)dx - \int_{\mathbb{R}^N} K(x)F(Mw_n)dx \\
\geq& MQ(u_n) - \int_{\mathbb{R}^N} K(x)F(Mw_n)dx,
\end{aligned}$$

where $Q : E \to \mathbb{R}$ is the function defined in (3.6). By definition of the sequence $(w_n)$, we have $\|\nabla w_n\|_\Phi \leq 1$ and $\|w_n\|_{V,\Phi} \leq 1$, for all $n \in \mathbb{N}$. Then,

$$\int_{\mathbb{R}^N} \Phi(|\nabla w_n|)dx \geq \|\nabla w_n\|^m_\Phi, \text{ and } \int_{\mathbb{R}^N} V(x)\Phi(|w_n|)dx \geq \|w_n\|^m_{V,\Phi}, \ \forall\, n \in \mathbb{N}.$$

So there is $C > 0$ such that

$$Q(w_n) \geq \|\nabla w_n\|^m_\Phi + \|w_n\|^m_{V,\Phi} \geq C(\|\nabla w_n\|_\Phi + \|w_n\|_{V,\Phi})^m, \ \ \forall\, n \in \mathbb{N}.$$

Thus

$$\begin{aligned}
J(t_n u_n) &\geq MC(\|w_n\|_E)^m - \int_{\mathbb{R}^N} K(x)F(Mw_n)dx \\
&= MC - \int_{\mathbb{R}^N} K(x)F(Mw_n)dx.
\end{aligned}$$

By Lemma 3.10,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)F(Mw_n)dx = 0,$$

therefore,

$$\liminf_{n\to\infty} J(t_n u_n) \geq M, \quad \forall\, M \geq 1.$$

which contradicts the Lemma 3.11, once that $(J(t_n u_n))$ is bounded from above. Therefore $(u_n)$ is bounded in E.

The cases $i)$ and $ii)$ are analogous to the case $iii)$. ■

Since that the Cerami sequence $(u_n)$ given in (3.17) is bounded in $E$, by Lemma 3.3, we can assume that for some subsequence, there is $u \in E$ such that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } D^{1,\Phi}(\mathbb{R}^N) \;\text{ and }\; u_n(x) \to u(x) \; a.e. \; \mathbb{R}^N \tag{3.36}$$

and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx \geq \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx. \tag{3.37}$$

Fix $v \in C_0^\infty(\mathbb{R}^N)$. By boundedness of Cerami sequence $(u_n)$, we have $J'(u_n)(v-u_n) = o_n(1)$, hence, since $\Phi$ is a convex function, it is possible to show that

$$Q(v) - Q(u_n) \geq \int_{\mathbb{R}^N} K(x)f(u_n)(v-u_n)dx + o_n(1), \tag{3.38}$$

where $Q : E \to \mathbb{R}$ is the function defined in (3.6). By (3.36), it follows from Fatou's Lemma that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)\Phi(|u_n|)dx \geq \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx. \tag{3.39}$$

Combining (3.37) and (3.39), we conclude that

$$\liminf_{n\to\infty} Q(u_n) \geq Q(u). \tag{3.40}$$

From (3.38) and (3.40) together with the Lemma 3.10, we get

$$Q(v) - Q(u) \geq \int_{\mathbb{R}^N} K(x)f(u)(v-u)dx, \quad \forall\, v \in C_0^\infty(\mathbb{R}^N).$$

As $E = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_E}$ and $\Phi \in (\Delta_2)$, we conclude that

$$Q(v) - Q(u) \geq \int_{\mathbb{R}^N} K(x)f(u)(v-u)dx, \quad \forall\, v \in E. \tag{3.41}$$

In other words, $u$ is a critical point of the $J$ functional. Its follows from Proposition 3.1, we can conclude that $u$ is a weak solution for $(P_1)$. Now, we substitute $v = u^+ := \max\{0, u(x)\}$ in (3.41) and use (3.12) to get

$$-\int_{\mathbb{R}^N} \Phi(|\nabla u^-|)dx - \int_{\mathbb{R}^N} V(x)\Phi(u^-)dx \geq \int_{\mathbb{R}^N} K(x)f(u)u^- dx = 0,$$

which leads to

$$\int_{\mathbb{R}^N} \Phi(|\nabla u^-|)dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)\Phi(u^-)dx = 0$$

whence it is readily inferred that $u^- = 0$, therefore, $u$ is a weak nonnegative solution.

Note that $u$ is nontrivial. Consider a sequence $(\varphi_k) \subset C_0^\infty(\mathbb{R}^N)$ such that $\varphi_k \to u$ in $D^{1,\Phi}(\mathbb{R}^N)$. Since that the Cerami sequence $(u_n)$ given in (3.17) is bounded in $E$ and $\Phi$ is convex, we can show that

$$Q(\varphi_k) - Q(u_n) \geq \int_{\mathbb{R}^N} K(x)f(u_n)(\varphi_k - u_n)dx + o_n(1)\|\varphi_k\| - o_n(1). \tag{3.42}$$

Since $(\|\varphi_k\|)_{k\in\mathbb{N}}$ is a bounded sequence, it follows from (3.42) and from Lemma 3.10 that

$$Q(\varphi_k) \geq \limsup_{n\to\infty} Q(u_n) + \int_{\mathbb{R}^N} K(x)f(u)(\varphi_k - u)dx.$$

Now, note that being $\Phi \in (\Delta_2)$ and $\varphi_k \to u$ in $E$, we conclude from the inequality above that

$$Q(u) \geq \limsup_{n\to\infty} Q(u_n). \tag{3.43}$$

From (3.40) and (3.43),

$$\lim_{n\to\infty} Q(u_n) = Q(u). \tag{3.44}$$

By Lemma 3.10, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)F(u_n)dx = \int_{\mathbb{R}^N} K(x)F(u)dx, \tag{3.45}$$

Therefore, by (3.17), (3.44) and (3.45), we conclude

$$0 < c = \lim_{n\to\infty} J(u_n) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx - \int_{\mathbb{R}^N} K(x)F(u)dx = J(u),$$

that is, $u \neq 0$.

### 3.2.1 Boundedness of nonnegative solutions of $(P_1)$ for the class $(V,K) \in \mathcal{K}_1$

To study the boundedness of nonnegative solutions to the problem $(P_1)$ in the case $(V,K) \in \mathcal{K}_1$, we define the $\mathcal{N}$-function $\Upsilon : \mathbb{R} \mapsto \mathbb{R}$ given by

$$\Upsilon(t) = \begin{cases} \Phi_*(t) & \text{if } \Phi \in \mathcal{C}_m \\ B(t) & \text{if } \Phi \notin \mathcal{C}_m \end{cases}. \tag{3.46}$$

It is clear that $\Upsilon \in C^1(\mathbb{R})$ and

$$\Upsilon'(t) = \begin{cases} t\phi_*(t) & \text{if } \Phi \in \mathcal{C}_m \\ tb(t) & \text{if } \Phi \notin \mathcal{C}_m \end{cases}.$$

**Remark 3.2** *The function $\Upsilon$ defined in (3.46) satisfies*

$$\xi_{0,\Upsilon}(t)\Upsilon(\rho) \geq \Upsilon(t\rho) \geq \xi_{1,\Upsilon}(t)\Upsilon(\rho), \quad \forall t, \rho > 0, \tag{3.47}$$

*where*

$$\xi_{0,\Upsilon}(t) = \begin{cases} \min\{t^{m^*}, t^{\ell^*}\} & \text{if } \Phi \in \mathcal{C}_m \\ \min\{t^{b_1{}^*}, t^{b_2{}^*}\} & \text{if } \Phi \notin \mathcal{C}_m \end{cases} \quad \text{and} \quad \xi_{1,\Upsilon}(t) = \begin{cases} \max\{t^{m^*}, t^{\ell^*}\} & \text{if } \Phi \in \mathcal{C}_m \\ \max\{t^{b_1{}^*}, t^{b_2{}^*}\} & \text{if } \Phi \notin \mathcal{C}_m \end{cases}.$$

To prove the following result in the cases where $\Phi \in \mathcal{C}_m$ or $\Phi \notin \mathcal{C}_m$ we will define the real number $\gamma$ given by

$$\gamma = \begin{cases} m & \text{if } \Phi \in \mathcal{C}_m \\ \ell & \text{if } \Phi \notin \mathcal{C}_m \end{cases}.$$

Note that if $\Phi \in C_m$, then according to Theorems 1.1 and 1.2 the nonlinearity $f$ satisfies the condition $(f_1)$. Now in the case where $\Phi \notin C_m$ the nonlinearity $f$ satisfies $(f_4)$, in both cases, given $\eta > 0$, there exists $C_\eta > 0$ such that

$$K(x)f(t) \leq \eta C_1 V(x) t\phi(t) + C_\eta \Upsilon'(t), \quad \forall t \geq 0 \text{ and } x \in \mathbb{R}^N, \tag{3.48}$$

where $C_1 = \|K\|_\infty$. Thus, we can begin by presenting a technical result that will be fundamental to guarantee the boundedness of the nonnegative solutions of the problem $(P_1)$. We would like to mention that the technique presented in the following lemma can be found in [5].

**Lemma 3.12** *Let $u \in E$ be a nonnegative solution of $(P_1)$, $x_0 \in \mathbb{R}^N$ and $R_0 > 0$. Then*

$$\int_{\mathcal{A}_{k,t}} |\nabla u|^{\gamma} dx \leq C \left( \int_{\mathcal{A}_{k,s}} \left| \frac{u-k}{s-t} \right|^{\gamma^*} dx + (k^{\gamma^*} + 1)|\mathcal{A}_{k,s}| \right)$$

*where $0 < t < s < R_0$, $k > 1$, $\mathcal{A}_{k,\rho} = \{x \in B_\rho(x_0) : u(x) > k\}$ and $C > 0$ is a constant that does not depend on $k$.*

**Proof.** Let $u \in E$ be a weak solution nonnegative of $(P_1)$ and $x_0 \in \mathbb{R}^N$. Moreover, fix $0 < t < s < R_0$ and $\zeta \in C_0^\infty(\mathbb{R}^N)$ verifying

$$0 \leq \zeta \leq 1, \quad supp(\zeta) \subset B_s(x_0), \quad \zeta \equiv 1 \text{ on } B_t(x_0) \text{ and } |\nabla \zeta| \leq \frac{2}{s-t}.$$

For $k > 1$, set $\varphi = \zeta^m (u-k)^+$ and

$$J = \int_{\mathcal{A}_{k,s}} \Phi(|\nabla u|)\zeta^m dx.$$

Using $\varphi$ as a test function and $\ell \Phi(t) \leq \phi(t)t^2$, we find

$$\begin{aligned} \ell J \leq m \int_{\mathcal{A}_{k,s}} \zeta^{m-1}(u-k)^+ \phi(|\nabla u|)|\nabla u||\nabla \zeta| dx - \int_{\mathcal{A}_{k,s}} V(x)\phi(u)u\zeta^m(u-k)^+ dx \\ + \int_{\mathcal{A}_{k,s}} K(x)f(u)\zeta^m(u-k)^+ dx. \end{aligned}$$

Considering $\eta = \frac{1}{C_1}$ in the inequality (3.48), there exists a constant $C_2 > 0$ such that

$$\ell J \leq m \int_{\mathcal{A}_{k,s}} \zeta^{m-1}(u-k)^+ \phi(|\nabla u|)|\nabla u||\nabla \zeta| dx + C_2 \int_{\mathcal{A}_{k,s}} \Upsilon'(u)\zeta^m(u-k)^+ dx. \tag{3.49}$$

For each $\tau \in (0,1)$, the Young's inequality gives

$$\phi(|\nabla u|)|\nabla u||\nabla \zeta|\zeta^{m-1}(u-k)^+ \leq \tilde{\Phi}(\phi(|\nabla u|)|\nabla u|\zeta^{m-1}\tau) + C_3 \Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big). \tag{3.50}$$

It follows from Lemma 2.18,

$$\tilde{\Phi}(\phi(|\nabla u|)|\nabla u|\zeta^{m-1}\tau) \leq C_4(\tau\zeta^{m-1})^{\frac{m}{m-1}}\Phi(|\nabla u|). \tag{3.51}$$

From (3.49), (3.50) and (3.51),

$$\begin{aligned} \ell J \leq & m C_4 \tau^{\frac{m}{m-1}} \int_{\mathcal{A}_{k,s}} \Phi(|\nabla u|)\zeta^m + mC_3 \int_{\mathcal{A}_{k,s}} \Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big) dx \\ & + C_2 \int_{\mathcal{A}_{k,s}} \Upsilon'(u)\zeta^m(u-k)^+ dx. \end{aligned}$$

Choosing $\tau \in (0,1)$ such that $0 < mC_4\tau^{\frac{m}{m-1}} < \ell$, we derive

$$J \leq C_5 \int_{\mathcal{A}_{k,s}} \Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big)dx + C_5 \int_{\mathcal{A}_{k,s}} \Upsilon'(u)\zeta^m(u-k)^+ dx. \tag{3.52}$$

By Young's inequality,

$$\Upsilon'(u)\zeta^m(u-k)^+ \leq C_6\Upsilon\left(\Big|\frac{u-k}{s-t}\Big|\right) + C_6\Upsilon(k). \tag{3.53}$$

Therefore, a combination of (3.52) and (3.53), yields

$$J \leq C_7 \int_{\mathcal{A}_{k,s}} \Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big)dx + C_7 \int_{\mathcal{A}_{k,s}} \Upsilon\Big(\Big|\frac{u-k}{s-t}\Big|\Big)dx + C_7 \int_{\mathcal{A}_{k,s}} \Upsilon(k)dx. \tag{3.54}$$

Now, using that $\ell \leq m < m^* \leq \gamma^*$ and applying the Lemma 2.16 and the Remark 3.2 for functions $\Phi$ and $\Upsilon$, respectivamente, we get

$$\Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big) \leq \Phi(1)\Big(\Big|\frac{u-k}{s-t}\Big|^{\gamma^*} + 1\Big), \quad \Upsilon\Big(\Big|\frac{u-k}{s-t}\Big|\Big) \leq \Upsilon(1)\Big(\Big|\frac{u-k}{s-t}\Big|^{\gamma^*} + 1\Big)$$

and

$$\Upsilon(k) \leq (k^{\gamma^*} + 1).$$

From (3.54) and the inequality above,

$$J \leq C_8\left(\int_{\mathcal{A}_{k,s}} \Big|\frac{u-k}{s-t}\Big|^{\gamma^*} dx + (k^{\gamma^*} + 1)|\mathcal{A}_{k,s}|\right).$$

By definition of $J$ and the function $\zeta$ we can conclude that

$$\int_{\mathcal{A}_{k,t}} \Phi(|\nabla u|)dx \leq C_8\left(\int_{\mathcal{A}_{k,s}} \Big|\frac{u-k}{s-t}\Big|^{\gamma^*} dx + (k^{\gamma^*} + 1)|\mathcal{A}_{k,s}|\right).$$

By Lemma 2.16, we have

$$\int_{\mathcal{A}_{k,t}} |\nabla u|^{\gamma} dx \geq \int_{\mathcal{A}_{k,t}} \Phi(|\nabla u|)dx.$$

Thus

$$\int_{\mathcal{A}_{k,t}} |\nabla u|^{\gamma} dx \leq C_8\left(\int_{\mathcal{A}_{k,s}} \Big|\frac{u-k}{s-t}\Big|^{\gamma^*} dx + (k^{\gamma^*} + 1)|\mathcal{A}_{k,s}|\right).$$

∎

**Lemma 3.13** *Let $u \in E$ be a nonnegative solution of $(P_1)$. Then, $u \in L^\infty_{loc}(\mathbb{R}^N)$.*

**Proof.** To begin with, consider $\Lambda$ a compact subset on $\mathbb{R}^N$. Fix $R_1 \in (0,1)$ and $x_0 \in \Lambda$. Given $M > 1$, define the sequences

$$\sigma_n = \frac{R_1}{2} + \frac{R_1}{2^{n+1}}, \quad \overline{\sigma}_n = \frac{\sigma_n + \sigma_{n+1}}{2} \quad \text{and} \quad K_n = \frac{M}{2}\left(1 - \frac{1}{2^{n+1}}\right), \ \forall n = 0, 1, 2, 3, \cdots.$$

For every $n \in \mathbb{N}$, we consider

$$J_n = \int_{\mathcal{A}_{K_n,\sigma_n}} \left((u - K_n)^+\right)^{\gamma^*} dx \ \text{ and } \ \xi_n = \xi\left(\frac{2^{n+1}}{R_1}\left(|x - x_0| - \frac{R_1}{2}\right)\right), \ x \in \mathbb{R}^N,$$

with $\xi \in C^1(\mathbb{R})$ satisfying

$$0 \le \xi \le 1, \quad \xi(t) = 1 \ \text{ if } \ t \le \frac{1}{2}, \quad \xi(t) = 0 \ \text{ if } \ t \ge \frac{3}{4} \quad \text{and} \quad |\xi'| < C.$$

By definition of $\xi_n$,

$$\xi_n = 1 \ \text{ in } \ B_{\sigma_{n+1}}(x_0) \quad \text{and} \quad \xi_n = 0 \ \text{ outside } \ B_{\overline{\sigma}_n}(x_0),$$

consequently

$$J_{n+1} \le \int_{B_{R_1}(x_0)} \left((u - K_{n+1})^+ \xi_n\right)^{\gamma^*} dx.$$

Note that,

$$|\nabla((u - K_{n+1})^+ \xi_n)|^\gamma \le 2^\gamma \left(|\nabla u|^\gamma \xi_n^\gamma + \frac{2^{\gamma(n+1)}}{R_1^\gamma}\left((u - K_{n+1})^+\right)^\gamma \chi_{B_{R_1}(x_0)}\right),$$

Since $W^{1,\Phi}(B_{\overline{\sigma}_n}(x_0)) \hookrightarrow W^{1,\gamma}(B_{\overline{\sigma}_n}(x_0))$ and $((u - K_{n+1})^+)\xi_n \in W^{1,\gamma}(\mathbb{R}^N)$, then $((u - K_{n+1})^+)\xi_n \in W^{1,\gamma}(\mathbb{R}^N)$. Therefore,

$$J_{n+1} \le C(N, \gamma, R_1) \left(\int_{\mathcal{A}_{K_{n+1},\overline{\sigma_n}}} |\nabla u|^\gamma dx + 2^{\gamma n} \int_{\mathcal{A}_{K_{n+1},\overline{\sigma_n}}} ((u - K_{n+1})^+)^\gamma dx\right)^{\frac{\gamma^*}{\gamma}}.$$

Applying the Lemma 3.12 to the previous inequality and then by the fact that $|\sigma_n - \overline{\sigma}_n| = \dfrac{R_1}{2^{n+3}}$ and $t^\gamma \le t^{\gamma^*} + 1$ for every $t \ge 0$, we obtain

$$J_{n+1} \le C(N, \gamma, R_1)\Big(2^{\gamma n} \int_{\mathcal{A}_{K_{n+1},\sigma_n}} ((u - K_{n+1})^+)^{\gamma^*} dx + (K^{\gamma^*} + 2^{\gamma n} + 1)|\mathcal{A}_{K_{n+1},\sigma_n}|\Big)^{\frac{\gamma^*}{\gamma}}$$

On the other hand, such as $K_{n+1} - K_n = \dfrac{M}{2^{n+3}}$

$$|\mathcal{A}_{K_{n+1},\sigma_n}| \le \frac{2^{\gamma^*(n+3)}}{K^{\gamma^*}} J_n, \tag{3.55}$$

which yields

$$\int_{\mathcal{A}_{K_{n+1},\sigma_n}} \left((u-K_{n+1})^+\right)^{\gamma^*} dx \le \int_{\mathcal{A}_{K_n,\sigma_n}} \left((u-K_n)^+\right)^{\gamma^*} dx + \left|K_{n+1}-K_n\right|^{\gamma^*} \left|\mathcal{A}_{K_{n+1},\sigma_n}\right|$$
$$\le 2J_n,$$

consequently, there exists a constant $C = C(N, \gamma, R_1) > 0$ such that

$$J_{n+1} \le CD^n J_n^{1+\omega}, \quad n = 0, 1, 2, \cdots,$$

where $D = 2^{(\gamma+\gamma^*)\frac{\gamma^*}{\gamma}}$ and $\omega = \frac{\gamma^*}{\gamma} - 1$.

Note that

$$J_0 = \int_{\mathcal{A}_{K_0,\sigma_0}} \left((u-K_0)^+\right)^{\gamma^*} dx \le \int_{B_{R_1}(x_0)} \left((u-K_0)^+\right)^{\gamma^*} dx. \tag{3.56}$$

Then, by the Lebesgue's Theorem, $\lim\limits_{K\to\infty} J_0 = 0$, from where it follows that

$$J_0 \le C^{-\frac{1}{\omega}} D^{-\frac{1}{\omega^2}}, \quad \text{for all} \;\; M \ge M^*$$

for some $M^* \ge 1$ that depends on $x_0$. Fix $M = M^*$. Thus, by [62, Lemma 4.7], we deduce that

$$J_n \to 0 \quad \text{as} \;\; n \to \infty.$$

On the other hand,

$$\lim_{n\to\infty} J_n = \lim_{n\to\infty} \int_{\mathcal{A}_{K_n,\sigma_n}} \left((u-K_0)^+\right)^{\gamma^*} dx = \int_{\mathcal{A}_{\frac{M^*}{2},\frac{R_1}{2}}} \left((u-\frac{M^*}{2})^+\right)^{\gamma^*} dx,$$

hence,

$$\int_{\mathcal{A}_{\frac{M^*}{2},\frac{R_1}{2}}} \left((u-\frac{M^*}{2})^+\right)^{\gamma^*} dx = 0,$$

leading to

$$u(x) \le \frac{M^*}{2}, \quad \text{a.e. in } B_{\frac{R_1}{2}}(x_0).$$

Since $x_0$ is arbitrary and $\Lambda$ is a compact subset, the last inequality ensures that

$$u(x) \le \frac{\Pi}{2} \quad \text{a.e. in } \Lambda$$

for some constant $\Pi > 0$. By the arbitrariness of $\Lambda$, we conclude that $u \in L^\infty_{loc}(\mathbb{R}^N)$. ■

The above results ensure that Theorem 1.1 and the first part of Theorem 1.3 are valid.

### 3.2.2 Regularity of nonnegative Solutions of $(P_1)$ for class $(V, K) \in \mathcal{K}_1$

At this point, in order to study the regularity of solutions to the problem $(P_1)$ we now require that the $\mathcal{N}$-fuction $\Phi$ satisfies the $(\phi_6)$. The following results are anchored in the regularity theory due to Lieberman [24, Theorem 1.7]. Here, we highlight that the hypothesis $(\phi_6)$ restricts the problem $(P_1)$ to the case in which $\tilde{\Phi} \in (\Delta_2)$, causing us to stray for a moment from the objective of our thesis (which is to study problems in which $\tilde{\Phi} \notin (\Delta_2)$).

**Lemma 3.14** *Under the hypotheses of Theorem 1.2 (or Theorem 1.3) if $u \in E \in L^{\infty}_{loc}(\mathbb{R}^N)$ be a nonnegative solution of $(P_1)$. Then $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$.*

**Proof.** It is enough to apply the regularity theorem due to Lieberman [24, Theorem 1.7]. And this is possible due to the condition $(\phi_6)$. ■

**Corollary 3.1** *Let $u \in E$ be a nonnegative solution of $(P_1)$. Then, $u$ is positive solution.*

**Proof.** If $\Omega \subset \mathbb{R}^N$ is a bounded domain, the Lemma 3.14 implies that $u \in C^1(\overline{\Omega})$. Using this fact, in the sequel, we fix $M_1 > \max\left\{\|\nabla u\|_{L^\infty(\overline{\Omega})}, 1\right\}$ and

$$\varphi(t) = \begin{cases} \phi(t) \quad , & \text{for } 0 < t \leq M_1 \\ \dfrac{\phi(M_1)}{M_1^{\beta-2}} t^{\beta-2} \ , & \text{for } t \geq M_1 \end{cases} ,$$

where $\beta$ is given in the hypothesis $(\phi_5)$. Still by the condition $(\phi_5)$, there are $\alpha_1, \alpha_2 > 0$ satisfying

$$\varphi(|y|)|y|^2 = \phi(|y|)|y|^2 \geq \alpha_1 |y|^\beta \ \text{ and } \ |\varphi(|y|)y| \leq \alpha_2 |y|^{\beta-1}, \ \forall y \in \mathbb{R}^N. \tag{3.57}$$

Now, consider the vector measurable functions $G_1 : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ given by $G_1(x, t, p) = \frac{1}{\alpha_1}\varphi(|p|)p$. From (3.57),

$$|G_1(x, t, p)| \leq \frac{\alpha_2}{\alpha_1}|p|^{\beta-1} \ \text{ and } \ |p|^{\beta-1} \leq pG_1(x, t, p), \tag{3.58}$$

for all $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. We next will consider the scalar measurable function $G_2 : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ given by $G_2(x,t,p) = \frac{1}{\alpha_1}\big(V(x)\phi(|t|)t - K(x)f(t)\big)$. Remember that from the inequality (3.10), there will be a constant $C_1 > 0$ satisfying

$$K(x)|f(t)| \leq C_1 V(x)\phi(|t|)|t| + C_1\phi_*(|t|)|t|, \quad \forall t \in \mathbb{R}^N \text{ and } x \in \mathbb{R}^N. \tag{3.59}$$

Fix $M \in (0,\infty)$. Through the condition $(\phi_5)$ and by a simple computation yields there exists $C_2 = C_2(M) > 0$ verifying

$$|G_2(x,t,p)| \leq C_2|t|^{\beta-1}, \text{ for every } (x,t,p) \in \Omega \times (-M,M) \times \mathbb{R}^N.$$

By the arbitrariness of $M$, we can conclude that functions $G_1$ and $G_2$ fulfill the structure required by Trudinger [61]. Also, as $u$ is a weak solution of $(P_1)$, we infer that $u$ is a quasilinear problem solution

$$-div\, G_1(x,u,\nabla u(x)) + G_2(x,u,\nabla u(x)) = 0 \text{ in } \Omega.$$

By [61, Theorem 1.1], we deduce that $u > 0$ in $\Omega$. As $\Omega$ arbitrary, we conclude that $u > 0$ in $\mathbb{R}^N$. ■

## 3.3 Existence of solution in the case $(V,K) \in \mathcal{K}_2$

To study this second class of problem where $(V,K) \in \mathcal{K}_2$, we will see some results that will be fundamental throughout this section.

**Remark 3.3** *The inequality (1.8) implies the following inequalities*

$$\xi_{0,A}(t)A(\rho) \leq A(\rho t) \leq \xi_{1,A}(t)A(\rho), \quad \forall \rho, t \geq 0$$

*when*

$$\xi_{0,A}(t) = \min\{t^{a_1}, t^{a_2}\} \quad \textit{and} \quad \xi_{1,A}(t) = \max\{t^{a_1}, t^{a_2}\}, \quad \forall t \geq 0.$$

*Besides by Lemma 2.16 and Lemma 2.20, we have*

$$\lim_{t\to 0} \frac{A(t)}{\Phi(t)} = 0 \quad \textit{and} \quad \lim_{|t|\to\infty} \frac{A(t)}{\Phi_*(t)} = 0.$$

**Proposition 3.4** *(Hardy Type Inequality) If $(V,K) \in \mathcal{K}_2$, then $E$ is compactly embedded in $L^A_K(\mathbb{R}^N)$ where $A$ is given in $(K_3)$.*

**Proof.** As $E$ is continuously embedded in $L^{\Phi_*}(\mathbb{R}^N)$, there exists $C_1 > 0$ such that

$$\|u\|_{\Phi_*} \leq C_1\|u\|_E, \quad \forall u \in E. \tag{3.60}$$

By $(K_3)$ given $\varepsilon > 0$ there is $r > 0$ large enough such that

$$K(x)A(t) \leq \varepsilon(V(x)\Phi(|t|) + \Phi_*(|t|)), \quad \forall t > 0 \text{ and } |x| \geq r. \tag{3.61}$$

On the other hand, by the Remark 3.3, there is a constant $C_2 > 0$ such that

$$A(t) \leq C_2\Phi(t) + C_2\Phi_*(t), \quad \forall t > 0.$$

Hence, for each $x \in B_r(0)$,

$$K(x)A(t) \leq C_2 \left\|\frac{K}{V}\right\|_{L^\infty(B_r(0))} V(x)\Phi(t) + C_2\|K\|_\infty \Phi_*(t), \ \forall t > 0. \tag{3.62}$$

Combining (3.61) and (3.62),

$$K(x)A(t) \leq C_3V(x)\Phi(t) + C_3\Phi_*(t), \quad \forall t > 0 \text{ and } x \in \mathbb{R}^N, \tag{3.63}$$

with $C_3 = \max\{1, C_2\|K\|_\infty, C_2 \left\|\frac{K}{V}\right\|_{L^\infty(B_r(0))}\}$. By the inequalities (3.60) and (3.63), we get

$$\int_{\mathbb{R}^N} K(x)A\left(\frac{|u|}{C_3\|u\|_E + C_1\|u\|_E}\right) dx \leq C_4,$$

where $C_4$ is a positive constant that does not depend on $u$. So we can conclude that $E \subset L^A_K(\mathbb{R}^N)$.

Now, consider $(v_n)$ a bounded sequence in $E$. To see that the operator $i : E \to L^A_K(\mathbb{R}^N)$ is compact just prove that $(v_n)$ has a convergent subsequence on $L^A_K(\mathbb{R}^N)$. Since $(v_n)$ is bounded in $E$, we have that $(v_n)$ is bounded in $D^{1,\Phi}(\mathbb{R}^N)$, so there is $u \in E$ such that $v_n \overset{*}{\rightharpoonup} v$ in $D^{1,\Phi}(\mathbb{R}^N)$, or equivalently $w_n \overset{*}{\rightharpoonup} 0$ in $D^{1,\Phi}(\mathbb{R}^N)$, where $w_n = v_n - v$. By the limitation of $(v_n)$ in $E$ and $\Phi, \Phi_* \in (\Delta_2)$, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)\Phi(|w_n|)dx \leq M_1 \text{ and } \int_{\mathbb{R}^N} \Phi_*(|w_n|)dx \leq M_1, \quad \forall n \in \mathbb{N}. \tag{3.64}$$

Thus, by (3.61) and (3.64), we obtain

$$\int_{B_r(0)^c} K(x)A(|w_n|)dx \leq \varepsilon M_1, \quad \forall n \in \mathbb{N}. \tag{3.65}$$

Again, we use the Corollary 2.2 and the fact that $E$ is compactly embedded in $L^{\Phi}_{loc}(\mathbb{R}^N)$ to ensure the existence of a subsequence of $(v_n)$, still denoted by itself, such that

$$v_n \to v \ \text{ in } \ L^{\Phi}(B_r(0)).$$

Thus, there is a subsequence of $(v_n)$, still denoted by itself, that such

$$v_n(x) \to v(x) \ \ a.e. \ \text{ in } \ B_r(0),$$

that is,

$$w_n(x) \to 0 \ \ a.e. \ \text{ in } \ B_r(0).$$

Consider the functions $P_1 : \mathbb{R} \to \mathbb{R}$ and $P_2 : \mathbb{R} \to \mathbb{R}$ given by

$$P_1(t) = A(t) \quad \text{and} \quad P_2(t) = \Phi_*(t).$$

Clearly $P_1$ and $P_2$ are continuous, moreover

$$\lim_{|t|\to+\infty} \frac{P_1(t)}{P_2(t)} = 0.$$

Finally, it follows from the boundedness of $(v_n)$ in $E$ that there is $C_1 > 0$, such that

$$\int_{\mathbb{R}^N} P_2(w_n)dx \le \int_{\mathbb{R}^N} \Phi_*(w_n)dx < C_1, \ \ \forall n \in \mathbb{N}.$$

Then, by compactness Lemma of Strauss [26, Theorem A.I, p. 338],

$$\int_{B_r(0)} P_1(w_n)dx \to 0.$$

Therefore,

$$\lim_{n\to\infty} \int_{B_r(0)} K(x)A(|w_n|)dx = 0. \tag{3.66}$$

By (3.65) e (3.66), we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} K(x)A(|w_n|)dx \le \varepsilon(CM_1 + 1).$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)A(|w_n|)dx = 0.$$

Since $A \in (\Delta_2)$, we can conclude that

$$v_n \to v \text{ in } L_K^A(\mathbb{R}^N),$$

which completes the proof. ■

Note that the condition $(f_6)$ implies that $\lim\limits_{t\to+\infty}\dfrac{f(t)}{\phi_*(t)t} = 0$. Then, by the conditions $(f_5)$ or $(f_6)$, given $\varepsilon > 0$ there exists $\delta_0 > 0$, $\delta_1 > 0$ and $C_\varepsilon > 0$ such that

$$K(x)|f(t)| \leq \varepsilon K(x)a(t)t + \varepsilon\|K\|_\infty\phi_*(t)t + C_\varepsilon K(x)\phi_*(t)t\chi_{[\delta_0,\delta_1]}(t), \tag{3.67}$$

for all $t \geq 0$ and $x \in \mathbb{R}^N$. This inequality together with the Proposition 3.4 yields that the functional $\mathcal{F} : E \to \mathbb{R}$, given by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} K(x)F(u)dx \tag{3.68}$$

is well defined and belongs to $C^1(E, \mathbb{R})$ with derivative

$$\mathcal{F}'(u)v = \int_{\mathbb{R}^N} K(x)f(u)vdx, \quad \forall u, v \in E.$$

Therefore, we can conclude that the energy functional $J : E \to \mathbb{R}$ associated to problem $(P_1)$, which is given by

$$J(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - \int_{\mathbb{R}^N} K(x)F(u)dx$$

is a continuous and Gateaux-differentiable functional such that $J' : E \to E^*$ given by

$$J'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u\nabla vdx + \int_{\mathbb{R}^N} V(x)\phi(|u|)uvdx - \int_{\mathbb{R}^N} K(x)f(u)vdx$$

is continuous from the norm topology of $E$ to the weak$^*$-topology of $E^*$. From (3.67) and $(f_3)$, it follows that $J$ satisfies the geometry of the mountain pass. Hence, there is a Cerami sequence $(u_n) \subset E$, such that,

$$J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|J'(u_n)\|_* \to 0 \tag{3.69}$$

where $c$ is the mountain pass level given by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0, 1], X) : \ \gamma(0) = 0 \ \text{ and } \ J(\gamma(1)) \leq 0\}.$$

As in the previous section, the above sequence is obtained from the Corollary A.1 in Appendix A.

In order to show that the Cerami sequence obtained in (3.69) is bounded, we present the following result.

