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Rigidity, uniqueness and nonexistence results in certain warped product spaces

by

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under the supervisor of

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Julho de 2023

Dedication

To God,
To my wife and sons, in
To my parents.

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Agradeço ao Senhor todo-poderoso, criador dos céus e da terra, que soberanamente me permitiu chegar até esta esfera de atuação, cumprindo o propósito insondável de Sua vontade.

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“Não há um único centímetro quadrado, em todos os domínios de nossa existência, sobre os quais Cristo, que é soberano sobre tudo, não clame: ‘É meu!’.”

Abraham Kuyper

Resumo

Esta tese apresenta o estudo de hipersuperfícies imersas em ambientes Lorentzianos e produtos warped Riemannianos. Na primeira parte, analisamos as hipersuperfícies que satisfazem condições sobre a curvatura média, obtendo resultados de rigidez e não existência para solitons do fluxo da curvatura média em espaços-tempo GRW e espaços estáticos padrão. Demonstramos aplicações desses resultados em ambientes como Einstein-de Sitter Spacetime, Steady State Type Spacetimes, Lorentz-Minkowski space, etc. Obtendo resultados tipo Calabi-Bernstein e destacando resultados de estabilidade de hipersuperfícies. Na segunda parte, estudamos hipersuperfícies two-sided imersas em produtos warped Riemannianos, estabelecendo resultados de existência, rigidez e não existência de solitons do fluxo da curvatura média, sujeitos a condições sobre a curvatura média e a função warping do ambiente. Demonstramos aplicações desses resultados em ambientes como Real projective space, pseudo-hyperbolic spaces, Schwarzchild space e Reissner-Nordstrõm spcace. Também dedicamos parte do estudo às subvariedades imersas em ambiente ponderados.

Palavras-chave: Espaço-tempo de Robertson-Walker generalizado; Solitons do Fluxo da curvatura média; subvariedade riemanianas; Espaços estaticos padrão.

Abstract

This thesis presents the study of hypersurfaces immersed in Lorentzian and warped Riemannian products ambient. In the first part, we analyze hypersurfaces that satisfy conditions on the mean curvature, obtaining rigidity and non-existence results for solitons of the mean curvature flow in GRW spacetimes and standard static spaces. We demonstrate applications of these results in ambient such as Einstein-de Sitter Spacetime, Steady State Type Spacetimes, Lorentz-Minkowski space, and more. We obtain Calabi-Bernstein type results and highlight stability results of hypersurfaces. In the second part, we study two-sided hypersurfaces immersed in warped Riemannian products, establishing results on existence, rigidity, and non-existence of solitons of the mean curvature flow, subject to conditions on the mean curvature and warping function of the ambient. We demonstrate applications of these results in ambient such as Real projective space, pseudo-hyperbolic spaces, Schwarzchild space, and Reissner-Nordström space. We also dedicate part of the study to submanifolds immersed in weighted ambient.

Keywords: Generalized Robertson-Walker spacetimes; mean curvature flow solitons; Riemannian submanifolds; Standard static spacetimes.

Contents

List of Figures

Introdução

Esta tese está dividida em duas partes independentes, como se segue:

Parte I: Unicidade e não existencia de hipersuperficies completas tipo-espaço

A teoria das imersões isométricas fornece as ferramentas adequadas para abordar alguns problemas importantes que envolvem singularidades e colapso gravitacional em espaços-tempo.

Os teoremas da singularidade provados na década de 1960 por Penrose [126] e Hawking [96] afirmam que a formação de singularidades é inevitável, caso assumirmos condições razoáveis sobre a curvatura do espaço tempo, e sobre a geometria extrínseca de certas hipersuperfícies e sobre a estrutura causal do colector Lorentziano. A existência de hipersuperfícies espaciais(ou seja, hipersuperfícies cuja métrica induzida é uma métrica Riemanniana) no espaço-tempo, em particular, é um requisito fundamental na formulação original dos teoremas de singularidade, bem como nas suas generalizações mais recentes (para mais detalhes, ver [136, 139] e respectivas referências).

Neste contexto, poderíamos fazer a seguinte pergunta:

Questionamento 1: Do ponto de vista matemático, existe alguma relevância para a investigação de hipersuperfícies espaciais em uma variedade lorentziana?

Para responder a esta pergunta, assumimos um ambiente que até certo ponto modela locais com singularidades e colapsos gravitacionais, desse modo investigamos a geometria de hipersuperfícies espaciais completas num espaço-tempo de Robertson-Walker (GRW) generalizado. Por espaço-tempo GRW, entendemos um produto lorentziano com deformação $-I \times_\rho M^n$ cuja fibra Riemanniana M^n e função warping $\rho \in C^\infty(I)$, onde $I \subset \mathbb{R}$ é um intervalo aberto.

Neste contexto, diante de várias características geométricas que poderiam ser trabalhadas, em particular sobre a curvatura média, poderíamos levantar uma segunda questão:

Questionamento 2: Quais são as restrições ideais na curvatura média de uma hipersuperfície completa tipo-espaço num espaço-tempo GRW, para obter resultados de singularidade e inexistência?

Recentemente, Aledo, Rubio e Salamanca [18] estudaram superfícies espaciais completas com curvatura média limitada nos espaços-tempo GRW $-I \times_\rho M^n$, cuja fibra Riemanniana M^n é assumida como sendo completa não compacta e com recobrimento universal parabólico. Como segue o enunciado abaixo:

Theorem (Theorem 3.1 of [18])**.** *Seja* $\overline{M} = -I \times_\rho M^n$ *um produto warped lorentiziano que possui recobrimento espacialmente parabólico cuja função warping não é globalmente constante e satisfaz* $(log\rho)" \leq \alpha(log\rho)'^2$ *para alguma constante real* $\alpha \geq 0$*. Seja* $\psi : M \longrightarrow \overline{M}$ *uma hipersuperficie completa com ângulo hiperbólico limitado e tal que* $\sup \rho(\tau) < \infty$ *e* $\inf \rho(\tau) > 0$*. Se*

$$H^2 \leq \frac{\rho'(\tau)}{\rho^2(\tau)} \cosh^2 \varphi,$$

então M *é um slice.*

Neste cenário, os autores supracitados utilizaram uma métrica adequada numa hipersuperfície tipo-espaço para fornecer resultados de rigidez. Como aplicação, obtiveram novos resultados do tipo Calabi-Bernstein relativos a gráficos espaciais definidos na fibra Riemanniana M^n.

Prosseguindo, lidamos com hipersuperfícies espaciais completas num espaço-tempo GRW $-I \times_\rho M^n$. Sob restrições adequadas sobre a curvatura seccional da fibra Riemanniana M^n, sobre a função de deformação ρ e sobre a curvatura média futura (isto é, a função curvatura média em relação ao mapa de Gauss da hipersuperfície tipo-espaço), trabalhamos com uma mudança conforme da métrica induzida (já utilizada por Aledo, Rubio e Salamanca in [18])) para provar que uma tal hipersuperfície tipo-espaço deve ser uma slice $\{t\} \times M^n$ do espaço-tempo ambiente. Também obtivemos resultados de não-existência e tipo Calabi-Bernstein relativos a gráficos espaciais inteiros definidos sobre a fibra Riemanniana M^n, bem como são dadas aplicações quando o ambiente é Einstein-de Sitter e ao steady state type espaço-tempo. A nossa abordagem se baseia no princípio do máximo generalizado de Omori-Yau e em certas propriedades de integrabilidade devidas ao Yau.