**Lemma 3.15** *Let $(v_n)$ be a bounded sequence in $E$ such that $v_n \overset{*}{\rightharpoonup} v$ in $D^{1,\Phi}(\mathbb{R}^N)$. Suppose that $f$ satisfies $(f_5)$ or $(f_6)$, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)F(v_n)dx = \int_{\mathbb{R}^N} K(x)F(v)dx, \tag{3.70}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)f(v_n)v_n dx = \int_{\mathbb{R}^N} K(x)f(v)v dx \tag{3.71}$$

*and*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)f(v_n)\psi dx = \int_{\mathbb{R}^N} K(x)f(v)\psi dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N). \tag{3.72}$$

**Proof.** As in (3.67), given $\varepsilon > 0$, there exists $\delta_0 > 0$, $\delta_1 > 0$ $C_1 > 0$ and $C_\varepsilon > 0$ such that

$$K(x)|f(t)| \leq C_1K(x)a(t)t + \varepsilon\|K\|_\infty\phi_*(t)t + C_\varepsilon K(x)\phi_*(t)t\chi_{[\delta_0,\delta_1]}(t), \tag{3.73}$$

for all $t \geq 0$ and $x \in \mathbb{R}^N$. By the condition $(K_3)$, there is $r_0 > 0$ sufficiently large satisfying

$$K(x)A(t) \leq \varepsilon\left(V(x)\Phi(t) + \Phi_*(t)\right), \quad \forall t > 0 \;\text{ and }\; |x| \geq r_0.$$

From the above inequalities, we have

$$K(x)F(t) \leq \varepsilon C_1V(x)\Phi(t) + \varepsilon C_2\Phi_*(t) + C_\varepsilon K(x)\Phi_*(\delta_1)\chi_{[\delta_0,\delta_1]}(t), \tag{3.74}$$

for all $t > 0$ and $|x| \geq r_0$. Repeating the same arguments used in the proof of Proposition 2.1, it follows that

$$\limsup_{n\to+\infty}\int_{B^c_{r_0}(0)} K(x)F(v_n)dx \leq \varepsilon C_3, \tag{3.75}$$

for some constant $C_3 > 0$ that does not depend on $n$ and $\varepsilon$. On the other hand, the compactness lemma of Strauss [26, Theorem A.I, p. 338], guarantees that

$$\lim_{n\to+\infty}\int_{B_{r_0}(0)} K(x)F(v_n)dx = \int_{B_{r_0}(0)} K(x)F(v)dx. \tag{3.76}$$

In light of this, we can conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)F(v_n)dx = \int_{\mathbb{R}^n} K(x)F(v)dx.$$

In the same way, we can get the limit (3.71). Related the limit (3.72), it follows directly from the condition $(f_5)$ or $(f_6)$ together with a version of the compactness lemma of Strauss for non-autonomous problem.(This version is an immediate consequence of [26, Theorem A.I, p. 338] where $K(x)dx$ is used as the new measure) ■

Repeating the same arguments used in the proof of Lemma 3.11 and of Proposition 3.3, it follows that the Cerami sequence $(u_n)$ given in (3.69) is bounded, up to some subsequence, we can assume that there is $u \in E$ such that

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } D^{1,\Phi}(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \to u(x) \;\; a.e. \;\; \mathbb{R}^N. \tag{3.77}$$

As in the previous section, we can conclude that $u \in E$ is a nonnegative solution for the problem $(P_1)$. By repeating the arguments presented in Subsections 3.2.1 and 3.2.2 we can guarantee the boundedness and regularity of the nonnegative solutions of $(P_1)$ for the case $(V, K) \in \mathcal{K}_2$, thereby proving the Theorems 1.4 and 1.5.

# Chapter 4

# A Generalized Choquard equation with weighted Stein-Weiss potential on a nonreflexive Orlicz-Sobolev Spaces

Continuing the study of the existence of positive solutions for a class of quasilinear Schrodinger equations with a potential vanishing at infinity on nonreflexive Orlicz-Sobolev spaces, in this chapter, we study the problem with an Stein-Weiss convolution term of the type:

$$
(P_2) \qquad \begin{cases} -\Delta_\Phi u + V(x)\phi(|u|)u = \dfrac{1}{|x|^\alpha}\left(\displaystyle\int_{\mathbb{R}^N} \dfrac{K(y)F(u(y))}{|x-y|^\lambda |y|^\alpha}dy\right)K(x)f(u(x)), \ x \in \mathbb{R}^N \\ u \in D^{1,\Phi}(\mathbb{R}^N) \end{cases}
$$

where $\alpha \geq 0$, $N \geq 2$, $\lambda > 0$, $V, K \in C(\mathbb{R}^N, [0,\infty))$ are nonnegative functions that may vanish to infinity, $F(t) = \int_0^t f(s)ds$ where the function $f \in C(\mathbb{R}, \mathbb{R})$ is quasicritical and $\tilde{\Phi} \notin (\Delta_2)$. Our main goal in this chapter is to study to prove the sequence of Theorems 1.6, 1.7 and 1.8. As in the Chapter 3, we assume that $\phi : (0,\infty) \to (0,\infty)$ is a $C^1$ function satisfying the conditions $(\phi_1) - (\phi_4)$(mentioned in the introduction), furthermore, we will assume that $0 \leq \alpha < \lambda$, $\lambda + 2\alpha \in (0, N) \cap (0, 2N - \frac{2N}{m})$ and consider the constant $\theta = \dfrac{2N}{2N-2\alpha-\lambda} > 0$ and observe that

$$1-\frac{1}{\theta}-\frac{\lambda}{N}<\frac{\theta}{N}<1-\frac{1}{\theta} \quad \text{and} \quad \frac{2}{\theta}-\frac{\lambda+2\alpha}{N}=2. \tag{4.1}$$

## 4.1 Existence of a solution in the case $(V,K)\in\mathcal{Q}_1$

Initially, we will assume the case $(V,K)\in\mathcal{Q}_1$ (mentioned in the introduction) and see some technical results that are fundamental to guarantee the existence of a non-trivial solution.

**Remark 4.1** *The inequality* (1.10) *and* (1.11) *implies the following inequalities*

$$\xi_{0,A}(t)A(\rho)\le A(\rho t)\le \xi_{1,A}(t)A(\rho), \quad \forall \rho,t\ge 0$$

$$\xi_{0,Z}(t)Z(\rho)\le Z(\rho t)\le \xi_{1,Z}(t)Z(\rho), \quad \forall \rho,t\ge 0$$

*when*

$$\xi_{0,A}(t)=\min\{t^{a_1},t^{a_2}\} \quad \textit{and} \quad \xi_{1,A}(t)=\max\{t^{a_1},t^{a_2}\}, \quad \forall t\ge 0.$$

$$\xi_{0,Z}(t)=\min\{t^{z_1},t^{z_2}\} \quad \textit{and} \quad \xi_{1,Z}(t)=\max\{t^{z_1},t^{z_2}\}, \quad \forall t\ge 0.$$

*Besides by Lemma 2.16 and Lemma 2.20, we have*

$$\limsup_{t\to 0}\frac{A(t)}{\Phi(t)}\le 1 \quad \textit{and} \quad \limsup_{|t|\to\infty}\frac{A(t)}{\Phi_*(t)}\le 1$$

$$\limsup_{t\to 0}\frac{Z(t)}{\Phi(t)}\le 1 \quad \textit{and} \quad \limsup_{|t|\to\infty}\frac{Z(t)}{\Phi_*(t)}\le 1.$$

**Lemma 4.1** *Consider $W\in L^\infty(\mathbb{R}^N)$ positive almost everywhere and $\Psi:\mathbb{R}\to\mathbb{R}$ an $\mathcal{N}$-function of the form*

$$\Psi(t)=\int_0^{|t|} s\psi(s)ds,$$

*where $\psi:(0,\infty)\to(0,\infty)$ is a $C^1$ function satisfying:*

$$t\longmapsto t\phi(t) \quad \textit{is increasing for} \quad t>0; \tag{$\psi_1$}$$

$$\lim_{t\to 0^+}t\phi(t)=0 \quad \textit{and} \quad \lim_{t\to+\infty}t\phi(t)=+\infty; \tag{$\psi_2$}$$

$(\psi_3)$ *There exist* $\tau_1, \tau_2 \in [1, N)$ *such that* $\tau_1 \leq \dfrac{\psi(t)t^2}{\Psi(t)} \leq \tau_2$ *for each* $t > 0$.

*Then the* $\mathcal{N}$*-function* $\Psi$ *satisfies*

$$\xi_{0,\Psi}(t)\Psi(\rho) \leq \Psi(\rho t) \leq \xi_{1,\Psi}(t)\Psi(\rho), \quad \forall \rho, t \geq 0 \tag{4.2}$$

*and*

$$\xi_{0,\Psi}(\|u\|_{L^{\Psi}_{W}(\mathbb{R}^N)}) \leq \int_{\Omega} W(x)\Psi(u)dx \leq \xi_{1,\Psi}(\|u\|_{L^{\Psi}_{W}(\mathbb{R}^N)}), \quad \forall u \in L^{\Psi}_{W}(\mathbb{R}^N) \tag{4.3}$$

*where*

$$\xi_{0,\Psi}(t) = \min\{t^{\tau_1}, t^{\tau_2}\} \quad \textit{and} \quad \xi_{1,\Psi}(t) = \max\{t^{\tau_1}, t^{\tau_2}\}, \quad \forall t \geq 0$$

*and*

$$L^{\Psi}_{W}(\mathbb{R}^N) = \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} W(x)\Psi\left(|u|\right) dx < +\infty \right\},$$

*is the Banach space endowed with the Luxemburg norm given by*

$$\|u\|_{L^{\Psi}_{W}(\mathbb{R}^N)} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^N} W(x)\Psi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

**Proof.** The inequality (4.2) is a consequence of Lemma 2.16. Now, Multiplying $W(x)$ on both sides of the inequality (4.2) and considering $t = \|u\|_{L^{\Psi}_{W}(\mathbb{R}^N)}$ and $\rho = \dfrac{|u|}{\|u\|_{L^{\Psi}_{W}(\mathbb{R}^N)}}$, we obtain

$$\xi_4(\|u\|_{L^{\Psi}_{W}(\mathbb{R}^N)}) \leq \int_{\Omega} W(x)\Psi(u)dx \leq \xi_5(\|u\|_{L^{\Psi}_{W}(\mathbb{R}^N)}), \quad \forall u \in L^{\Psi}_{W}(\mathbb{R}^N).$$

Showing the inequality (4.3). ■

Consider $E$ the energy space defined as in (3.1). The following immersion follows directly from the above limits and its proof will be omitted.

**Proposition 4.1** *(Hardy-type inequality) If* $(V, K) \in \mathcal{Q}_1$*, then the space* $E$ *is continuous embedded in* $L^{A}_{K^\theta}(\mathbb{R}^N)$ *and* $L^{Z}_{K^\theta}(\mathbb{R}^N)$.

To state the result below, we will define the functions $H : \mathbb{R} \to [0, \infty)$ and $P : \mathbb{R} \to [0, \infty)$ given by $H(t) = A(|t|^{1/\theta})$ and $P(t) = Z(|t|^{1/\theta})$. Through the assumptions imposed under $A$ and $Z$ it is possible to show that $H$ and $P$ are $\mathcal{N}$-functions, in addition, the functions $h : (0, \infty) \to (0, \infty)$ defined by $h(t)t = \dfrac{1}{\theta}a(t^{1/\theta})t^{(2/\theta)-1}$ and $p(t)t = \dfrac{1}{\theta}z(t^{1/\theta})t^{(2/\theta)-1}$ are increasing and satisfy

$$H(w) = \int_0^{|w|} th(t)dt, \quad \text{and} \quad P(w) = \int_0^{|w|} tp(t)dt. \tag{4.4}$$

**Lemma 4.2** *Suppose that $(V, K) \in \mathcal{Q}_1$ and $(f_1')$ holds. For each $u \in E$, there is a constant $C_1 > 0$ that does not depend on $u$, such that*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\Big| \leq C_1\left[(\max\{\|u\|_E^{a_1},\|u\|_E^{a_2}\})^{\frac{2}{\theta}}+(\max\{\|u\|_E^{z_1},\|u\|_E^{z_2}\})^{\frac{2}{\theta}}\right].$$

*Furthermore, for $u \in E$, there is a constant $C_2 > 0$, which does not depend on $u$, such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\right| \leq C_2C_u\|v\|_E, \quad \forall v \in E, \tag{4.5}$$

*where*

$$C_u := (\max\{\|u\|_E^{a_1},\|u\|_E^{a_2}\}+\max\{\|u\|_E^{z_1},\|u\|_E^{z_2}\})^{\frac{1}{\theta}}\left(\|a(|u|)|u|^{2-\theta}\|_{L_{K^\theta}^{\tilde{H}}(\mathbb{R}^N)}+\|z(|u|)|u|^{2-\theta}\|_{L_{K^\theta}^{\tilde{P}}(\mathbb{R}^N)}\right)^{\frac{1}{\theta}}$$

*and $\tilde{H}$ and $\tilde{P}$ are the complementary functions of $H$ and $P$, respectively.*

**Proof.** By $(f_1')$, there is a constant $C > 0$ such that

$$|f(t)|^\theta \leq C(a(t)t^{2-\theta}+z(t)t^{2-\theta}), \quad \forall t \in \mathbb{R}. \tag{4.6}$$

For each $t \geq 0$, we have

$$|F(t)| \leq \int_0^t |f(\tau)|d\tau \leq \left[\int_0^t |f(\tau)|^\theta d\tau\right]^{\frac{1}{\theta}}\left[\int_0^t d\tau\right]^{\frac{\theta-1}{\theta}} \leq t^{\frac{\theta-1}{\theta}}\left[\int_0^t |f(\tau)|^\theta d\tau\right]^{\frac{1}{\theta}}.$$

Thus,

$$\begin{aligned}
|F(t)|^\theta &\leq t^{\theta-1}C\left(\int_0^t \big(a(\tau)\tau^{2-\theta}+z(\tau)\tau^{2-\theta}\big)d\tau\right)\\
&\leq Ct^\theta\big(a(t)t^{2-\theta}+z(t)t^{2-\theta}\big) \qquad (\tau a(\tau) \text{ and } \tau z(\tau) \text{ are increasing in } (0,\infty))\\
&\leq C\big(A(|t|)+Z(|t|)\big), \ \ \forall t \geq 0.
\end{aligned} \tag{4.7}$$

Similarly,

$$|F(t)|^\theta \leq C\big(A(|t|)+Z(|t|)\big), \ \ \forall t \leq 0$$

Therefore,

$$|F(t)|^\theta \leq C\big(A(|t|)+Z(|t|)\big), \ \ \forall t \in \mathbb{R}$$

that is,

$$\int_{\mathbb{R}^N} K(x)^\theta|F(u)|^\theta dx \leq C\int_{\mathbb{R}^N} K(x)^\theta A(|u|)dx + C\int_{\mathbb{R}^N} K(x)^\theta Z(|u|)dx < \infty, \tag{4.8}$$

for all $u \in E$. By the Proposition 1.1 (Stein-Weiss inequality), it follows that

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\Big| &\leq C\left|\int_{\mathbb{R}^N}K(x)^\theta|F(u)|^\theta dx\right|^{\frac{2}{\theta}}\\ &\leq C\left(\int_{\mathbb{R}^N}K(x)^\theta A(|u|)dx+\int_{\mathbb{R}^N}K(x)^\theta Z(|u|)dx\right)^{\frac{2}{\theta}}\\ &\leq C\left[\left(\int_{\mathbb{R}^N}K(x)^\theta A(|u|)dx\right)^{\frac{2}{\theta}}+\left(\int_{\mathbb{R}^N}K(x)^\theta Z(|u|)dx\right)^{\frac{2}{\theta}}\right],\end{aligned}$$

for every $u \in E$ and $C > 0$ is a positive constant that does not depend on $u$. It follows from Lemma 4.1 that

$$\int_{\mathbb{R}^N}K(x)^\theta A(|u|)dx \leq \max\{\|u\|^{a_1}_{L^A_{K^\theta}(\mathbb{R}^N)}, \|u\|^{a_2}_{L^A_{K^\theta}(\mathbb{R}^N)}\}.$$

and

$$\int_{\mathbb{R}^N}K(x)^\theta Z(|u|)dx \leq \max\{\|u\|^{z_1}_{L^A_{K^\theta}(\mathbb{R}^N)}, \|u\|^{z_2}_{L^A_{K^\theta}(\mathbb{R}^N)}\}.$$

Since $E$ is continuously embedded in $L^A_{K^\theta}(\mathbb{R}^N)$ and $L^Z_{K^\theta}(\mathbb{R}^N)$, we can conclude that the previous inequalities sums up to

$$\int_{\mathbb{R}^N}K(x)^\theta A(|u|)dx \leq C\max\{\|u\|^{a_1}_E, \|u\|^{a_2}_E\}, \tag{4.9}$$

and

$$\int_{\mathbb{R}^N}K(x)^\theta Z(|u|)dx \leq C\max\{\|u\|^{z_1}_E, \|u\|^{z_2}_E\}, \tag{4.10}$$

for some constant $C > 0$. Thus,

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\Big| \leq C\left[(\max\{\|u\|^{a_1}_E, \|u\|^{a_2}_E\})^{\frac{2}{\theta}}+(\max\{\|u\|^{z_1}_E, \|u\|^{z_2}_E\})^{\frac{2}{\theta}}\right],$$

for every $u \in E$ and $C > 0$ is a positive constant that does not depend on $u$.

Now, consider $u, v \in E$, from (4.6) we have

$$\begin{aligned}\tau_{uv} = \int_{\mathbb{R}^N}K(x)^\theta|f(u)|^\theta|v|^\theta dx \leq & C\int_{\mathbb{R}^N}K(x)^\theta a(|u|)|u|^{2-\theta}|v|^\theta dx\\ &+C\int_{\mathbb{R}^N}K(x)^\theta z(|u|)|u|^{2-\theta}|v|^\theta dx.\end{aligned}$$

Consider the functions $H : \mathbb{R} \to [0, \infty)$ and $P : \mathbb{R} \to [0, \infty)$ defined in (4.4), it follows from hypotheses $(B_3)$ and $(A_3)$ that $H$ and $P$ satisfy the $\Delta_2$-condition. Knowing this, we can obtain

$$\begin{aligned}\int_{\mathbb{R}^N}K(x)^\theta\tilde{H}(a(|u|)|u|^{2-\theta})dx &= \int_{\mathbb{R}^N}K(x)^\theta\tilde{H}(\theta|u|^\theta h(|u|^\theta))dx\\ &\leq C_1\int_{\mathbb{R}^N}K(x)^\theta H(|u|^\theta)dx = C_1\int_{\mathbb{R}^N}K(x)^\theta A(|u|)dx\end{aligned}$$

and

$$\int_{\mathbb{R}^N} K(x)^\theta \tilde{P}(z(|u|)|u|^{2-\theta})dx = \int_{\mathbb{R}^N} K(x)^\theta \tilde{P}(\theta|u|^\theta p(|u|^\theta))dx$$
$$\leq C_1 \int_{\mathbb{R}^N} K(x)^\theta P(|u|^\theta)dx = C_1 \int_{\mathbb{R}^N} K(x)^\theta Z(|u|)dx.$$

By (4.9) and (4.10), we have that

$$\int_{\mathbb{R}^N} K(x)^\theta \tilde{H}(a(|u|)|u|^{2-\theta})dx \leq C_2 \max\{\|u\|_E^{a_1}, \|u\|_E^{a_2}\},$$

and

$$\int_{\mathbb{R}^N} K(x)^\theta \tilde{P}(z(|u|)|u|^{2-\theta})dx \leq C_2 \max\{\|u\|_E^{a_1}, \|u\|_E^{a_2}\}.$$

Moreover, we have

$$\int_{\mathbb{R}^N} K(x)^\theta H\left(\frac{|v|^\theta}{|v|^\theta_{L^A_{Q^\theta}(\mathbb{R}^N)}}\right)dx = \int_{\mathbb{R}^N} K(x)^\theta A\left(\frac{|v|}{|v|_{L^A_{Q^\theta}(\mathbb{R}^N)}}\right)dx = 1$$

and

$$\int_{\mathbb{R}^N} K(x)^\theta P\left(\frac{|v|^\theta}{|v|L^Z_{K^\theta}(\mathbb{R}^N)^\theta}\right)dx = \int_{\mathbb{R}^N} K(x)^\theta Z\left(\frac{|v|}{|v|_{L^Z_{K^\theta}(\mathbb{R}^N)}}\right)dx = 1.$$

With this, we conclude that $a(|u|)|u|^{2-\theta} \in L^{\tilde{H}}_{K^\theta}(\mathbb{R}^N)$, $z(|u|)|u|^{2-\theta} \in L^{\tilde{P}}_{K^\theta}(\mathbb{R}^N)$, $|v|^\theta \in L^H_{K^\theta}(\mathbb{R}^N)$ and $|v|^\theta \in L^P_{K^\theta}(\mathbb{R}^N)$. Furthermore,

$$\||v|^\theta\|_{L^H_{K^\theta}(\mathbb{R}^N)} = \|v\|^\theta_{L^A_{K^\theta}(\mathbb{R}^N)} \leq C_3\|v\|^\theta_E$$

and

$$\||v|^\theta\|_{L^P_{K^\theta}(\mathbb{R}^N)} = \|v\|^\theta_{L^Z_{K^\theta}(\mathbb{R}^N)} \leq C_3\|v\|^\theta_E,$$

where $C_i > 0$, $i = 1, 2$, are positives constants that does not depend on $v$. By Proposition 4.1, it follows that

$$\begin{aligned} \tau_{uv} \leq & C_4\|a(|u|)|u|^{2-\theta}\|_{L^{\tilde{H}}_{K^\theta}(\mathbb{R}^N)}\||v|^\theta\|_{L^H_{K^\theta}(\mathbb{R}^N)} + C_4\|Z(|u|)|u|^{2-\theta}\|_{L^{\tilde{P}}_{K^\theta}(\mathbb{R}^N)}\||v|^\theta\|_{L^P_{K^\theta}(\mathbb{R}^N)} \\ \leq & C_5(\|a(|u|)|u|^{2-\theta}\|_{L^{\tilde{H}}_{K^\theta}(\mathbb{R}^N)}\|v\|^\theta_E + \|z(|u|)|u|^{2-\theta}\|_{L^{\tilde{P}}_{K^\theta}(\mathbb{R}^N)}\|v\|^\theta_E), \end{aligned} \tag{4.11}$$

where $C_i > 0$, $i = 3, 4$, are positives constants that does not depend on $u$ and $v$. From (4.11), (4.8) together with the Proposition 1.1 (Stein-Weiss inequality), it follows that

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\right| \leq C\left|\int_{\mathbb{R}^N} K(x)^\theta|F(u)|^\theta dx\right|^{\frac{1}{\theta}} \left|\int_{\mathbb{R}^N} K(x)^\theta|f(u)|^\theta|v|^\theta dx\right|^{\frac{1}{\theta}}$$
$$\leq C_6\left(\max\{\|u\|_E^{a_1}, \|u\|_E^{a_2}\} + \max\{\|u\|_E^{z_1}, \|u\|_E^{z_2}\}\right)^{\frac{1}{\theta}} \left(\|a(|u|)|u|^{2-\theta}\|_{L^{\tilde{H}}_{K^\theta}(\mathbb{R}^N)} + \|z(|u|)|u|^{2-\theta}\|_{L^{\tilde{P}}_{K^\theta}(\mathbb{R}^N)}\right)^{\frac{1}{\theta}}\|v\|_E.$$

■

**Lemma 4.3** *Assume that $(V,K) \in \mathcal{Q}_1$ and $(f_1')$ holds. Let $(u_n)$ be a bounded sequence in $E$, and consider $u \in E$ such that $u_n \xrightarrow{*} u$ in $E$. We will show the following limits*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)^\theta |F(u_n) - F(u)|^\theta dx = 0, \tag{4.12}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)^\theta |f(u_n)u_n - f(u)u|^\theta dx = 0 \tag{4.13}$$

*and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x)^\theta |f(u_n)\varphi - f(u)\varphi|^\theta dx = 0. \tag{4.14}$$

**Proof.** By remark 4.1, $\limsup_{t\to 0} \dfrac{a(t)t^{2-\theta}}{\phi(t)t^{2-\theta}} \leq 1$ and $\limsup_{t\to\infty} \dfrac{z(t)t^{2-\theta}}{\phi_*(t)t^{2-\theta}} \leq 1$, then from $(f_1')$, given $\varepsilon > 0$ there exist $\delta_0 > 0$, $\delta_1 > 0$ and $C_\varepsilon > 0$ such that

$$|f(t)|^\theta \leq \varepsilon\Big(\phi(t)t^{2-\theta} + \phi_*(t)t^{2-\theta}\Big) + C_\varepsilon \phi_*(t)t^{2-\theta}\chi_{[\delta_0,\delta_1]}(t). \tag{4.15}$$

In the same way as (4.7),

$$|F(t)|^\theta \leq \varepsilon\Big(\frac{a_2}{\theta}\Phi(t) + \frac{m^*}{\theta}\Phi_*(t)\Big) + \frac{C_\varepsilon m^*}{\theta}\Phi_*(t)\chi_{[\delta_0,\delta_1]}(t) \tag{4.16}$$

From (4.16), Proposition 4.1 and the Sobolev inequality, it follows that the sequence $(K(\cdot)F(u_n))$ is bounded in $L^\theta(\mathbb{R}^N)$. It is clear that $K(x)F(u_n(x)) \to K(x)F(u(x))$ a.e. in $\mathbb{R}^N$ in the sense of subsequence. Then, by the Brézis-Lieb Lemma [56, Lemma 1.32] we obtain

$$\int_{\mathbb{R}^N} K(x)^\theta |F(u_n) - F(u)|^\theta dx = \int_{\mathbb{R}^N} K(x)^\theta |F(u_n)|^\theta dx - \int_{\mathbb{R}^N} K(x)^\theta |F(u)|^\theta dx + o_n(1). \tag{4.17}$$

In view of this fact, to verify (4.12), we only need to prove that the right side of (4.17) is a quantity $o_n(1)$. Note that $F_n = \{x \in \mathbb{R}^N : |v_n(x)| \geq \delta_0\}$ is such that

$$\Phi_*(\delta_0)|F_n| \leq \int_{F_n} \Phi_*(|v_n(x)|)dx \leq \int_{\mathbb{R}^N} \Phi_*(|v_n(x)|)dx \leq C_1,$$

for some constant $C_1 > 0$ that does not depend on $n$. Thus, $\sup_{n\in\mathbb{N}} |F_n| < +\infty$. From $(Q_1)$, we have

$$\lim_{r\to+\infty} \int_{F_n \cap B_r^c(0)} K(x)^\theta dx = 0, \quad \text{uniformly in } n \in \mathbb{N},$$

thus, there is $r_0 > 0$, so that

$$\int_{F_n \cap B^c_{r_0}(0)} K(x)^\theta dx < \frac{\varepsilon}{\Phi_*(\delta_1) C_\varepsilon}, \quad \forall n \in \mathbb{N}.$$

Moreover, as $(v_n)$ is bounded in $E$, there is a constant $M_1 > 0$ satisfying

$$\int_{\mathbb{R}^N} V(x)\Phi(|u_n|)dx \le M_1 \quad \text{and} \quad \int_{\mathbb{R}^N} \Phi_*(|u_n|)dx \le M_1, \quad \forall n \in \mathbb{N}.$$

By (4.16), it follows that

$$\begin{aligned}\int_{B^c_{r_0}(0)} K(x)^\theta |F(u_n)|^\theta dx \le & \varepsilon C_1 \left( \int_{B^c_{r_0}(0)} V(x)\Phi(|u_n|)dx + \int_{B^c_{r_0}(0)} \Phi_*(|u_n|)dx \right) \\ & + C_\varepsilon \Phi_*(\delta_1) \int_{F_n \cap B^c_{r_0}(0)} K(x)^\theta dx \\ \le & \varepsilon(C_1 M_1 + 1),\end{aligned}$$

for all $n \in \mathbb{N}$ where $C_1 > 0$ does not depend of $\varepsilon > 0$. Therefore

$$\limsup_{n\to+\infty} \int_{B^c_{r_0}(0)} K(x)^\theta |F(u_n)|^\theta dx \le \varepsilon(C_1 M_1 + 1). \tag{4.18}$$

On the other hand, using the compactness Lemma of Strauss [26, Theorem A.I, p. 338], it follows that

$$\lim_{n\to+\infty} \int_{B_{r_0}(0)} K(x)^\theta |F(u_n)|^\theta dx = \int_{B_{r_0}(0)} K(x)^\theta |F(u)|^\theta dx. \tag{4.19}$$

In light of this, we can conclude that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)^\theta |F(u_n)|^\theta dx = \int_{\mathbb{R}^n} K(x)^\theta |F(u)|^\theta dx.$$

Through this limit together with (4.17), we will get (4.12). Similarly, the limit (4.13) is shown. Related the limit (4.14), it follows directly from the condition $(f_1')$ together with a version of the compactness Lemma of Strauss for non-autonomous problem.(This version is an immediate consequence of [26, Theorem A.I, p. 338] where $K(x)dx$ is used as the new measure) ■

The following result is an immediate consequence of Stein-Weiss inequality and Lemma 4.3.

**Lemma 4.4** *Assume that $(V, K) \in \mathcal{Q}_1$ and $(f_1')$ holds. Let $(u_n)$ be a sequence bounded in $E$ and $u \in E$ such that $u_n \overset{*}{\rightharpoonup} u$ in $D^{1,\Phi}(\mathbb{R}^N)$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))F(u_n(y))}{|x|^\alpha |x-y|^\lambda |y|^\alpha} dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha |x-y|^\lambda |y|^\alpha} dxdy, \tag{4.20}$$

(4.21)
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u_n(x))f(u_n(y))u_n(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))u(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy$$

*and*

(4.22)
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u_n(x))f(u_n(y))\varphi(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))\varphi(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy$$

*for all* $\varphi\in C_0^\infty(\mathbb{R}^N)$.

By the inequality (4.6), together with all the results presented above, it is verified that the function

$$\Psi(u)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,\quad u\in E$$

is well defined, is continuously differentiable and the Gateaux derivative $\Psi':E\to E^*$ is given by

$$\Psi'(u)v=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,\quad \forall u,v\in E.$$

This fact is proved similarly to Lemma 3.2, found in [13].

From the results presented in Section 3.1, we can conclude that the energy function associated with $(P_2)$ given by

$$J(u)=\int_{\mathbb{R}^N}\Phi(|\nabla u|)dx+\int_{\mathbb{R}^N}V(x)\Phi(|u|)dx-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,$$

for $u\in E$ is a continuous and Gateaux-differentiable functional such that $J':E\to E^*$ given by

$$\begin{aligned}J'(u)v=&\int_{\mathbb{R}^N}\phi(|\nabla u|)\nabla u\nabla v dx+\int_{\mathbb{R}^N}V(x)\phi(|u|)uvdx\\&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\end{aligned}$$

is continuous from the norm topology of $E$ to the weak$^*$-topology of $E^*$.

As in (3.1) that $u\in E$ is a critical point for the functional $J$ if

(4.23)
$$Q(v)-Q(u)\geq\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))(v(y)-u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,\quad\forall v\in E$$

where the functional $Q : E \mapsto \mathbb{R}$ is defined by

$$Q(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx.$$

As with Proposition 3.1, a critical point $u$ in the sense (4.23) is a weak solution for $(P_2)$, that is,

(4.24)
$$\int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)\phi(|u|)uv dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy = 0,$$

for each $v \in E$.

**Lemma 4.5** *Suppose that $(V, K) \in \mathcal{Q}_1$ and $(f_1')$ hold. Then there are $\rho, \eta > 0$ such that $J(u) \geq \eta$ for all $u \in E \cap \partial B_\rho(0)$.*

**Proof.** By Lemma 4.2, there exists a positive constant $C > 0$ satisfying

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x-y|^\lambda}dxdy\right| \leq C\left(\max\{\|u\|_E^{a_1}, \|u\|_E^{a_2}\}\right)^{\frac{2}{\theta}} + \left(\max\{\|u\|_E^{z_1}, \|u\|_E^{z_2}\}\right)^{\frac{2}{\theta}},$$

for all $u \in E$. Hence, by defining the functional $J$ together with Proposition 4.1, we get

$$\begin{aligned} J(u) \geq& \xi_0(\|\nabla u\|_\Phi) + \xi_0(\|u\|_{V,\Phi}) - C\left(\max\{\|u\|_E^{a_1}, \|u\|_E^{a_2}\}\right)^{\frac{2}{\theta}} - C\left(\max\{\|u\|_E^{z_1}, \|u\|_E^{z_2}\}\right)^{\frac{2}{\theta}} \\ \geq& \|\nabla u\|_\Phi^m + \|u\|_{V,\Phi}^m - C\big(\|u\|_E\big)^{\frac{2a_1}{\theta}} - C\big(\|u\|_E\big)^{\frac{2z_1}{\theta}}, \end{aligned}$$

for $u \in E$ with $\|u\|_E \leq 1$ where $\xi_0(t) = \min_{t>0}\{t^\ell, t^m\}$. By the classical inequality

$$(x+y)^\sigma \leq 2^{\sigma-1}(x^\sigma + y^\sigma), \quad x, y > 0, \text{ and } \sigma > 1,$$

we get for $u \in E$ with $\|u\|_E \leq 1$ that

$$J(u) \geq C\big(\|\nabla u\|_\Phi + \|u\|_{V,\Phi}\big)^m - C\big(\|u\|_E^{\frac{2a_1}{\theta}} + \|u\|_E^{\frac{2z_1}{\theta}}\big) \geq C\|u\|_E^m - C\big(\|u\|_E^{\frac{2a_1}{\theta}} + \|u\|_E^{\frac{2z_1}{\theta}}\big),$$

for some constant $C > 0$. As $\dfrac{2}{\theta} > 1$, then $m < \dfrac{2a_1}{\theta}$ and $m < \dfrac{2z_1}{\theta}$. Hence, setting $\rho = \|u\|$ small enough,

$$J(u) \geq C\|u\|_E^m - C\big(\|u\|_E^{\frac{2a_1}{\theta}} + \|u\|_E^{\frac{2z_1}{\theta}}\big) := \eta > 0.$$

Which completes the proof.

■

By a standard argument, the following lemma follows from the condition $(f_4')$.

**Lemma 4.6** *There is $e \in E$ with $\|u\|_E > \rho$ and $J(e) < 0$.*

The previous lemmas establish the mountain pass geometry for the functional $J$ in both cases. In what follows, let us denote by $c > 0$ the mountain pass level associated with $J$, that is,

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0 \ \text{ and } \ \gamma(1) = e\}.$$

Associated with $c$, we have a Cerami sequence $(u_n) \subset E$, that is,

$$J(u_n) \longrightarrow c \quad \text{ and } \quad (1 + \|u_n\|)\|J'(u_n)\|_* \longrightarrow 0. \tag{4.25}$$

The above sequence is obtained from the Corollary A.1 in Appendix A.

Now, we are able to prove that the Cerami sequence given in (4.25) is bounded in $E$.

**Lemma 4.7** *Let $(u_n)$ the Cerami sequence given in (4.25). There is a constant $M > 0$ such that $J(tu_n) \leq M$ for every $t \in [0,1]$ and $n \in \mathbb{N}$.*

**Proof.** Let $t_n \in [0,1]$ be such that $J(t_n u_n) = \max\limits_{t \in [0,1]} J(tu_n)$. If $t_n = 0$ and $t_n = 1$, we are done. Thereby, we can assume $t_n \in (0,1)$, and so $J'(t_n u_n)u_n = 0$. From this,

$$\begin{aligned}
mJ(t_n u_n) =& mJ(t_n u_n) - J'(t_n u_n)(t_n u_n) \\
=& \int_{\mathbb{R}^N} \big(m\Phi(|\nabla(t_n u_n)|) - \phi(|\nabla(t_n u_n)|)|\nabla(t_n u_n)|^2\big)dx \\
&+ \int_{\mathbb{R}^N} V(x)\big(m\Phi(|t_n u_n|) - \phi(|t_n u_n|)|t_n u_n|^2\big)dx \\
+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & \frac{K(x)K(y)\big[F(t_n u_n(x))f(t_n u_n(y))t_n u_n(y) - \frac{m}{2}F(t_n u_n(x))F(t_n u_n(y))\big]}{|x|^\alpha |x-y|^\lambda |y|^\alpha} dxdy.
\end{aligned}$$

The conditions $(f_2')$ and $(f_3')$ guarantee that the functions $f(t)t - \dfrac{m}{2}F(t)$ and $F(t)$ are nondecreasing for $t > 0$. The condition $(\phi_4)$ ensures that the function $m\Phi(t) - \phi(t)t^2$ is increasing for $t > 0$. Thus,

$$mJ(t_n u_n) \leq mJ(u_n) - J'(u_n)u_n = mJ(u_n) - o_n(1).$$

Since $(J(u_n))$ is bounded, there is $M > 0$ such that

$$J(tu_n) \leq M, \quad \forall t \in [0,1] \ \text{ and } \ n \in \mathbb{N}.$$

■

**Proposition 4.2** *The Cerami sequence $(u_n)$ given in (4.25) is bounded.*

**Proof.** Suppose by contradiction that $\|u_n\|_E \to \infty$, then we have the following cases:

*i)* $\|\nabla u_n\|_\Phi \to +\infty$ and $(\|u_n\|_{V,\Phi})$ is bounded

*ii)* $\|u_n\|_{V,\Phi} \to \infty$ and $(\|\nabla u_n\|_\Phi)$ is bounded

*iii)* $\|\nabla u_n\|_\Phi \to +\infty$ and $\|u_n\|_{V,\Phi} \to +\infty$.

In the case *iii)*, consider

$$w_n = \frac{u_n}{\|u_n\|_E}, \qquad \forall n \in \mathbb{N}.$$

Since $\|w_n\|_E = 1$, by Lemma 3.3, there exists $w \in E$ such that $w_n \xrightarrow{*} w$ in $D^{1,\Phi}(\mathbb{R}^N)$. There are two possible cases: $w = 0$ or $w \neq 0$.

*Case:* $w = 0$

Note that for every constant $\sigma > 1$ there is $n_0 \in \mathbb{N}$ such that $\dfrac{\sigma}{\|u_n\|_E} \in [0,1]$, for $n \geq n_0$. Given this, we get

$$\begin{aligned}
J(t_n u_n) \geq & J\left(\frac{\sigma}{\|\nabla u_n\|_\Phi} u_n\right) \\
= & J(\sigma w_n) \\
= & \int_{\mathbb{R}^N} \Phi(\sigma|\nabla w_n|)dx + \int_{\mathbb{R}^N} V(x)\Phi(\sigma|w_n|)dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(\sigma w_n(x))F(\sigma w_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy \\
\geq & \sigma Q(w_n) - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(\sigma w_n(x))F(\sigma w_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy
\end{aligned}$$

By definition of the sequence $(w_n)$, we have $\|\nabla w_n\|_\Phi \leq 1$ and $\|w_n\|_{V,\Phi} \leq 1$, for all $n \in \mathbb{N}$. Then,

$$\int_{\mathbb{R}^N} \Phi(|\nabla w_n|)dx \geq \|\nabla w_n\|_\Phi^m \ \text{ and } \ \int_{\mathbb{R}^N} V(x)\Phi(|w_n|)dx \geq \|w_n\|_{V,\Phi}^m.$$

So there is $C > 0$ such that

$$Q(w_n) \geq \|\nabla w_n\|_\Phi^m + \|w_n\|_{V,\Phi}^m \geq C(\|\nabla w_n\|_\Phi + \|w_n\|_{V,\Phi})^m, \quad \forall\, n \in \mathbb{N}.$$

Thus

$$\begin{aligned}J(t_nu_n) \geq& \sigma C(\|w_n\|_E)^m - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(\sigma w_n(x))F(\sigma w_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\\ =&\sigma C - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(\sigma w_n(x))F(\sigma w_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\end{aligned}$$

If $w = 0$, it follows from (4.20) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(\sigma w_n(x))F(\sigma w_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy = 0,$$

therefore,

$$\liminf_{n\to\infty} J(t_nu_n) \geq C\sigma, \quad \forall\, \sigma \geq 1.$$

which constitutes a contradiction with Lemma 4.7, once that $(J(t_nu_n))$ is bounded from above.

*Case:* $w \neq 0$

Recalling that

$$|u_n| = |w_n|\|u_n\|_E \quad \text{and} \quad \frac{u_n(x)}{\|u_n\|_E} = w_n(x) \to w(x), \ a.e. \text{ in } \mathbb{R}^N$$

we will get that

$$|w_n(x)| \to |w(x)|, \ \ a.e. \text{ in } \mathbb{R}^N.$$

Furthermore, from the fact that $\|u_n\|_E \to +\infty$, we can conclude that

$$|u_n(x)| = |w_n(x)|\|u_n\|_E \to +\infty, \ \text{ as } \ n\to\infty \ \text{ for } x \in \{y \in \mathbb{R}^N : w(y) \neq 0\}.$$

By (4.25),

$$0 = \limsup_{n\to\infty}\frac{c}{\|u_n\|_E^m} = \limsup_{n\to\infty}\frac{J(u_n)}{\|u_n\|_E^m}. \tag{4.26}$$

As $\|u_n\|_\Phi \geq 1$ and $\|u_n\|_{V,\Phi} \geq 1$ for every $n \geq n_0$,

$$\int_{\mathbb{R}^N}\Phi(|\nabla u_n|)dx \leq \|\nabla u_n\|_\Phi^m \ \text{ and } \int_{\mathbb{R}^N} V(x)\Phi(|u_n|)dx \leq \|u_n\|_{V,\Phi}^m, \ \forall n \geq n_0. \tag{4.27}$$

Thus, it follows from $(f_4')$, (4.26), (4.27) and Fatou's Lemma that

$$
\begin{aligned}
0 =& \limsup_{n\to\infty} \frac{J(u_n)}{\|u_n\|_E^m} \\
\leq& \limsup_{n\to\infty} \left[ \frac{1}{\|u_n\|_E^m} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx + \frac{1}{\|u_n\|_E^m} \int_{\mathbb{R}^N} V(x)\Phi(|u_n|)dx \right] \\
&- \liminf_{n\to\infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)K(y)}{|x|^\alpha |x-y|^\lambda |y|^\alpha} \frac{F(u_n(x))F(u_n(y))}{\|u_n\|_E^m} dxdy \right] \\
\leq& 2 - \frac{1}{2} \liminf_{n\to\infty} \left[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)K(y)}{|x|^\alpha |x-y|^\lambda |y|^\alpha} \frac{F(u_n(x))}{|u_n(x)|^{\frac{m}{2}}} |w_n(x)| \frac{F(u_n(y))}{|u_n(y)|^{\frac{m}{2}}} |w_n(y)| dxdy \right] \\
=& -\infty
\end{aligned}
$$

which is a contradiction. This shows that $(u_n)$ is bounded in $E$.

The cases $i)$ and $ii)$ are analogous to the case $iii)$.

■

Since that the Cerami sequence $(u_n)$ given in (4.25) is bounded in $E$, by Lemma 3.3, we can assume that for some subsequence, there is $u \in E$ such that

$$
u_n \overset{*}{\rightharpoonup} u \quad \text{in } D^{1,\Phi}(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \longrightarrow u(x) \;\; a.e. \;\; \mathbb{R}^N. \tag{4.28}
$$

and

$$
\liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx \geq \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx. \tag{4.29}
$$

Fix $v \in C_0^\infty(\mathbb{R}^N)$. By boundedness of Cerami sequence $(u_n)$, we have $J'(u_n)(v-u_n) = o_n(1)$, hence, since $\Phi$ is a convex function, it is possible to show that

$$
Q(v) - Q(u_n) \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))f(u_n(y))(v(y)-u_n(y))}{|x|^\alpha |x-y|^\lambda |y|^\alpha} dxdy + o_n(1). \tag{4.30}
$$

By (4.28), it follows from Fatou's Lemma that

$$
\liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)\Phi(|u_n|)dx \geq \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx. \tag{4.31}
$$

Combining (4.29) and (4.31), we conclude that

$$
\liminf_{n\to\infty} Q(u_n) \geq Q(u). \tag{4.32}
$$

From (4.30) and (4.32) together with the limits (4.20) and (4.22), we get

$$Q(v) - Q(u) \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))f(u(y))(v(y)-u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy.$$

As $E = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_E}$ and $\Phi \in (\Delta_2)$, we conclude that

(4.33)
$$Q(v) - Q(u) \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))f(u(y))(v(y)-u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy, \quad \forall\, v \in E.$$

In other words, $u$ is a critical point of the $J$ functional. By (4.24), we can conclude that $u$ is a weak solution for $(P_2)$. Now, we substitute $v = u^+ := \max\{0, u(x)\}$ in (4.33) and we get

$$-\int_{\mathbb{R}^N} \Phi(|\nabla u^-|)dx - \int_{\mathbb{R}^N} V(x)\Phi(u^-)dx \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))f(u(y))(u^-(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy = 0,$$

which leads to

$$\int_{\mathbb{R}^N} \Phi(|\nabla u^-|)dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)\Phi(u^-)dx = 0$$

whence it is readily inferred that $u^- = 0$, therefore, $u$ is a weak nonnegative solution.

Note that $u$ is nontrivial. In the sense, consider a sequence $(\varphi_k) \subset C_0^\infty(\mathbb{R}^N)$ such that $\varphi_k \to u$ in $D^{1,\Phi}(\mathbb{R}^N)$. Since $(u_n)$ is bounded, we get $J'(u_n)(\varphi_k - u_n) = o_n(1)\|\varphi_k\| - o_n(1)$. As $\Phi$ is convex, we can show that

$$I(\varphi_k) - I(u_n) - o_n(1) \geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))f(u_n(y))(\varphi_k(y)-u_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy. \tag{4.34}$$

Since $(\|\varphi_k\|)_{k\in\mathbb{N}}$ is a bounded sequence, it follows from (4.34) and from limits (4.21) and (4.22) that

$$Q(\varphi_k) - \limsup_{n\to\infty} Q(u_n) \geq \limsup_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))f(u_n(y))(\varphi_k(y)-u_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,$$

for every $k \in \mathbb{N}$. Notice that $\Phi \in (\Delta_2)$ and $\varphi_k \to u$ in $E$, we conclude from the inequality above that

$$Q(u) \geq \limsup_{n\to\infty} Q(u_n). \tag{4.35}$$

From (4.32) and (4.35),

$$Q(u) = \lim_{n\to\infty} Q(u_n). \tag{4.36}$$

By (4.20), we have

(4.37)
$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))F(u_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u_n(x))F(u_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,$$

Therefore

$$
\begin{aligned}
0 < c = & \lim_{n\to\infty} J(u_n)\\
= & \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\\
= & J(u),
\end{aligned}
$$

that is, $u \neq 0$.

Now, we prove that the solution obtained is a ground state solution. Let us recall the definition of a ground state solution:

**Definition 4.1** *A weak solution $u \in E$ of $(P_2)$ is called a ground state solution if it has the least energy, i.e., we say, the solution $u$ is ground state solution of $(P_2)$ if*

$$J(u) = b = \inf_{u\in\mathcal{S}} J(u) \tag{4.38}$$

*where $\mathcal{S}$ is the set of all critical points of the functional $J$.*

In order to prove the result below, we will use the following continuity result:

**Lemma 4.8** *The function $u \mapsto J'(u)\cdot u$ is continuous from $E$ to $\mathbb{R}$.*

The above lemma is immediate whenever $J \in C^1(E, \mathbb{R})$.

**Lemma 4.9** *Assume that $(V, K) \in \mathcal{Q}_1$ and $f$ satisfies $(f_1')-(f_4')$. For each $v \in E\backslash\{0\}$ the function $\psi_v(s) = J(sv)$ has the following properties:*

($\psi_1$) *there is a bounded closed interval $[a_v, b_v]$ (which can be degenerate) such that $0 < a_v$ and $J'(sv)\cdot v > 0$, for all $s < a_v$*

($\psi_2$) *$0 < \max_{s>0} J(sv) = J(\tau v)$, for all $\tau \in [a_v, b_v]$, $J(sv) > 0$ in $s \in (0, a_v)$*

($\psi_3$) *$J(\tau v) < \max_{s>0} J(sv)$, for all $\tau \notin [a_v, b_v]$*

($\psi_4$) *There are $s_v > b_v$ and $\delta_v > 0$ such that $J'(su)\cdot u < 0$ and $J(su) < 0$, for all $s \geq s_v$ and $u \in B_{\delta_v}(v)$.*

**Proof.** Fixed $v \neq 0$, the function $h(t) = J(tv)$ has derivative $h'(t) = J'(tv)\cdot v$. As in the Lemma 4.5, there will be $r > 0$ such that $J'(v)\cdot v > 0$, for all $0 < \|v\| \leq R$. Hence,

$$J(sv) = \int_0^s J'(tv)\cdot v dt > 0, \text{ for } 0 < s < \frac{r}{\|v\|}. \tag{4.39}$$

From $(f_4')$, there exists $s_v > 0$ such that

$$J(sv) < 0, \text{ for all } s > s_v \tag{4.40}$$

thus, $\max\limits_{s>0} J(sv) = J(\tau v) > 0$ for some $\tau \in (0, s_v)$. By definition of the $J$ functional, we have

$$\begin{aligned}\frac{J'(\tau v)v}{\tau^{m-1}} &= \int_{\mathbb{R}^N} \frac{\phi(|\nabla tv|)|\nabla v|^2}{t^{m-2}} dx + \int_{\mathbb{R}^N} V(x)\frac{\phi(|tv|)|v|^2}{t^{m-2}} dx \\ &- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(tv(x))f(tv(y))v(y)}{t^{\frac{m}{2}}|x|^\alpha|x-y|^\lambda|y|^\alpha t^{\frac{m}{2}-1}} dxdy.\end{aligned}$$

Using the hypothesis $(\phi_4)$, we can conclude that the function

$$t \mapsto \int_{\mathbb{R}^N} \frac{\phi(|\nabla tv|)|\nabla v|^2}{t^{m-2}} dx + \int_{\mathbb{R}^N} V(x)\frac{\phi(|tv|)|v|^2}{t^{m-2}} dx$$

is nonincreasing, since the hypothesis $(f_2')$ guarantees that the function

$$t \mapsto \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(tv(x))f(tv(y))v(y)}{t^{\frac{m}{2}}|x|^\alpha|x-y|^\lambda|y|^\alpha t^{\frac{m}{2}-1}} dxdy$$

is nondecreasing. Therefore, by (4.39) and (4.40) there will be an interval $[a_v, b_v]$ such that $h'(\tau) > 0$ in $t < a_v$, $h'(\tau) < 0$ in $\tau > b_v$ and $h'(\tau) = 0$ in the interval $[a_v, b_v]$. The conclusion $(\psi_1)$, $(\psi_2)$ and $(\psi_3)$ is immediate. The property $(\psi_4)$ follows from the previous items together with (4.40) and with the continuity of $J$ and $v \mapsto J'(v)v$.

■

In the proof of the lemma below, we have adapted the ideas presented by Willem, which can be found in Theorem 4.2 in [56].

**Proposition 4.3** *If $u \in E$ is a nontrivial solution for $(P_2)$ such that $J(u) = c$, where $c$ is the level given in (4.25). Then $c = \inf_{u\in\mathcal{S}} J(u)$ where $\mathcal{S}$ is the set of all critical points of the functional $J$.*

**Proof.** By condition $(f_4')$, we can fix without losing generality $e \in E$ such that $J(e) < 0$ and $J'(e) \cdot e < 0$. Consider the following sets:

$$\Gamma = \{\gamma : [0,1] \to E : \gamma(0) = 0,\ \gamma(1) = e\},\quad \Gamma_0 = \{\gamma : [0,1] \to E : \gamma(0) = 0,\ J(\gamma(1)) < 0\}$$

and

$$\mathcal{N} = \{v \in E \setminus \{0\} : J'(v) \cdot v = 0\},\quad \mathcal{S} = \{v \in E \setminus \{0\} : J'(v) = 0\}.$$

We will compare the following numbers:

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)), \quad c_0 = \inf_{\gamma\in\Gamma_o} \max_{t\in[0,1]} J(\gamma(t)), \quad d = \inf_{v\neq 0} \max_{s>0} J(sv),$$

and

$$a = \inf_{v\in\mathcal{N}} J(v), \quad b = \inf_{v\in\mathcal{S}} J(v).$$

Let us see some immediate inequalities:

$(i)$ It is obvious that $c_0 \leq c$ and $a \leq b$;

$(ii)$ Let us see $c_0 \leq d$. Note that if $v \neq 0$, then the path $\gamma(t) = ts_v v$ is such that $\gamma(0) = 0$ and $J(\gamma(1)) = J(s_v v) < 0$. Therefore, $\gamma \in \Gamma_0$ and

$$c_0 \leq \max_{t\in[0,1]} J(\gamma(t)) = \max_{t>0} J(sv),$$

that is, $c_0$ is a lower bound for the definition of $d$. The affirmation is justified.

$(iii)$ Let us show that $a \leq c$. In fact, fix $\gamma \in \Gamma$. Just check that $J'(\gamma(1))\cdot\gamma(1) = J'(e)\cdot e < 0 < J'(\gamma(t))\cdot\gamma(t)$ for $t > 0$ small enough. Having the Lemma 4.8 true, we can use the Intermediate Value Theorem to guarantee the existence of $t_1 \in (0,1)$ such that $J'(\gamma(t_1))\cdot\gamma(t_1) = 0$, so $\gamma(t_1) \in \mathcal{N}$. Thus $a \leq J(\gamma(t_1)) \leq \max_{t\in[0,1]} J(\gamma(t)) < c+\varepsilon$, and therefore the inequality $a \leq c$ is shown.

$(iv)$ Now $c \leq c_0$. Let $\gamma \in \Gamma$. The idea is to define a function $\tilde{\gamma} \in \Gamma$ such that

$$\max_{t\in[0,1]} J(\gamma(t)) = \max_{t\in[0,1]} J(\tilde{\gamma}(t)).$$

For this, define $\tilde{\gamma} : [0,1] \to E$ as follows:

$$\tilde{\gamma}(t) = \gamma(2t), \text{ for } t \in [0, \frac{1}{2}].$$

It remains to define the function $\tilde{\gamma}$ to values $t \in \left[\frac{1}{2}, 1\right]$. Remember that $J(e) < 0$ and $J(\tilde{\gamma}(\frac{1}{2})) = J(\gamma(1)) < 0$. Being $\gamma(1) = v_1$, consider any point $u$ of the segment $[e, v_1]$. We cover this compact segment with a finite number of balls $B_{\delta_u}(u)$ obtained through the property $(\psi_4)$, that is, $[e, v_1] \subset B_{\delta_{u_1}}(u_1) \cup B_{\delta_{u_2}}(u_2) \cup \cdots \cup B_{\delta_{u_n}}(u_n)$. Consider $\lambda = \max\{s_{u_1}, s_{u_2}, \cdots, s_{u_2}\}$, numbers given by (4.40) and $(\psi_4)$. Set the $\tilde{\gamma} : \left[\frac{1}{2}, 1\right] \to E$ the polygonal line segment from $v_1$ going to $\lambda v_1$, then connecting $\lambda v_1$ to $\lambda e$ and finally, connecting $\lambda e$ to the point $e$. It is easily shown that $\tilde{\gamma} \in \Gamma$, $J(\tilde{\gamma}(t)) < 0$ for all $t \in \left[\frac{1}{2}, 1\right]$ and therefore, $\max_{t\in[0,1]} J(\gamma(t)) = \max_{t\in[0,1]} J(\tilde{\gamma}(t))$. Showing that $c_0 \geq c$.

$(v)$ Let us see that $d \leq a$. Fix $v \in \mathcal{N}$ so that $J(v) < a+\varepsilon$. In the proof of the Lemma 4.9 the function defined by $\psi_v(t) = J(tv)$ satisfies $\psi_v'(t) = 0$ only if $t \in [a_v, b_v]$. Consider $v \in \mathcal{N}$ and note that $\psi_v'(1) = J'(v).v = 0$, this implies that $1 \in [a_v, b_v]$. Knowing that the function $\psi_v$ reaches a maximum in the interval $[a_v, b_v]$, we will obtain $\psi_v(t) \leq \psi_v(1)$ for all $t > 0$, because $\psi_v$ is constant in $[a_v, b_v]$. In light of this,

$$d \leq \max_{s>0} J(sv) \leq J(v) < a + \varepsilon.$$

If $\varepsilon$ is arbitrary, we have $d \leq a$.

Finally, consider $u \in \mathcal{S}$ satisfying $J(u) = c$. By the inequalities above we can conclude that $a = b = c = c_o = d$.

■

### 4.1.1 Boundedness of nonnegative solutions of $(P_2)$ for the class $(V, K) \in \mathcal{Q}_1$

Assuming the assumptions of Theorem 1.6, the above argument guarantees the existence of a nonnegative ground state solution for problem $(P_2)$, thus showing the first part of Theorem 1.6. Now, to show to study the boundedness of nonnegative solutions of the problem $(P_2)$ we will make heavy use of hypothesis $2\alpha + \lambda < 2\ell$.

Now, we begin by presenting a technical result, which is an adaptation of a result that can be found in [5].

**Lemma 4.10** *Let $u \in E$ be a nonnegative solution of $(P_2)$, $x_0 \in \mathbb{R}^N$ and $R_0 > 0$. Then*

$$\int_{\mathcal{A}_{k,t}} |\nabla u|^\ell dx \leq C\left(\int_{\mathcal{A}_{k,s}} \left|\frac{u-k}{s-t}\right|^{\ell^*} dx + (k^{\ell^*}+1)|\mathcal{A}_{k,s}|\right) + C\left(\int_{\mathcal{A}_{k,s}} \left|\frac{u-k}{s-t}\right|^{\ell^*} dx + (k^{\ell^*}+1)|\mathcal{A}_{k,s}|\right)^{\frac{1}{\theta}}$$

*where $0 < t < s < R_0$, $k > 1$, $\mathcal{A}_{k,\rho} = \{x \in B_\rho(x_0) : u(x) > k\}$ and $C > 0$ is a constant that does not depend on $k$.*

**Proof:** Let $u \in E$ be a weak solution nonnegative of $(P_2)$ and $x_0 \in \mathbb{R}^N$. Moreover, fix $0 < t < s < R_0$ and $\zeta \in C_0^\infty(\mathbb{R}^N)$ verifying

$$0 \leq \zeta \leq 1, \quad supp(\zeta) \subset B_s(x_0), \quad \zeta \equiv 1 \text{ on } B_t(x_0) \text{ and } |\nabla \zeta| \leq \frac{2}{s-t}.$$

For $k > 1$, set $\varphi = \zeta^m (u-k)^+$ and

$$J = \int_{\mathcal{A}_{k,s}} \Phi(|\nabla u|)\zeta^m dx,$$

Using $\varphi$ as a test function and $\ell \Phi(t) \leq \phi(t)t^2$, we find

$$\begin{aligned}
\ell J \leq & m \int_{\mathcal{A}_{k,s}} \zeta^{m-1}(u-k)^+ \phi(|\nabla u|)|\nabla u||\nabla \zeta| dx - \int_{\mathcal{A}_{k,s}} V(x)\phi(u)u\zeta^m (u-k)^+ dx \\
& + \int_{\mathcal{A}_{k,s}} \int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))f(u(y))\zeta^m(y)(u(y)-k)^+}{|x|^\alpha |x-y|^\lambda |y|^\alpha} dxdy \\
\leq & \int_{\mathcal{A}_{k,s}} \zeta^{m-1}(u-k)^+ \phi(|\nabla u|)|\nabla u||\nabla \zeta| dx \\
& + C_1(\theta,\lambda,N) \left| \int_{\mathbb{R}^N} K(x)^\theta |F(u)|^\theta dx \right|^{\frac{1}{\theta}} \left| \int_{\mathbb{R}^N} K(x)^\theta |f(u)|^\theta |\zeta^m (u-k)^+|^\theta dx \right|^{\frac{1}{\theta}}
\end{aligned}$$

By $(f_1')$, given $\eta > 0$, there exists $C_\varepsilon > 0$ such that

$$K(x)^\theta f(t)^\theta \leq \frac{\varepsilon}{\theta} K(x)^\theta a(t)t^{2-\theta} + C_\varepsilon K(x)^\theta z(t)t^{2-\theta}, \quad \forall t \geq 0 \text{ and } x \in \mathbb{R}^N.$$

Thus,

(4.41)

$$\begin{aligned}
\ell J \leq & m \int_{\mathcal{A}_{k,s}} \zeta^{m-1}(u-k)^+ \phi(|\nabla u|)|\nabla u||\nabla \zeta| dx \\
& + C_2 \left[ \int_{\mathcal{A}_{k,s}} Q^\theta(x) a(|u|)u^{2-\theta}(\zeta^m(u-k)^+)^\theta dx + \int_{\mathcal{A}_{k,s}} Q^\theta(x) z(|u|)u^{2-\theta}(\zeta^m(u-k)^+)^\theta dx \right]^{\frac{1}{\theta}}
\end{aligned}$$

where $C_2 = C_1(\theta,\lambda,N) \left( \int_{\mathbb{R}^N} Q(x)^\theta |F(u)|^\theta dx \right)^{\frac{1}{\theta}}$. For each $\tau \in (0,1)$, the Young's inequalities gives

$$\phi(|\nabla u|)|\nabla u||\nabla \zeta|\zeta^{m-1}(u-k)^+ \leq \tilde{\Phi}(\phi(|\nabla u|)|\nabla u|\zeta^{m-1}\tau) + C_3 \Phi\Big( \Big| \frac{u-k}{s-t} \Big| \Big). \tag{4.42}$$

It follows from Lemma 2.18,

$$\tilde{\Phi}(\phi(|\nabla u|)|\nabla u|\zeta^{m-1}\tau) \leq C_4 (\tau \zeta^{m-1})^{\frac{m}{m-1}} \Phi(|\nabla u|). \tag{4.43}$$

From (4.41), (4.42) and (4.43),

$$\begin{aligned}
\ell J \leq & m C_4 \tau^{\frac{m}{m-1}} \int_{\mathcal{A}_{k,s}} \Phi(|\nabla u|)\zeta^m + m C_3 \int_{\mathcal{A}_{k,s}} \Phi\Big( \Big| \frac{u-k}{s-t} \Big| \Big) dx \\
& + C_2 \left[ \int_{\mathcal{A}_{k,s}} K^\theta(x) a(|u|)u^{2-\theta}(\zeta^m(u-k)^+)^\theta dx + \int_{\mathcal{A}_{k,s}} K^\theta(x) z(|u|)u^{2-\theta}(\zeta^m(u-k)^+)^\theta dx \right]^{\frac{1}{\theta}}
\end{aligned}$$

Choosing $\tau \in (0,1)$ such that $0 < mC_4\tau^{\frac{m}{m-1}} < \ell$, we derive

$$J \leq C_5 \int_{\mathcal{A}_{k,s}} \Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big)dx + C_2\left[\int_{\mathcal{A}_{k,s}} K^\theta(x)a(|u|)u^{2-\theta}(\zeta^m(u-k)^+)^s dx + \int_{\mathcal{A}_{k,s}} K^\theta(x)z(|u|)u^{2-\theta}(\zeta^m(u-k)^+)^\theta dx\right]^{\frac{1}{\theta}} \tag{4.44}$$

By Young's inequalities,

$$z(u)u^{2-\theta}(\zeta^m(u-k)^+)^\theta \leq C_6 Z\left(\left|\frac{u-k}{s-t}\right|\right) + C_6 Z(k). \tag{4.45}$$

and

$$a(u)u^{2-\theta}(\zeta^m(u-k)^+)^\theta \leq C_6 A\left(\left|\frac{u-k}{s-t}\right|\right) + C_6 A(k). \tag{4.46}$$

Therefore, a combination of (4.44) and (4.45), yields

$$J \leq C_7 \int_{\mathcal{A}_{k,s}} \Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big)dx + C_7\left[\int_{\mathcal{A}_{k,s}} A\Big(\Big|\frac{u-k}{s-t}\Big|\Big)dx + \int_{\mathcal{A}_{k,s}} A(k)dx + \int_{\mathcal{A}_{k,s}} Z\Big(\Big|\frac{u-k}{s-t}\Big|\Big)dx + \int_{\mathcal{A}_{k,s}} Z(k)dx\right]^{\frac{1}{\theta}}. \tag{4.47}$$

Now, using that $\ell \leq m < a_2 < \ell^*$ and applying the Lemmas 2.16, 2.20 and the Remark 4.1 for functions $\Phi$, $A$ and $\Phi_*$, respectively, we get

$$\Phi\Big(\Big|\frac{u-k}{s-t}\Big|\Big) \leq \Phi(1)\Big(\Big|\frac{u-k}{s-t}\Big|^{\ell^*} + 1\Big),$$

$$A\Big(\Big|\frac{u-k}{s-t}\Big|\Big) \leq A(1)\Big(\Big|\frac{u-k}{s-t}\Big|^{\ell^*} + 1\Big) \text{ and } A(k) \leq (k^{\ell^*} + 1).$$

and

$$Z\Big(\Big|\frac{u-k}{s-t}\Big|\Big) \leq Z(1)\Big(\Big|\frac{u-k}{s-t}\Big|^{\ell^*} + 1\Big) \text{ and } Z(k) \leq (k^{\ell^*} + 1).$$

From (4.47) and the inequality above,

$$J \leq C_8\left(\int_{\mathcal{A}_{k,s}} \Big|\frac{u-k}{s-t}\Big|^{\ell^*} dx + (k^{\ell^*}+1)|\mathcal{A}_{k,s}|\right) + C_8\left(\int_{\mathcal{A}_{k,s}} \Big|\frac{u-k}{s-t}\Big|^{\ell^*} dx + (k^{\ell^*}+1)|\mathcal{A}_{k,s}|\right)^{\frac{1}{\theta}}.$$

**Lemma 4.11** *Let $u \in E$ be a nonnegative solution of $(P_2)$. Then, $u \in L^\infty_{loc}(\mathbb{R}^N)$.*

**Proof:** To begin with, consider $U$ a compact subset on $\mathbb{R}^N$. Fix $R_1 \in (0,1)$, $x_0 \in U$ and define the sequences

$$\sigma_n = \frac{R_1}{2} + \frac{R_1}{2^{n+1}} \quad \text{and} \quad \overline{\sigma}_n = \frac{\sigma_n + \sigma_{n+1}}{2} \quad \text{for } n = 0, 1, 2, \cdots .$$

Note that

$$\sigma_n \downarrow \frac{R_1}{2} \quad \text{and} \quad \sigma_{n+1} < \overline{\sigma}_n < \sigma_n < R_1.$$

Since $E$ is continuously embedded in $W^{1,\ell}_{loc}(\mathbb{R}^N)$, it follows from Lebesgue dominated convergence theorem,

$$\lim_{M \to \infty} \int_{B_{R_1}(x_0)} \left((u-P)^+\right)^{\ell^*} dx = 0, \tag{4.48}$$

hence, there is $P^* \geq 1$ which depends on $x_0$ and $R_1$, such that

$$\int_{B_{R_1}(x_0)} \left((u-P)^+\right)^{\ell^*} dx \leq 1, \quad \text{for } \ P \geq P^*. \tag{4.49}$$

Now, consider $M > 4P^*$ and for every $n \in \mathbb{N}$ define

$$K_n = \frac{M}{2}\left(1 - \frac{1}{2^{n+1}}\right) \quad \text{and} \quad J_n = \int_{\mathcal{A}_{K_n,\sigma_n}} \left((u-K_n)^+\right)^{\ell^*} dx, \quad \text{for } n = 0, 1, 2, \cdots .$$

and

$$\xi_n = \xi\left(\frac{2^{n+1}}{R_1}\left(|x - x_0| - \frac{R_1}{2}\right)\right), \quad x \in \mathbb{R}^N \text{ and } n = 0, 1, 2, \cdots ,$$

where $\xi \in C^1(\mathbb{R})$ satisfies

$$0 \leq \xi \leq 1, \quad \xi(t) = 1 \ \text{ for } \ t \leq \frac{1}{2} \quad \text{and} \quad \xi(t) = 0 \ \text{ for } \ t \geq \frac{3}{4}.$$