É claro que podemos expandir um pouco o nosso universo de elementos para trabalhar com uma classe maior de objetos geométricos. Isto pode ser feito da seguinte forma: tome $\mathbb{R}^{n+1}_1$ um espaço de Minkowski $(n+1)$-dimensional $(\mathbb{R}^{n+1}_1, \bar{g})$ com a sua métrica Lorentziana padrão

$$\bar{g} = -dx_1^2 + \sum_{i=2}^{n+1} dx_i^2.$$

Seja $\psi : \Sigma^n \to \mathbb{R}^{n+1}_1$ uma imersão espacial (o que significa que tem uma métrica Riemanniana induzida) no espaço Minkowski. O fluxo de curvatura média do espaço-tempo associado a ψ é uma família de imersões espaciais suaves $\Psi_t = \Psi(t, \cdot) : \Sigma^n \to \mathbb{R}^{n+1}_1$ com imagens correspondentes $\Sigma^n_t = \Psi_t(\Sigma^n)$ satisfazendo a seguinte equação de evolução

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} = \vec{H} \\ \Psi(0, x) = \psi(x) \end{cases}$$

em algum intervalo de tempo, onde $\vec{H}$ representa o vector de curvatura média (não-normalizado) da subvariedade espacial Σ^n_t em $\mathbb{R}^{n+1}_1$. As soluções da equação de evolução anterior são chamados de sólitons.

Mais uma vez, chegamos à seguinte questão:

Questionamento 3: Qual é a relevância de estudar os sólitons do fluxo de curvatura média nos espações-tempo GRW?

O fluxo da curvatura média no espaço de Minkowski e, mais geralmente, em uma variedade Lorentziana tem sido estudado extensivamente por vários autores (ver, por exemplo, [1,80–83,85, 86,102,103,105,106,108,141]) e, segundo [81], uma justificação importante para este interesse é o fato de que os sólitons de translação tipo-espaço podem ser considerados como uma forma natural de folhear o espaço-tempo por hipersuperfícies. Exemplos particulares podem dar uma visão da estrutura de certos espaços-tempo no infinito nulo e ter possíveis aplicações na Relatividade Geral (para mais detalhes, ver [81]).

Mais recentemente, Lambert e Lotay [111] provaram uma existência para tempos longos e resultados de convergência para soluções espaciais que são fluxo da curvatura média no espaço pseudo-euclidiano n-dimensional $\mathbb{R}^n_m$ de índice m, que são inteiros ou definidos em domínios delimitados e satisfazem as condições de fronteira do tipo Neumann ou Dirichlet. Em [94], Guilfoyle e Klingenberg provaram a existência para tempos longos de um fluxo da curvatura média de uma subvariedade tipo-espaço $m + n$-dimensional cuja métrica satisfaz a chamada condição de curvatura temporal.

Em [69], Colombo, Mari e Rigoli também estudaram algumas propriedades dos sólitons do fluxo curvatura média em geral, nas variedades Riemannianas e em produtos warped, concentrando-se nos resultados de classificação e rigidez sob várias condições geométricas, desde a estabilidade do sólitons até ao fato da imagem do mapa Gauss estar contida em regiões adequadas da esfera. Além disso, investigaram também o caso de gráficos inteiros do fluxo da curvatura média, como podemos verificar no seguinte enunciado:

Theorem (Theorem 3.4 of [69])**.** *Seja $\psi : \Sigma^n \longrightarrow \overline{M}^{n+1} = -I \times_\rho M^n$ um sóliton do fluxo da curvatura média com respeito $K = \rho(t)\partial_t$ conexo, completo e estável e seja c a constante de sóliton*[1]*. Assumindo que $\overline{M}$ é completo e que a curvatura seccional $\overline{k}$, satisfaz*

$$c\rho'(\pi_I \circ \psi) \leq n\overline{k} \text{ on } \Sigma. \tag{1}$$

Seja $\Psi = II - \langle\ ,\ \rangle_\Sigma \otimes H$ o tensor de umbilicidade de ψ e suponha que

$$|\Psi| \in L^2(\Sigma, e^{c\eta}) \tag{2}$$

com $\eta = \frac{|\Psi|^2}{2}$. Então ocorre um dos seguintes casos:

(i) ψ é totalmente geodésico (e se $c \neq 0$ então $\psi(\Sigma)$ é invariante pelo fluxo de K), ou

[1]Uma hipersuperficie tipo-espaço $\psi : \Sigma^n \to \overline{M}^{n+1}$ imersa em um GRW espaço-tempo $\overline{M}^{n+1} = -I \times_\rho M^n$ é dita ser *sóliton do fluxo da curvatura média* com respeito a $\mathcal{K} = \rho(t)\partial_t$ e tem *constante de sóliton* $c \in \mathbb{R}$ se, e somente se, a função curvatura média(não normalizada) futura satisfaz $H = c\rho(h)\Theta$.

(ii) $I = \mathbb{R}$, ρ *é constante em* $\mathbb{R}$, Σ *é isométrico ao produto* $\mathbb{R} \times M$ *com* M *uma variedade flat completa e* Σ *é também flat. Introduzindo os recobrimentos universais* $\pi_\Sigma : \mathbb{R}^n \longrightarrow \Sigma$, $\pi_M : \mathbb{R}^n \longrightarrow M$ *e* $\pi_{\overline{M}} = id_{\mathbb{R}} \times \pi_M : \mathbb{R}^{n+1} \longrightarrow \overline{M}$, *o mapa* ψ *passa a ser uma imersão* $\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n$ *satisfazendo* $\pi_{\overline{M}} \circ \psi = \psi \circ \pi_\Sigma$, *que é uma isometria de* $\mathbb{R}^n$ *e uma translação ao longo do fator* $\mathbb{R}$ *de* $\mathbb{R}^{n+1}$ *é dada por*

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n, \quad (x^1, x^2, \ldots, x^n) \mapsto (\sigma_1(x^1), \sigma_2(x^1)x^2, \ldots, x^n)$$

onde $\gamma = (\sigma_1, \sigma_2) : \mathbb{R} \longrightarrow \mathbb{R}^2$ *é a curva do coletor com imagem*

$$\sigma(\mathbb{R}) = \left\{ (x, y) \in \mathbb{R}^2 : x = -\frac{1}{c\rho_0} \log(\cos(c\rho_0 y)), |y| < \frac{2}{\pi|c|\rho_0} \right\}$$

e ρ_0 *é o valor constante de* ρ *on* $\mathbb{R}$. *Além disso, existe uma submersão Riemanniana* $\pi_\Omega : \Sigma \longrightarrow \Omega$ *numa variedade flat e compacta* Ω *com fibras geodésicas não compactas e unidimensionais do tipo* $\pi_M(\mathbb{R} \times \{(x^2, \ldots, x^n)\})$, *para constante* $(x^2, \ldots, x^n) \in \mathbb{R}^{n-1}$. *Tal fibra é mapeada por* ψ *na curva* $\pi_{\overline{M}}(\sigma(\mathbb{R}) \times \{(x^2, \ldots, x^n)\})$.

Além disso, qualquer um dos sólitons em (ii) é estável, enquanto um sóliton em (i) é estável se e só se $L = \Delta_{-c\eta} + (n\overline{k} - c\rho')$ *é não negativo.*

Quando o espaço ambiente é um produto Lorentziano, Batista e de Lima [40] estabeleceram resultados de não-existência para sólitons de translação completos tipo-espaço sob restrições de curvatura adequadas sobre as curvaturas da base. Em particular, obtiveram resultados do tipo Calabi-Bernstein para gráficos completos construídos sobre esta base Riemanniana. Para tal, provaram uma versão do princípio do máximo de Omori-Yau para sólitons. Além disso, também construíram novos exemplos de sólitons de translação tipo-espaço rotacionalmente simétricos embutidos num espaço ambiente deste tipo.

Prosseguindo, no Capítulo 3, estendemos as técnicas desenvolvidas em [18, 27, 31, 40, 69] para estudar sólitons do fluxo da curvatura média completos imersos num espaço-tempo generalizado Robertson-Walker (GRW), ou seja, um produto warped lorentziano $-I \times_\rho M^n$ com uma base 1-dimensional negativa definida I e fibra n-dimensional Riemanniana M^n. Sob restrições adequadas sobre a função warping ρ e sobre a curvatura de M^n, aplicamos alguns princípios do máximos adequados a fim de obter resultados de inexistência e rigidez relativamente a estes sólitons. São dadas aplicações no espaço-tempo padrão GRW como, por exemplo, os espaços-tempo do tipo Einstein-de Sitter e steady state type espaço-tempo. Além disso, estabelecemos novos resultados do tipo Calabi-Bernstein relacionados com os gráficos do fluxo da curvatura média de todo o espaço-tempo, construídos sobre a fibra Riemanniana.