From definition of $\xi_n$,

$$\xi_n = 1 \ \text{ in } \ B_{\sigma_{n+1}}(x_0) \quad \text{and} \quad \xi_n = 0 \ \text{ outside } \ B_{\overline{\sigma}_n}(x_0),$$

consequently

$$\begin{aligned}
J_{n+1} &\leq \int_{B_{R_1}(x_0)} \left((u-K_{n+1})^+\xi_n\right)^{\ell^*} dx \\
&\leq C_1 \left(\int_{\mathcal{A}_{K_{n+1},\overline{\sigma}_n}} |\nabla((u-K_{+1})^+\xi_n)|^\ell dx\right)^{\frac{\ell^*}{\ell}} \\
&\leq C_2 \left(\int_{\mathcal{A}_{K_{n+1},\overline{\sigma}_n}} |\nabla u|^\ell dx + 2^{\ell n} \int_{\mathcal{A}_{K_{n+1},\overline{\sigma}_n}} ((u-K_{n+1})^+)^\ell dx\right)^{\frac{\ell^*}{\ell}},
\end{aligned} \tag{4.50}$$

for some constant $C_2 = C(N, \ell, R_1) > 0$. Applying the Lemma 4.10 to the previous inequality, we get

$$J_{n+1}^{\frac{\ell}{\ell^*}} \leq C_3\Big(\int_{\mathcal{A}_{K_{n+1},\sigma_n}} \Big|\frac{u-K_{n_k+1}}{\sigma_n-\overline{\sigma}_n}\Big|^{\ell^*} dx + (K_{n+1}^{\ell^*}+1)|\mathcal{A}_{K_{n+1},\sigma_n}| + 2^{\ell n}\int_{\mathcal{A}_{K_{n+1},\overline{\sigma}_n}} ((u-K_{n+1})^+)^\ell dx\Big)$$
$$+ C_3\Big(\int_{\mathcal{A}_{K_{n+1},\sigma_n}} \Big|\frac{u-K_{n+1}}{\sigma_n-\overline{\sigma}_n}\Big|^{\ell^*} dx + (K_{n+1}^{\ell^*}+1)|\mathcal{A}_{K_{n+1},\sigma_n}|\Big)^{\frac{1}{\theta}}, \tag{4.51}$$

where $C_3 > 0$ is a constant that depends only on $N$, $\ell$ and $R_1$. Being $|\sigma_n - \overline{\sigma}_n| = \dfrac{R_1}{2^{n+3}}$, we conclude that

$$\begin{aligned} J_{n+1}^{\frac{\ell}{\ell^*}} \leq C_4(N,\ell,R_1)\Big(2^{\ell n}\int_{\mathcal{A}_{K_{n+1},\sigma_n}} ((u-K_{n+1})^+)^{\ell^*} dx + (M^{\ell^*}+1)|\mathcal{A}_{K_{n+1},\sigma_n}| \\ + 2^{\ell n}\int_{\mathcal{A}_{K_{n+1},\overline{\sigma}_n}} ((u-K_{n+1})^+)^\ell dx\Big) \\ + C_4(N,\ell,R_1)\Big(2^{\ell n}\int_{\mathcal{A}_{K_{n+1},\sigma_n}} ((u-K_{n+1})^+)^{\ell^*} dx + (M^{\ell^*}+1)|\mathcal{A}_{K_{n+1},\sigma_n}|\Big)^{\frac{1}{\theta}}. \end{aligned} \tag{4.52}$$

Combined the inequality above with $t^\ell \leq t^{\ell^*} + 1$, for $t \geq 0$ and using that $\overline{\sigma}_n < \sigma_n$, we get that

$$\begin{aligned} J_{n+1}^{\frac{\ell}{\ell^*}} \leq C_4(N,\ell,R_1)\Big(2^{\ell n}\int_{\mathcal{A}_{K_{n+1},\sigma_{n_k}}} ((u-K_{n+1})^+)^{\ell^*} dx + (M^{\ell^*}+2^{\ell n}+1)|\mathcal{A}_{K_{n+1},\sigma_n}|\Big) \\ + C_4(N,\ell,R_1)\Big(2^{\ell n}\int_{\mathcal{A}_{K_{n+1},\sigma_n}} ((u-K_{n+1})^+)^{\ell^*} dx + (M^{\ell^*}+2^{\ell n}+1)|\mathcal{A}_{K_{n+1},\sigma_n}|\Big)^{\frac{1}{\theta}}. \end{aligned} \tag{4.53}$$

On the other hand, since $K_{n+1} - K_n = \dfrac{M}{2^{n+3}}$,

$$\begin{aligned} \left(\frac{M}{2^{n+3}}\right)^{\ell^*} \big|\mathcal{A}_{K_{n+1},\sigma_n}\big| &= (K_{n+1}-K_n)^{\ell^*}\big|\mathcal{A}_{K_{n+1},\sigma_n}\big| \\ &\leq \int_{\mathcal{A}_{K_{n+1},\sigma_n}} (K_{n+1}-K_n)^{\ell^*} dx \\ &\leq \int_{\mathcal{A}_{K_{n+1},\sigma_n}} \Phi_*((u-K_n)^+)\chi_{\mathcal{A}_{K_{n+1},\sigma_n}}(x) \leq J_n, \end{aligned} \tag{4.54}$$

which yields

$$\big|\mathcal{A}_{K_{n+1},\sigma_n}\big| \leq \frac{1}{\left(\frac{M}{2^{n+3}}\right)^{\ell^*}} J_n. \tag{4.55}$$

Thus,

$$\begin{aligned} \int_{\mathcal{A}_{K_{n+1},\sigma_n}} \big((u-K_{n+1})^+\big)^{\ell^*} dx &\leq \int_{\mathcal{A}_{K_{n+1},\sigma_n}} \big((u-K_n)^+\big)^{\ell^*} dx + \int_{\mathcal{A}_{K_{n+1},\sigma_n}} \big(K_{n+1}-K_n\big)^{\ell^*} dx \\ &\leq \int_{\mathcal{A}_{K_n,\sigma_n}} \big((u-K_n)^+\big)^{\ell^*} dx + \big|K_{n+1}-K_n\big|^{\ell^*}\big|\mathcal{A}_{K_{n+1},\sigma_n}\big| \\ &\leq 2J_n \end{aligned}$$

and consequently

$$J_{n+1}^{\frac{\ell}{\ell^*}} \leq C_5(N,\ell,R_1)\Big(2^{\ell n+1}+2^{n(\ell+\ell^*)}+(2^{\ell n}+1)2^{n(\ell+\ell^*)}\Big)J_n + C_5(N,\ell,R_1)\Big(2^{\ell n+1}+2^{n(\ell+\ell^*)}+(2^{\ell n}+1)2^{n(\ell+\ell^*)}\Big)^{\frac{1}{\theta}}J_n^{\frac{1}{\theta}} \tag{4.56}$$

Due to the fact that $M > 4M^*$, we conclude from the inequality (4.49) that

$$J_n \leq \int_{B_{R_1}(x_0)} \big((u-M^*)^+\big)^{\ell^*}dx \leq 1, \quad \text{for } n = 0,1,2,\cdots,$$

hence,

$$J_{n+1} \leq CD^n J_n^{1+\omega}, \tag{4.57}$$

where $C = 2C_5(N,\ell,R_1)$, $D = 2^{2(\ell+\ell^*)\frac{\ell^*}{\ell}}$ and $\omega = \frac{\ell^*}{\theta\ell} - 1$.

We claim that

$$J_0 \leq C^{-\frac{1}{\omega}}D^{-\frac{1}{\omega^2}}, \quad \text{for} \quad M \geq M^*.$$

Indeed, note that,

$$J_0 = \int_{\mathcal{A}_{K_0,\sigma_0}} \big((u-K_0)^+\big)^{\ell^*}dx \leq \int_{B_{R_1}(x_0)} \big((u-K_0)^+\big)^{\ell^*}dx \tag{4.58}$$

Since $E$ is continuously embedded in $W^{1,\ell}_{loc}(\mathbb{R}^N)$, it follows from Lebesgue dominated convergence theorem,

$$\lim_{M\to\infty}\int_{B_{R_1}(x_0)} \big((u-K_0)^+\big)^{\ell^*}dx = 0.$$

Therefore, there exists $M \geq 5P^*$ that depends on $x_0$, such that

$$\int_{B_{R_1}(x_0)} \big((u-K_0)^+\big)^{\ell^*}dx \leq C^{-\frac{1}{\omega}}D^{-\frac{1}{\omega^2}}, \quad \text{for} \quad M \geq M^*. \tag{4.59}$$

From (4.58) and (4.59),

$$J_0 \leq C^{-\frac{1}{\omega}}D^{-\frac{1}{\omega^2}}, \quad \text{for} \quad M \geq M^*. \tag{4.60}$$

Fix $M = M^*$, by [Lemma 4.7, 62], we deduce that

$$J_n \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand,

$$\lim_{n\to\infty} J_n = \lim_{n\to\infty}\int_{\mathcal{A}_{K_n,\sigma_n}} \big((u-K_n)^+\big)^{\ell^*}dx = \int_{\mathcal{A}_{\frac{M^*}{2},\frac{R_1}{2}}} \big((u-\frac{M^*}{2})^+\big)^{\ell^*}dx$$

Hence,

$$\int_{\mathcal{A}_{\frac{M^*}{2},\frac{R_1}{2}}} \left((u-\frac{M^*}{2})^+\right)^{\ell^*} dx = 0,$$

leading to

$$u(x) \leq \frac{M^*}{2} \text{ a.e. in } B_{\frac{R_1}{2}}(x_0).$$

Since $x_0$ is arbitrary and $U$ is a compact subset, the last inequality ensures that

$$u(x) \leq \frac{\Lambda}{2} \text{ a.e. in } \Lambda \tag{4.61}$$

for some constant $\Lambda > 0$. By the arbitrariness of $U$, we conclude that $u \in L^\infty_{loc}(\mathbb{R}^N)$.

These Lemmas guarantae that the Theorem 1.6 is valid.

### 4.1.2 Regularity of nonnegative Solutions of $(P_2)$ for the class $(V, K) \in \mathcal{Q}_1$

Let us consider the hypotheses of Theorem 1.7. By the argument presented in this chapter, we can infer from Theorem 1.6 that there exists possesses a nonnegative ground state solution, locally bounded $u \in E$, to problem $(P_2)$. Therefore, to conclude the proof of Theorem 1.7, it suffices to examine the regularity of this solution. It will be crucial here to assume that $\alpha = 0$. The regularity will be divided into the following lemmas:

**Lemma 4.12** *$u \in C^{1,\gamma}_{loc}(\mathbb{R}^N)$, for some $\gamma \in (0,1)$.*

**Proof.** By $(f_1')$ together with the Remark 4.1, there exists $C_1 > 0$ such that

$$|F(t)|^\theta \leq C_1 \Phi_*(|t|), \ \ \forall t \geq 1$$

Hence, by Hölder inequality

$$
\begin{aligned}
\int_{[u\geq 1]} \frac{K(y)F(u(y))}{|x-y|^{\lambda}}dy \leq & \left(\int_{[u\geq 1]} \frac{K(y)^{\frac{\theta}{\theta-1}}}{|x-y|^{\frac{\lambda(\theta-1)}{\theta}}}dy\right)^{\frac{\theta-1}{\theta}} \left(\int_{[u\geq 1]} F(u(y))^{\theta}dy\right)^{\frac{1}{\theta}} \\
\leq & C_1 \left(\int_{\mathbb{R}^N} \frac{K(y)^{\frac{\theta}{\theta-1}}}{|x-y|^{\frac{\lambda(\theta-1)}{\theta}}}dy\right)^{\frac{\theta-1}{\theta}} \left(\int_{\mathbb{R}^N} \Phi_*(u)dy\right)^{\frac{1}{\theta}} \\
\leq & C_1 \left(\int_{\mathbb{R}^N} \Phi_*(u)dy\right)^{\frac{1}{\theta}} \left(\int_{|x-y|<1} \frac{K(y)^{\frac{\theta}{\theta-1}}}{|x-y|^{\frac{\lambda(\theta-1)}{\theta}}}dy + \int_{|x-y|\geq 1} \frac{K(y)^{\frac{\theta}{\theta-1}}}{|x-y|^{\frac{\lambda(\theta-1)}{\theta}}}dy\right)^{\frac{\theta-1}{\theta}} \\
\leq & C_1 \left(\int_{\mathbb{R}^N} \Phi_*(u)dy\right)^{\frac{1}{\theta}} \left(\|K\|_{\infty}^{\theta}\int_{|x-y|<1} \frac{1}{|x-y|^{\frac{\lambda(\theta-1)}{\theta}}}dy + \int_{|x-y|\geq 1} K(y)^{\frac{\theta}{\theta-1}}dy\right)^{\frac{\theta-1}{\theta}} \\
\leq & C_1 \left(\int_{\mathbb{R}^N} \Phi_*(u)dy\right)^{\frac{1}{\theta}} \left(\|K\|_{\infty}^{\theta}\int_0^1 r^{N-1-\frac{\lambda(\theta-1)}{\theta}}dy + \|K\|_{L^{\frac{\theta-1}{\theta}}(\mathbb{R}^N)}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta-1}{\theta}}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_{[u\leq 1]} \frac{K(y)F(u(y))}{|x-y|^{\lambda}}dy \leq & \|F\|_{L^{\infty}([0,1])}\int_{\mathbb{R}^N} \frac{K(y)}{|x-y|^{\lambda}}dy \\
\leq & \|F\|_{L^{\infty}([0,1])}\left(\int_{|x-y|<1} \frac{K(y)}{|x-y|^{\lambda}}dy + \int_{|x-y|\geq 1} \frac{K(y)}{|x-y|^{\lambda}}dy\right) \\
\leq & \|F\|_{L^{\infty}([0,1])}\left(\|K\|_{\infty}\int_0^1 r^{N-1-\lambda}dy + \|K\|_{L^1(\mathbb{R}^N)}\right),
\end{aligned}
$$

that is,

$$
\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^{\lambda}}dy \leq C_2,
$$

where

$$
\begin{aligned}
C_2 = & \Bigg\{ C_1 \left(\int_{\mathbb{R}^N} \Phi_*(u)dy\right)^{\frac{1}{\theta}} \left(\|K\|_{\infty}^{\theta}\int_0^1 r^{N-1-\frac{\lambda(\theta-1)}{\theta}}dy + \|K\|_{L^{\frac{\theta-1}{\theta}}(\mathbb{R}^N)}^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta-1}{\theta}}; \\
& \|F\|_{L^{\infty}([0,1])}\left(\|K\|_{\infty}\int_0^1 r^{N-1-\lambda}dy + \|K\|_{L^1(\mathbb{R}^N)}\right)\Bigg\},
\end{aligned}
$$

showing that

$$
\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^{\lambda}}dy \in L^{\infty}(\mathbb{R}^N). \tag{4.62}
$$

Let $\Omega \subset \mathbb{R}^N$ be an open set and $M > 0$ the constant satisfying (4.61). Define the scalar measurable function $\mathcal{Z} : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ given by

$$
Z(x,t,p) = V(x)\varphi(|t|)t - \left(\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^{\lambda}}dy\right) K(x)\tilde{f}(t)
$$

with

$$\varphi(t) = \begin{cases} \phi(t) \ , & \text{for } 0 < t \leq M/2 \\ \phi(M/2) \ , & \text{for } t \geq M/2 \end{cases} \quad \text{and} \quad \tilde{f}(t) = \begin{cases} f(t) \ , & \text{for } 0 < t \leq M/2 \\ f(M/2) \ , & \text{for } t \geq M/2 \end{cases} ,$$

where $M > 0$ is the constant satisfying (4.61). By (4.62), there exists a constant $C_3$ such that

$$|\mathcal{Z}(x,t,p)| \leq C_3, \ \forall x \in \Omega, \ p \in \mathbb{R}^N \text{ and } t \in [0, M/2].$$

This fact together with the hypothesis $(\phi_6)$ allows us to apply the theorem of regularity due to Lieberman [24, Theorem 1.7]. Thus showing the result

∎

**Corollary 4.1** *Let $u \in E$ be a nonnegative solution of $(P_2)$. Then, $u$ is positive solution.*

**Proof:** If $\Omega \subset \mathbb{R}^N$ is a bounded domain, the Lemma 4.12 implies that $u \in C^1(\overline{\Omega})$. Using this fact, in the sequel, we fix $M_1 > \max\left\{\|\nabla u\|_{L^\infty(\overline{\Omega})}, 1\right\}$ and

$$\varphi(t) = \begin{cases} \phi(t) \ , & \text{for } 0 < t \leq M_1 \\ \dfrac{\phi(M_1)}{M_1^{\beta-2}} t^{\beta-2} \ , & \text{for } t \geq M_1 \end{cases} ,$$

where $\beta$ is given in the hypothesis $(\phi_5)$. Still by condition $(\phi_5)$, there are $\alpha_1, \alpha_2 > 0$ satisfying

$$\varphi(|y|)|y|^2 = \phi(|y|)|y|^2 \geq \alpha_1 |y|^\beta \quad \text{and} \quad |\varphi(|y|)y| \leq \alpha_2 |y|^{\beta-1}, \quad \forall y \in \mathbb{R}^N. \tag{4.63}$$

Now, consider the vector measurable functions $G : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ given by $G(x,t,p) = \frac{1}{\alpha_1}\varphi(|p|)p$. From (4.63) ,

$$|G(x,t,p)| \leq \frac{\alpha_2}{\alpha_1}|p|^{\beta-1} \quad \text{and} \quad pG(x,t,p) \geq |p|^{\beta-1}, \tag{4.64}$$

for all $(x,t,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. We next will consider the scalar measurable function $L : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ given by

$$L(x,t,p) = \frac{1}{\alpha_1}\big(V(x)\phi(|t|)t - \left(\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^\lambda}dy\right) K(x)f(t)\big).$$

By $(f_1')$, there will be a constant $C_1 > 0$ satisfying

$$K(x)|f(t)| \leq C_1 K(x) a(|t|)|t| + C_1 \phi_*(|t|)|t|, \quad \forall t \in \mathbb{R}^N \text{ and } x \in \mathbb{R}^N. \tag{4.65}$$

Fix $M \in (0, \infty)$. Through the condition $(\phi_5)$ and by a simple computation yields there exists $C_2 = C_2(M) > 0$ verifying

$$|L(x,t,p)| \leq C_2 |t|^{\beta - 1}, \text{ for every } (x,t,p) \in \Omega \times (-M, M) \times \mathbb{R}^N.$$

By the arbitrariness of $M$, we can conclude that functions $G$ and $L$ fulfill the structure required by Trudinger [61]. Also, as $u$ is a weak solution of $(P_2)$, we infer that $u$ is a quasilinear problem solution

$$-div\, G(x, u, \nabla u(x)) + L(x, u, \nabla u(x)) = 0 \text{ in } \Omega.$$

By [61, Theorem 1.1], we deduce that $u > 0$ in $\Omega$. By the arbitrariness of $\Omega$ , we conclude that $u > 0$ in $\mathbb{R}^N$.

## 4.2 Existence of a solution in the case $(V, K) \in \mathcal{Q}_2$

To study this second class of problem where $(V, K) \in \mathcal{Q}_2$, we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $(f_2')$, $(f_3')$ and $(f_4')$. Furthermore, for this case, we replace the condition $(f_1')$ with the following condition:

$$(f_5') \qquad \limsup_{t \to 0} \frac{f(t)}{\left(\frac{1}{\theta} b(|t|)|t|^{2-\theta}\right)^{1/\theta}} < \infty \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(t)}{\left(\frac{1}{\theta} \phi_*(|t|)|t|^{2-\theta}\right)^{1/\theta}} = 0$$

where $\phi_*(t)t$ is such that the Sobolev conjugate function $\Phi_*$ of $\Phi$ is its primitive, that is, $\Phi_*(t) = \displaystyle\int_0^{|t|} \phi_*(s) s ds$.

Our first main result of this subsection can be stated as follows. Under these conditions, the next result of the existence of a nonnegative solution has the following statement:

**Theorem 4.1** *Assume that $\Phi$ satisfies $(\phi_1) - (\phi_4)$, $0 \leq \alpha < \lambda$ and $\lambda + 2\alpha \in (0, N) \cap (0, 2N - \frac{2N}{m})$. Suppose that $(V, K) \in \mathcal{Q}_2$, $(B_1) - (B_4)$ and $(f_2')$, $(f_3')$, $(f_4')$, $(f_5')$ hold. If $\Phi_*(|t|^{1/\theta})$ is convex in $\mathbb{R}$, then the problem $(P_2)$ possesses a nonnegative ground state solution.*

**Remark 4.2** *The inequality* $(B_3)$ *implies the following inequalities*

$$\xi_{0,B}(t)B(\rho) \leq B(\rho t) \leq \xi_{1,B}(t)B(\rho), \quad \forall \rho, t \geq 0$$

*when*

$$\xi_{0,B}(t) = \min\{t^{b_1}, t^{b_2}\} \quad \textit{and} \quad \xi_{1,B}(t) = \max\{t^{b_1}, t^{b_2}\}, \quad \forall t \geq 0.$$

*Besides by Lemma 2.16 and Lemma 2.20, we have*

$$\limsup_{t \to 0} \frac{B(t)}{\Phi(t)} = 0 \quad \textit{and} \quad \limsup_{|t| \to \infty} \frac{B(t)}{\Phi_*(t)} = 0$$

**Proposition 4.4** *(Hardy-type inequality) If* $(V, K) \in \mathcal{Q}_2$*, then the space* $E$ *is continuous embedded in* $L^B_{K^\theta}(\mathbb{R}^N)$*.*

**Proof.** Now, let us assume that $(V, K) \in \mathcal{Q}_2$. As $E$ is continuously embedded in $L^{\Phi_*}(\mathbb{R}^N)$, there exists $C_1 > 0$ such that

$$\|u\|_{\Phi_*} \leq C_1 \|u\|_E, \quad \forall u \in E. \tag{4.66}$$

Given condition $(Q_3)$, for any $0 < \varepsilon < 1$, there exists a positive real number $r$ such that

$$\frac{K(x)^\theta}{H(x)} < \varepsilon, \quad \forall |x| \geq r,$$

where $H(x)$ is defined as $H(x) = \min_{\tau > 0} \left\{ V(x) \frac{\Phi(\tau)}{B(\tau)} + \frac{\Phi_*(\tau)}{B(\tau)} \right\}$. Consequently, we deduce that

$$K(x)^\theta B(t) \leq V(x)\Phi(t) + \Phi_*(t), \quad \forall t > 0 \text{ and } |x| \geq r. \tag{4.67}$$

On the other hand, by the Remark 4.1, there is a constant $C_2 > 0$ such that

$$B(t) \leq C_2 \Phi(t) + C_2 \Phi_*(t), \quad \forall t > 0.$$

Hence, for each $x \in B_r(0)$,

$$K(x)^\theta B(t) \leq C_2 \left\| \frac{K^\theta}{V} \right\|_{L^\infty(B_r(0))} V(x)\Phi(t) + C_2 \|K^\theta\|_\infty \Phi_*(t), \quad \forall t > 0. \tag{4.68}$$

Combining (4.67) and (4.68),

$$K(x)^\theta B(t) \leq C_3 V(x)\Phi(t) + C_3 \Phi_*(t), \quad \forall t > 0 \text{ and } x \in \mathbb{R}^N, \tag{4.69}$$

with $C_3 = \max\{1, C_2\|K^\theta\|_\infty, C_2 \left\|\frac{K^\theta}{V}\right\|_{L^\infty(B_r(0))}\}$. Applying the inequality (4.66), we obtain

$$\int_{\mathbb{R}^N} Q(x)B\left(\frac{|u|}{C_3\|u\|_E+C_1\|u\|_E}\right)dx \leq C_3\int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u|}{\|u\|_{V,\Phi}}\right)dx + C_3\int_{\mathbb{R}^N}\Phi_*\left(\frac{|u|}{\|u\|_{\Phi_*}}\right)dx \leq C_4$$

where $C_4$ is a positive constant that does not depend on $u$. So we can conclude that $E \subset L^B_{K^\theta}(\mathbb{R}^N)$. Furthermore, there is a constant $C_5 > 0$ that does not depend on $u$, so $\|u\|_{L^B_{K^\theta}(\mathbb{R}^N)} \leq C_5\|u\|_E$. Concluding that $E$ is continuous embedded in $L^B_{K^\theta}(\mathbb{R}^N)$. ■

Note that the condition $(f_5')$ implies that there exists $\delta_0 > 0$, $\delta_1 > 0$ and $C_\varepsilon > 0$ such that

$$|f(t)|^\theta \leq Cb(t)t^{2-\theta} + \frac{\varepsilon}{\theta}\phi_*(t)t^{2-\theta} + C_\varepsilon\phi_*(t)t^{2-\theta}\chi_{[\delta_0,\delta_1]}(t), \quad \forall t > 0 \tag{4.70}$$

where $C > 0$ is a constant that does not depend of $\varepsilon > 0$. Assuming that $\Phi_*(|t|^{1/\theta})$ is convex in $\mathbb{R}$, we can repeat the same arguments used in the proof of Lemma 4.2, we can state the following results.

Here we will define the functions $H_1 : \mathbb{R} \to [0,\infty)$ and $P_1 : \mathbb{R} \to [0,\infty)$ given by $H_1(t) = A(|t|^{1/\theta})$ and $P_1(t) = \Phi_*(|t|^{1/\theta})$. Through the assumptions imposed under $B$ and $\Phi_*$ it is possible to show that $H_1$ and $P_1$ are $\mathcal{N}$-functions, in addition, the functions $h_1, p_1 : (0,\infty) \to (0,\infty)$ defined by $h_1(t)t = \dfrac{1}{\theta}b(t^{1/\theta})t^{(2/\theta)-1}$ and $p_1(t)t = \dfrac{1}{\theta}\phi_*(t^{1/\theta})t^{(2/\theta)-1}$ are increasing and satisfy

$$H_1(w) = \int_0^{|w|} th_1(t)dt, \quad \text{and} \quad P_1(w) = \int_0^{|w|} tp_1(t)dt. \tag{4.71}$$

**Lemma 4.13** *Suppose that $(V,K) \in \mathcal{Q}_2$ and $(f_5')$ holds. For each $u \in E$, there is a constant $C_1 > 0$ that does not depend on $u$, such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\right| \leq C_1\left[\left(\max\{\|u\|_E^{b_1}, \|u\|_E^{b_2}\}\right)^{\frac{2}{\theta}} + \left(\max\{\|u\|_E^{\ell^*}, \|u\|_E^{m^*}\}\right)^{\frac{2}{\theta}}\right].$$

*Furthermore, for $u \in E$, there is a constant $C_2 > 0$, which does not depend on $u$, such that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy\right| \leq C_2C_u\|v\|_E, \ \ \forall v \in E, \tag{4.72}$$

*where*

$$C_u := \left(\max\{\|u\|_E^{b_1}, \|u\|_E^{b_2}\} + \max\{\|u\|_E^{\ell^*}, \|u\|_E^{m^*}\}\right)^{\frac{1}{\theta}} \left(\||b(|u|)|u|^{2-\theta}\|_{L^{\tilde{H}_1}_{K^\theta}(\mathbb{R}^N)} + \|\phi_* z(|u|)|u|^{2-\theta}\|_{L^{\tilde{P}_1}_{K^\theta}(\mathbb{R}^N)}\right)^{\frac{1}{\theta}}$$

*and $\tilde{H}_1$ and $\tilde{P}_1$ are the complementary functions of $H_1$ and $P_1$, respectively.*

**Lemma 4.14** *Suppose that $(V,K)\in\mathcal{Q}_2$ and $(f_5')$ holds. Let $(u_n)$ be a bounded sequence in $E$, and consider $u\in E$ such that $u_n \xrightharpoonup{*} u$ in $E$. We will show the following limits*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)^\theta |F(u_n)-F(u)|^\theta dx = 0, \tag{4.73}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)^\theta |f(u_n)u_n-f(u)u|^\theta dx = 0 \tag{4.74}$$

*and*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)^\theta |f(u_n)\varphi-f(u)\varphi|^\theta dx = 0. \tag{4.75}$$

**Proof.** Due to similarity, it suffices to verify (4.73). By (4.70), for any $\varepsilon>0$ there exists $\delta_0>0$, $\delta_1>0$ and $C_\varepsilon>0$ such that

$$|F(t)|^\theta \le \varepsilon\Big(\frac{b_2}{\theta}B(t)+\frac{m^*}{\theta}\Phi_*(t)\Big)+\frac{C_\varepsilon m^*}{\theta}\Phi_*(t)\chi_{[\delta_0,\delta_1]}(t). \tag{4.76}$$

By the condition $(Q_3)$, there is $r_0>0$ sufficiently large satisfying

$$K(x)B(t)\le \varepsilon\left(V(x)\Phi(t)+\Phi_*(t)\right),\quad \forall t>0 \;\text{ and }\; |x|\ge r_0.$$

From the above inequalities, we have

$$K(x)F(t)\le \varepsilon C_1 V(x)\Phi(t)+\varepsilon C_2\Phi_*(t)+C_\varepsilon Q(x)\Phi_*(\delta_1)\chi_{[\delta_0,\delta_1]}(t),$$

for all $t>0$ and $|x|\ge r_0$. From (4.76), Proposition 4.4 and the Sobolev inequality, it follows that the sequence $(K(\cdot)F(u_n))$ is bounded in $L^\theta(\mathbb{R}^N)$. It is clear that $K(x)F(u_n(x))\to K(x)F(u(x))$ a.e. in $\mathbb{R}^N$ in the sense of subsequence. Then, by the Brézis-Lieb Lemma [56, Lemma 1.32] we obtain

$$\int_{\mathbb{R}^N} K(x)^\theta|F(u_n)-F(u)|^\theta dx = \int_{\mathbb{R}^N} K(x)^\theta|F(u_n)|^\theta dx - \int_{\mathbb{R}^N} K(x)^\theta|F(u)|^\theta dx + o_n(1) \tag{4.77}$$

In view of this fact, to verify (4.73), we only need to prove that the right side of (4.77) is $o_n(1)$. Repeating the same arguments used in the proof of Lemma 4.3, it follows that

$$\limsup_{n\to+\infty}\int_{B_{r_0}^c(0)} K(x)^\theta|F(u_n)|^\theta dx \le \varepsilon(C_1M_1+1). \tag{4.78}$$

On the other hand, using $(f_5')$ and the compactness Lemma of Strauss [26, Theorem A.I, p. 338], it is guaranteed that

$$\lim_{n\to+\infty}\int_{B_{r_0}(0)} K(x)^\theta|F(u_n)|^\theta dx = \int_{B_{r_0}(0)} K(x)^\theta|F(u)|^\theta dx. \tag{4.79}$$

In light of this, we can conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)^\theta|F(u_n)|^\theta dx = \int_{\mathbb{R}^n} K(x)^\theta|F(u)|^\theta dx.$$

Through this limit together with (4.77), we will get (4.73). Similarly, we show the limit (4.74). Related the limit (4.75), it follows directly from the condition $(f_5')$ together with a version of the compactness lemma of Strauss for non-autonomous problem.

■

**Corollary 4.2** *Assume that $(V,Q)\in\mathcal{Q}_2$ and $(f_5')$ holds. Let $(u_n)$ be a sequence bounded in $E$ and $u\in E$ such that $u_n \overset{*}{\rightharpoonup} u$ in $D^{1,\Phi}(\mathbb{R}^N)$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u_n(x))F(u_n(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy, \tag{4.80}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u_n(x))f(u_n(y))u_n(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))u(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy \tag{4.81}$$

*and*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u_n(x))f(u_n(y))\varphi(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))\varphi(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy \tag{4.82}$$

*for all $\varphi\in C_0^\infty(\mathbb{R}^N)$.*

By the inequalities (4.70) and Proposition 4.4, together with all the results presented above, it is verified that the function

$$\Psi(u)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,\quad u\in E$$

is well defined, is continuously differentiable and the Gateaux derivative $\Psi':E\to E^*$ is given by

$$\Psi'(u)v=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha|x-y|^\lambda|y|^\alpha}dxdy,\quad \forall u,v\in E.$$

From the results presented in Section 3.1, we can conclude that the energy function $J : E \to \mathbb{R}$ associated with $(P_2)$ given by

$$J(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(x)\Phi(|u|)dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))F(u(y))}{|x|^\alpha |x-y|^\lambda |y|^\alpha} dxdy,$$

is continuous and Gateaux-differentiable and the Gateaux derivative $J' : E \to E^*$ given by

$$J'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)\phi(|u|)uvdx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(x)K(y)F(u(x))f(u(y))v(y)}{|x|^\alpha |x-y|^\lambda |y|^\alpha} dxdy$$

is continuous from the norm topology of $E$ to the weak$^*$-topology of $E^*$.

Now, by using Lemmas 4.5 and 4.6, the functional $J$ verify the mountain pass geometry. In what follows, let us denote by $c > 0$ the mountain pass level associated with $J$, that is,

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0 \ \text{ and } \ \gamma(1) = e\}.$$

Associated with $c$, we have a Cerami sequence $(u_n) \subset E$, that is,

$$J(u_n) \to c \quad \text{ and } \quad (1 + \|u_n\|)\|J'(u_n)\|_* \to 0. \tag{4.83}$$

The above sequence is obtained from the Corollary A.1 in Appendix A..

Repeating the same arguments used in the proof of Lemma 4.7 and of Proposition 4.2, it follows that the Cerami sequence $(u_n)$ given in (4.83) is bounded, up to some subsequence, we can assume that there is $u \in E$ such that

$$u_n \overset{*}{\rightharpoonup} u \ \text{ in } D^{1,\Phi}(\mathbb{R}^N) \ \text{ and } \ u_n(x) \to u(x) \ \ a.e. \ \mathbb{R}^N.$$

As in the previous section, we can conclude that $u$ is a nonnegative ground state solution for the problem $(P_2)$.

# Chapter 5

# Quasilinear systems on nonreflexive Orlicz-Sobolev spaces

In this chapter, we study the existence of solutions for the class of quasilinear systems in Orlicz-Sobolev spaces of the type:

$$(S) \qquad \begin{cases} -\Delta_{\Phi_1} u = F_u(x,u,v) + \lambda R_u(x,u,v) \ \text{ in } \Omega \\ -\Delta_{\Phi_2} v = -F_v(x,u,v) - \lambda R_v(x,u,v) \ \text{ in } \Omega \\ \ u = v = 0 \ \text{ on } \partial\Omega \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $\lambda > 0$ and $\Delta_{\Phi_i} u = div(\phi_i(|\nabla u|)\nabla u)$, $i = 1, 2$. Furthermore, we assume that $\Phi_i : \mathbb{R} \mapsto [0,\infty)$ are $\mathcal{N}$-functions of the type

$$(5.1) \qquad \Phi_i(t) = \int_0^{|t|} s\phi_i(s)ds, \ \ t \in \mathbb{R}$$

with $\phi_i \in C^1(0,\infty)$ and $\Phi_i \in (\Delta_2)$ or $\tilde{\Phi}_i \in (\Delta_2)$.

## 5.1 The $\mathcal{N}$-functions $\Phi_1$ and $\Phi_2$ may not verify the $\Delta_2$-condition.

In this section, we study the quasilinear system $(S)$ assuming that $\lambda = 1$ and $F = 0$ in $\Omega$, i.e., we study the quasilinear system of the type:

$$(S_1) \qquad \begin{cases} -\Delta_{\Phi_1} u = R_u(x, u, v) \ \text{ in } \Omega \\ -\Delta_{\Phi_2} v = -R_v(x, u, v) \ \text{ in } \Omega \\ \ u = v = 0 \ \text{ on } \partial\Omega \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$. To show the veracity of Theorem 1.9, we will assume that $\phi_i (i = 1, 2) \in C^1(0, +\infty)$ are two functions which satisfy:

$(\phi_{i,1})$ $\qquad t \mapsto t\phi_i(t)$ are stricly increasing and $t \mapsto t^2\phi_i(t)$ is convex in $(0, \infty)$;

$(\phi_{i,2})$ $$t\phi_i(t) \to 0 \ \text{ as } \ t \to 0 \ \text{ and } t\phi_i(t) \to +\infty \ \text{ as } \ t \to +\infty;$$

$(\phi_{i,3})$ $$1 < \ell_i \leq \frac{t^2\phi_i(t)}{\Phi_i(t)}, \ \text{ where } \ \Phi_i(t) = \int_0^{|t|} s\phi_i(s)ds, \ t \in \mathbb{R};$$

$(\phi_{i,4})$ $$\liminf_{t\to+\infty} \frac{\Phi_i(t)}{t^{q_i}} > 0, \ \text{ for some } q_i > N;$$

$(\phi_{i,5})$ $$\left| 1 - \frac{\Phi_1(t)}{t^2\phi_1(t)} \left( 1 + \frac{t\phi_1'(t)}{\phi_1(t)} \right) \right| \leq 1, \ \ \forall t > 0.$$

The hypothesis $(\phi_{i,5})$ first appears in the paper [8] and here it will be fundamental to prove that sequences $(PS)$ are bounded (See for example the Lemma 5.3). The assumption $(\phi_{i,4})$ implies that the embedding

$$W_0^{1,\Phi_i}(\Omega) \hookrightarrow W^{1,q_i}(\Omega)$$

for some $q_i > N$ is continuous. Hence,

$$W_0^{1,\Phi_i}(\Omega) \hookrightarrow C^{0,\alpha_i}(\overline{\Omega})$$

is continuous for some $\alpha_i \in (0,1)$ and

$$W_0^{1,\Phi_i}(\Omega) \hookrightarrow C(\overline{\Omega}) \tag{5.2}$$

is compact. In what follows, we denote by $\Lambda_i > 0$ the best constant that satisfies

$$\|u\|_{C(\overline{\Omega})} \leq \Lambda_i \|u\|_i, \quad \forall u \in W_0^{1,\Phi_i}(\Omega), \tag{5.3}$$

where $\|\cdot\|_i = \|\nabla \cdot\|_{L^{\Phi_i}(\Omega)}$.

Let $d$ twice the diameter of $\Omega$, then we will assume that there exists $\delta \geq 0$ such that

$$\frac{t^2}{d^2} \leq \Phi_1(t/d), \quad \forall |t| \geq \delta \tag{$\phi_{i,6}$}$$

Regarding the function $R$, let us assume that:

$(R_1')$ $\quad R \in C^1(\overline{\Omega} \times \mathbb{R}^2)$ and $R_v(x,u,0) \neq 0$ for all $(x,u) \in \Omega \times \mathbb{R}$;

$(R_2')$ $\quad R(x,u,0) \leq \frac{1}{2}\Phi_1(u/d) + \frac{1}{2d^2}|u|^2$, for all $(x,u) \in \Omega \times \mathbb{R}$;

$(R_3')$ $\quad R(x,0,v) \geq -\frac{1}{2}\Phi_2(v/d) - Mv$, for all $(x,v) \in \Omega \times \mathbb{R}$, for some constant $M > 0$;

$(R_4')$ There are $\nu > 0$, $\mu > 1$ and $0 < \beta < 1$ such that

$$\frac{1}{\mu}h(u)R_u(x,u,v)u + \frac{1}{\nu}R_v(x,u,v)v - R(x,u,v) \geq 0, \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2 \tag{$i$}$$

and

$$\beta R(x,u,v) - \frac{1}{\mu}h(u)R_u(x,u,v)u \geq 0, \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2 \tag{$ii$}$$

where $h(u) = \dfrac{\Phi_1(u)}{u^2\phi_1(u)}$.

Under the assumptions $(\phi_{i,1}) - (\phi_{i,6})$ it is well known in the literature that the $\mathcal{N}$-functions $\Phi_1$ and $\Phi_2$ might not satisfy the $\Delta_2$-condition, and as a consequence, $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$ might not be reflexive anymore (See Lemma 2.24). Another important fact we can highlight is that under these conditions, it is well known that there are $u \in W_0^{1,\Phi_1}(\Omega)$ and $v \in W_0^{1,\Phi_2}(\Omega)$ such that

$$\int_\Omega \Phi_1(|\nabla u|)dx = \infty \quad \text{and} \quad \int_\Omega \Phi_2(|\nabla v|)dx = \infty.$$

In order to avoid this problem, we will work with the space $W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega)$, because in this space the functional $Q : W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega) \longrightarrow \mathbb{R}$ given by

$$Q(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx$$

belongs to $C^1(W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega), \mathbb{R})$ (See Lemma 3.4 in [48]). However, independent of $\Delta_2$-condition, the compact embedding $W_0^{1,\Phi_i}(\Omega) \hookrightarrow C(\overline{\Omega})$ guarantees that the functional $H : W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \longrightarrow \mathbb{R}$ given by

$$H(u,v) = \int_\Omega R(x,u,v)dx$$

belongs to $C^1(W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega), \mathbb{R})$. In particular, $H|_{W_0^1E^{\Phi_1}(\Omega)\times W_0^1E^{\Phi_2}(\Omega)}$ is also of class $C^1$. That is, the energy functional $J : W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega) \longrightarrow \mathbb{R}$ associated to the system $(S_1)$ given by

$$J(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega R(x,u,v)dx$$

belongs to $C^1(W_0^1E^{\Phi_1}(\Omega) \times W_0^1E^{\Phi_2}(\Omega), \mathbb{R})$.

In order to apply the Saddle-point theorem, in the next one we fix some notations. Since $W_0^1E^{\Phi_2}(\Omega)$ is separable (See Lemma 2.7), there exists a sequence $(e_n) \subset W_0^1E^{\Phi_2}(\Omega)$ such that

$$W_0^1E^{\Phi_2}(\Omega) = \overline{span\{e_n : n \in \mathbb{N}\}}. \tag{5.4}$$

Hereafter, for each $n \in \mathbb{N}$ we denote by $V_n$, $X_n$ and $X_n'$ the following spaces

$$V_n = span\{e_j : j = 1, \cdots, n\}, \quad X_n = W_0^1E^{\Phi_1}(\Omega) \times V_n \quad \text{and} \quad X_n' = W_0^{1,\Phi_1}(\Omega) \times V_n.$$

The restriction of $J$ to $X_n$ will be denoted by $J_n$. From the regularity of $J$, it follows that $J_n$ belongs to $C^1(X_n, \mathbb{R})$ with

$$\begin{aligned} J_n'(u,v)(w_1,w_2) = &\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla w_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla w_2 dx \\ &- \int_\Omega R_u(x,u,v)w_1 dx - \int_\Omega R_v(x,u,v)w_2 dx, \end{aligned}$$

for all $(w_1, w_2) \in X_n$.

In the following, we prove that $J_n$ satisfies the hypotheses of Saddle-point theorem for Gateaux-differentiable functionals (See Theorem A.3).

**Lemma 5.1** *Under the space $Z = W_0^1 E^{\Phi_1}(\Omega) \times \{0\}$ the functional $J_n$ is bounded from below.*

**Proof.** By the condition $(R_2')$,

$$J_n(u,0) \geq \int_\Omega \Phi_1(|\nabla u|)dx - \frac{1}{2}\int_\Omega \Phi_1(|u|/d)dx - \frac{1}{2d^2}\int_\Omega |u|^2 dx. \tag{5.5}$$

Hence, using the Poincaré inequality (See Lemma (2.23)) together with the hypothesis $(\phi_{i,6})$ on the inequality (5.5), we obtain

$$J_n(u,0) \geq -\frac{1}{2d^2}\int_{[|u|\leq\delta]} |u|^2 \geq -\frac{\delta^2}{2d^2}|\Omega|, \quad \forall u \in W_0^1 E^{\Phi_1}(\Omega).$$

This finishes the proof. ∎

**Lemma 5.2** *If $\|v\|_2 \to \infty$, then $J(0,v) \to -\infty$.*

**Proof.** Let $v \in W_0^1 E^{\Phi_2}(\Omega)$ with $\|v\|_2 \geq 1$. The assumption $(R_3')$ together with the Poincaré inequality (See Lemma (2.23)) implies that

$$J(0,v) \leq -\frac{1}{2}\int_\Omega \Phi_2(|\nabla v|)dx + M\int_\Omega |v|dx. \tag{5.6}$$

From $(\phi_{i,3})$,

$$\frac{d}{ds}ln(\Phi_2(rs)) = \frac{\psi_2(rs)r^2 s}{\Phi_2(rs)} \geq \frac{\ell_2}{s}, \quad \forall s, r > 0$$

thus,

$$\int_1^t \frac{d}{ds}ln(\Phi_2(rs))ds \geq \ell_2 \int_1^t \frac{1}{s}ds, \quad \forall t \geq 1.$$

Therefore,

$$ln\frac{\Phi_2(rt)}{\Phi_2(r)} \geq ln(t^{\ell_2}), \quad \forall t \geq 1.$$

Because of the monotonicity of the logarithmic function,

$$\frac{\Phi_2(rt)}{\Phi_2(r)} \geq t^{\ell_2}, \quad \forall t \geq 1.$$

And as a consequence of this inequality, we have

$$\int_\Omega \Phi_2(|\nabla v|)dx \geq \|v\|_2^{\ell_2} \quad \text{for} \quad \|v\|_2 \geq 1. \tag{5.7}$$

By combining the inequalities (5.6) and (5.7), we conclude that

$$J(0,v) \leq -\|v\|_2^{\ell_2} + M|\Omega|\Lambda_2\|v\|_2.$$

Since $1 < \ell_2$, the result follows. ∎

**Corollary 5.1** *If* $\|v\|_2 \to \infty$, *then* $J_n(0,v) \to -\infty$.

**Corollary 5.2** *There is* $M > 0$ *such that* $\inf_Z J_n > \max_{\partial \mathcal{M}_n} J_n := b_n$ *where* $\mathcal{M}_n = B_M(0) \cap Y_n$.

**Proof.** By the above Corollary $J_n(0,v) \to -\infty$ as $\|v\|_2 \to +\infty$ in $Y$, then, fix $M > 1$ such that $J_n(0,v) < \inf_Z J_n$ for $\|v\|_2 = M$ and $v \in Y_n$. Since $dim Y_n < \infty$, we can conclude $\inf_Z J_n > \max_{\mathcal{N}_n} J_n$. ■

Then, by results above, we can apply the Saddle-point theorem (See Theorem A.3) to functional $J_n$ using the sets

$$Y_n = \{0\} \times V_n, \quad Z = W_0^1 E^{\Phi_1}(\Omega) \times \{0\}, \quad \text{and} \quad \mathcal{M}_n = B_M(0) \cap Y_n,$$

where $M > 0$ is obtained from Corollary 5.2. Thus, there exists a sequence $(u_k, v_k) \subset X_n$ with

$$J_n(u_k, v_k) \longrightarrow c_n \quad \text{and} \quad J_n'(u_k, v_k) \longrightarrow 0 \text{ as } k \to +\infty. \tag{5.8}$$

where

$$c_n = \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}_n} J_n(\gamma(u)), \tag{5.9}$$

with

$$\Gamma = \{\gamma \in C(\mathcal{M}_n, X_n) : \ \gamma|_{\mathcal{N}_n} = Id\}.$$

**Lemma 5.3** *The sequence* $(u_k, v_k)$ *satisfying* (5.8) *is bounded in* $X_n$.

**Proof.** Define the function

$$\eta(t) = \begin{cases} \dfrac{\Phi_1(t)}{t\phi_1(t)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and consider the sequence

$$g_k(x) = \eta(u_k(x)), \quad x \in \Omega.$$

A direct computation leads to

$$\nabla g_k = \left[1 - \frac{\Phi_1(u_k)}{u_k^2 \phi_1(u_k)}\left(1 + \frac{u_k \phi_1'(u_k)}{\phi_1(u_k)}\right)\right] \nabla u_k.$$

Furthermore, considering the hypothesis $(\phi_{1,5})$ and using the Lemma 2.7(item 4) we can conclude that $g_k \in W_0^1E^{\Phi_1}(\Omega)$ and $\|g_k\|_1 \leq \|u_k\|_1$ for each $k \in \mathbb{N}$. Being $(u_k, v_k)$ a sequence $(PS)_{c_n}$, then by $(R_4')(i)$ and $(\phi_{i,3})$,

$$
\begin{aligned}
& c_n + 1 + o_k(1)\|(u_k,v_k)\| \geq J_n(u_k,v_k) - J_n'(u_k,v_k)(\frac{1}{\mu}g_k, \frac{1}{\nu}v_k) \\
&= \int_\Omega \Phi(|\nabla u_k|)dx - \frac{1}{\mu}\int_\Omega \phi(|\nabla u_k|)|\nabla u_k|^2 S(u_k)dx - \int_\Omega \Psi(|\nabla v_k|)dx + \frac{1}{\nu}\int_\Omega \psi(|\nabla v_k|)|\nabla v_k|^2 dx \\
&\quad + \frac{1}{\mu}\int_\Omega R_u(x,u_k,v_k)u_k h(u_k)dx + \frac{1}{\nu}\int_\Omega R_v(x,u_k,v_k)v_k dx - \int_\Omega R(x,u_k,v_k)dx \\
&\geq \int_\Omega \Phi_1(|\nabla u_k|)dx - \frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2 S(u_k)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla v_k|)dx,
\end{aligned} \tag{5.10}
$$

where $h(t) = \frac{\Phi_1(t)}{t^2\phi_1(t)}$ and $S(t) = 1 - \frac{\Phi_1(t)}{t^2\phi_1(t)}\left(1 + \frac{t\phi_1'(t)}{\phi_1(t)}\right)$.(The functions $S$ and $h$ were introduced by Alves et al in [8]) On the other hand, it follows from $(R_4')(ii)$ that

$$
\begin{aligned}
c_n + 1 + o_k(1)\|g_k\|_1 &\geq -\beta J_n(u_k,v_k) + J_n'(u_k,v_k)(\frac{1}{\mu}g_k, 0) \\
&= -\beta\int_\Omega \Phi_1(|\nabla u_k|)dx + \frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2 S(u_k)dx + \beta\int_\Omega \Phi_2(|\nabla v_k|)dx \\
&\quad - \frac{1}{\mu}\int_\Omega R_u(x,u_k,v_k)u_k h(u_k)dx + \beta\int_\Omega R(x,u_k,v_k)dx, \\
&\geq -\beta\int_\Omega \Phi_1(|\nabla u_k|)dx + \frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2 S(u_k)dx,
\end{aligned}
$$

i.e,

$$
-\frac{1}{\mu}\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2 S(u_k)dx \geq -c_n - 1 - o_k(1)\|u_k\|_1 - \beta\int_\Omega \Phi_1(|\nabla u_k|)dx. \tag{5.11}
$$

From (5.10) and (5.11),

$$
2(c_n+1) + o_k(1)\|(u_k,v_k)\| \geq (1-\beta)\int_\Omega \Phi_1(|\nabla u_k|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla v_k|)dx. \tag{5.12}
$$

Suppose for contradiction that, up to a subsequence, $\|(u_k,v_k)\| \to +\infty$ as $k \to +\infty$. In this way, we need to study the following situations:

$(i)$ $\|u_k\|_1 \to +\infty$ and $\|v_k\|_2 \to \infty$

$(ii)$ $\|u_k\|_1 \to +\infty$ and $\|v_k\|_2$ is bounded

$(iii)$ $\|v_k\|_2 \to \infty$ and $\|u_k\|_1$ is bounded

In the first case, there is $k_0 \in \mathbb{N}$ such that

$$
\int_\Omega \Phi_1(|\nabla u_k|)dx \geq \|u_k\|_1 \quad \text{and} \quad \int_\Omega \Phi_2(|\nabla v_k|)dx \geq \|v_k\|_2, \ \forall k \geq k_0.
$$

Hence, the inequality (5.12) is reduced to

$$2c_n^2 + o_k(1)\|(u_k, v_k)\|^2 \geq (1-\beta)^2 \|u_k\|_1^2 + \left(\frac{\ell_2}{\nu} - 1\right)^2 \|v_k\|_2^2, \ \forall k \geq k_0.$$

Which is absurd.

In case $(ii)$, there is $k_0 \in \mathbb{N}$ such that

$$\int_\Omega \Phi_1(|\nabla u_k|)dx \geq \|u_k\|_1, \ \forall k \geq k_0.$$

Thus, the inequality (5.12) is reduced to

$$2c_n^2 + C_1 + o_k(1)\|u_k\|_1 \geq (1-\beta)^2 \|u_k\|_1^2, \ \forall k \geq k_0.$$

which is absurd. The last case is similar to the case $(ii)$. The above analysis shows that $(u_k, v_k)$ is now a bounded sequence in $X_n$.

■

From Lemmas 5.3 and 2.21, we may assume that there exists a subsequence of $(u_k, v_k)$, still denoted by itself, and $(w_n, y_n) \in X_n'$ such that

$$u_k \overset{*}{\rightharpoonup} w_n \ \text{ weakly in } W_0^{1,\Phi_1}(\Omega) \quad \text{and} \quad v_k \overset{*}{\rightharpoonup} y_n \ \text{ weakly in } V_n, \ \text{ as } k \to \infty. \tag{5.13}$$

Here, we highlight that the pair $(w_n, y_n)$ may not belong to the space $X_n$, because whenever $\Phi_1$ does not satisfy the $\Delta_2$-condition the space $X_n$ is not a weak$^*$ closed subspace of $X_n'$.

The results below will be used to ensure that the sequence $(w_n, y_n)$ is bounded in $W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$, moreover, we will do some results that will be fundamental.

**Lemma 5.4** *The sequence $(u_k, v_k)$ obtained in (5.8) satisfies*

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi dx = \int_\Omega R_u(x, u_k, v_k)\varphi dx + o_k(1), \ \ \forall k \in \mathbb{N} \ \ \textit{and} \ \ \varphi \in W_0^{1,\Phi_1}(\Omega).$$

**Proof.** From (5.8),

$$J_n'(u_k, v_k)(\varphi, 0) = o_k(1)\|\varphi\|_1, \ \ \forall \varphi \in W_0^1 E^{\Phi_1}(\Omega). \tag{5.14}$$

By definition, the space $W_0^{1,\Phi_1}(\Omega)$ is the weak$^*$ closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi_1}(\Omega)$, thus, given $\varphi \in W_0^{1,\Phi_1}(\Omega)$ there will be a sequence $(\varphi_m)$ in $C_0^\infty(\Omega)$ such that

$$\varphi_m \overset{*}{\rightharpoonup} \varphi \ \text{ in } W_0^{1,\Phi_1}(\Omega). \tag{5.15}$$

It is clear that $(\|\varphi_m\|_1)$ is bounded in $\mathbb{R}$, so by (5.14),

$$o_k(1) = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi_m dx - \int_\Omega R_u(x, u_k, v_k)\varphi_m dx, \;\; \forall k \in \mathbb{N}$$

Using the fact that $\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_i} \in E^{\tilde{\Phi}_1}(\Omega)$ along with the limit (5.15), we will get

$$\lim_{m\to\infty} \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi_m dx = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi dx.$$

Therefore, since the spaces $W_0^{1,\Phi_1}(\Omega)$, $W_0^{1,\Phi_2}(\Omega)$ are embedded in $C(\overline{\Omega})$, we can conclude that

$$o_k(1) = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi dx - \int_\Omega R_u(x, u_k, v_k)\varphi dx, \;\; \forall k \in \mathbb{N}.$$

■

Before proceeding with the results, we need to make the following definitions:

- We will denote by $D(J_{\Phi_i}) \subset W_0^{1,\Phi_i}(\Omega)$, the following set:

$$D(J_{\Phi_i}) = \left\{ u \in W_0^{1,\Phi_i}(\Omega) : \int_\Omega \Phi_i(|\nabla u|)dx < +\infty \right\}$$

- We will denote by $dom(\phi_i(t)t) \subset W_0^{1,\Phi}(\Omega)$, the following set:

$$dom(\phi_i(t)t) = \left\{ u \in W_0^{1,\Phi_i}(\Omega) : \int_\Omega \tilde{\Phi}_i\big(\phi_i(|\nabla u|)|\nabla u|\big)dx < +\infty \right\}$$

**Lemma 5.5** *Let be $(w_n)$ the sequence obtained in (5.13). Then $(w_n) \subset D(J_{\Phi_1}) \cap dom(\phi_1(t)t)$, furthermore,*

$$c_n = \lim_{k\to\infty} J_n(u_k, v_k) = J_n(w_n, y_n)$$

*and*

$$\begin{aligned} &\int_\Omega \Phi_1(|\nabla \varphi_1|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla(\varphi_2 - y_n)dx \\ &\qquad \geq \int_\Omega R_u(x, w_n, y_n)(\varphi_1 - w_n)dx + \int_\Omega R_v(x, w_n, y_n)(\varphi_2 - y_n)dx, \end{aligned} \tag{5.16}$$

*for all $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$.*

**Proof.** Using the fact that $J_n'(u_k, v_k) \to 0$ as $k \to \infty$ together with Lemma 5.4, we can conclude that

(5.17)
$$\begin{aligned} \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla \varphi_1 dx - \int_\Omega \phi_2(|\nabla v_k|)\nabla v_k \nabla \varphi_2 dx = &\int_\Omega R_u(x, u_k, v_k)\varphi_1 dx \\ &+ \int_\Omega R_v(x, u_k, v_k)\varphi_2 dx + o_k(1), \end{aligned}$$

for each $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$ and $k \in \mathbb{N}$. Since $\Phi_1$ is convex, we have

$$\int_\Omega \Phi_1(|\nabla \eta_1|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla(\eta_1 - u_k)dx,$$

for all $\eta_1 \in W_0^{1,\Phi_1}(\Omega)$. Hence, considering $\varphi_1 = \eta_1 - u_k$ in (5.17) and using the inequality above, we get

(5.18)
$$\int_\Omega \Phi_1(|\nabla \eta_1|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx - \int_\Omega \phi_2(|\nabla v_k|)\nabla v_k \nabla \varphi_2 dx \\ \geq \int_\Omega R_u(x, u_k, v_k)(\eta_1 - u_k)dx + \int_\Omega R_v(x, u_k, v_k)\varphi_2 dx + o_k(1),$$

for every $(\eta_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$ and $k \in \mathbb{N}$. Since $u_k \xrightarrow{*} w_n$ in $W_0^{1,\Phi_1}(\Omega)$, it follows from Lemma 2.22 that

$$\int_\Omega \Phi_1(|\nabla w_n|)dx \leq \lim_{k\to\infty} \int_\Omega \Phi_1(|\nabla u_k|)dx, \tag{5.19}$$

Remember that $dimV_n = n$, so $v_k \to y_n$ in $V_n$. Hence,

(5.20)
$$\int_\Omega \Phi_1(|\nabla \eta_1|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 dx \\ \geq \int_\Omega R_u(x, w_n, y_n)(\eta_1 - w_n)dx + \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx,$$

for each $(\eta_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$. Justifying the inequality (5.16).

Considering $(\eta_1, \varphi_2) = (w_n, 0)$ in the inequality (5.18), we get

$$\int_\Omega \Phi_1(|\nabla w_n|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq \int_\Omega R_u(x, u_k, v_k)(w_n - u_k)dx + o_k(1).$$

Thus,

$$\int_\Omega \Phi_1(|\nabla w_n|)dx \geq \lim_{k\to\infty} \int_\Omega \Phi_1(|\nabla u_k|)dx. \tag{5.21}$$

Combining (5.19) and (5.21),

$$\lim_{k\to\infty} \int_\Omega \Phi_1(|\nabla u_k|)dx = \int_\Omega \Phi_1(|\nabla w_n|)dx.$$

Therefore, we can conclude that

$$c_n = \lim_{k\to\infty} J_n(u_k, v_k) = J_n(w_n, y_n).$$

Finally, we will show that $w_n \in dom(\phi_1(t)t)$. By the inequality (5.18),

$$\int_\Omega \Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq -\frac{1}{k}\int_\Omega R_u(x,u_k,v_k)u_k dx + o_k(1),$$

i.e,

$$\int_\Omega \frac{\left(\Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|) - \Phi_1(|\nabla u_k|)\right)}{-\frac{1}{k}}dx \leq \int_\Omega R_u(x,u_k,v_k)u_k dx + o_k(1).$$

As $(u_k)$ and $(v_k)$ are bounded in $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$, respectively, there will be $M > 0$ such that

$$\int_\Omega \Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \leq M, \quad \forall k \in \mathbb{N}.$$

Since $\Phi_1$ is in $C^1$ class, there exists $\theta_k(x) \in [0,1]$ such that

$$\frac{\Phi_1(|\nabla u_k - \frac{1}{k}\nabla u_k|) - \Phi_1(|\nabla u_k|)}{-\frac{1}{k}} = \phi_1(|\big(1 - \frac{\theta_k}{k}(x)\big)\nabla u_k|)\big(1 - \frac{\theta_k(x)}{k}\big)|\nabla u_k|^2.$$

Recalling that $0 < 1 - \frac{\theta_k(x)}{k} \leq 1$, we know that $1 - \frac{\theta_k(x)}{k} \geq \big(1 - \frac{\theta_k(x)}{k}\big)^2$ which leads to

$$\int_\Omega \phi_1(|\big(1 - \frac{\theta_k}{k}(x)\big)\nabla u_k|)\big(1 - \frac{\theta_k(x)}{k}\big)^2|\nabla u_k|^2 dx \leq M, \quad \forall k \in \mathbb{N}.$$

As $\nabla u_k \xrightarrow{*} \nabla w_n$ in $\big(L^{\Phi_1}(\Omega)\big)^{N-1}$, we also have $\big(1 - \frac{\theta_k(x)}{k}\big)\nabla u_k \xrightarrow{*} \nabla w_n$ in $\big(L^{\Phi_1}(\Omega)\big)^{N-1}$ as $k \to \infty$. Then, by using the fact that $\phi_1(t)t^2$ is convex, we can apply [30, Theorem 2.1, Chapter 8] to get

$$\liminf_{k\to\infty} \int_\Omega \phi_1(|\big(1 - \frac{\theta_k}{k}(x)\big)\nabla u_k|)\big(1 - \frac{\theta_k(x)}{k}\big)^2|\nabla u_k|^2 \geq \int_\Omega \phi_1(|\nabla w_n|)|w_n|^2 dx$$

and so,

$$\int_\Omega \phi_1(|\nabla w_n|)|w_n|^2 dx \leq M.$$

By Lemma 2.3(item 3) ,

$$\phi_1(t)t^2 = \Phi_1(t) + \tilde{\Phi}_1(\phi_1(t)t), \quad \forall t \in \mathbb{R}$$

thus

$$\phi_1(|\nabla w_n|)|\nabla w_n|^2 = \Phi_1(|\nabla w_n|) + \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|)$$

which leads to

$$\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx = \int_\Omega \Phi_1(|\nabla w_n|)dx + \int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|^2)dx.$$

Since $\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx$ is finite, so from the above identity we see that $\int_\Omega \Phi_1(|\nabla w_n|)dx$ and $\int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|^2)dx$ are also finite, showing that $w_n \in D(J_{\Phi_1}) \cap dom(\phi_1(t)t)$. This finishes the proof. ■

**Lemma 5.6** *For each $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times V_n$, the following equality holds*

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla\varphi_1 dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla\varphi_2 dx = \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx + \int_\Omega R_u(x, w_n, y_n)\varphi_2 dx.$$

**Proof.** Given $\varepsilon \in (0, 1/2)$ and $\varphi_1 \in C_0^\infty(\Omega)$, we set the function

$$v_\varepsilon = \frac{1}{1-\frac{\varepsilon}{2}}((1-\varepsilon)w_n + \varepsilon\varphi_1).$$

Consider $\varphi_2 \in V_n$ and apply $(v_\varepsilon, \varepsilon\varphi_2 + y_n)$ on the inequality (5.16), hence

$$\int_\Omega \Phi_1(|\nabla v_\varepsilon|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx - \varepsilon\int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla\varphi_2 dx \geq \int_\Omega R_u(x, w_n, y_n)(v_\varepsilon - w_n)dx + \varepsilon\int_\Omega R_v(x, w_n, y_n)\varphi_2 dx,$$

and so,

$$\frac{\int_\Omega \Phi_1(|\nabla v_\varepsilon|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx}{\varepsilon} - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla\varphi_2 dx \geq \int_\Omega R_u(x, w_n, y_n)\Big(\frac{v_\varepsilon - w_n}{\varepsilon}\Big)dx + \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx.$$

Note that

$$\frac{\varepsilon\varphi_1}{1-\frac{\varepsilon}{2}} = 2\left(1 - \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\right)\varphi_1.$$

Hence, by the convexity of $\Phi_1$,

$$\Phi_1\Big(\frac{1}{1-\frac{\varepsilon}{2}}\big((1-\varepsilon)\nabla w_n + \varepsilon\nabla\varphi_1\big)\Big) \leq \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\Phi_1(|\nabla w_n|) + \big(1 - \frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\big)\Phi_1(2|\nabla\varphi_1|).$$

Thus, by Lebesgue dominated convergence theorem, we get

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n(\nabla w_n - \frac{\nabla w_n}{2})dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla\varphi_2 dx \geq \int_\Omega R_u(x, w_n, y_n)\Big(\varphi_1 - \frac{w_n}{2}\Big)dx + \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx. \tag{5.22}$$

Therefore

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla\varphi_1 - \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx \geq A, \quad \forall\varphi_1 \in C_0^\infty(\Omega), \tag{5.23}$$

where

$$A = \frac{1}{2}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx - \frac{1}{2}\int_\Omega R_u(x, w_n, y_n) w_n dx.$$

As $C_0^\infty(\Omega)$ is a vector space, the last inequality gives

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla \varphi_1 - \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx = 0, \quad \forall \varphi_1 \in C_0^\infty(\Omega). \tag{5.24}$$

We know that $W_0^{1,\Phi_1}(\Omega)$ is the weak* closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi_1}(\Omega)$, then using the fact that $\phi_1(|\nabla w_n|)|\nabla w_n| \in L^{\tilde{\Phi}_1}(\Omega)$ we can conclude that

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla \varphi_1 dx - \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx = 0, \ \forall \varphi_1 \in W_0^{1,\Phi_1}(\Omega). \tag{5.25}$$

Still by (5.22), we have

$$-\int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 \geq \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx, \ \forall \varphi_2 \in V_n.$$

Since $V_n$ is a vector space, the above inequality gives

$$\int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 = -\int_\Omega R_v(x, w_n, y_n)\varphi_2 dx, \ \forall \varphi_2 \in V_n. \tag{5.26}$$

From (5.25) and (5.26),

$$\begin{aligned}\int_\Omega \phi_2(|\nabla w_n|)\nabla w_n \nabla \varphi_1 dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 = \int_\Omega R_u(x, w_n, y_n)\varphi_1 dx \\ + \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx,\end{aligned}$$

for any $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_2}(\Omega) \times V_n$. ■

**Lemma 5.7** *The sequence $(w_n, y_n)$ is bounded in $X$.*

**Proof.** Consider the sequence

$$g_n(x) = \eta(w_n(x)), \ \ x \in \Omega.$$

where $\eta$ is given in Lemma 5.3. A direct computation leads to

$$\nabla g_n = \left[1 - \frac{\Phi_1(w_n)}{w_n^2 \phi_1(w_n)}\left(1 + \frac{w_n \phi_1'(w_n)}{\phi_1(w_n)}\right)\right] \nabla w_n.$$

The last identity together with $(\phi_{i,5})$ implies that

$$|\nabla g_n| \leq |\nabla w_n|, \ \ \forall n \in \mathbb{N} \tag{5.27}$$

On the other hand, $(\phi_{i,3})$ also gives

$$|g_n(x)| \leq \frac{1}{\ell_1}|\nabla w_n(x)|, \quad \forall x \in \Omega. \tag{5.28}$$

From (5.27) and (5.28), $g_n \in D(J_{\Phi_1})$ with

$$\|g_n\|_1 \leq \|w_n\|_1, \ \forall n \in \mathbb{N}.$$

By the Lemmas 5.5 and 5.6,

$$\begin{aligned} c_n = & \int_\Omega \Phi_1(|\nabla w_n|)dx - \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx \\ & + \frac{1}{\nu}\int_\Omega \phi_2(|\nabla y_n|)|\nabla y_n|^2 dx + \frac{1}{\mu}\int_\Omega R_u(x, w_n, v_k)w_n h(w_n)dx \\ & + \frac{1}{\nu}\int_\Omega R_v(x, w_n, y_n)y_n dx - \int_\Omega R(x, w_n, y_n)dx, \end{aligned}$$

where $h(t) = \frac{\Phi_1(t)}{t^2\phi_1(t)}$ and $S(t) = 1 - \frac{\Phi(t)}{t^2\phi_1(t)}\left[1 + \frac{t\phi_1'(t)}{\phi_1(t)}\right]$. By $(R_4')(i)$ together with $(\phi_{i,3})$,

(5.29)
$$c_n \geq \int_\Omega \Phi_1(|\nabla w_n|)dx - \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla y_n|)dx.$$

On the other hand, the Lemmas 5.5 and 5.6 together with $(R_4')(ii)$ imply that

$$\begin{aligned} -\beta c_n = & -\beta\int_\Omega \Phi_1(|\nabla w_n|)dx + \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx + \beta\int_\Omega \Phi_2(|\nabla y_n|)dx \\ & - \frac{1}{\mu}\int_\Omega R_u(x, w_n, y_n)w_n h(w_n)dx + \beta\int_\Omega R(x, w_n, y_n)dx, \\ \geq & -\beta\int_\Omega \Phi_1(|\nabla w_n|)dx + \frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx, \end{aligned}$$

i.e,

$$-\frac{1}{\mu}\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 S(w_n)dx \geq \alpha c_n - \alpha\int_\Omega \Phi_1(|\nabla w_n|)dx. \tag{5.30}$$

From (5.29) and (5.30),

$$(1-\beta)c_n \geq (1-\beta)\int_\Omega \Phi_1(|\nabla w_n|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla y_n|)dx.$$

As a consequence of Lemma 5.2, we have that $(c_n)$ is bounded. Therefore the sequences $\left(\int_\Omega \Phi_1(|\nabla w_n|)dx\right)$ and $\left(\int_\Omega \Phi_2(|\nabla y_n|)dx\right)$ are bounded and consequently $(w_n, y_n)$ is bounded at $W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$. ■

Since $(w_n, y_n)$ is bounded, it follows from Lemmas 2.21 and 2.22 that

$$(w_n, y_n) \xrightarrow{*} (u, v) \quad \text{in } W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega) \quad \text{as} \quad n \to \infty, \tag{5.31}$$

$$\int_\Omega \Phi_1(|\nabla u|)dx \le \liminf_{n\to\infty} \int_\Omega \Phi_1(|\nabla w_n|)dx \tag{5.32}$$

and

$$\int_\Omega \Phi_2(|\nabla v|)dx \le \liminf_{n\to\infty} \int_\Omega \Phi_2(|\nabla y_n|)dx. \tag{5.33}$$

These limits will be fundamental to the argument used below.