Também no Capítulo 3, destacaremos alguns resultados obtidos na área da estabilidade das hipersuperfícies espaciais. Recordemos que a noção de estabilidade relativa a hipersuperfícies de curvatura média constante dos espaços ambientais Riemannianos foi estudada pela primeira vez por Barbosa e do Carmo in [36], e por Barbosa, do Carmo e Eschenburg in [37], onde provaram que as esferas são os únicos pontos críticos estáveis do funcional área para variações que preservam o volume. Posteriormente, trabalhando no contexto Lorentziano, Barbosa e Oliker [38] obtiveram um resultado análogo provando que as hipersuperfícies de curvatura média constante nas variedades Lorentzianos são também pontos críticos do funcional área para variações que mantêm o volume constante. Mais tarde, Barros, Brasil e Caminha [39] estudaram o problema de estabilidade forte(ou seja, estabilidade em relação a variações mas que não possuem volume preservados necessariamente) das hipersuperfícies com curvatura média constante num espaço-tempo Robertson-Walker (GRW) generalizado com curvatura seccional constante, dando uma caracterização para as hipersuperfícies máxima e slices tipo-espaço de tal espaço ambiente.

Nesta altura, o leitor pode pensar:

Questionamento 4: Em qual classe de espaços-tempo podemos ainda obter resultados semelhantes aos dos exemplos do Capítulo 2 e 3?

Para esta última questão desta primeira parte da tese, trabalhamos com o objectivo de investigar a rigidez e a inexistência de sólitons do fluxo de curvatura média do espaço em relação ao campo vetorial Killing temporal K de um *espaço-tempo estático padrão*, que (de acordo com a Definição 12.36 of [123]) pode ser considerado como um produto deformado $M^n \times_\rho \mathbb{R}_1$ cuja base Riemanniana M^n é uma folha arbitrariamente fixa da distribuição ortogonal a K e com função warping $\rho \in C^\infty(M)$ dado por $\rho = |K|$. A importância do espaço-tempo estático padrão provém do fato de incluírem alguns espaços-tempo clássicos, como o espaço-tempo Lorentz-Minkowski, o universo estático de Einstein, bem como modelos que descrevem um universo onde existe apenas uma massa esférica simétrica não rotativa, como uma estrela ou um buraco negro, como o espaço-tempo exterior de Schwarzschild (ver os exemplos citados na subseção 1.3).

Esta parte da tese é dedicada a generalizar e melhorar alguns dos resultados acima citados.

Parte II: Rigidez de hipersuperfícies em certos produtos warped e resultados para subvariedades em produtos ponderados

Na segunda parte desta tese, dedicamo-nos a expor os resultados em variedades Riemannianas. Embora exista certa similaridade com a primeira parte, onde foram obtidos resultados semelhantes em ambientes Lorentzianos, também abordamos temas distintos que contribuíram para a literatura acadêmica em outras frentes. Ao trabalhar com variedades Riemannianas, é possível explorar propriedades geométricas e métricas das variedades sem a restrição do caráter Lorentziano. Isso permite investigar questões específicas relacionadas à curvatura, geodésicas, volume, entre outras características intrínsecas das variedades Riemannianas.

Embora os resultados possam ser semelhantes aos encontrados para ambientes Lorentzianos em termos de técnicas e métodos utilizados, os contextos diferem e podem levar a conclusões distintas. Além disso, ao abordar temas diferentes na segunda parte da tese, contribui-se para a diversificação do conhecimento na literatura acadêmica. Essa abordagem complementar entre a primeira e segunda parte da tese, explorando ambientes Lorentzianos e Riemannianos, pode enriquecer a compreensão das propriedades das variedades e fornecer uma visão mais abrangente sobre o assunto estudado.

A investigação sobre a rigidez de hipersuperfícies imersas num espaço Riemanniano é certamente um tópico relevante em análise geométrica, e podemos afirmar que este ramo de investigação teve início com o teorema clássico de Bernstein [50] (depois emendado por Hopf em [99]), no qual afirma que os únicos gráficos inteiros mínimos em $\mathbb{R}^3$ são os planos. O teorema de Bernstein foi estendido a $\mathbb{R}^n$, para $n \leq 7$, com os trabalhos de Fleming [88], de Giorgi [90] e Simons [137]. Mas Bombieri, de Giorgi e Giusti [51] inferiram que o teorema de Bernstein não é válido para $n \geq 8$. Por outro lado, Moser [120] mostrou que os hiperplanos são os únicos gráficos mínimos inteiros de funções $u \in C^2(\mathbb{R}^n)$ cujo gradiente Du tem norma limitada em $\mathbb{R}^n$, para todos os valores de n. Já em 2015, Lima e Oliveira [121] obteve novos resultados do tipo Moser relacionados com grafos inteiros de curvatura média constante construídos sobre a fibra M^n de um espaço produto $\mathbb{R} \times M^n$.

Quando o espaço ambiente é um produto warped do tipo $I \times_\rho M^n$, onde $I \subset \mathbb{R}$ representa um intervalo aberto e ρ é uma função suave positiva definida em I, Montiel [115] estudou a rigidez de hipersuperfícies compactas de curvatura média constante. Neste contexto mais geral, usou o fato de um tal produto warped ser dotado de um campo de vetores Killing globalmente conforme definido dado por $\rho\partial_t$ (onde ∂_t representa o campo de vectores unitário tangente a $I \subset \mathbb{R}$) para provar que estas hipersuperfícies devem ser fatias $\{t\} \times M^n$, sob a hipótese de que são localmente grafos na fibra M^n.

Mais tarde, Alías e Dajczer [25] obtiveram os resultados de Montiel [115] considerando hipersuperfícies completas, não necessariamente compactas, imersas em $\mathbb{R} \times_\rho M^n$. Posteriormente, de Lima juntamente com Aquino [20] e Caminha [62], obtiveram resultados de rigidez para gráficos verticais completos com curvatura média constante em $I \times_\rho M^n$, assumindo restrições apropriadas sobre os valores da curvatura média e da norma do gradiente da função altura h. Em seguida, supondo que o gradiente de h é integrável e que a função curvatura média toma valores no intervalo $(0, 1]$, o segundo autor em conjunto com Camargo e Caminha [61] aplicou uma técnica de Yau [148] para provar que hipersuperfícies completas situadas em uma faixa de um espaço pseudo-hiperbólico $\mathbb{R} \times_{e^t} M^n$ devem ser slices.

Motivados por estes trabalhos, tratamos no Capítulo 6 de hipersuperfícies completas two-sided (isto é, hipersuperfícies completas com fibrado normal trivial) imersas num produto warped do tipo $I \times_\rho M^n$. Sob restrições adequadas na função warping ρ, na curvatura seccional da fibra M^n e na curvatura média de uma tal hipersuperfície Σ^n, aplicamos alguns princípios de máximo para mostrar que Σ^n tem de ser uma slice de $I \times_\rho M^n$. É também feito um estudo de grafos inteiros construídos sobre M^n, bem como são dadas aplicações a espaços pseudo-hiperbólicos $I \times_{e^t} M^n$.

Por outro lado, Alías, de Lira e Rigoli [27] introduziram a definição geral de soluções auto-similares do fluxo da curvatura média numa variedade Riemanniana $\overline{M}^{n+1}$ dotada de um campo vetorial K e estabeleceram a correspondente noção de soliton do fluxo da curvatura média. Em particular, quando $\overline{M}^{n+1}$ é um produto warped Riemanniano do tipo $I \times_\rho M^n$ e $K = \rho(t)\partial_t$, aplicaram princípios de máximos fracos para garantir que um sóliton do fluxo da curvatura média completo é uma slice de $\overline{M}^{n+1}$. De modo similar ao realizado no Capítulo 3, obtivemos resultados de rigidez e não existência de sólitons do fluxo da curvatura média em modelos Riemannianos warped.

Em alguns dos resultados obtidos no Capítulo 3, utilizamos o operador Laplaciano ponderado como um mecanismo analítico para obter os resultados desejados, independentemente da presença de uma função ponderadora no ambiente. No entanto, na última seção desta tese, dedicamos-nos a apresentar resultados obtidos em ambientes ponderados por uma função φ positiva e integrável.