**Proof of Theorem 1.9** Fixing $k, n \in \mathbb{N}$ with $n \ge k$, we have $X_k' \subset X_n'$. Thus, for $(\varphi_1, \varphi_2) \in X_k'$, it follows from Lemma 5.6 that

$$\begin{aligned}\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n \nabla \varphi_1 dx - \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla \varphi_2 dx = &\int_\Omega R_u(x, w_n, y_n)\varphi_1 dx \\ &+ \int_\Omega R_v(x, w_n, y_n)\varphi_2 dx,\end{aligned} \tag{5.34}$$

for all $n \ge k$. By the above equality together with the convexity of $\Phi_1$, we will obtain

$$\int_\Omega \Phi_1(|\nabla \varphi_1|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx \ge \int_\Omega R_u(x, w_n, y_n)(\varphi_1 - w_n)dx, \tag{5.35}$$

for each $\varphi_1 \in W_0^{1,\Phi_1}(\Omega)$. From this inequality, we can conclude that

$$\int_\Omega \Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx \ge -\frac{1}{n}\int_\Omega R_u(x, w_n, y_n)w_n dx,$$

i.e,

$$\int_\Omega \frac{\left(\Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|) - \Phi_1(|\nabla w_n|)\right)}{-\frac{1}{n}}dx \le \int_\Omega R_u(x, w_n, y_n)w_n dx.$$

As $(w_n)$ and $(y_n)$ are bounded in $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$, respectively, there will be $M > 0$ such that

$$\int_\Omega \frac{\Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|) - \Phi_1(|\nabla w_n|)}{-\frac{1}{n}}dx \le M, \quad \forall n \in \mathbb{N}.$$

Since $\Phi_1$ is in $C^1$ class, there exists $\theta_n(x) \in [0, 1]$ such that

$$\frac{\Phi_1(|\nabla w_n - \frac{1}{n}\nabla w_n|) - \Phi_1(|\nabla w_n|)}{-\frac{1}{n}} = \phi_1(|(1 - \frac{\theta_n(x)}{n})\nabla w_n|)(1 - \frac{\theta_n(x)}{n})|\nabla w_n|^2.$$

Recalling that $0 < 1 - \frac{\theta_n(x)}{n} \leq 1$, we know that $1 - \frac{\theta_n(x)}{n} \geq \left(1 - \frac{\theta_n(x)}{n}\right)^2$ which leads to

$$\int_\Omega \phi_1(|\big(1 - \frac{\theta_n(x)}{n}\big)\nabla w_n|)\big(1 - \frac{\theta_n(x)}{n}\big)^2|\nabla w_n|^2 dx \leq M, \;\; \forall n \in \mathbb{N}.$$

As $\nabla w_n \xrightarrow{*} \nabla u$ in $\left(L^{\Phi_1}(\Omega)\right)^{N-1}$, we also have $\left(1 - \frac{\theta_n(x)}{n}\right)\nabla w_n \xrightarrow{*} \nabla u$ in $\left(L^{\Phi_1}(\Omega)\right)^{N-1}$ as $n \to \infty$. Then, by using the fact that $\phi_1(t)t^2$ is convex, we can apply [30, Theorem 2.1, Chapter 8] to get

$$\liminf_{n\to\infty} \int_\Omega \phi_1(|\big(1 - \frac{\theta_n(x)}{n}\big)\nabla w_n|)\big(1 - \frac{\theta_n(x)}{n}\big)^2|\nabla w_n|^2 \geq \int_\Omega \phi_1(|\nabla u|)|\nabla u|^2 dx$$

and so,

$$\int_\Omega \phi_1(|\nabla w_n|)|w_n|^2 dx \leq M.$$

By Lemma 2.3(item 3),

$$\phi_1(t)t^2 = \Phi_1(t) + \tilde{\Phi}_1(\phi_1(t)t), \;\; \forall t \in \mathbb{R}$$

thus

$$\phi_1(|\nabla w_n|)|\nabla w_n|^2 = \Phi_1(|\nabla w_n|) + \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|)$$

which leads to

$$\int_\Omega \phi_1(|\nabla w_n|)|\nabla w_n|^2 dx = \int_\Omega \Phi_1(|\nabla w_n|)dx + \int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla w_n|)|\nabla w_n|^2)dx.$$

Since $\int_\Omega \phi_1(|\nabla u|)|\nabla u|^2 dx$ is finite, we see that $\int_\Omega \Phi_1(|\nabla u|)dx$ and $\int_\Omega \tilde{\Phi}_1(\phi_1(|\nabla u|)|\nabla u|^2)dx$ are also finite, showing that $u \in D(J_{\Phi_1})$ and $u \in dom(\phi_1(t)t)$. Furthermore, it follows from (5.32) and (5.35) that

(5.36)
$$\int_\Omega \Phi_1(|\nabla \varphi_1|)dx - \int_\Omega \Phi_1(|\nabla u|)dx \geq \int_\Omega R_u(x,u,v)(\varphi_1 - u)dx, \;\; \forall \varphi_1 \in W_0^{1,\Phi_1}(\Omega).$$

On the other hand, it follows from the equality (5.34) that

(5.37)
$$\int_\Omega \Phi_2(|\nabla \varphi_2|)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx \geq -\int_\Omega R_v(x,w_n,y_n)(\varphi_2 - y_n)dx, \;\; \forall \varphi_2 \in V_k.$$

From this inequality, we can conclude that

$$\int_\Omega \Phi_2(|\nabla y_n - \frac{1}{n}\nabla w_n|)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx \geq \frac{1}{n}\int_\Omega R_v(x,w_n,y_n)y_n dx,$$

i.e,

$$\int_\Omega \frac{\Phi_2(|\nabla y_n - \frac{1}{n}\nabla y_n|) - \Phi_2(|\nabla y_n|)}{-\frac{1}{n}} dx \leq -\int_\Omega R_v(x, w_n, y_n) y_n dx.$$

As $(w_n)$ and $(y_n)$ are bounded in $W_0^{1,\Phi_1}(\Omega)$ and $W_0^{1,\Phi_2}(\Omega)$, respectively, there will be $M > 0$ such that

$$\int_\Omega \frac{\Phi_2(|\nabla y_n - \frac{1}{n}\nabla y_n|) - \Phi_2(|\nabla y_n|)}{-\frac{1}{n}} dx \leq M, \quad \forall n \in \mathbb{N}.$$

Since $\Phi_2$ is in $C^1$ class, there exists $\theta_n(x) \in [0,1]$ such that

$$\frac{\Phi_2(|\nabla y_n - \frac{1}{n}\nabla y_n|) - \Phi_2(|\nabla y_n|)}{-\frac{1}{n}} = \phi_2(|(1 - \frac{\theta_n(x)}{n})\nabla y_n|)(1 - \frac{\theta_n(x)}{n})|\nabla y_n|^2.$$

Recalling that $0 < 1 - \frac{\theta_n(x)}{n} \leq 1$, we know that $1 - \frac{\theta_n(x)}{n} \geq \left(1 - \frac{\theta_n(x)}{n}\right)^2$ which leads to

$$\int_\Omega \phi_2(|(1 - \frac{\theta_n(x)}{n})\nabla y_n|)(1 - \frac{\theta_n(x)}{n})^2 |\nabla y_n|^2 dx \leq M, \quad \forall n \in \mathbb{N}.$$

As $\nabla y_n \xrightharpoonup{*} \nabla v$ in $\left(L^{\Phi_2}(\Omega)\right)^{N-1}$, we also have $\left(1 - \frac{\theta_n(x)}{n}\right)\nabla y_n \xrightharpoonup{*} \nabla v$ in $\left(L^{\Phi_2}(\Omega)\right)^{N-1}$ as $n \to \infty$. Then, by using the fact that $\phi_2(t)t^2$ is convex, we can apply [30, Theorem 2.1, Chapter 8] to get

$$\liminf_{n\to\infty} \int_\Omega \phi_2(|(1 - \frac{\theta_n(x)}{n})\nabla y_n|)(1 - \frac{\theta_n(x)}{n})^2 |\nabla y_n|^2 \geq \int_\Omega \phi_2(|\nabla v|)|\nabla v|^2 dx$$

and so,

$$\int_\Omega \phi_2(|\nabla v|)|\nabla v|^2 dx \leq M.$$

By Lemma 2.3(item 3),

$$\phi_2(t)t^2 = \Phi_2(t) + \tilde{\Phi}_2(\phi_2(t)t), \quad \forall t \in \mathbb{R}$$

thus

$$\int_\Omega \phi_2(|\nabla v|)|\nabla v|^2 dx = \int_\Omega \Phi_2(|\nabla v|) dx + \int_\Omega \tilde{\Phi}_2(\phi_2(|\nabla v|)|\nabla v|^2) dx.$$

Since $\int_\Omega \phi_2(|\nabla v|)|\nabla v|^2 dx$ is finite, we see that $\int_\Omega \Phi_2(|\nabla v|)dx$ and $\int_\Omega \tilde{\Phi}_2(\phi_2(|\nabla u|)|\nabla v|^2)dx$ are also finite, showing that $v \in D(J_{\Phi_2})$ and $v \in dom(\phi_2(t)t)$.

Now, for $\varphi \in W_0^1 E^{\Phi_2}(\Omega)$, there exists $\chi_m \in V_m$ such that

$$\lim_{m\to\infty} \chi_m = \varphi \quad \text{in } W_0^1 E^{\Phi_2}(\Omega). \tag{5.38}$$

From (5.34),

$$-\int_\Omega \phi_2(|\nabla y_n|)\nabla y_n \nabla(\chi_m - y_n)dx = \int_\Omega R_v(x, w_n, y_n)(\chi_m - y_n)dx, \ \ \forall n \geq m.$$

The convexity of $\Phi_2$ implies that

(5.39)

$$\int_\Omega \Phi_2(|\nabla \chi_m|)dx - \int_\Omega \Phi_2(|\nabla y_n|)dx \geq -\int_\Omega R_v(x, w_n, y_n)(\chi_m - y_n)dx, \ \ \forall n \geq m.$$

Thus, by the limit (5.33) we have

$$\int_\Omega \Phi_2(|\nabla \chi_m|)dx - \int_\Omega \Phi_2(|\nabla v|)dx \geq -\int_\Omega R_v(x, u, v)(\chi_m - v)dx. \tag{5.40}$$

Now we use (5.38) in the above inequality to get

$$\int_\Omega \Phi_2(|\nabla \varphi|)dx - \int_\Omega \Phi_2(|\nabla v|)dx \geq -\int_\Omega R_v(x, u, v)(\varphi - v)dx. \tag{5.41}$$

Repeating the arguments used in Lemma 5.6, the inequalities (5.36) and (5.41) imply

$$\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla \varphi_1 dx = \int_\Omega R_u(x, u, v)\varphi_1 dx, \ \ \forall \varphi_2 \in W_0^{1,\Phi_1}(\Omega),$$

$$\int_\Omega \phi_2(|\nabla v|)\nabla v \nabla \varphi_2 dx = -\int_\Omega R_v(x, u, v)\varphi_2 dx, \ \ \forall \varphi_2 \in W_0^1 E^{\Phi_2}(\Omega).$$

Finally, the fact that $\phi_2(|\nabla v|)|\nabla v| \in L^{\tilde{\Phi}_2}(\Omega)$ together with the density weak$^*$ of $C_0^\infty(\Omega)$ in $W_0^{1,\Phi_2}(\Omega)$ given

$$\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla \varphi_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla \varphi_2 dx = \int_\Omega R_u(x, u, v)\varphi_1 dx + \int_\Omega R_v(x, u, v)\varphi_2 dx,$$

for every $(\varphi_1, \varphi_2) \in W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$. To conclude, the hypothesis $(R_1')$ guarantees that $(u, v)$ is a nontrivial solution for $(S_1)$, and the proof of Theorem 1.9 is complete.

## 5.2 The $\mathcal{N}$-functions $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ may not verify the $\Delta_2$-condition.

Continuing the study of systems $(S)$ in non-reflexive Orlicz-Sobolev spaces, in this section, we study the existence of solutions for the following class of quasilinear systems of the type:

$$(S_2) \qquad \begin{cases} -\Delta_{\Phi_1} u = F_u(x, u, v) + \lambda R_u(x, u, v) \ \text{ in } \Omega \\ -\Delta_{\Phi_2} v = -F_v(x, u, v) - \lambda R_v(x, u, v) \ \text{ in } \Omega \\ \ u = v = 0 \ \text{ on } \partial\Omega \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and $\lambda > 0$. Our main goal in this section is to prove of Theorem 1.10. For this, we assume that $\phi_i (i = 1, 2) : (0, \infty) \to (0, \infty)$ are two functions which satisfy:

$(\phi'_{1,i})$ $$\phi_i \in C^1(0, +\infty) \text{ and } t \mapsto t\phi_i(t) \text{ are stricly increasing;}$$

$(\phi'_{2,i})$ $$t\phi_i(t) \to 0 \text{ as } t \to 0;$$

$(\phi'_{3,i})$ $1 \leq \ell_i = \inf_{t>0} \dfrac{t^2\phi_i(t)}{\Phi_i(t)} \leq \sup_{t>0} \dfrac{t^2\phi_i(t)}{\Phi_i(t)} = m_i < N$, where $\Phi_i(t) = \int_0^{|t|} s\phi_i(s)ds$ and $\ell_i < m_i < \ell_i^*$.

As mentioned in the introduction of this thesis, in this problem $(S_2)$, we will assume that $F(x, u, v) = \Phi_{1*}(u) + G(v)$ where $\Phi_{1*}$ denotes the Sobolev conjugate function of $\Phi_1$ and that $G$ is a function satisfying the following conditions:

$(G_1)$ There are $C > 0$, $G \in C^1(\mathbb{R}, \mathbb{R})$, $a_1, a_2 \in (1, \infty)$ and a $\mathcal{N}$-function $A(t) = \int_0^{|t|} sa(s)ds$ satisfying

$(i)$ $$m_2 < a_1 \leq \frac{a(t)t^2}{A(t)} \leq a_2, \quad \forall t > 0$$

and

$(ii)$ $$|g(s)| \leq a_1 Ca(|s|)|s|, \quad \text{for all } s \in \mathbb{R}$$

where $g(s) = G'(s)$. If $a_2 \geq \ell_2^*$, we add that

$(iii)$ $$(g(t) - g(s))(t - s) \geq Ca(|t - s|)|t - s|^2, \quad \text{for all } t, s \in \mathbb{R}.$$

$(G_2)$ There exists $\nu \in (0, \ell_1)$ such that

$$0 \leq \nu G(s) \leq sg(s), \quad \text{for all } s \in \mathbb{R}.$$

Furthermore, we will assume that the $R$ function meets the conditions below:

$(R_1)$ $R \in C^1(\overline{\Omega} \times \mathbb{R}^2)$, $R_u(x, 0, 0) = 0$, $R_v(x, 0, 0) = 0$, $R(x, u, v) \geq 0$ and $R_u(x, u, v)u \geq 0$, for all $(x, u, v) \in \overline{\Omega} \times \mathbb{R}^2$.

$(R_2)$ There are $\mathcal{N}$-functions $B(t) = \int_0^{|t|} sb(s)ds$, $P(t) = \int_0^{|t|} sp(s)ds$, $Q(t) = \int_0^{|t|} sq(s)ds$ and $Z(t) = \int_0^{|t|} sz(s)ds$ satisfying

$(i)$ $$m_1 < p_1 \leq \frac{p(t)t^2}{P(t)} \leq p_2 < \ell_1^*$$

$(ii)$ $$m_1 < b_1 \leq \frac{b(t)t^2}{B(t)} \leq b_2 < \ell_1^*$$

$(iii)$ $$m_2 < q_1 \leq \frac{q(t)t^2}{Q(t)} \leq q_2 < \ell_2^*$$

$(iv)$ $$m_2 < z_1 \leq \frac{z(t)t^2}{Z(t)} \leq z_2 < \ell_2^*,$$

with $\max\{b_2, q_2\} < \min\{\ell_1^*, \ell_2^*\}$ such that

$$|R_u(x,u,v)| \leq C(p(|u|)u + q(|v|)v) \quad \text{and} \quad |R_v(x,u,v)| \leq C(b(|u|)u + z(|v|)v), \tag{5.42}$$

for all $(x,u,v) \in \Omega \times \mathbb{R}^2$ and for some constant $C > 0$.

$(R_3)$ There exists $\mu \in (m_1, \ell_1^*)$ such that

$$\frac{1}{\mu}R_u(x,u,v) + \frac{1}{\nu}R_v(x,u,v) - R(x,u,v) \geq 0, \quad \text{for all } x \in \Omega \text{ and } (u,v) \in \mathbb{R}^2,$$

where $\nu$ is given by condition $(G_2)$.

$(R_4)$ There exists $s \in (m_1, \max\{p_2, b_2\}]$, a nonempty open subset $\Omega_0 \subset \Omega$ and a constant $\omega > 0$ such that

$$R(x,u,v) \geq \omega |u|^s \quad \text{for all } x \in \Omega_0 \text{ and } (u,v) \in \mathbb{R}^2.$$

**Example 5.2.1** *Fix $p \in (m_1, \ell_1^*)$ and $q \in (m_2, \ell_2^*)$. The function $R(u,v) = |u|^p + C|v|^q + \varepsilon \sin|u|^p \sin|v|^q$ satisfies $(R_1) - (R_4)$ with $P(t) = B(t) = |t|^p/p$, $Q(t) = Z(t) = |t|^q/q$, $C > 0$ and $\varepsilon > 0$ small enough.*

In what follows, fix some notations. In the sequel $V_A$ stands for the space $W_0^{1,\Phi_2}(\Omega) \cap L^A(\Omega)$ endowed with the norm

$$\|v\|_A = \|v\|_{W_0^{1,\Phi_2}(\Omega)} + |v|_A,$$

where $\|v\|_{W_0^{1,\Phi_2}(\Omega)}$ and $|v|_A$ denote the usual norms in in $W_0^{1,\Phi_2}(\Omega)$ and $L^A(\Omega)$, respectively.

We write $X$ for the space $W_0^{1,\Phi_1}(\Omega) \times V_A$ endowed with the norm

$$\|(u,v)\|^2 = \|u\|^2_{W_0^{1,\Phi_1}(\Omega)} + \|v\|^2_A,$$

where $\|u\|_{W_0^{1,\Phi_1}(\Omega)}$ denotes the usual norm in $W_0^{1,\Phi_1}(\Omega)$. Under the assumptions $(G_1)$ and $(R_2)$, the functional $\mathcal{H}_\lambda$ given by

$$\mathcal{H}_\lambda(u,v) = \int_\Omega H(x,u,v)dx. \tag{5.43}$$

is well defined, belongs to $C^1(X,\mathbb{R})$ and

$$\mathcal{H}'_\lambda(u,v)(w_1,w_2) = \int_\Omega H_u(x,u,v)w_1dx + \int_\Omega H_v(x,u,v)w_2dx, \tag{5.44}$$

for all $(u,v),(w_1,w_2) \in X$. Now, we consider the functional $Q : X \to \mathbb{R}$ which is given by

$$Q(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx, \tag{5.45}$$

It is well known in the literature that $Q \in C^1(E,\mathbb{R})$ when $\Phi_1$, $\Phi_2$, $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ satisfy the $\Delta_2$-condition and this occurs when we have the condition satisfied to $\ell_1 > 1$, $\ell_2 > 1$ and $m_1 < \infty,\ m_2 < \infty$. When $\ell_1 = 1$ (or $\ell_2 = 1$), we know that $\tilde{\Phi}_1 \notin (\Delta_2)$ (or $\tilde{\Phi}_2 \notin (\Delta_2)$) and therefore cannot guarantee the differentiability of functional $Q$. However, following the ideas presented in Chapter 3, it is clear that the functional $Q$ is continuous and Gateaux-differentiable with derivative $Q' : X \to X^*$ given by

$$Q'(u,v)(w_1,w_2) = \int_\Omega \phi_1(|\nabla u|)\nabla u\nabla w_1dx - \int_\Omega \phi_2(|\nabla v|)\nabla v\nabla w_2dx,$$

continuous from the norm topology of $X$ to the weak$^*$-topology of $X^*$. Therefore, we can conclude that the energy functional $J_\lambda : X \to \mathbb{R}$ associated with the system $(S_2)$ given by

$$J_\lambda(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega H(x,u,v)dx.$$

is continuous and Gateaux-differentiable with derivative $J'_\lambda : X \to X^*$ defined by

$$\begin{aligned} J'_\lambda(u,v)(w_1,w_2) = &\int_\Omega \phi_1(|\nabla u|)\nabla u\nabla w_1dx - \int_\Omega \phi_2(|\nabla v|)\nabla v\nabla w_2dx \\ &- \int_\Omega H_u(x,u,v)w_1dx - \int_\Omega H_v(x,u,v)w_2dx \end{aligned}$$

continuous from the norm topology of $X$ to the weak$^*$-topology of $X^*$.

Since $J'_\lambda(0,0) = 0$, we say that $(u,v) \in X$ is a nontrivial solution of $(S_2)$ when $J'_\lambda(u,v)(w_1,w_2) = 0$, for all $(w_1,w_2) \in X$ and satisfies $J_\lambda(u,v) \neq 0$.

In order to apply the linking theorem for Gateaux-differentiable functionals (See Theorem A.2), we introduce one more piece of notation. Since $(V_A, \|\cdot\|_A)$ is separable, then there exists a sequence $(e_n) \subset V_A$ such

$$V_A = \overline{span\{e_n : n \in \mathbb{N}\}}. \tag{5.46}$$

Hereafter, for each $n \in \mathbb{N}$ we denote by $V_A^n$ and $X_n$ the following spaces

$$V_A^n = span\{e_j : j = 1, \cdots, n\} \quad \text{and} \quad X_n = W_0^{1,\Phi_1}(\Omega) \times V_A^n.$$

The restriction of $J_\lambda$ to $X_n$ will be denoted by $J_{\lambda,n}$. Then $J_{\lambda,n} : X_n \to \mathbb{R}$ is the functional given by

$$J_{\lambda,n}(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega H(x,u,v)dx.$$

is continuous and Gateaux-differentiable with derivative $J'_{\lambda,n} : X_n \longrightarrow X_n^*$ given by

$$\begin{aligned} J'_{\lambda,n}(u,v)(w_1,w_2) = &\int_\Omega \phi_1(|\nabla u|)\nabla u \nabla w_1 dx - \int_\Omega \phi_2(|\nabla v|)\nabla v \nabla w_2 dx \\ &- \int_\Omega H_u(x,u,v)w_1 dx - \int_\Omega H_v(x,u,v)w_2 dx \end{aligned}$$

continuous from the norm topology of $X_n$ to the weak$^*$-topology of $X_n^*$.

In the following, we prove that $J_{\lambda,n}$ satisfies the hypotheses of linking theorem for Gateaux-differentiable functionals (See Theorem A.2).

**Lemma 5.8** *Assume that $(G_1) - (G_2)$ and $(R_1) - (R_4)$ hold. For every $\lambda > 0$, there exist $\sigma > 0$ and $\rho > \sigma$ such that if $u_* \in W_0^{1,\Phi_1}(\Omega)$ satisfies $\|u_*\|_{W_0^{1,\Phi_1}(\Omega)} = 1$, then*

$$d_n := \sup_{\mathcal{M}_{u_*}^n} J_{\lambda,n} \geq b_n := \inf_{\mathcal{N}_n} J_{\lambda,n} > 0 = \max_{\partial \mathcal{M}_{u_*}^n} J_{\lambda,n},$$

*where*

$$\mathcal{M}_{u_*}^n = \{(\theta u_*, v) \in X_n : \|(\theta u_*, v)\|^2 \leq \rho^2,\ \theta \geq 0\} \text{ and } \mathcal{N}_n = \{(u,0) \in X_n : \|u\|_{W_0^{1,\Phi_1}(\Omega)} = \sigma\}.$$

**Proof.** By definition of the functional $J_{\lambda,n}$,

$$J_{\lambda,n}(u,0) = \int_\Omega \Phi(|\nabla u|)dx - \int_\Omega \Phi_*(|u|)dx - \lambda \int_\Omega R(x,u,0)dx.$$

Note that, integrating the first inequality in (5.42), from 0 to $t$, we obtain

$$|R(x,t,s) - R(x,0,s)| \leq C(P(|t|) + uq(|s|)s) \text{ for all } (x,t,s) \in \Omega \times \mathbb{R}^2$$

Hence, setting $s = 0$, it follows from $(R_1)$ that the inequality above reduces to

$$|R(x,t,0)| \leq CP(|t|), \quad \text{for all } (x,t,0) \in \Omega \times \mathbb{R}^2,$$

for some constant $C > 0$. Thus, by $(R_2)$,

$$J_{\lambda,n}(u,0) \geq \xi^0_{\Phi}(\|u\|_{1,\Phi}) - \xi^1_{\Phi_*}(\|u\|_{\Phi_*}) - C\lambda\xi^1_P(\|u\|_P),$$

where

$$\xi^0_{\Phi_1}(t) = \min\{t^{\ell_1}, t^{m_1}\}, \quad \xi^1_{\Phi_{1*}}(t) = \max\{t^{\ell_1^*}, t^{m_1^*}\} \quad \text{and} \quad \xi^1_P(t) = \max\{t^{p_1}, t^{p_2}\}.$$

Now, remember that by the assumption $(R_2)(i)$ it is possible to show the following limits:

$$\lim_{t\to 0} \frac{P(|t|)}{\Phi_1(|t|)} = 0 \quad \text{and} \quad \lim_{|t|\to+\infty} \frac{P(|t|)}{\Phi_{1*}(|t|)} = 0.$$

Through these two limits we can guarantee the existence of a constant $C_1 > 0$ that does not depend on $u$ so that

$$\|u\|_P \leq C_1\|u\|_{1,\Phi_1}, \quad \forall u \in W_0^{1,\Phi_1}(\Omega).$$

Another important inequality was proved by Donaldson and Trudinger [70], which establishes the existence of a constant $S_0 > 0$ that depends on $N$ such that

$$\|u\|_{\Phi_{1*}} \leq S_0\|u\|_{1,\Phi_1}, \quad \forall u \in W_0^{1,\Phi_1}(\Omega). \tag{5.47}$$

Thus,

$$J_{\lambda,n}(u,0) \geq \|u\|^{m_1}_{1,\Phi_1} - \|u\|^{\ell_1^*}_{1,\Phi_1} - C\lambda\|u\|^{p_1}_{1,\Phi_1}, \quad \text{for } \|u\|_{1,\Phi} < 1$$

Since $m_1 < \ell_1^*$ and $m_1 < p_1$, choose $\rho > 0$ sufficiently small such that

$$J_{\lambda,n}(u,0) \geq C\sigma^{\ell_2}, \quad \text{for } \|u\|_{1,\Phi} = \rho, \tag{5.48}$$

therefore

$$b_n := \inf_{\mathcal{N}_n} J_{\lambda,n} \geq C\sigma^{\ell_2}, \quad \forall n \in \mathbb{N}. \tag{5.49}$$

Now, from $(G_2)$ and $(R_1)$,

$$J_{\lambda,n}(0,v) \leq 0, \quad \forall v \in V^n_A. \tag{5.50}$$

Consider $u_* \in W_0^{1,\Phi_1}(\Omega)$ with $\|u_*\|_{1,\Phi_1}= 1$, by assumptions $(G_2)$ and $(R_1)$,

$$J_{\lambda,n}(\theta u_*, v) \le \int_\Omega \Phi_1(|\nabla(\theta u_*)|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega \Phi_{1*}(|\theta u_*|)dx$$
$$\le \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_*\|_{1,\Phi_1}) - \xi^0_{\Phi_2}(\|v\|_{1,\Phi_2}) - \xi^0_{\Phi_{1*}}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}})$$

for each $\theta > 0$ and $v \in V_A^n$, where

$$\xi^1_{\Phi_1}(t) = \max\{t^{\ell_1}, t^{m_1}\}, \quad \xi^0_{\Phi_{1*}}(t) = \min\{t^{\ell_1^*}, t^{m_1^*}\} \quad \text{and} \quad \xi^0_{\Phi_2}(t) = \max\{t^{\ell_2}, t^{m_2}\}.$$

If $a_2 < m_1^*$, then $A$ increases essentially more slowly than $\Phi_{2*}$ near infinity. From Lemma 2.14 it follows that $L^{\Phi_{2*}}(\Omega)$ is continuously embedded in $L^A(\Omega)$, consequently $W_0^{1,\Phi_1}(\Omega) = V_A$ and as norms $\|\cdot\|_A$ and $\|\cdot\|_{1,\Phi_2}$ are equivalent. Given this, there is a constant $C > 0$ such that

$$J_{\lambda,n}(\theta u_*, v) \le \xi^1_{\Phi_1}(\theta) - C\xi^0_{\Phi_2}(\|v\|_A) - \xi^0_{\Phi_*}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}}),$$

for each $\theta > 0$ and $v \in V_A^n$.

Note that $\|(\theta u_*, v)\|^2 = \theta^2 + \|v\|_A^2 = \rho^2$ implies that

$$\theta^2 \ge \frac{\rho^2}{2} \quad \textit{or} \quad \|v\|_A^2 \ge \frac{\rho^2}{2}.$$

Assume that $\theta^2 \ge \rho^2/2$ occurs, then for $\rho > 0$ large enough, we have

$$\xi^1_{\Phi_1}(\theta) - C\xi^0_{\Phi_2}(\|v\|_A) - \xi^0_{\Phi_{1*}}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}}) = \theta^{m_1} - C\xi^0_{\Phi_2}(\|v\|_A) - \theta^{\ell_1^*}\xi^0_{\Phi_*}(\|u_*\|_{\Phi_*}) < 0,$$

because $m_1 < \ell_1^*$. Similar property happens when $\|v\|_A^2 \ge \rho^2/2$. Therefore, we conclude that there exists $\rho > \sigma$ such that

$$J_{\lambda,n}(\theta u_*, v) \le 0, \tag{5.51}$$

for all $(\theta u_*, v) \in X_n$ so that $\|\theta u_*\|_{1,\Phi_1} + \|v\|_A^2 = \rho^2$ and $\theta > 0$. By (5.50) and (5.51), we have $\max_{\partial \mathcal{M}^n_{u_*}} J_n = 0$, since $(0,0) \in \partial\mathcal{M}^n_{u_*}$, and the proof is complete in this case.

Now, if $a_2 \ge m_1^*$ , from $(G_1)(ii)$ there is a positive constant $C$ such that

$$J_{\lambda,n}(\theta u_*, v) \le \int_\Omega \Phi_1(|\nabla(\theta u_*)|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega \Phi_{1*}(|\theta u_*|)dx - C\int_\Omega A(|v|)dx$$
$$\le \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_*\|_{1,\Phi}) - \xi^0_{\Phi_2}(\|v\|_{1,\Phi_2}) - \xi^0_{\Phi_{1*}}(\theta)\xi^0_{\Phi_{1*}}(\|u_*\|_{\Phi_{1*}}) - \xi^0_A(|v|_A),$$

where

$$\xi_A^0(t) = \min\{t^{a_1}, t^{a_2}\}.$$

Observing that $\|(\theta u_*, v)\|^2 = \theta^2 + \|v\|_A^2 = \rho^2$ implies that

$$\theta^2 \geq \frac{\rho^2}{2}, \ \ \|v\|_{1,\Phi_2}^2 \geq \frac{\rho^2}{4} \quad or \quad |v|_A^2 \geq \frac{\rho^2}{4},$$

the same argument used in the former case implies that for $\rho > 0$ large enough

$$J_n(\theta u_*, v) \leq 0, \tag{5.52}$$

for all $(\theta u_*, v) \in X_n$ so that $\|\theta u_*\|_{1,\Phi_1} + \|v\|_A^2 = \rho^2$ and $\theta > 0$. Therefore, the lemma is proved. ∎

In order to prove the Theorem 1.10, we need to consider that $\Omega_0 \subset \Omega$ be an open set satisfying $(R_4)$ and $u_0 \in W_0^{1,\Phi_1}(\Omega)$ such that

$$u_0 \geq 0, \ \ u_0 \neq 0, \ \ supp(u_0) \subset \Omega_0 \ \text{ and } \ \|u_0\|_{W_0^{1,\Phi_1}(\Omega)} = 1. \tag{5.53}$$

Then, by Lemma 5.8, we can apply the linking theorem A.2 to functional $J_{\lambda,n}$ using a point $z_n = (u_0, 0)$ and the sets

$$Y_n = \{0\} \times V_r^n, \quad Z = W_0^{1,\Phi_1}(\Omega) \times \{0\} \quad \text{and} \quad \mathcal{N}_n = \{(u,0) \in X_1 : \|u\|_{1,\Phi_1} = \sigma\}.$$

Then, there are sequences $(u_k, v_k) \subset X_n$ such that

$$J_{\lambda,n}(u_k, v_k) \to c_{\lambda,n} \quad \text{and} \quad J'_{\lambda,n}(u_k, v_k) \to 0 \ \text{ as } k \to \infty \tag{5.54}$$

where

$$b_n \leq c_{\lambda,n} := \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}_{u_0}^n} J_{\lambda,n}(\gamma(u)), \tag{5.55}$$

$$\Gamma = \{\gamma \in C(\mathcal{M}_{u_0}^n, X_n) : \ \gamma|_{\partial \mathcal{M}_{u_0}^n} = Id_{\partial \mathcal{M}_{u_0}^n}\}.$$

**Lemma 5.9** *The sequence $(u_k, v_k)$ is bounded in $X_n$.*

**Proof.** From (5.54)

$$J_{\lambda,n}(u_k, v_k) - J'_n(u_k, v_k)(\frac{1}{\mu} u_k, \frac{1}{\nu} v_k) = c_{\lambda,n} + o_k(1)\|(u_k, v_k)\|$$

By $(G_2)$, $(R_3)$, $(\phi'_{i,3})$,

$$J_{\lambda,n}(u_k,v_k) - J'_{\lambda,n}(u_k,v_k)(\frac{1}{\mu}u_k,\frac{1}{\nu}v_k) \geq \left(1-\frac{m_1}{\mu}\right)\xi^0_{\Phi_1}(\|u_k\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu}-1\right)\xi^0_{\Phi_2}(\|v_k\|_{1,\Phi_2}),$$

where

$$\xi^0_{\Phi_1}(t) = \min\{t^{\ell_1}, t^{m_1}\} \text{ and } \xi^0_{\Phi_2}(t) = \min\{t^{\ell_2}, t^{m_2}\}.$$

Since $V^n_A$ is a finite dimensional space, the norms $\|\cdot\|_{1,\Phi_2}$ and $\|\cdot\|_A$ are equivalent, hence, from the above inequalities

$$c_{\lambda,n} + o_k(1)\|(u_k,v_k)\| \geq \left(1-\frac{m_1}{\mu}\right)\xi^0_{\Phi_1}(\|u_k\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu}-1\right)\xi^0_{\Phi_2}(C\|v_k\|_A), \tag{5.56}$$

for some $C = C(n) > 0$. Suppose for contradiction that, up to a subsequence, $\|(u_k,v_k)\| \to +\infty$ as $k \to +\infty$. This way, we need to study the following situations:

$(i)$ $\|u_k\|_{1,\Phi_1} \to +\infty$ and $\|v_k\|_A \to \infty$

$(ii)$ $\|u_k\|_{1,\Phi_1} \to +\infty$ and $\|v_k\|_A$ is bounded

$(iii)$ $\|v_k\|_A \to \infty$ and $\|u_k\|_{1,\Phi_1}$ is bounded

In the first case, the inequality (5.56) implies that

$$2c^2_{\lambda,n} + o_k(1)\|(u_k,v_k)\|^2 \geq \left(1-\frac{m_1}{\mu}\right)^2\|u_k\|^{2\ell_1}_{1,\Phi_1} + \left(\frac{\ell_2}{\nu}-1\right)^2\|v_k\|^{2\ell_2}_A.$$

for $k$ large enough. Which is absurd, because $\ell_1 \geq 1$, $\ell_2 \geq 1$ and $o_k(1) \to 0$.

In case $(ii)$, we have for $k$ large enough

$$2c^2_{\lambda,n} + C_1 + o_k(1)\|u_k\|_{1,\Phi} \geq \left(1-\frac{m_1}{\mu}\right)^2\|u_k\|^{2\ell_1}_{1,\Phi_1}$$

an absurd. The last case is similar to the case $(iii)$. ■

**Corollary 5.3** *The following sequences*

$$\{\|u_k\|_{\Phi_{1*}}\}_{k\in\mathbb{N}}, \quad \left\{\int_\Omega \Phi_1(|\nabla u_k|)dx\right\} \quad \textit{and} \quad \left\{\int_\Omega \Phi_{1*}(|\nabla u_k|)dx\right\}$$

*are bounded.*

From Lemma 2.21, Corollary 2.1 and the Lemma 2.14, we may assume that there exists a subsequence of $(u_k,v_k)$, still denoted by itself, and $(w_n,y_n) \in X_n$ such that

$$u_k \xrightarrow{*} w_n \quad \text{in } W^{1,\Phi_1}_0(\Omega) \quad \text{and} \quad v_k \xrightarrow{*} y_n \quad \text{in } V_n, \quad \text{as} \quad k \to \infty \tag{5.57}$$

$$u_k \longrightarrow w_n \quad \text{in } L^{\Phi_{1*}}(\Omega) \tag{5.58}$$

$$\frac{\partial u_k}{\partial x_i} \xrightarrow{*} \frac{\partial w_n}{\partial x_i} \quad \text{in } L^{\Phi_1}(\Omega),\ i \in \{1,\cdots,N\}. \tag{5.59}$$

$$(5.60) \qquad u_k \longrightarrow w_n \quad \text{in } L^{\Phi_1}(\Omega),$$

and

$$(5.61) \qquad u_k(x) \longrightarrow w_n(x) \quad \text{a.e. } \Omega.$$

In view of (5.57) and Corollary 5.3, for each $n \in \mathbb{N}$ we may assume that there exist nonnegative functions $\mu_n, \nu_n \in \mathcal{M}(\mathbb{R}^N)$, the space of Radon measures, such that

$$(5.62) \quad \Phi(|\nabla u_k|) \overset{*}{\rightharpoonup} \mu_n \quad \text{in } \mathcal{M}(\mathbb{R}^N) \text{ and } \Phi_*(|u_k|) \overset{*}{\rightharpoonup} \nu_n \quad \text{in } \mathcal{M}(\mathbb{R}^N) \text{ as } k \to \infty.$$

The result below is known as second concentration-compactness lemma of P. L. Lions. We would like to point out that also this lemma holds for nonreflexive Orlicz-Sobolev space. The proof of this fact can be seen [60 Proposition 4.3].

**Lemma 5.10** *(i) For every $n \in \mathbb{N}$ and $\lambda > 0$, there exist an at most countable set $I_\lambda$, a family $\{x_i\}_{i \in I_\lambda}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_i\}_{i \in I_\lambda}$ of constant $\nu_i > 0$ such that*

$$(5.63) \qquad \nu_n = \Phi_{1*}(|w_n|) + \sum_{i \in I_\lambda} \nu_i \delta_{x_i}.$$

*(ii) In addition we have*

$$(5.64) \qquad \mu_n \geq \Phi_1(|\nabla w_n|) + \sum_{i \in I_\lambda} \mu_i \delta_{x_i},$$

*for some $\mu_j > 0$ satisfying*

$$(5.65) \qquad 0 < \nu_j \leq \max\{S_0^{\ell_1^*} \mu_i^{\ell_1^*/\ell_1}, S_0^{m_1^*} \mu_i^{m_1^*/\ell_1}, S_0^{\ell_1^*} \mu_i^{\ell_1^*/m_1}, S_0^{m_1^*} \mu_i^{m_1^*/m_1}\}$$

*for all $i \in I_\lambda$, where $\delta_{x_i}$ is the Dirac measure of mass 1 concentrated at $x_i$.*

**Lemma 5.11** *The set $\{x_i\}_{i \in I_\lambda}$ in Lemma 5.10 is a finite set.*

**Proof.** Let an $x_i$ be fixed. Take $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$0 \leq \varphi \leq 1, \quad \varphi(x) = 1 \text{ in } B_1(0) \quad \text{and} \quad \varphi(x) = 0 \text{ in } \mathbb{R}^N \backslash B_2(0)$$

and put $\varphi_\varepsilon(x) = \varphi((x - x_i)/\varepsilon)$ for $\varepsilon > 0$. It is clear that $\{\varphi_\varepsilon u_k\}_{k \in \mathbb{N}}$ is bounded in $W_0^{1,\Phi_1}(\Omega)$, thus $J'_{\lambda,n}(u_k, v_k)(\varphi_\varepsilon u_k, 0) = o_k(1)$, that is,

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \cdot \nabla(\varphi_\varepsilon u_k) dx = \int_\Omega \phi_{1*}(|u_k|) u_k (\varphi_\varepsilon u_k) dx + \lambda \int_\Omega R_u(x, u_k, v_k)(\varphi_\varepsilon u_k) dx + o_k(1).$$

Knowing that

$$\int_\Omega R_u(x,u_k,v_k)(\varphi_\varepsilon u_k)dx \longrightarrow \int_\Omega R_u(x,w_n,y_n)(\varphi_\varepsilon w_n)dx \ \text{ as } k\to\infty,$$

we can conclude that

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\cdot\nabla(\varphi_\varepsilon u_k)dx \le m_1^*\int_\Omega \Phi_{1*}(|u_k|)\varphi_\varepsilon dx + \lambda\int_\Omega R_u(x,w_n,y_n)(\varphi_\varepsilon w_n)dx + o_k(1). \tag{5.66}$$

By $(\phi'_{i,3})$,

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\cdot\nabla(\varphi_\varepsilon u_k)dx \ge \ell_1\int_\Omega \Phi_1(|\nabla u_k|)\varphi_\varepsilon dx + \int_\Omega \phi_1(|\nabla u_k|)u_k\nabla u_k\nabla\varphi_\varepsilon dx.$$

Therefore,

$$\begin{aligned}\ell_1\int_\Omega \Phi_1(|\nabla u_k|)\varphi_\varepsilon dx + \int_\Omega \phi_1(|\nabla u_k|)u_k\nabla u_k\nabla\varphi_\varepsilon dx \le & m_1^*\int_\Omega \Phi_{1*}(|u_k|)\varphi_\varepsilon dx \\ & + \lambda\int_\Omega R_u(x,w_n,y_n)(\varphi_\varepsilon w_n)dx + o_k(1)\end{aligned} \tag{5.67}$$

Since the sequence $(\phi_1(|\nabla u_k|)\dfrac{\partial u_k}{\partial x_j})_{k\in\mathbb{N}}$ is limited to $L^{\tilde{\Phi}_1}(\Omega)$, there is $\omega_j\in L^{\tilde{\Phi}_1}(\Omega)$ such that

$$\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j} \overset{*}{\rightharpoonup} \omega_j \ \text{ in } L^{\tilde{\Phi}_1}(\Omega), \quad j\in\{1,\cdots,N\} \tag{5.68}$$

for some subsequence. Keep in mind that

$$u_k\frac{\partial\varphi_\varepsilon}{\partial x_j} \longrightarrow w_n\frac{\partial\varphi_\varepsilon}{\partial x_j} \ \text{ in } L^{\Phi_1}(\Omega), \quad j\in\{1,\cdots,N\}$$

we conclude that

$$\int_\Omega u_k\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j}\frac{\partial\varphi_\varepsilon}{\partial x_j}dx \longrightarrow \int_\Omega w_n\phi_1(|\nabla w_n|)\frac{\partial w_n}{\partial x_j}\frac{\partial\varphi_\varepsilon}{\partial x_j}dx, \quad j\in\{1,\cdots,N\}. \tag{5.69}$$

Therefore, considering $\omega=(\omega_1,\cdots,\omega_N)$ we get

$$\int_\Omega \phi_1(|\nabla u_k|)u_k\nabla u_k\nabla\varphi_\varepsilon dx - \int_\Omega u_n\omega\nabla\varphi_\varepsilon dx = o_k(1). \tag{5.70}$$

From (5.67) and (5.70),

$$\begin{aligned} m_1^*\int_\Omega \Phi_{1*}(|u_k|)\varphi_\varepsilon dx + \lambda\int_\Omega R_u(x,u_n,v_n)(\varphi_\varepsilon u_n)dx \ge & \ell_1\int_\Omega \Phi_1(|\nabla u_k|)\varphi_\varepsilon dx \\ & + \int_\Omega u_n\omega\nabla\varphi_\varepsilon dx + o_k(1).\end{aligned} \tag{5.71}$$

By (5.62), taking the limit of $k \to +\infty$, we get

$$(5.72)\qquad \begin{aligned} m_1^* \int_\Omega \varphi_\varepsilon d\nu_n + \lambda \int_\Omega R_u(x, u_n, v_n)\varphi_\varepsilon u_n dx \geq & \ell_1 \int_\Omega \varphi_\varepsilon d\mu_n \\ & + \int_\Omega u_n w \nabla \varphi_\varepsilon dx + o_k(1). \end{aligned}$$

On the other hand, given $v \in W_0^{1,\Phi_1}(\Omega)$, it follows from (5.54) that

$$o_k(1) = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla v dx - \int_\Omega \phi_{1*}(|u_k|)u_k v dx - \lambda \int_\Omega R_u(x, u_k, v_k) v dx,$$

since the sequence $(\phi_{1*}(|u_k|)u_k)$ is bounded in $L^{\tilde{\Phi}_{1*}}(\Omega)$, there is $\eta_n \in L^{\tilde{\Phi}_{1*}}(\Omega)$ such that

$$(5.73)\qquad \phi_{1*}(|u_k|)u_k \rightharpoonup \eta_n \text{ in } L^{\tilde{\Phi}_{1*}}(\Omega) \text{ as } k \to \infty$$

so, from (5.68) and (5.73),

$$\int_\Omega \omega \nabla v dx - \int_\Omega \eta_n v dx - \lambda \int_\Omega R_u(x, u_n, v_n) v dx = 0, \quad \forall v \in W_0^{1,\Phi_1}(\Omega).$$

In particular, for $v = u_n \varphi_\varepsilon$, we have

$$(5.74)\qquad \int_\Omega u_n \omega \nabla \varphi_\varepsilon dx = \int_\Omega \eta_n u_n \varphi_\varepsilon dx + \lambda \int_\Omega R_u(x, u_n, v_n) u_n \varphi_\varepsilon dx - \int_\Omega \varphi_\varepsilon \omega \nabla u_n dx,$$

Therefore,

$$(5.75)\qquad \lim_{\varepsilon \to 0} \int_\Omega u_n(\omega \nabla \varphi_\varepsilon) dx = 0$$

It follows from (5.72) and (5.75) that

$$(5.76)\qquad \ell_1 \mu_i \leq m_1^* \nu_i, \quad i \in I_\lambda,$$

and by Lemma 5.10,

$$(5.77)\qquad \ell_1 \mu_i \leq m_1^* S_0^\beta \mu_i^\alpha,$$

for some $\alpha$ and $\beta$ verifying

$$(5.78)\qquad 1 < \alpha \in \left\{ \frac{\ell_1^*}{\ell_1}, \frac{m_1^*}{\ell_1}, \frac{\ell_1^*}{m_1}, \frac{m_1^*}{m_1} \right\} \quad \text{and} \quad \beta \in \{\ell_1^*, m_1^*\}.$$

Thereby,

$$0 < \frac{\ell_1}{m_1^* S_0^\beta} \leq \mu_i^{\alpha - 1}, \quad i \in I_\lambda,$$

showing that

$$\mu_i \geq \Big(\frac{\ell_1}{m_1^* S_0^\beta}\Big)^{\frac{1}{\alpha-1}}, \quad i \in I_\lambda. \tag{5.79}$$

By (5.76) and (5.79)

$$\nu_i \geq \Big(\frac{\ell_1}{m_1^*}\Big)^{\frac{\alpha}{\alpha-1}} S_0^{-\frac{\beta}{\alpha-1}}, \quad i \in I_\lambda.$$

Since $\nu_n$ is a finite measure, the last inequality yields $I_\lambda$ is finite. ■

In fact we will see that the set $\{x_i\}_{i\in I_\lambda}$ is an empty set for $\lambda > 0$ large enough. For this, we will establishes an important estimate from above of the mountain level of functional $J_{\lambda,n}$.

**Lemma 5.12** *Let $n \in \mathbb{N}$ be arbitrary and consider $u_0$ given in (5.53). Then, there is $\lambda_0 > 0$ such that if $\lambda > \lambda_0$, we have that*

$$\max_{\mathcal{M}_{u_0}^n} J_{\lambda,n} < \omega,$$

*where*

$$\omega := \left(1 - \frac{m_1}{\mu}\right) \min\left\{ \left(\frac{\ell_1}{m_1^* S_0^\beta}\right)^{\frac{1}{\alpha-1}} : \alpha \in \left\{\frac{\ell_1^*}{\ell_1}, \frac{m_1^*}{\ell_1}, \frac{\ell_1^*}{m_1}, \frac{m_1^*}{m_1}\right\} \quad \textit{and} \quad \beta \in \{\ell_1^*, m_1^*\} \right\}.$$

*Consequently,*

$$c_{\lambda,n} < \left(1 - \frac{m_1}{\mu}\right) \min\left\{ \left(\frac{\ell_1}{m_1^* S_0^\beta}\right)^{\frac{1}{\alpha-1}} : \alpha \in \left\{\frac{\ell_1^*}{\ell_1}, \frac{m_1^*}{\ell_1}, \frac{\ell_1^*}{m_1}, \frac{m_1^*}{m_1}\right\} \quad \textit{and} \quad \beta \in \{\ell_1^*, m_1^*\} \right\}, \tag{5.80}$$

*where $c_{\lambda,n}$ is given in (5.55).*

**Proof.** By $(R_4)$, given $\theta \geq 0$ and $v \in V_A^n$, we have

$$\begin{aligned} J_{\lambda,n}(\theta u_0, v) &\leq \xi_{\Phi_1}^1(\theta)\xi_{\Phi_1}^1(\|u_0\|_{1,\Phi_1}) - \xi_{\Phi_{1*}}^0(\theta)\xi_{\Phi_1}^1(\|u_0\|_{\Phi_1}) - \lambda \int_{\Omega_0} R(x, \theta u_0, v)dx \\ &\leq \xi_{\Phi_1}^1(\theta)\xi_{\Phi_1}^1(\|u_0\|_{1,\Phi_1}) - \xi_{\Phi_{1*}}^0(\theta)\xi_{\Phi_1}^1(\|u_0\|_{\Phi_1}) - \lambda\omega \int_{\Omega_0} |\theta u_0|^s dx \end{aligned}$$

This inequality implies that

$$0 < b_n \leq \inf_{\gamma\in\Gamma} \max_{u\in\mathcal{M}_{u_0}^n} J_{\lambda,n}(\gamma(u)) \leq \max_{\mathcal{M}_{u_0}^n} J_{\lambda,n} \leq \max_{\theta\geq 0} \mathcal{V}_\lambda(\theta) \tag{5.81}$$

where

$$\mathcal{V}_\lambda(\theta) = \xi^1_{\Phi_1}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \xi^0_{\Phi_{1*}}(\theta)\xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) - \lambda\omega \int_{\Omega_0} |\theta u_0|^s dx. \tag{5.82}$$

In what follows, we denote by $\theta_\lambda > 0$ the real number verifying

$$\max_{\theta\geq 0} \mathcal{V}_\lambda(\theta) = \mathcal{V}_\lambda(\theta_\lambda) \tag{5.83}$$

Let us see that $\mathcal{V}_\lambda(\theta_\lambda) \to 0$ as $\lambda \to \infty$. For that, consider $(\lambda_m)$ a sequence verifying

$$\lambda_m \longrightarrow \infty \quad \text{as} \quad m \to \infty.$$

We claim that $(\theta_{\lambda_m})$ is a bounded sequence. Indeed, assuming by contradiction that $(\theta_{\lambda_m})$ is not bounded, we have that for a subsequence, still denote by itself,

$$\theta_{\lambda_m} \to \infty \quad \text{as} \quad m \to \infty.$$

According to (5.82), (5.81) and (5.83),

$$0 < \max_{\theta\geq 0} \mathcal{V}_{\lambda_m}(\theta) = \mathcal{V}_{\lambda_m}(\theta_{\lambda_m}) \leq \theta^{m_1}_{\lambda_m}\xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \theta^{\ell_1^*}_{\lambda_m}\xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) \to -\infty \quad \text{as} \quad m \to \infty,$$

which is an absurd, and so, $(\theta_{\lambda_m})$ is bounded. We claim that $\theta_{\lambda_m} \to 0$ as $m \to \infty$. If the above limit does not hold, we can assume by contradiction, that for some subsequence, still denote by $(\theta_{\lambda_m})$, there is $k_0 > 0$ such that

$$\theta_{\lambda_m} > k_0 > 0, \quad \forall n \in \mathbb{N}$$

Then, by (5.82) and (5.83),

$$0 < \max_{\theta\geq 0} \mathcal{V}_{\lambda_m}(\theta) = \mathcal{V}_{\lambda_m}(\theta_{\lambda_m}) \leq \xi^1_{\Phi_1}(\theta_{\lambda_m})\xi^1_{\Phi_1}(\|u_0\|_{1,\Phi_1}) - \xi^0_{\Phi_{1*}}(\theta_{\lambda_m})\xi^1_{\Phi_1}(\|u_0\|_{\Phi_1}) - \lambda_m\omega \int_{\Omega_0} |\theta_{\lambda_m} u_0|^s dx,$$

thus

$$0 < \max_{\theta\geq 0} \mathcal{V}_{\lambda_m}(\theta) \to -\infty \quad \text{as} \quad m \to \infty,$$

which is a contradiction. Hence,

$$t_{\lambda_m} \to 0 \quad \text{as} \quad m \to \infty,$$

which leads to

$$\max_{\theta\geq 0}\mathcal{V}_{\lambda_m}(\theta)=\mathcal{V}_{\lambda_m}(\theta_{\lambda_m})\to 0 \quad \text{as} \quad m\to\infty,$$

and by (5.81) it follows that

$$c_{\lambda_m,n}\to 0 \quad \text{as} \quad m\to\infty.$$

■

**Lemma 5.13** *For every $n\in\mathbb{N}$ and $\lambda>\lambda_0$ the set $I_\lambda$ is empty, where $\lambda_0$ was given in Lemma 5.12.*

**Proof.** Let $(u_k,u_k)\subset X_n$ the $(PS)_{c_{\lambda,n}}$ sequence obtained in (5.54). In view of assumptions $(G_2)$, $(R_3)$, $(\phi'_{i,3})$, we have that

$$c_{\lambda,n}+o_k(1)\geq J_n(u_k,v_k)-J'_{\lambda,n}(u_k,v_k)(\frac{1}{\mu}u_k,\frac{1}{\nu}v_k)\geq\left(1-\frac{m_1}{\mu}\right)\int_\Omega\Phi_1(|\nabla u_k|)dx.$$

Fixing a function $\varphi\in C_0^\infty(\mathbb{R}^N)$ with $\varphi(x)=1$ on $\overline{\Omega}$, we derive that

$$c_{\lambda,n}+o_k(1)\geq\left(1-\frac{m_1}{\mu}\right)\int_{\mathbb{R}^N}\Phi_1(|\nabla u_k|)\varphi dx.$$

Taking the limit of $k\to+\infty$, we get

$$c_{\lambda,n}\geq\left(1-\frac{m_1}{\mu}\right)\int_{\mathbb{R}^N}\varphi d\mu_n\geq\left(1-\frac{m_1}{\mu}\right)\mu_n(\overline{\Omega}).$$

Supposing that $I_\lambda$ is not empty, there is $i\in I_\lambda$, and so,

$$c_{\lambda,n}\geq\left(1-\frac{m_1}{\mu}\right)\mu_i.$$

In (5.79) we show that

$$\mu_i\geq(\frac{\ell_1}{m_1^*S_0^\beta})^{\frac{1}{\alpha-1}},\ i\in I_\lambda,$$

where $\alpha$ is given in (5.77) and (5.78). Therefore, we can conclude that

$$c_{\lambda,n}\geq\left(1-\frac{m_1}{\mu}\right)\min\left\{\left(\frac{\ell_1}{m_1^*S_0^\beta}\right)^{\frac{1}{\alpha-1}}:\alpha\in\left\{\frac{\ell_1^*}{\ell_1},\frac{m_1^*}{\ell_1},\frac{\ell_1^*}{m_1},\frac{m_1^*}{m_1}\right\}\ \text{ and }\ \beta\in\{\ell_1^*,m_1^*\}\right\}.$$

Then, if $\lambda\geq\lambda_0$, the last inequality together with Lemma 5.12 yields $I_\lambda=\emptyset$.

■

**Lemma 5.14** *For $\lambda \geq \lambda_0$, the sequence $(u_k)$ is strongly convergent for its weak limit $w_n$ in $L^{\Phi_{1*}}(\Omega)$ as $k \to \infty$.*

**Proof.** Let $\varphi \in C^\infty(\mathbb{R}^N)$ be a function verifying $\varphi(x) = 1$, for $x \in \Omega$. In this case,

$$\lim_{k\to\infty} \int_\Omega \Phi_{1*}(u_k)dx = \lim_{k\to\infty} \int_{\mathbb{R}^N} \Phi_{1*}(u_k)\varphi dx = \int_{\mathbb{R}^N} \varphi d\nu_n$$

The Lemma 5.10(item i) combined with Lemma 5.13 gives

$$\lim_{k\to\infty} \int_\Omega \Phi_{1*}(u_k)dx = \int_{\mathbb{R}^N} \Phi_{1*}(u_n)\varphi dx = \int_\Omega \Phi_{1*}(u_n)dx.$$

Since $\Phi_{1*}$ is a convex function, it follows from a result due to Brezis and Lieb [28] that

$$\lim_{k\to\infty} \int_\Omega \{\Phi_{1*}(|u_k|) - \Phi_{1*}(|u_k - u_n|) - \Phi_{1*}(|u_n|)\}dx = 0.$$

Then,

$$\lim_{k\to\infty} \int_\Omega \Phi_{1*}(|u_k - u_n|)dx = 0,$$

we can conclude that $(u_k)$ converges strongly for $u_n$ in $L^{\Phi_{1*}}(\Omega)$. ■

**Lemma 5.15** *Consider $\lambda > \lambda_0$ and $(u_k) \subset W_0^{1,\Phi_1}(\Omega)$ the sequence obtained in (5.54). Then, for some subsequence, still denoted by itself,*

$$u_k \to w_n \ \ \textit{in } W_0^{1,\Phi_1}(\Omega) \ \ \textit{as} \ \ k \to \infty.$$

**Proof.** Since $(u_k)$ is a bounded sequence in $W_0^{1,\Phi_1}(\Omega)$, then

$$o_k(1) = \int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla(v-u_k)dx - \int_\Omega \phi_{1*}(|u_k|)u_k(v-u_k)dx - \lambda \int_\Omega R_u(x,u_k,v_k)(v-u_k)dx.$$

Given $v \in W_0^{1,\Phi_1}(\Omega)$, by the convexity of $\Phi_1$ it follows that

$$\Phi_1(|\nabla v|) - \Phi_1(|\nabla u_k|) \geq \phi_1(|\nabla u_k|)\nabla u_k \nabla(v - u_k),$$

thus,

$$\begin{aligned}\int_\Omega \Phi_1(|\nabla v|)dx - \int_\Omega \Phi_1(|\nabla u_k|)dx \geq & \int_\Omega \phi_{1*}(|u_k|)u_k(v-u_k)dx \\ & - \lambda \int_\Omega R_u(x,u_k,v_k)(v-u_k)dx + o_k(1).\end{aligned} \tag{5.84}$$

Through boundedness da sequence $(u_k)$ in $W_0^{1,\Phi_1}(\Omega)$ together with the limits

$$u_k(x) \longrightarrow w_n \quad \text{a.e. in } \Omega \quad \text{and} \quad \frac{\partial u_k}{\partial x_i} \rightharpoonup \frac{\partial w_n}{\partial x_i} \quad \text{in } L^1(\Omega),$$

we can apply the Lemma 2.22 to get

$$\liminf_{k\to\infty}\int_\Omega \Phi_1(|\nabla u_k|)dx \geq \int_\Omega \Phi_1(|\nabla w_n|)dx.$$

Furthermore, since $(u_k)$ strongly converges to $u_n$ in $L^{\Phi_{1*}}(\Omega)$,

$$\int_\Omega \phi_{1*}(|u_k|)u_k(v-u_k)dx \to \int_\Omega \phi_{1*}(|w_n|)w_n(v-w_n)dx, \quad \text{as} \quad k\to\infty.$$

Therefore, it follows from (5.84) that

$$\begin{aligned}\int_\Omega \Phi_1(|\nabla v|)dx - \int_\Omega \Phi_1(|\nabla w_n|)dx \geq &\int_\Omega \phi_{1*}(|w_n|)w_n(v-w_n)dx\\ &+\lambda\int_\Omega R_u(x,w_n,y_n)(v-w_n)dx.\end{aligned}$$

By arbitrariness $v$ we can conclude that

$$\begin{aligned}\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n\nabla(w_n-u_k)dx = &\int_\Omega \phi_{1*}(|w_n|)w_n(w_n-u_k)dx\\ &+\lambda\int_\Omega R_u(x,w_n,y_n)(w_n-u_k)dx,\end{aligned}$$

implying that

$$\int_\Omega \phi_1(|\nabla w_n|)\nabla w_n\nabla(w_n-u_k)dx = o_k(1). \tag{5.85}$$

On the other hand, since $(u_k,v_k)$ is a sequence $(PS)_{c_{\lambda,n}}$,

$$\begin{aligned}o_k(1) =&J'_{\lambda,n}(u_k,v_k)(w_n-u_k,0)\\ =&\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\nabla(w_n-u_k)dx - \int_\Omega \phi_{1*}(|u_k|)u_k(w_n-u_k)dx\\ &-\lambda\int_\Omega R_u(x,u_k,v_k)(w_n-u_k)dx,\end{aligned}$$

Therefore,

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k\nabla(w_n-u_k)dx = o_k(1). \tag{5.86}$$

From (5.85) and (5.86),

$$\int_\Omega \big(\phi_1(|\nabla u_k|)\nabla u_k - \phi_1(|\nabla w_n|)\nabla w_n\big)\big(\nabla u_k-\nabla w_n\big)dx \longrightarrow 0 \;\; \text{as } k\to\infty.$$

Now, applying a result due to Dal Maso and Murat [22], it follows that

$$\nabla u_k(x) \longrightarrow \nabla w_n(x) \;\; \text{a.e. in } \Omega \;\; \text{as } k\to\infty. \tag{5.87}$$

Since $(u_k)$ is bounded in $W_0^{1,\Phi_1}(\Omega)$ and $\Phi_1 \in (\Delta_2)$, then the sequence $(\phi_1(|\nabla u_k|)\dfrac{\partial u_k}{\partial x_j})_{k\in\mathbb{N}}$ is bounded by $L^{\tilde{\Phi}_1}(\Omega)$, for each $j \in \{1, \cdots, N\}$. Furthermore, by (5.87), it follows that

$$\phi_1(|\nabla u_k(x)|)\frac{\partial u_k(x)}{\partial x_j} \to \phi_1(|\nabla w_n(x)|)\frac{\partial w_n(x)}{\partial x_j} \quad \text{a.e. in } \Omega \quad \text{as} \quad k \to \infty.$$

Thus, by Lemma 2.5 in [11],

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla v dx \to \int_\Omega \phi_1(|\nabla u_n|)\nabla u_n \nabla v dx, \quad v \in C_0^\infty(\Omega) \quad \text{as} \quad k \to \infty. \tag{5.88}$$

Still due to the boundedness of the sequence $(\phi_1(|\nabla u_k|)\dfrac{\partial u_k}{\partial x_j})_{k\in\mathbb{N}}$ in $L^{\tilde{\Phi}_1}(\Omega)$, there will be $v_j \in L^{\tilde{\Phi}_1}(\Omega)$ such that

$$\phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j} \overset{*}{\rightharpoonup} v_j \quad \text{in } L^{\tilde{\Phi}_1}(\Omega) \quad \text{as} \quad k \to \infty,$$

i.e,

$$\int_\Omega \phi_1(|\nabla u_k|)\frac{\partial u_k}{\partial x_j} w dx \to \int_\Omega v_j w dx, \ \forall w \in E^{\Phi_1}(\Omega) = L^{\Phi_1}(\Omega) \quad \text{as} \quad k \to \infty. \tag{5.89}$$

By (5.88) and (5.89), it follows that $v_j = \phi_1(|\nabla u_n|)\dfrac{\partial u_n}{\partial x_j}$, for each $j \in \{1, \cdots, N\}$. Still from (5.89),

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla w dx \to \int_\Omega \phi_1(|\nabla u_n|)\nabla u_n \nabla w dx, \quad \forall w \in W_0^{1,\Phi_1}(\Omega) \quad \text{as} \quad k \to \infty.$$

We know from (5.86) that

$$\int_\Omega \phi_1(|\nabla u_k|)\nabla u_k \nabla(u_n - u_k) dx = o_k(1),$$

then

$$\int_\Omega \phi_1(|\nabla u_k|)|\nabla u_k|^2 dx \to \int_\Omega \phi_1(|\nabla u_n|)|\nabla u_n|^2 dx \quad \text{as} \quad k \to \infty.$$

Given this, we can conclude that

$$\phi_1(|\nabla u_k|)|\nabla u_k|^2 \to \phi_1(|\nabla u_n|)|\nabla u_n|^2 \quad \text{in } L^1(\Omega) \quad \text{as} \quad k \to \infty.$$

By $(\phi'_{i,3})$ together with the $\Delta_2$-condition, it follows that

$$u_k \to u_n \quad \text{in } W_0^{1,\Phi_1}(\Omega) \quad \text{as} \quad k \to \infty.$$

This finishes the proof. ■

**Lemma 5.16** *For $\lambda > \lambda_0$, the sequence $(w_n, y_n)$ is bounded in $X$. Moreover*

$$J_{\lambda,n}(w_n, y_n) = c_{\lambda,n} \quad \textit{and} \quad J'_{\lambda,n}(w_n, y_n) = 0 \ \textit{in}\ X_n^*. \tag{5.90}$$

**Proof.** Since $V_A^n$ is a finite dimensional space, $(v_k)$ converges strongly to $(y_n)$ in $V_A^n$. Therefore,

$$(u_k, v_k) \longrightarrow (w_n, y_n) \quad \text{in } X_n \quad \text{as} \quad k \to \infty.$$

which implies

$$J_{\lambda,n}(w_n, y_n) = c_{\lambda,n} \in [b_n, d_n] \quad \text{and} \quad J'_{\lambda,n}(w_n, y_n) = 0 \quad \text{in } X_n^*.$$

In a first moment, let us assume that $a_2 < \ell_2^*$. By hypothesis $m_2 < a_1$, then $W_0^{1,\Phi_2}(\Omega)$ is continuously embedded in $L^A(\Omega)$, thus, there will be $C > 0$ such

$$\|v\|_A \leq C\|v\|_{1,\Phi_2}, \quad \forall v \in V_A^n. \tag{5.91}$$

By $(G_2)(ii)$, $(R_3)$ and (5.91),

$$\begin{aligned} c_{\lambda,n} =& J_{\lambda,n}(w_n, y_n) - J'_{\lambda,n}(w_n, y_n)(\frac{1}{\mu}w_n, \frac{1}{\nu}y_n) \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\int_\Omega \Phi_1(|\nabla w_n|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla y_n|)dx + \left(\frac{\ell_1^*}{\mu} - 1\right)\int_\Omega \Phi_{1*}(|\nabla w_n|)dx \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\xi_\Phi^0(\|w_n\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu} - 1\right)\xi_{\Phi_2}^0(\frac{1}{C}\|y_n\|_A). \end{aligned} \tag{5.92}$$

It follows from the inequalities (5.80) and (5.92) that $(w_n, y_n)$ is bounded in $X$.

Now, let us assume that $a_2 \geq \ell_2^*$. By $(R_3)$, $(G_2)$, (5.91) and from items $(i) - (iii)$ of $(G_1)$, it follows that

$$\begin{aligned} c_{\lambda,n} =& J_{\lambda,n}(u_n, v_n) - J'_{\lambda,n}(u_n, v_n)(\frac{1}{\mu}u_n, \frac{1}{\nu}v_n) \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\int_\Omega \Phi_1(|\nabla u_n|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla v_n|)dx \\ &+ \frac{C}{\nu}\int_\Omega a(|v_n|)|v_n|^2 dx - Ca_1\int_\Omega A(v_n)dx \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\int_\Omega \Phi_1(|\nabla u_n|)dx + \left(\frac{\ell_2}{\nu} - 1\right)\int_\Omega \Phi_2(|\nabla v_n|)dx + \left(\frac{a_1 C}{\nu} - a_1 C\right)\int_\Omega A(|v_n|)dx \\ \geq& \left(1 - \frac{m_1}{\mu}\right)\xi_{\Phi_1}^0(\|u_n\|_{1,\Phi_1}) + \left(\frac{\ell_2}{\nu} - 1\right)\xi_{\Phi_2}^0(\|v_n\|_{1,\Phi_2}) + \left(\frac{a_1 C}{\nu} - a_1 C\right)\xi_A^0(|v_n|_A). \end{aligned}$$

From the above inequality together with (5.80), we can conclude that $(w_n, y_n)$ is bounded by $X$. ∎

### 5.2.1 Proof of Theorem 1.10

The proof of Theorem 1.10 will be carried out in three lemmas. We start observing that since $(w_n, y_n)$ is bounded, there is no loss of generality in assuming that

$$(w_n, y_n) \xrightarrow{*} (u, v) \quad \text{in } X \ \text{ as } \ n \to \infty. \tag{5.93}$$

The same arguments used in the proof of Lemma 5.15 can be repeated to show that

$$u_n \to u \quad \text{in } W_0^{1,\Phi_1}(\Omega) \ \text{ as } \ n \to \infty. \tag{5.94}$$

By the limit (5.93), it follows that

$$y_n \xrightarrow{*} v \quad \text{in } L^A(\Omega) \tag{5.95}$$

and

$$y_n \xrightarrow{*} v \quad \text{in } W_0^{1,\Phi_2}(\Omega). \tag{5.96}$$

**Lemma 5.17** *For $\lambda > \lambda_0$, the sequence $(y_n)$ verifies the following limit $y_n \to v$ in $L^A(\Omega)$.*

**Proof.** From (5.46), there is $(\xi_k) \subset V_A$ such that

$$\xi_k \to v \ \text{ in } \ V_A \tag{5.97}$$

and

$$\xi_k = \sum_{i=1}^{j(k)} \alpha_i e_i \in V_A^{j(k)},$$

where $j(k) \in \mathbb{N}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, it follows that

$$V_A^{j(k)} \subset V_A^n \ \text{ for all } \ n \geq n_0,$$

for some $n_0 \geq j(k)$.