No ramo da análise geométrica, muitos problemas levam-nos a considerar variedades Riemannianas dotadas de uma medida que tem uma densidade positiva suave em relação à medida Riemanniana. Isto acaba por ser compatível com a estrutura métrica da variedade e os espaços resultantes são *variedades weighted*, que também são chamadas variedades com densidade ou espaços de medida métrica suave na literatura atual. Mais precisamente, dada uma variedade Riemanniana completa n-dimensional (M^n, g) e uma função suave $\varphi : M^n \to \mathbb{R}$, a variedade ponderada M^n_φ associada a M^n e φ é a tripla $(M^n, g, d\mu = e^{-\varphi} dM)$, onde dM denota o elemento de volume padrão de M^n.

Aparecendo naturalmente no estudo de self-shrinkers, sólitons de Ricci, fluxos do calor harmónicos e muitos outros, as variedades ponderadas provaram ser importantes generalizações das variedades Riemannianas e, hoje em dia, há várias investigações geométricas a seu respeito. Para um breve panorâmica dos resultados neste domínio, remetemos para os artigos de Morgan [119] e Wei-Wylie [146].

Salientamos que uma teoria da curvatura de Ricci para variedades ponderadas remonta a Lichnerowicz [112, 113] e foi posteriormente desenvolvida por Bakry e Émery no seu trabalho seminal [45]. Neste contexto, como ingrediente crucial para compreender a geometria de uma variedade ponderada M^n_φ, introduzimos o chamado *Bakry-Émery-Ricci tensor* Ric_φ como sendo a seguinte extensão do tensor de Ricci padrão Ric de M^n:

$$\mathrm{Ric}_\varphi = \mathrm{Ric} + \mathrm{Hess}\,\varphi. \tag{3}$$

Consequentemente, é natural estender os resultados enunciados em termos da curvatura de Ricci a resultados análogos para o tensor de Bakry-Émery-Ricci.

Por outro lado, sabe-se que os campos vetoriais Killing conformes são objetos importantes que têm sido amplamente utilizados para compreender a geometria de subvariedades imersas em espaços Riemannianos. Neste contexto, Montiel [115] estudou hipersuperfícies compactas de curvatura média constante imersas em produtos warped do tipo $\mathbb{R} \times_\rho M^n$ e $\mathbb{S}^1 \times_\rho M^n$. Observamos que esta classe de produtos warped é dotada de um campo vetorial Killing globalmente conforme definido por $\rho\partial_t$, onde ∂_t representa o campo vectorial unitário tangente a $\mathbb{R}$ ou $\mathbb{S}^1$. Supondo que tais hipersuperfícies são localmente grafos em M^n, Montiel provou que (até casos excepcionais bem compreendidos) têm de ser slices $\{t\} \times M^n$.

Mais tarde, este tema foi revisitado em [25] por Alías e Dajczer, onde generalizaram os resultados de Montiel considerando hipersuperfícies completas, não necessariamente compactas, imersas em $\mathbb{R} \times_\rho M^n$. Posteriormente, Henrique de Lima juntamente com Caminha [62] e posteriormente com Aquino [20], investigaram a unicidade de gráficos verticais completos com curvatura média constante num produto deformado $I \times_\rho M^n$. Sob restrições adequadas aos valores da curvatura média e à norma do gradiente da função altura, obtiveram teoremas de unicidade relativos a tais gráficos. Em seguida, Rosenberg, Schulze e Spruck [134] mostraram que um grafo inteiro mínimo com função altura não negativa num espaço produto $\mathbb{R} \times M^n$, cuja fibra M^n é completa com curvatura de Ricci não negativa e curvatura seccional limitada por baixo, tem de ser um slice. Posteriormente, Henrique de Lima et al. [19, 74] obtiveram algumas outras condições suficientes que asseguram que uma hipersuperfície completa de two-sided imersa num espaço produto $\mathbb{R} \times M^n$, cuja fibra M^n tem curvatura seccional limitada por baixo, é uma slice do espaço ambiente, desde que a sua função angular tenha algum comportamento adequado.

Mais recentemente, Araujo, de Lima e Velasquez em [29] investigaram subvariedades n-dimensionais imersas em $I \times_\rho M^{n+p}$, cuja função de deformação ρ tem logaritmo convexo. Assumindo que uma tal subvariedade $\psi : \Sigma^n \to I \times_\rho M^{n+p}$ é fechada, estocasticamente completa ou completa com curvatura de Ricci não negativa, e que a sua função suporte $\langle \vec{H}, \partial_t \rangle$ é constante (onde $\vec{H}$ representa o campo vetorial de curvatura média de ψ), provaram que $\psi(\Sigma)$ tem de estar contido numa fatia do espaço ambiente. Como consequência dos seus resultados de rigidez, quando $p = 1$ obtiveram resultados de não existência relativos a subvariedades mínimas imersas num tal espaço ambiente.

Finalizamos esta tese dedicando-nos ao estudo de subvariedades completas imersas em um produto warped ponderado do tipo $I \times_\rho M_\varphi^{n+p}$, onde a função de warping ρ é logaritmicamente convexa e a função de peso φ não depende do parâmetro real $t \in I$. Ao assumir a constância de uma função de suporte apropriada que envolve o campo vetorial de curvatura média φ de uma subvariedade Σ^n, juntamente com restrições adequadas sobre o tensor de Bakry-Émery-Ricci de Σ^n, provamos que ela deve estar contida em um slice do espaço ambiente. Como resultado, obtivemos reduções de codimensão e resultados do tipo Bernstein para multigrafos completos φ-minimal bounded construídos sobre o espaço Gaussiano n-dimensional. Nossa abordagem baseia-se no princípio do máximo fraco generalizado de Omori-Yau e em resultados do tipo Liouville para o drift Laplaciano.

No decorrer desta tese, serão apresentados todos os resultados obtidos, os quais foram detalhados em um total de 11 artigos científicos:

[1] J.G. Araújo, de Lima, H.F., W.F. Gomes and M.A.L. Velásquez, *Submanifolds immersed in a warped product with density.* Bull. Belg. Math. Soc. Simon Stevin **27** (2020) 683-696. https://doi.org/10.36045/j.bbms.200126

[2] J.G. Araújo, de Lima, H.F. and W.F. Gomes, *Uniqueness and nonexistence of complete spacelike hypersurfaces, Calabi-Bernstein type results and applications to Einstein-de Sitter and steady state type spacetimes*, Rev. Mat. Complut. **34** (2021), 653–673. https://doi.org/10.1007/s13163-020-00375-7

[3] J.G. Araújo, H.F. de Lima and W.F. Gomes, *Rigidity of hypersurfaces and Moser?Bernstein type results in*

certain warped products, with applications to pseudo-hyperbolic spaces. Aequat. Math. **96**, (2022), 1159-1177. https://doi.org/10.1007/s00010-022-00914-1

[4] J.G. Araújo, H.F. de Lima and W.F. Gomes, *On the rigidity of mean curvature flow solitons in certain semi-riemannian warped products*, Kodai Math. J. **46** (2023), 62-74. https://doi.org/10.2996/kmj46105

[5] J.G. Araújo, H.F. de Lima and W.F. Gomes, *Spacelike mean curvature flow solitons in standard static spacetimes and new calabi-bernstein type results*, Ricerche mat. (2023). 10.1007/s11587-023-00775-z

[6] J.G. Araújo, H.F. de Lima and W.F. Gomes, *On the mean curvature flow solitons in Riemannian spaces endowed with a Killing vector field*, preprint.

[7] M. Batista, G.M. Bisci, H.F. de Lima and W.F Gomes, *Solitons of the spacelike mean curvature flow in a generalized Robertson-Walker spacetime*, New York J. Math. **29** (2023) 554-579.

[8] M. Batista, H.F. de Lima and W.F Gomes *Rigidity of mean curvature flow solitons and uniqueness of solutions of the mean curvature flow soliton equation in certain warped products*, Mediterr. J. Math. **20**, 199 (2023). https://doi.org/10.1007/s00009-023-02407-0

[9] M. Batista, G.M. Bisci, H.F. de Lima and W.F Gomes, *Nonexistence of mean curvature flow solitons with polynomial volume growth immersed in certain semi-Riemannian warped products*, preprint.

[10] M. Batista, H.F. de Lima and W.F Gomes *Mean curvature flow solitons in certain warped products: Nonexistence, rigidity and Moser-Bernstein type results*, preprint.

[11] H.F. de Lima, W.F. Gomes, M.S. Santos and M.A.L. Velásquez, On the Geometry of spacelike mean curvature flow solitons immersed in a grw spaceetime, recommended for publication in Journal of the Australian Mathematical Society.

Introduction

This thesis is divided into two independent parts as follows:

Part I: Uniqueness and nonexistence of complete spacelike hypersurfaces

The theory of isometric immersions provides the adequate tools to approach some important problems involving spacetime singularities and gravitational collapse.

The singularity theorems proved in the 1960s by Penrose [126] and Hawking [96] state that the formation of singularities is unavoidable, if one assumes reasonable conditions on the curvature of the spacetime, on the extrinsic geometry of certain hypersurfaces and on the causal structure of the Lorentzian manifold. The existence of *spacelike hypersurfaces* (that is, hypersurfaces whose induced metric is a Riemannian metric) in the spacetime, in particular, is a key requirement in the original formulation of the singularity theorems as well as in their more recent generalizations (for more details, see [136, 139] and references therein).

In this context we could ask the following question:

Question 1: From a mathematical point of view, is there any relevance of the investigation of spacelike hypersurfaces in a Lorentzian manifold?

To answer this question, we assume an environment that to some extent locales models with singularities and gravitational collapses, so that we have investigate the geometry of complete spacelike hypersurfaces in a *generalized Robertson-Walker* (GRW) spacetime. By a GRW spacetime, we mean a Lorentzian warped product $-I \times_\rho M^n$ with Riemannian fiber M^n and warping function $\rho \in C^\infty(I)$, where $I \subset \mathbb{R}$ is an open interval.

In this context, in view of several geometrical features that could be worked on, in particular about mean curvature, we could ask a second question:

Question 2: What are the optimal constraints on the average curvatures of a complete hypersurfaces spacelike in a GRW spacetime, to obtain singularity and nonexistence results?

Recently, Aledo, Rubio and Salamanca [18] studied complete spacelike hypersurfaces with functionally bounded mean curvature in GRW spacetime $-I \times_\rho M^n$, whose Riemannian fiber M^n is assumed to be complete noncompact and with parabolic universal covering. As follows the statement below:

Theorem (Theorem 3.1 of [18]). *Let $\overline{M} = -I \times_\rho M^n$ be a spatially parabolic covered Lorentzian warped product whose warping function is not globally constant and satisfies $(log\rho)" \leq \alpha(log\rho)'^2$ for a certain real constant $\alpha \geq 0$. Let $\psi : M \longrightarrow \overline{M}$ be a complete hypersurface with bounded hyperbolic angle and such that* $\sup \rho(\tau) < \infty$ *and* $\inf \rho(\tau) > 0$. *If*

$$H^2 \leq \frac{\rho'(\tau)}{\rho^2(\tau)} \cosh^2 \varphi,$$

then M is a spacelike slice.

In this setting, the previous authors they used a suitable metric conformal to that induced on a spacelike hypersurface to provide rigidity results. As application, they obtained new Calabi-Bernstein type results concerning spacelike graphs defined on the Riemannian fiber M^n.

Proceeding with this picture, we deal with complete spacelike hypersurfaces in a GRW spacetime $-I \times_\rho M^n$. Under suitable constraints on the sectional curvature of the Riemannian fiber M^n, on the warping function ρ and on the future mean curvature (that is, the mean curvature function with respect to the future-pointing Gauss map of the spacelike hypersurface), we work with a conformal change of the induced metric (already used by Aledo, Rubio and Salamanca in [18]) to prove that such a spacelike hypersurface must be a slice $\{t\} \times M^n$ of the ambient spacetime. Nonexistence and Calabi-Bernstein type results concerning entire spacelike graphs constructed over the Riemannian fiber M^n we have obtained, as well as applications to the Einstein-de Sitter and steady state type spacetimes are given. Our approach is based on the so-called Omori-Yau's generalized maximum principle and on certain integrability properties due to Yau.

Of course we can expand our universe of elements a bit to work with a larger class of geometric objects. This can be done as follows: with $\mathbb{R}_1^{n+1}$ be the $(n+1)$-dimensional Minkowski space $(\mathbb{R}_1^{n+1}, \bar{g})$ with its standard Lorentzian metric

$$\bar{g} = -dx_1^2 + \sum_{i=2}^{n+1} dx_i^2.$$

Let $\psi : \Sigma^n \to \mathbb{R}_1^{n+1}$ be a spacelike immersion (which means that it has a Riemannian induced metric) in the Minkowski space. The *spacelike mean curvature flow* associated to ψ is a family of smooth spacelike immersions $\Psi_t = \Psi(t, \cdot) : \Sigma^n \to \mathbb{R}_1^{n+1}$ with corresponding images $\Sigma_t^n = \Psi_t(\Sigma^n)$ satisfying the following evolution equation

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} = \vec{H} \\ \Psi(0, x) = \psi(x) \end{cases}$$

on some time interval, where $\vec{H}$ stands for the (non-normalized) mean curvature vector of the spacelike submanifold Σ_t^n in $\mathbb{R}_1^{n+1}$. The solutions of the previous evolution equation are called solitons.

Again we come to the following question:

Question 3: What is the relevance of studying mean curvature flow solitons in GRW?

Mean curvature flow in the Minkowski space and, more generally, in a Lorentzian manifold has been extensively studied by several authors (see, for instance, [1, 80–83, 85, 86, 102, 103, 105, 106, 108, 141]) and, according to [81], an important justification for this interest is the fact that spacelike translating solitons can be regarded as a natural way of foliating spacetimes by almost null like hypersurfaces. Particular examples may give insight into the structure of certain spacetimes at null infinity and have possible applications in General Relativity (for more details, see [81]).

More recently, Lambert and Lotay [111] proved long-time existence and convergence results for spacelike solutions to mean curvature flow in the n-dimensional pseudo-Euclidean space $\mathbb{R}^n_m$ of index m, which are entire or defined on bounded domains and satisfying Neumann or Dirichlet boundary conditions. In [94], Guilfoyle and Klingenberg proved the longtime existence for mean curvature flow of a smooth n-dimensional spacelike submanifold of an $(n+m)$-dimensional manifold whose metric satisfies the so-called timelike curvature condition. Meanwhile, Alías, de Lira and Rigoli [27] introduced the general definition of self-similar mean curvature flow in a Riemannian manifold $\overline{M}^{n+1}$ endowed with a vector field K and establishing the corresponding notion of mean curvature flow soliton.

In [69], Colombo, Mari and Rigoli also studied some properties of mean curvature flow solitons in general Riemannian manifolds and in warped products, focusing on splitting and rigidity results under various geometric conditions, ranging from the stability of the soliton to the fact that the image of its Gauss map be contained in suitable regions of the sphere, as we can see from the following statement:

Theorem (Theorem 3.4 of [69]). *Let $\psi : \Sigma^n \longrightarrow \overline{M}^{n+1} = -I \times_\rho M^n$ be a connected, complete, stable mean curvature flow soliton with respect to $K = \rho(t)\partial_t$ with soliton constant c[2]. Assume that $\overline{M}$ is complete and has constant sectional curvature $\overline{k}$, with*

$$c\rho'(\pi_I \circ \psi) \leq n\overline{k} \text{ on } \Sigma. \tag{4}$$

Let $\Psi = II - \langle\ ,\ \rangle_\Sigma \otimes H$ be the umbilicity tensor of ψ and suppose that

$$|\Psi| \in L^2(\Sigma, e^{c\eta}) \tag{5}$$

with $\eta = \frac{|\Psi|^2}{2}$. Then one of the following cases occurs:

(i) ψ is totally geodesic (and if $c \neq 0$ then $\psi(\Sigma)$ is invariant by the flow of X), or

(ii) $I = \mathbb{R}$, ρ is constant on $\mathbb{R}$, Σ is isometric to the product $\mathbb{R} \times M$ with M a complete flat manifold and Σ is also flat. By introducing the universal coverings $\pi_\Sigma : \mathbb{R}^n \longrightarrow \Sigma$, $\pi_M : \mathbb{R} \longrightarrow M$ and $\pi_{\overline{M}} = id_\mathbb{R} \times \pi_M : \mathbb{R}^{n+1} \longrightarrow \overline{M}$, the map ψ lifts to an immersion

[2] a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ is said a *spacelike mean curvature flow soliton* with respect to $\mathcal{K} = \rho(t)\partial_t$ and with *soliton constant* $c \in \mathbb{R}$ if its (non-normalized) future mean curvature function satisfies $H = c\rho(h)\Theta$.