If $a_2 \geq \ell_2^*$, from $(G_1)$, we have that there is $C > 0$ such that

(5.98)

$$a_1 C \int_\Omega A(|y_n - \xi_k|)dx \leq C \int_\Omega a(|y_n - \xi_k|)|y_n - \xi_k|^2 dx \leq \int_\Omega (g(y_n) - g(\xi_k))(y_n - \xi_k)dx.$$

Since $J'_{\lambda,n}(u_n, v_n) = 0$ in $X_n^*$, we derive that

(5.99)
$$\int_\Omega (g(y_n) - g(\xi_k))(y_n - \xi_k)dx = \int_\Omega \phi_2(|\nabla y_n|)\nabla y_n(\nabla \xi_k - \nabla y_n)dx - \lambda \int_\Omega R_v(x, w_n, y_n)y_n dx$$
$$+ \lambda \int_\Omega R_v(x, w_n, y_n)\xi_k dx - \int_\Omega g(\xi_k)(y_n - \xi_k)dx.$$

Due to the convexity of $\Phi_2$, we have

$$\Phi_2(|\nabla \xi_k|) - \Phi_2(|\nabla y_n|) \geq \phi_2(|\nabla y_n|)\nabla y_n \nabla(\xi_k - y_n), \quad n \in \mathbb{N}. \tag{5.100}$$

It follows from the above inequalities that

(5.101)
$$a_1 C \int_\Omega A(|y_n - \xi_k|)dx \leq \int_\Omega \Phi_2(|\nabla \xi_k|)dx - \int_\Omega \Phi_2(|\nabla w_n|)dx - \lambda \int_\Omega R_v(x, w_n, y_n)y_n dx$$
$$+ \lambda \int_\Omega R_v(x, w_n, y_n)\xi_k dx - \int_\Omega g(\xi_k)(y_n - \xi_k)dx.$$

Knowing that

$$y_n(x) \longrightarrow v(x) \ \text{ a.e. in } \Omega \quad \text{and} \quad \frac{\partial y_n}{\partial x_i} \rightharpoonup \frac{\partial v}{\partial x_i} \ \text{ in } L^1(\Omega),$$

we can apply the Lemma 2.22 to get

$$\liminf_{n\to\infty} \int_\Omega \Phi_2(|\nabla v_n|)dx \geq \int_\Omega \Phi_2(|\nabla v|)dx. \tag{5.102}$$

Taking as limit $n \to \infty$, it follows that

(5.103)
$$\limsup_{n\to\infty} \left( a_1 C \int_\Omega A(|y_n - \xi_k|)dx \right) \leq \int_\Omega \Phi_2(|\nabla \xi_k|)dx - \int_\Omega \Phi_2(|\nabla v|)dx - \int_\Omega R_v(x, u, v)v dx$$
$$+ \int_\Omega R_v(x, u, v)\xi_k dx - \int_\Omega g(\xi_k)(v - \xi_k)dx.$$

By the limit (5.97), given $\delta > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\frac{1}{a_1 C}\Bigg[ \int_\Omega \Phi_2(|\nabla \xi_k|)dx - \int_\Omega \Phi_2(|\nabla u|)dx - \int_\Omega R_v(x, u, v)v dx$$
$$+ \int_\Omega R_v(x, u, v)\xi_k dx - \int_\Omega g(\xi_k)(v - \xi_k)dx \Bigg] < \frac{\delta}{2},$$

for each $k \geq k_0$. Hence,

$$\limsup_{n\to\infty} \int_\Omega A(|y_n - \xi_k|)dx \leq \frac{\delta}{2}, \quad \text{for all } k \geq k_0. \tag{5.104}$$

Given $0 < \varepsilon < 4$, for $\delta$ sufficiently small, it follows that

$$\limsup_{n\to\infty} \int_{\Omega} A(|y_n - \xi_k|)dx \leq \frac{\varepsilon}{4}, \quad \text{for all } k \geq k_0. \tag{5.105}$$

Fixing $k \geq k_0$ sufficiently large such that

$$|\xi_k - v|_A < \left(\frac{\varepsilon}{4}\right)^{1/a_1} \tag{5.106}$$

it follows from $(G_1)(i)$ that

(5.107)

$$\int_{\Omega} A\left(|y_n - v|\right) dx \leq C \int_{\Omega} A\left(|y_n - \xi_k|\right) dx + C|\xi_k - v|_A^{a_1} \leq C \int_{\Omega} A\left(|y_n - \xi_k|\right) dx + \frac{\varepsilon C}{4},$$

for some constant $C > 0$ that does not depend on $n$ and $k$. By (5.105) and (5.107), we have

$$\limsup_{n\to\infty} \int_{\Omega} A\left(|y_n - v|\right) dx < \frac{\varepsilon C}{2}$$

and by the arbitrariness of $\varepsilon > 0$,

$$\lim_{n\to\infty} \int_{\Omega} A\left(|y_n - v|\right) dx = 0.$$

Therefore,

$$y_n \to v \quad \text{in } L^A(\Omega).$$

Now, let us consider $a_2 < \ell_2^*$, then $A$ increases essentially more slowly than $\Phi_{2*}$ near infinity. In this case, the space $W_0^{1,\Phi_2}(\Omega)$ is compactly embedded in $L^A(\Omega)$, therefore, the desired limit follows directly from that compact embedding. ∎

The following lemma is made using similar arguments to those given in Lemma 5.15. Therefore, we will omit its proof.

**Lemma 5.18** *For $\lambda > \lambda_0$, the sequence $(y_n)$ verifies the following limit $y_n \to v$ in $W_0^{1,\Phi_2}(\Omega)$.*

From the above lemmas, we can conclude that

$$y_n \to v \ \text{ in } \ V_A. \tag{5.108}$$

In view of the above facts, it is possible to obtain the proof of the Theorem 1.10 as can be seen in the following result.

**Lemma 5.19** *For $\lambda > \lambda_0$, the pair $(u,v)$ satisfies $J'(u,v) = 0$ in $X$ and $J(u,v) \neq 0$.*

**Proof.** Fixing $k, n \in \mathbb{N}$ with $n \geq k$, we have $X_k \subset X_n$. Thus, for $(\varphi_1, \varphi_2) \in X_k$, it follows that

$$J'_{\lambda,n}(w_n, y_n)(\varphi_1, \varphi_2) = 0, \quad \forall n \geq k,$$

because, by Lemma 5.16, $J'_{\lambda,n}(w_n, y_n) = 0$. Combining (5.108) with (5.94) we get

$$J'_\lambda(u,v)(\varphi_1, \varphi_2) = 0, \quad \text{for all } (\varphi_1, \varphi_2) \in X_k. \tag{5.109}$$

We claim that

$$J'_\lambda(u,v)(\varphi_1, \varphi_2) = 0, \quad \text{for all } (\varphi_1, \varphi_2) \in X. \tag{5.110}$$

In fact, we start observing that for all $\varphi_1 \in W_0^{1,\Phi_1}(\Omega)$, the pair $(\varphi_1, 0) \in X_k$ for all $k$. Hence, $J'_\lambda(u,v)(\varphi_1, 0) = 0$. On the other hand, for $\varphi_2 \in V_A$, there exists $\chi_n \in V_A^{k(n)}$ such that

$$\lim_{n \to \infty} \chi_n = \varphi_2, \quad \text{in } V_A.$$

From (5.109),

$$J'_\lambda(u,v)(0, \chi_n) = 0, \quad \text{for all } n \in \mathbb{N}, \tag{5.111}$$

which implies after passage to the limit as $n \to \infty$ that

$$J'_\lambda(u,v)(0, \varphi_2) = 0, \quad \text{for all } \varphi_2 \in V_A. \tag{5.112}$$

Thus, (5.110) is proved. Using the fact that $(w_n, y_n) \to (u,v)$ in $X$ and that $J'_\lambda(w_n, y_n) \geq b_n \geq C\sigma^{\ell_2} > 0$, for all $n \in \mathbb{N}$, for some constant $C > 0$ which does not depend on $n$, we have that $J'_\lambda(u,v) \geq C\sigma^{\ell_2} > 0$, from where it follows that $(u,v)$ is a nontrivial solution for $(S_2)$, and the proof is complete. ■

# Appendix A

# Results on the critical point theory for locally Lipschitz functionals

We recall some few notations and results on the critical point theory for locally Lipschitz functionals defined on a real Banach space X with norm $\|\cdot\|_X$. All the results we will list below can be found in [15, 29, 44] and in references therein.

Let $I : X \to \mathbb{R}$ be a locally Lipschitz functional ($I \in Lip_{loc}(X, \mathbb{R})$), that is, for each $x \in X$, there exist an open neighborhood $N(x)$ of $x$ and a constant $k(x) > 0$, such that

$$|I(y_1) - I(y_2)| \leq k(x)\|y_1 - y_2\|,$$

for all $y_1$ and $y_2$ in $N(x)$.

A generalized directional derivative of a locally Lipschitz functional $I : X \to \mathbb{R}$ at $x \in X$ in the direction $v \in X$, denoted by $I^0(x; v)$, is defined by

$$I^0(x; v) = \limsup_{h\to 0\ \lambda\to 0^+} \frac{I(x + h + \lambda v) - I(x + h)}{\lambda}$$

and the generalized gradient of $I$ at $x$ is the set

$$\partial I(x) = \{\mu \in X^* : \langle \mu, v\rangle \leq I^0(x; v),\ v \in X\}.$$

Let $Q$ be a compact metric space and let $Q_*$ be a nonempty closed subset strictly contained in $Q$. We set

$$\mathcal{P} = \{p \in C(Q, X)\colon p = p_* \text{ on } Q_*\}, \tag{A.1}$$

where $p_*$ is a fixed continuous map on $Q_*$ and

$$c = \inf_{c\in\mathcal{P}} \max_{x\in Q} I(p(x)). \tag{A.2}$$

So

$$c \geq \max_{x\in Q_*} I(p_*(x)). \tag{A.3}$$

We say that the subset $A \subset X$ *links with the pair* $(Q, Q_*)$ if $p_*(Q_*) \cap A = \emptyset$ and for each $p \in \mathcal{P}$, $p(Q) \cap A \neq \emptyset$.

**Theorem A.1** *(See [44]) Let $I \in Lip_{loc}(X, \mathbb{R})$ and $A \subset I_c = \{x \in X : I(x) \geq c\}$ be a closed subset which links with the pair $(Q, Q_*)$. Then there exists a sequence $(x_n) \subset X$ satisfying*

$$\lim_{n\to\infty} d(x_n, A) = 0, \quad \lim_{n\to\infty} I(x_n) = c \quad e \quad \lim_{n\to\infty} \lambda_I(x_n) = 0,$$

*with*

$$\lambda_I(x_n) = \min\{\|\mu\|_{X^*}\colon \mu \in \partial I(x_n)\}.$$

**Proposition A.1** *(See [44]) Let $I : X \to \mathbb{R}$ be a continuous and Gateaux-differentiable functional such that $I' : X \to X^*$ is continuous from the norm topology of $X$ to the weak*-topology of $X^*$. Then $I \in Lip_{loc}(X, \mathbb{R})$ and $\partial I(x) = \{I'(x)\}$, $\forall x \in X$.*

The Theorem A.1 together with Proposition A.1 allows us to propose a linking theorem for Gateaux-differentiable functionals. This result will be fundamental to study the class of system proposed in Chapter 6.

**Theorem A.2 (The linking theorem)** *(See [44]) Let $X$ be a real Banach space with $X = Y \oplus Z$, where $Y$ is finite dimensional. Suppose that $I : X \to \mathbb{R}$ is continuous and Gateaux-differentiable with derivative $I' : X \to X^*$ continuous from the norm topology of $E$ to the weak*-topology of $X^*$ satisfying:*

$(I_1)$ *There is $\sigma > 0$ such that if $\mathcal{N} = \{u \in Z : \|u\| \leq \sigma\}$, then $b \doteq \inf_{\partial\mathcal{N}} I > 0$.*

$(I_2)$ *There are* $z_* \in Z \cap \partial B_1$ *and* $\rho > \sigma$ *such that*

$$0 = \sup_{\partial \mathcal{M}} I < d \doteq \sup_{\mathcal{M}} I,$$

*where*

$$\mathcal{M} = \{u = \lambda z_* + y : \|u\| \leq \rho,\ \lambda \geq 0,\ y \in Y\}$$

*If*

$$c = \inf_{\gamma \in \Gamma} \max_{x \in \mathcal{N}} I(\gamma(t)),$$

*with*

$$\Gamma = \{\gamma \in C(\mathcal{N}, X) : \gamma|_{\partial \mathcal{N}} = Id_{\partial \mathcal{N}}\}.$$

*Then,* $b \leq c$ *and there is a sequence* $(u_n) \subset X$ *such that*

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

**Proof.** The result follows from Proposition 2.2 and Theorem 4.7 with $\mathcal{P} = \Gamma$, $Q = \mathcal{N}$, $Q_* = \partial \mathcal{N}$, $p_* = Id_{Q_*}$ e $A = \{x \in Z + Y : I(x) \geq c\}$. ■

For the last section of this paper we will use the already known saddle-point theorem of Rabinowitz without Palais-Smale condition. The proof of this result also follows from Theorem A.1 along with Proposition A.1.

**Theorem A.3 (Saddle-point theorem)** *(See [44]) Let* $X$ *be a real Banach space with* $X = Y \oplus Z$*, where* $Y$ *is finite dimensional. Suppose that* $I : X \to \mathbb{R}$ *is continuous and Gateaux-differentiable with derivative* $I' : X \to X^*$ *continuous from the norm topology of* $E$ *to the weak*-topology of* $X^*$ *satisfying:*

$(I_1)$ *there are constants* $\rho > 0$ *and* $\alpha_1 \in \mathbb{R}$ *such that if* $\mathcal{M} = \{u \in Y : \|u\| \leq \rho\}$*, then* $I|_{\partial \mathcal{M}} \leq \alpha_1$.

$(I_2)$ *there is a constant* $\alpha_2 > \alpha_1$ *such that* $I|_Z \geq \alpha_2$*. If*

$$c = \inf_{\gamma \in \Gamma} \max_{x \in \mathcal{M}} I(\gamma(t)),$$

*with*

$$\Gamma = \{\gamma \in C(\mathcal{M}, X) : \gamma|_{\partial \mathcal{M}} = Id|_{\partial \mathcal{M}}\}.$$

*Then, there is* $(u_n) \subset X$ *such that*

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

**Definition A.1 (Cerami Sequence)** *Let $(X, \|\cdot\|_X)$ be a Banach space and $I : X \to \mathbb{R}$ a continuous, Gateaux-differentiable function, such that $I' : X \to X^*$ is continuous from the norm topology of $X$ to the weak* topology of $X^*$. We say that $(x_n) \subset X$ is a Cerami sequence at the level $c \in \mathbb{R}$, denoted by $(C)_c$, when*

$$I(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|_X)\|I'(x_n)\|_{X^*} \to 0, \quad \text{when} \quad n \to \infty.$$

*We will say that $I$ verifies the Cerami condition, or simply the $(C)$ condition, when every sequence $(C)_c$ for $c \in \mathbb{R}$, admits a subsequence that converges strongly on $X$.*

Next we state a result of the mentha step due to Ghoussoub-Preiss Theorem. This result produces Cerami sequences even if the functional is not of class $C^1$. The Your proof can be found in [29, Theorem 6] or [15, Theorem 5.46].

**Theorem A.4 (Ghoussoub-Preiss)** *Let $(X, \|\cdot\|_X)$ be a Banach space and $I : X \to \mathbb{R}$ a continuous, Gateaux-differentiable function, such that $I' : X \to X^*$ is continuous from the norm topology of $X$ to the weak* topology of $X^*$. Set $z_0, z_1 \in X$ and consider*

$$\Gamma = \{\sigma \in C\big([0,1], X\big) : \ \sigma(0) = z_0 \text{ and } \sigma(1) = z_1\}.$$

*Set the number $c$ given by*

$$c := \inf_{\sigma \in \Gamma} \max_{0 \le t \le 1} L(\sigma(t)).$$

*Assume that there is a $F$ subset of $X$ such that $\{x \in F : \ L(x) \ge c\}$ separates $z_0$ and $z_1$. So, there exists a sequence $(x_n)$ in $X$ such that*

$$\delta(x_n, F) \to 0, \ \ I(x_n) \to c \quad \text{and} \quad (1 + \|x_n\|)\|L'(x_n)\|_* \to 0,$$

*where*

$$\delta(x_n, F) := \inf_{x \in F} \delta(x_n, x)$$

*and*

$$\delta(x_n, x) := \Big\{ \int_0^1 \frac{\|\gamma'(t)\|}{1 + \|\gamma(t)\|} dt : \gamma \in C\big([0,1], X\big), \ \gamma(0) = z_0 \ e \ \gamma(1) = z_1 \Big\}.$$

**Corollary A.1** *(Mountain Pass Theorem) Let* $(X, \|\cdot\|_X)$ *be a Banach space and* $I : X \to \mathbb{R}$ *a continuous, Gateaux-differentiable function, such that* $I' : X \to X^*$ *is continuous from the norm topology of* $X$ *to the weak*$^*$ *topology of* $X^*$*. In addition, assume that* $I$ *verifies the mountain pass geometry, that is:*

*i)* $I|_{\partial B_\rho} \geq \eta$*, for some constants* $\rho, \eta > 0$

*ii)* $I(e) < \eta$*, for some* $e \notin \overline{B_\rho(0)}$

*If*

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \eta$$

*where*

$$\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = 0 \ \ \textit{and} \ \ \gamma(1) = e\}.$$

*Then there exists a sequence* $(x_n)$ *in* $X$ *such that*

$$I(x_n) \to c \quad \textit{and} \quad (1 + \|x_n\|)\|I'(x_n)\|_* \to 0.$$

**Proof.** It is enough to apply Theorem A.4 with $F = E$ [if $c > \inf\limits_{B_\rho(0)} I$] or $F = \partial B_\rho(0)$ [if $c = \inf\limits_{B_\rho(0)} I$]. ∎

# Appendix B

# Construction of examples of potentials satisfying the conditions $(V,K) \in \mathcal{Q}_1$ and $(V,K) \in \mathcal{Q}_2$

To construct such functions, it is crucial to recognize that $(Q_1)$ (mentioned in the introduction) is less restrictive than any of the conditions listed below:

$(a)$ There are $r \geq 1$ and $\rho \geq 0$ such that $K \in L^r(\mathbb{R}^N \setminus B_\rho(0))$;

$(b)$ $K(x) \to 0$ as $|x| \to \infty$;

$(c)$ $K = H_1 + H_2$, with $H_1$ and $H_2$ verifying $(a)$ and $(b)$ respectively.

Now, for every $n \in \mathbb{N}$, fix $z_n = (n, 0, \cdots, 0)$. Consider $\left\{B_{\frac{1}{2^n}}(z_n)\right\}_{n\in\mathbb{N}}$ the disjoint sequence of open balls in $\mathbb{R}^N$ and the nonnegative function $H_1 : \mathbb{R}^N \to \mathbb{R}$ given by

$$0 \leq H_1(x) \leq 1,\ \forall x \in \mathbb{R}^N, \quad H_1(z_n) = 1,\ \forall n \in \mathbb{N}, \quad H_1 \equiv 0 \ \text{ in } \ \mathbb{R}^N \setminus \left\{\bigcup_{n\in\mathbb{N}} B_{\frac{1}{2^n}}(z_n)\right\}$$

and

$$\int_{B_{\frac{1}{2^n}}(z_n)} H_1(x)dx \leq \frac{1}{2^n}, \quad \forall n \in \mathbb{N}.$$

Thus, without difficulty we can see that the functions

$$V(x) = K(x) = H_1(x) + \frac{1}{ln(2+|x|)} \tag{B.1}$$

verify the condition $(Q_1)$. Furthermore, clearly $V$ and $K$ satisfy the conditions $(Q_0)$ and $(Q_2)$. However, these functions do not verify the $(Q_3)$ condition.

Now, consider the $\mathcal{N}$-function $B(t) = |t|^p$ with $p \in (m, \ell^*)$ and define the

$$K(x) = H_1(x) + \frac{1}{ln(2+|x|)} \tag{B.2}$$

and

$$V(x) = (|x|H_1(x))^{\frac{\theta(m^*-m)}{m^*-p}} + \left(\frac{1}{ln(2+|x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}}. \tag{B.3}$$

It is clear that $B$ satisfies the conditions $(B_1)-(B_4)$, moreover, as mentioned previously, these functions check $(Q_0)$ and $(Q_1)$. We will show that $(Q_3)$ is satisfied with $B(t) = |t|^p$ where $p \in (m, \ell^*)$. In effect, we state that there is $C > 0$ depending on $p, m$ and $\ell$, so that, for each $x \in \mathbb{R}^N$, we have

$$G(x) = C\min\{V(x)^{\frac{\ell^*-p}{\ell^*-\ell}}, V(x)^{\frac{m^*-p}{m^*-m}}\} \leq H(x) := \min_{t>0}\left\{V(x)\frac{\Phi(t)}{t^p} + \frac{\Phi_*(t)}{t^p}\right\}. \tag{B.4}$$

In fact, for every $x \in \mathbb{R}^N$ fixed, the function

$$g(t) = V(x)t^{m-p} + t^{m^*-p}, \quad \forall t > 0$$

has $C_pV(x)^{\frac{m^*-p}{m^*-m}}$ as minimum value, where

$$C_p = \left[\left(\frac{p-m}{m^*-p}\right)^{\frac{m-p}{m^*-m}} + \left(\frac{p-m}{m^*-p}\right)^{\frac{m^*-p}{m^*-m}}\right].$$

So,

$$C_pV(x)^{\frac{m^*-p}{m^*-m}} \leq V(x)|t|^{m-p} + |t|^{m^*-p}, \quad \forall x \in \mathbb{R}^N$$

and all $t \in \mathbb{R}$. In particular

$$C_pV(x)^{\frac{m^*-p}{m^*-m}} \leq V(x)|t|^{m-p} + |t|^{m^*-p}, \quad \forall x \in \mathbb{R}^N$$

and $|t| \leq 1$. I.e,

$$C_p|t|^pV(x)^{\frac{m^*-p}{m^*-m}} \leq C(V(x)\Phi(|t|) + \Phi_*(|t|)), \quad \forall x \in \mathbb{R}^N$$

and $|t| \leq 1$. Therefore,

(B.5)

$$C_p\min\{V(x)^{\frac{\ell^*-p}{\ell^*-\ell}}, V(x)^{\frac{m^*-p}{m^*-m}}\} \leq C\left(V(x)\frac{\Phi(|t|)}{|t|^p} + \frac{\Phi_*(|t|)}{|t|^p}\right), \ \forall x \in \mathbb{R}^N \ \text{ e } \ |t| \leq 1.$$

Now, for each $x \in \mathbb{R}^N$ fixed, the function

$$h(t) = V(x)t^{\ell-p} + t^{\ell^*-p}, \quad \forall t > 0$$

has $D_pV(x)^{\frac{\ell^*-p}{\ell^*-\ell}}$ as minimum value, where

$$D_p = \left[\left(\frac{p-\ell}{\ell^*-p}\right)^{\frac{\ell-p}{\ell^*-\ell}} + \left(\frac{p-\ell}{\ell^*-p}\right)^{\frac{\ell^*-p}{\ell^*-\ell}}\right].$$

So,

$$D_pV(x)^{\frac{\ell^*-p}{\ell^*-\ell}} \leq V(x)|t|^{\ell-p} + |t|^{\ell^*-p}, \quad \forall x \in \mathbb{R}^N$$

and all $t \in \mathbb{R}$. In particular

$$D_pV(x)^{\frac{\ell^*-p}{\ell^*-\ell}} \leq V(x)|t|^{\ell-p} + |t|^{\ell^*-p}, \quad \forall x \in \mathbb{R}^N$$

and $|t| > 1$. I.e,

$$D_p|t|^pV(x)^{\frac{\ell^*-p}{\ell^*-\ell}} \leq C(V(x)\Phi(|t|) + \Phi_*(|t|)), \quad \forall x \in \mathbb{R}^N$$

and $|t| > 1$. Therefore,

(B.6)

$$D_p \min\{V(x)^{\frac{\ell^*-p}{\ell^*-\ell}}, V(x)^{\frac{m^*-p}{m^*-m}}\} \leq C\left(V(x)\frac{\Phi(|t|)}{|t|^p} + \frac{\Phi_*(|t|)}{|t|^p}\right), \ \forall x \in \mathbb{R}^N \text{ and } |t| \geq 1.$$

From (B.5) and (B.6), we obtain

$$C'_p \min\{V(x)^{\frac{\ell^*-p}{\ell^*-\ell}}, V(x)^{\frac{m^*-p}{m^*-m}}\} \leq CH(x), \quad \forall x \in \mathbb{R}^N \ \text{ and } \ t \in \mathbb{R},$$

where $C'_p = \min\{C_p, D_p\}$. Proving (B.4). Now, consider $x \in \mathbb{R}^N \setminus \bigcup_{n\in\mathbb{N}} B_{\frac{1}{2n}}(z_n)$, hence, $H_1(x) = 0$, therefore

$$\begin{aligned}
\frac{K(x)^\theta}{V(x)^{\frac{m^*-p}{m^*-m}}} =& \frac{\left[\dfrac{1}{ln(2+|x|)}\right]^\theta}{\left[\left(\dfrac{1}{ln(2+|x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}} + \left(\dfrac{1}{ln(2+|x|)}\right)^{\frac{\theta(\ell^*-\ell)}{\ell^*-q}}\right]^{\frac{m^*-p}{m^*-m}}} \\
\leq& \frac{\left[\dfrac{1}{ln(2+|x|)}\right]^\theta}{\left[\dfrac{1}{ln(2+|x|)}\right]^{\frac{\theta(m^*-p)}{m^*-q}}} \\
=& \left[ln(2+|x|)\right]^{\frac{\theta(m^*-p)}{m^*-q}-\theta}.
\end{aligned}$$

Similarly, for each $x \in \mathbb{R}^N \setminus \bigcup_{n\in\mathbb{N}} B_{\frac{1}{2n}}(z_n)$ , we have

$$\frac{K(x)^\theta}{V(x)^{\frac{\ell^*-p}{\ell^*-\ell}}} \le \big[ln(2+|x|)\big]^{\frac{\theta(\ell^*-p)}{\ell^*-q}-\theta}.$$

Therefore,

$$\text{(B.7)}\qquad \frac{K(x)^\theta}{G(x)} \le C_1\left(\big[ln(2+|x|)\big]^{\frac{\theta(\ell^*-p)}{\ell^*-q}-\theta} + \big[ln(2+|x|)\big]^{\frac{\theta(m^*-p)}{m^*-q}-\theta}\right),$$

for all $x \in \mathbb{R}^N \setminus \bigcup_{n\in\mathbb{N}} B_{\frac{1}{2n}}(z_n)$. On the other hand, if $x \in \bigcup_{n\in\mathbb{N}} B_{\frac{1}{2n}}(z_n)$, we have $x \neq 0$ and $H_1(x) \neq 0$. So,

$$\begin{aligned}
\frac{K(x)^\theta}{V(x)^{\frac{m^*-p}{m^*-m}}} \le & \frac{\left[H_1(x) + \dfrac{1}{ln(2+|x|)}\right]^\theta}{\left[(|x|H_1(x))^{\frac{\theta(m^*-m)}{m^*-p}} + \left(\dfrac{1}{ln(2+|x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}}\right]^{\frac{m^*-p}{m^*-m}}} \\
= & C_2 \frac{H_1(x)^\theta}{\left[(|x|H_1(x))^{\frac{\theta(m^*-m)}{m^*-p}} + \left(\dfrac{1}{ln(2+|x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}}\right]^{\frac{m^*-p}{m^*-m}}} \\
& + C_2 \frac{\left[\dfrac{1}{ln(2+|x|)}\right]^\theta}{\left[(|x|H_1(x))^{\frac{\theta(m^*-m)}{m^*-p}} + \left(\dfrac{1}{ln(2+|x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}}\right]^{\frac{m^*-p}{m^*-m}}} \\
\le & C_2\left(\frac{H_1(x)^\theta}{|x|^\theta H_1(x)^\theta} + \frac{\left[\dfrac{1}{ln(2+|x|)}\right]^\theta}{\left[\dfrac{1}{ln(2+|x|)}\right]^{\frac{\theta(m^*-p)}{m^*-q}}}\right).
\end{aligned}$$

Similarly, for each $x \in \bigcup_{n\in\mathbb{N}} B_{\frac{1}{2n}}(z_n)$, we have

$$\frac{K(x)^\theta}{V(x)^{\frac{\ell^*-p}{\ell^*-\ell}}} \le C_3\left(\frac{H_1(x)^\theta}{|x|^\theta H_1(x)^\theta} + \frac{\left[\dfrac{1}{ln(2+|x|)}\right]^\theta}{\left[\dfrac{1}{ln(2+|x|)}\right]^{\frac{\theta(\ell^*-p)}{\ell^*-q}}}\right).$$

Therefore, for all $x \in \bigcup\limits_{n\in\mathbb{N}} B_{\frac{1}{2n}}(z_n)$, it follows

(B.8)
$$\frac{K(x)^\theta}{G(x)} \leq C_2 \left( \frac{1}{|x|^\theta} + [ln(2+|x|)]^{\frac{\theta(m^*-p)}{m^*-q}-\theta} \right) + C_3 \left( \frac{1}{|x|^\theta} + [ln(2+|x|)]^{\frac{\theta(\ell^*-p)}{\ell^*-q}-\theta} \right).$$

By (B.7) and (B.8), we conclude that

$$\frac{K(x)^\theta}{G(x)} \leq C_4 \left( \frac{1}{|x|^\theta} + [ln(2+|x|)]^{\frac{\theta(m^*-p)}{m^*-q}-\theta} + [ln(2+|x|)]^{\frac{\theta(\ell^*-p)}{\ell^*-q}-\theta} \right),$$

for all $x \in \mathbb{R}^N \setminus \{0\}$. As $q < p$, then $\frac{\theta(m^*-p)}{m^*-q} - \theta$, $\frac{\theta(\ell^*-p)}{\ell^*-q} - \theta < 0$. That said,

$$\frac{K(x)^\theta}{G(x)} \longrightarrow 0, \quad \text{as} \quad |x| \to +\infty.$$

Knowing that $G(x) \leq H(x)$, for all $x \in \mathbb{R}^N$, then

$$\frac{K(x)^\theta}{H(x)} \longrightarrow 0, \quad \text{as} \quad |x| \to +\infty.$$

Now, let us see that $V$ and $Q$ do not verify the condition $(Q_2)$. Consider $x \in \mathbb{R}^N \setminus \bigcup B_{\frac{1}{2n}}(z_n)$, then $H_1(x) = 0$. So,

(B.9)
$$\frac{K(x)}{V(x)} = \frac{\dfrac{1}{ln(2+|x|)}}{\left(\dfrac{1}{ln(2+|x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}} + \left(\dfrac{1}{ln(2+|x|)}\right)^{\frac{\theta(\ell^*-\ell)}{\ell^*-q}}}.$$

Note that the function $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\omega(t) = \frac{\dfrac{tN}{N-t} - t}{\dfrac{tN}{N-t} - q} = \frac{tN - t(N-t)}{tN - q(N-t)}$$

is decreasing in the interval $[0, m]$. In fact, note that

$$\begin{aligned} \omega'(t) =& \frac{(N - [(N-t) - t])(tN - q(N-t)) - (tN - t(N-t))(N+q)}{[tN - q(N-t)]^2} \\ =& \frac{2t(tN - q(N-t)) - (tN - t(N-t))(N+q)}{[tN - q(N-t)]^2}. \end{aligned}$$

Clearly, for all $t \in [0, m]$, we have

$$2t(tN - q(N-t)) - (tN - t(N-t))(N+q) < 0.$$

Therefore, $\omega'(t) < 0$, for all $t \in [0, m]$. Showing the decrease of the function $\omega$ in the interval $[0, m]$, in view of this, we come to the conclusion that

$$\frac{\theta(m^* - m)}{m^* - q} < \frac{\theta(\ell^* - \ell)}{\ell^* - q}.$$

Hence, for $|x| \to +\infty$, we have

$$\left(\frac{1}{ln(2 + |x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}} > \left(\frac{1}{ln(2 + |x|)}\right)^{\frac{\theta(\ell^*-\ell)}{\ell^*-q}}. \tag{B.10}$$

By (B.9) and (B.10),

$$\begin{aligned} \frac{K(x)}{V(x)} &\geq \frac{\dfrac{1}{ln(2 + |x|)}}{2\left(\dfrac{1}{ln(2 + |x|)}\right)^{\frac{\theta(m^*-m)}{m^*-q}}} \\ &= \frac{1}{2}[ln(2 + |x|)]^{\frac{\theta(m^*-m)}{m^*-q}-1}. \end{aligned}$$

Since $m < q$ and $1 \leq s$, it follows that $\frac{\theta(m^*-m)}{m^*-q} - 1 > 0$. Therefore, for all $x \in \mathbb{R}^N \setminus \bigcup B_{\frac{1}{2n}}(z_n)$ such that $|x| \to +\infty$, it follows that

$$\frac{K(x)}{V(x)} \to +\infty.$$

As a consequence of the limit above, we can state that the functions $V$ and $K$ defined in (B.2) and (B.3) do not verify $(Q_2)$.

**Remark B.1** *In an analogous way, one can construct an example of functions $V$ and $K$ satisfying the conditions $(V, K) \in \mathcal{K}_1$ and $(V, K) \in \mathcal{K}_2$ worked in chapter 3.*

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