$\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n$ satisfying $\pi_{\overline{M}} \circ \psi = \psi \circ \pi_\Sigma$, which up to an isometry of $\mathbb{R}^n$ and a translation along the $\mathbb{R}$ factor of $\mathbb{R}^{n+1}$ is given by

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n, \quad (x^1, x^2, \ldots, x^n) \mapsto (\sigma_1(x^1), \sigma_2(x^1)x^2, \ldots, x^n)$$

where $\gamma = (\sigma_1, \sigma_2) : \mathbb{R} \longrightarrow \mathbb{R}^2$ is the grim reaper curve with image

$$\sigma(\mathbb{R}) = \left\{ (x, y) \in \mathbb{R}^2 : x = -\frac{1}{c\rho_0} \log(\cos(c\rho_0 y)), |y| < \frac{2}{\pi |c| \rho_0} \right\}$$

and ρ_0.is the constant value of ρ on $\mathbb{R}$. Furthermore, there exists a Riemannian submersion $\pi_\Omega : \Sigma \longrightarrow \Omega$ onto a compact, flat manifold Ω with 1-dimensional, noncompact geodesic fibers of the type $\pi_M(\mathbb{R} \times \{(x^2, \ldots, x^n)\})$, for constant $(x^2, \ldots, x^n) \in \mathbb{R}^{n-1}$. Such fiber is mapped by ψ into the grim reaper curve $\pi_{\overline{M}}(\sigma(\mathbb{R}) \times \{(x^2, \ldots, x^n)\})$.

Furthermore, any of the solitons in (ii) is stable, while a soliton in (i) is stable if and only if $L = \Delta_{-c\eta} + (n\overline{k} - c\rho')$ is non-negative.

Moreover, they also investigated the case of entire mean curvature flow graphs. When the ambient space is a Lorentzian product space, the Batista and de Lima [40] established nonexistence results for complete spacelike translating solitons under suitable curvature constraints on the curvatures of the Riemannian base of the ambient space. In particular, they obtained Calabi-Bernstein type results for entire translating graphs constructed over this Riemannian base. For this, they proved a version of the Omori-Yau's maximum principle for complete spacelike translating solitons. Besides, they also constructed new examples of rotationally symmetric spacelike translating solitons embedded in such an ambient space.

Proceeding with this retrospective commentary, in Chapter 3, extend the techniques developed in [18, 27, 31, 40, 69] to study complete spacelike mean curvature flow solitons immersed in a generalized Robertson-Walker (GRW) spacetime, that is, a Lorentzian warped product $-I \times_\rho M^n$ with 1-dimensional negative definite base I and n-dimensional Riemannian fiber M^n. Under suitable constraints on the warping function ρ and on the curvatures of M^n, we apply suitable maximum principles in order to obtain nonexistence and uniqueness results concerning these solitons. Applications to standard GRW spacetimes as, for instance, the Einstein-de Sitter and steady state type spacetimes, are given. Furthermore, we establish new Calabi-Bernstein type results related to entire spacelike mean curvature flow graphs constructed over the Riemannian fiber of the ambient spacetime.

Also in Chapter 3, we will highlight some results obtained in the area of stability of hypersurfaces spacelike. Recall that the notion of stability concerning hypersurfaces of constant mean curvature of Riemannian ambient spaces was first studied by Barbosa and do Carmo in [36], and by Barbosa, do Carmo and Eschenburg in [37], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations. Afterwards, working in the Lorentzian context, Barbosa and Oliker [38] obtained an analogous result proving that constant mean curvature spacelike hypersurfaces in Lorentzian manifolds are also critical points of the area functional for variations that keep the volume constant. Later on, Barros, Brasil and Caminha [39] studied the problem of strong stability (that is, stability with respect to not necessarily volume-preserving variations) for spacelike hypersurfaces with constant mean curvature in a generalized Robertson-Walker (GRW) spacetime of constant sectional curvature, giving a characterization for the maximal spacelike hypersurfaces and spacelike slices of such an ambient space.

At this point, the reader might think:

Question 4: Anyorder class of spacetimes that we can obtain similar results those examples in Chapter 2 and 3?

For this last question of this first part of the thesis, we work with the objective our purpose is to investigate the uniqueness and nonexistence of solitons of the spacelike mean curvature flow with respect to the timelike Killing vector field K of a *standard static spacetime*, which (according to Definition 12.36 of [123]) can be regarded as a warped product $M^n \times_\rho \mathbb{R}_1$ whose Riemannian base M^n is an arbitrarily fixed spacelike integral leaf of the distribution orthogonal to K and with warping function $\rho \in C^\infty(M)$ given by $\rho = |K|$. The importance of standard static spacetimes comes from the fact that they include some classical spacetimes, such as Lorentz-Minkowski spacetime, Einstein static universe as well as models that describe an universe where there is only a spherically symmetric non-rotating mass, as a star or a black hole, like exterior Schwarzschild spacetime (see the examples quoted in Subsection 1.3).

This part of the thesis is devoted to generalize and improve some of the above cited notorious results.

Part II: Rigidity of hypersurfaces in certain warped products and results for submanifolds in weighted productss

In the second part of this thesis, we devoted ourselves to presenting the results in Riemannian manifolds. Although there is some similarity with the first part, where similar results were obtained in Lorentzian settings, we also addressed distinct topics that contributed to the academic literature in other directions. Working with Riemannian manifolds allows us to explore their geometric and metric properties without the restriction of Lorentzian character. This enables the investigation of specific issues related to curvature, geodesics, volume, and other intrinsic characteristics of Riemannian manifolds.

While the results may be similar to those found in Lorentzian settings in terms of techniques and methods used, the contexts differ and can lead to distinct conclusions. Furthermore, by addressing different topics in the second part of the thesis, we contribute to the diversification of knowledge in the academic literature. This complementary approach between the first and second parts of the thesis, exploring Lorentzian and Riemannian settings, can enhance the understanding of manifold properties and provide a more comprehensive view of the studied subject

The investigation concerning the rigidity of hypersurfaces immersed in a Riemannian space is certainly a relevant topic in geometric analysis, and we can affirm that this research branch started with Bernstein's classical theorem [50] (after amended by Hopf in [99]), which says that the only entire minimal graphs in $\mathbb{R}^3$ are the planes. Bernstein's theorem was extended to $\mathbb{R}^n$, for $n \leq 7$, with the works of Fleming [88], de Giorgi [90] and Simons [137]. But, Bombieri, de Giorgi and Giusti [51] inferred that Bernstein's theorem does not hold for $n \geq 8$. On the other hand, Moser [120] showed that the hyperplanes are the only entire minimal graphs of functions $u \in C^2(\mathbb{R}^n)$ whose gradient Du has bounded norm on $\mathbb{R}^n$, for all values of n. In 2015, Lima and Oliveira [121] obtained new Moser-type results related to entire constant mean curvature graphs constructed over the fiber M^n of a product space $\mathbb{R} \times M^n$.

When the ambient space is a warped product of the type $I \times_\rho M^n$, where $I \subset \mathbb{R}$ stands for an open interval and ρ is a positive smooth function defined on I, Montiel [115] studied the rigidity of constant mean curvature compact hypersurfaces. In this more general context, he used the fact that such a warped product is endowed with a globally defined conformal Killing vector field given by $\rho\partial_t$ (where ∂_t stands for the unit vector field tangent to $I \subset \mathbb{R}$) to prove that these hypersurfaces must be slices $\{t\} \times M^n$, under the assumption that they are locally graphs on the fiber M^n.

Later on, Alías and Dajczer [25] reobtained Montiel's results [115] considering complete, not necessarily compact, hypersurfaces immersed in $\mathbb{R} \times_\rho M^n$. Afterwards, de Lima jointly with Aquino [20] and Caminha [62] obtained rigidity results for complete vertical graphs with constant mean curvature in $I \times_\rho M^n$, assuming appropriate restrictions on the values of the mean curvature and the norm of the gradient of the height function h. Next, supposing that the gradient of h is Lebesgue integrable and that the mean curvature function takes values in the interval $(0, 1]$, the second author jointly with Camargo and Caminha [61] applied a technique of Yau [148] to prove that complete hypersurfaces lying in a slab of a pseudo-hyperbolic space $\mathbb{R} \times_{e^t} M^n$ must be slices.

Motivated by these works, here we deal in Chapter 6 with complete two-sided hypersurfaces (that is, complete hypersurfaces having trivial normal bundle) immersed in a warped product of the type $I \times_\rho M^n$. Under suitable constraints on the warping function ρ, on the sectional curvature of the fiber M^n and on the mean curvature of such a hypersurface Σ^n, we apply some maximum principles in order to show that Σ^n must be a slice of $I \times_\rho M^n$. A study of entire graphs constructed over M^n is also made, as well as applications to pseudo-hyperbolic spaces $I \times_{e^t} M^n$ are given.

On the other hand, Alías, de Lira and Rigoli [27] introduced the general definition of self-similar mean curvature flow in a Riemannian manifold $\overline{M}^{n+1}$ endowed with a vector field K and establishing the corresponding notion of mean curvature flow soliton. In particular, when $\overline{M}^{n+1}$ is a Riemannian warped product of the type $I \times_\rho M^n$ and $K = \rho(t)\partial_t$, they applied weak maximum principles to guarantee that a complete n-dimensional mean curvature flow soliton is a slice of $\overline{M}^{n+1}$. Similar to what was done in Chapter 3, we have obtained results regarding rigidity and the non-existence of solitons for the mean curvature flow in warped Riemannian models.

In some of the results obtained in Chapter 3, we used the weighted Laplacian operator as an analytical mechanism to achieve the desired outcomes, regardless of the presence of a weight function in the environment. However, in the last section of this thesis, we focused on presenting results obtained in environments weighted by a positive and integrable function φ.

Into the branch of the geometric analysis, many problems lead us to consider Riemannian manifolds endowed with a measure that has a smooth positive density with respect to the Riemannian one. This turns out to be compatible with the metric structure of the manifold and the resulting spaces are the *weighted manifolds*, which are also called manifolds with density or smooth metric measure spaces in the current literature. More precisely, given a complete n-dimensional Riemannian manifold (M^n, g) and a smooth function $\varphi : M^n \to \mathbb{R}$, the weighted manifold M^n_φ associated to M^n and φ is the triple $(M^n, g, d\mu = e^{-\varphi} dM)$, where dM denotes the standard volume element of M^n.

Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many others, weighted manifolds are proved to be important nontrivial generalizations of Riemannian manifolds and, nowadays, there are several geometric investigations concerning them. For a brief overview of results in this scope, we refer the articles of Morgan [119] and Wei-Wylie [146].

We point out that a theory of Ricci curvature for weighted manifolds goes back to Lichnerowicz [112, 113] and it was later developed by Bakry and Émery in their seminal work [45]. In this setting, as a crucial ingredient to understand the geometry of a weighted manifold M^n_φ, they introduced the so-called *Bakry-Émery-Ricci tensor* Ric_φ as being the following extension of the standard Ricci tensor Ric of M^n:

$$\mathrm{Ric}_\varphi = \mathrm{Ric} + \mathrm{Hess}\,\varphi. \tag{6}$$

Consequently, it is natural to try to extend results stated in terms of the Ricci curvature to analogous results for the Bakry-Émery-Ricci tensor.

On the other hand, it is well known that conformal Killing vector fields are important objects which have been widely used in order to understand the geometry of submanifolds immersed in Riemannian spaces. In this setting, Montiel [115] studied constant mean curvature compact hypersurfaces immersed in warped products of the type $\mathbb{R} \times_\rho M^n$ and $\mathbb{S}^1 \times_\rho M^n$. We observe that such class of warped products are endowed with a globally defined conformal Killing vector field given by $\rho\partial_t$, where ∂_t stands for the unit vector field tangent to either $\mathbb{R}$ or $\mathbb{S}^1$. By supposing that such hypersurfaces are locally graphs on M^n, Montiel proved that (up to exceptional well-understood cases) they must be slices $\{t\} \times M^n$.

Later on, this thematic was revisited in [25] by Alías and Dajczer, where they generalized Montiel's results considering complete, not necessarily compact, hypersurfaces immersed in $\mathbb{R} \times_\rho M^n$. Afterwards, Henrique de Lima together with Caminha [62] and later Aquino [20] investigated the uniqueness of complete vertical graphs with constant mean curvature in a warped product $I \times_\rho M^n$. Under suitable restrictions on the values of the mean curvature and the norm of the gradient of the height function, they obtained uniqueness theorems concerning to such graphs. Next, Rosenberg, Schulze and Spruck [134] showed that an entire minimal graph with nonnegative height function in a product space $\mathbb{R} \times M^n$, whose fiber M^n is complete with nonnegative Ricci curvature and sectional curvature bounded from below, must be a slice. Afterwards, the Henrique de Lima et al. [19, 74] obtained some other sufficient conditions which assure that a complete two-side hypersurface immersed in a product space $\mathbb{R} \times M^n$, whose fiber M^n has sectional curvature bounded from below, is a slice of the ambient space, provided that its angle function has some suitable behavior.

More recently, Araujo, de Lima and Velasquez in [29] investigated n-dimensional submanifolds immersed in $I \times_\rho M^{n+p}$, whose warping function ρ has convex logarithm. Assuming that such a submanifold $\psi : \Sigma^n \to I \times_\rho M^{n+p}$ is either closed, stochastically complete or complete with nonnegative Ricci curvature, and that its support function $\langle \vec{H}, \partial_t \rangle$ is constant (where $\vec{H}$ stands for the mean curvature vector field of ψ), they proved that $\psi(\Sigma)$ must be contained in a slice of the ambient space. As a consequence of their rigidity results, when $p = 1$ they obtained nonexistence results concerning minimal submanifolds immersed in such an ambient space.

We conclude this thesis by dedicating ourselves to the study of complete n-dimensional submanifolds immersed in a weighted warped product of the form $I \times_\rho M^{n+p}_\varphi$, where the warping function ρ is logarithmically convex and the weight function φ does not depend on the real parameter $t \in I$. Assuming the constancy of an appropriate support function involving the mean curvature vector field φ of such a submanifold Σ^n, along with suitable constraints on the Bakry-Émery-Ricci tensor of Σ^n, we prove that it must be contained in a slice of the ambient space. As applications, we obtain codimension reductions and Bernstein-type results for complete φ-minimal bounded multigraphs constructed on the n-dimensional Gaussian space. Our approach relies on the generalized weak maximum principle of Omori-Yau and Liouville-type results for the drift Laplacian.

Throughout this thesis, all the results obtained will be presented, which have been detailed in a total of 11 scientific articles:

[1] J.G. Araújo, de Lima, H.F., W.F. Gomes and M.A.L. Velásquez, *Submanifolds immersed in a warped product with density.* Bull. Belg. Math. Soc. Simon Stevin **27** (2020) 683-696. https://doi.org/10.36045/j.bbms.200126

[2] J.G. Araújo, de Lima, H.F. and W.F. Gomes, *Uniqueness and nonexistence of complete spacelike hypersurfaces, Calabi-Bernstein type results and applications to Einstein-de Sitter and steady state type spacetimes*, Rev. Mat. Complut. **34** (2021), 653–673. https://doi.org/10.1007/s13163-020-00375-7

[3] J.G. Araújo, H.F. de Lima and W.F. Gomes, *Rigidity of hypersurfaces and Moser?Bernstein type results in certain warped products, with applications to pseudo-hyperbolic spaces.* Aequat. Math. **96**, (2022), 1159-1177. https://doi.org/10.1007/s00010-022-00914-1

[4] J.G. Araújo, H.F. de Lima and W.F. Gomes, *On the rigidity of mean curvature flow solitons in certain semi-riemannian warped products*, Kodai Math. J. **46** (2023), 62-74. https://doi.org/10.2996/kmj46105

[5] J.G. Araújo, H.F. de Lima and W.F. Gomes, *Spacelike mean curvature flow solitons in standard static spacetimes and new calabi-bernstein type results*, Ricerche mat. (2023). 10.1007/s11587-023-00775-z

[6] J.G. Araújo, H.F. de Lima and W.F. Gomes, *On the mean curvature flow solitons in Riemannian spaces endowed with a Killing vector field*, preprint.

[7] M. Batista, G.M. Bisci, H.F. de Lima and W.F Gomes, *Solitons of the spacelike mean curvature flow in a generalized Robertson-Walker spacetime*, New York J. Math. **29** (2023) 554-579.

[8] M. Batista, H.F. de Lima and W.F Gomes *Rigidity of mean curvature flow solitons and uniqueness of solutions of the mean curvature flow soliton equation in certain warped products*, Mediterr. J. Math. **20**, 199 (2023). https://doi.org/10.1007/s00009-023-02407-0

[9] M. Batista, G.M. Bisci, H.F. de Lima and W.F Gomes, *Nonexistence of mean curvature flow solitons with polynomial volume growth immersed in certain semi-Riemannian warped products*, preprint.

[10] M. Batista, H.F. de Lima and W.F Gomes *Mean curvature flow solitons in certain warped products: Nonexistence, rigidity and Moser-Bernstein type results*, preprint.

[11] H.F. de Lima, W.F. Gomes, M.S. Santos and M.A.L. Velásquez, On the Geometry of spacelike mean curvature flow solitons immersed in a grw spaceetime, recommended for publication in Journal of the Australian Mathematical Society.

Part I

Uniqueness and nonexistence of complete spacelike hypersurfaces

Chapter 1

Preliminaries for Part I

The Generalized Robertson-Walker (GRW) spacetime is one of the simplest and most important models in modern cosmology. It is a generalization of the Robertson-Walker (RW) model, which describes a homogeneous and isotropic universe on a large scale. The GRW model introduces a warping function, which describes the spatial curvature of the universe on a smaller scale than the cosmological scale. This warping function is responsible for describing the warped product structures present in the GRW model.

Since its introduction, the GRW model has been widely studied in cosmology and general relativity theory. In this section, we will present some of the basic concepts of the GRW spacetime, with an emphasis on its warped product structure. The main theoretical results will be discussed, including exact solutions for the GRW model, as well as a detailed analysis of its geometric and physical properties. In this chapter, for the sake of clarity we shall introduce several useful the nitions and notations that will appear throughout Part I of this thesis.

1.1 Generalized Robertson Walker spacetimes

In this setting, we beginning by establishing the notations which will appear in forthcomings Chapters 2, 3 and 4. Let (M^n, g_M) be a connected, n-dimensional, oriented Riemannian manifold, $I \subset \mathbb{R}$ an open interval and $\rho : I \to \mathbb{R}$ a positive smooth function. Also, in the product manifold $\overline{M}^{n+1} = I \times M^n$ furnished with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + \rho^2(\pi_I)\pi_M^*(g_M),$$

where π_I and π_M are the projections onto the factors I and M^n, respectively, is a Lorentzian warped product with warping function ρ and fiber M. Along this work, we will simply write

$$\overline{M}^{n+1} = -I \times_\rho M^n. \tag{1.1}$$

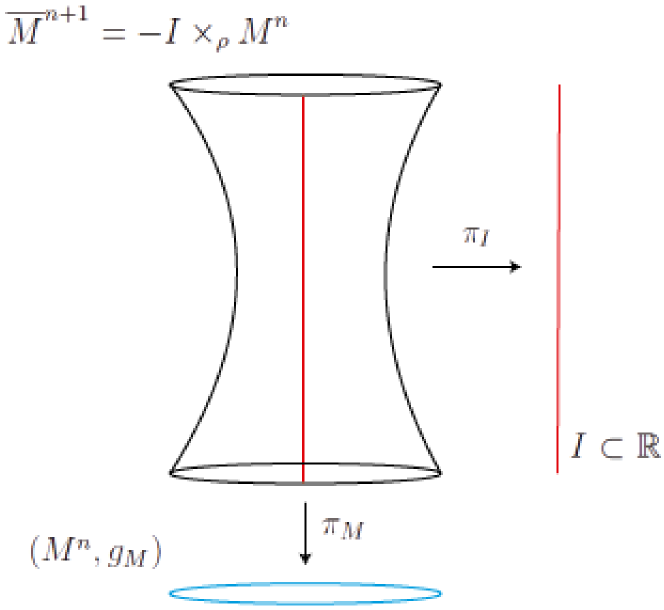

Figure 1.1: Representation of the Lorentzian warped product

A standard computation shows that $\mathcal{K} = \rho(t)\partial_t$, where $\pi_I(p) = t$, is a conformal closed vector field globally defined on $\overline{M}$, where ∂_t stands for the coordinate timelike vector field tangent to I (see [123] for details).

According to the nomenclature established in [9], we say that $\overline{M}^{n+1}$ is a *generalized Robertson Walker* (GRW) spacetime with warping function ρ and Riemannian fiber M^n. When M^n has constant sectional curvature, (1.1) has been known in the mathematical literature as a Robertson-Walker (RW) spacetime, an allusion to the fact that, for $n = 3$ and certain cases of warping functions ρ, it is an exact solution of Einstein's field equations (see, for instance, [53, Corollary 9.107] or [123, Chapter 12]).

Let Σ^n be an n-dimensional connected manifold. A smooth immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ is said to be a *spacelike hypersurface* if Σ^n, furnished with the metric g induced from $\overline{g}$ via ψ, is a Riemannian manifold. We will denote by ∇ the Levi-Civita connection of g. Since $\overline{M}$ is time-orientable, it follows from the connectedness of Σ^n that one can uniquely choose a globally defined timelike unit vector field N is normal, having the same time-orientation of ∂_t, that is, such that $\overline{g}(N, \partial_t) < 0$. In this case, one says that N is the *future-pointing Gauss map* of Σ^n and will always assume such a timelike orientation for Σ^n. From the inverse Cauchy-Schwarz inequality (see [123, Proposition 5.30]), we have that $\overline{g}(N, \partial_t) \leq -1$, with the equality holding at a point $p \in \Sigma^n$ if, and only if, $N = \partial_t$ at p. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of I and M^n (see [123, Proposition 7.35]), it follows that

$$\overline{\nabla}_V \mathcal{K} = \rho'(\pi_I)V, \tag{1.2}$$

for all $V \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$ defined in (1.1).

Throughout this paper, given a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ and its future-pointing Gauss map N, we will consider the Weingarten operator $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which is defined by $AX = -\overline{\nabla}_X N$, and the mean curvature function $H = -\frac{1}{n}\text{trace}(A)$, which will be called the *future mean curvature* of Σ^n.

Remark 1.1.1. *For a fixed* $t_0 \in I$*, we orient the* slice $\Sigma^n_{t_0} = \{t_0\} \times M^n$ *using the field of normal vectors* ∂_t*. According to Example* 5.6 *in* [10] *we have that the* slice *has constant future mean curvature* $H = \frac{\rho'(t_0)}{\rho(t_0)}$ *with respect to* $N = \partial_t$*.*

Now, we consider two particular functions naturally attached to a spacelike hypersurface Σ^n into a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$, namely, the height function denoted by h, is the restriction of the projection $\pi_I(t, y) = t$ to Σ^n, that is, $h : \Sigma^n \to I$ is given by

$$h = \pi_I|_{\Sigma^n} = \pi_I \circ \psi. \tag{1.3}$$

Thus, the hyperbolic angle Θ of Σ^n verifies

$$\Theta = \langle N, \partial_t \rangle \leq -1, \tag{1.4}$$

where N denotes the future-pointing Gauss map of Σ^n.

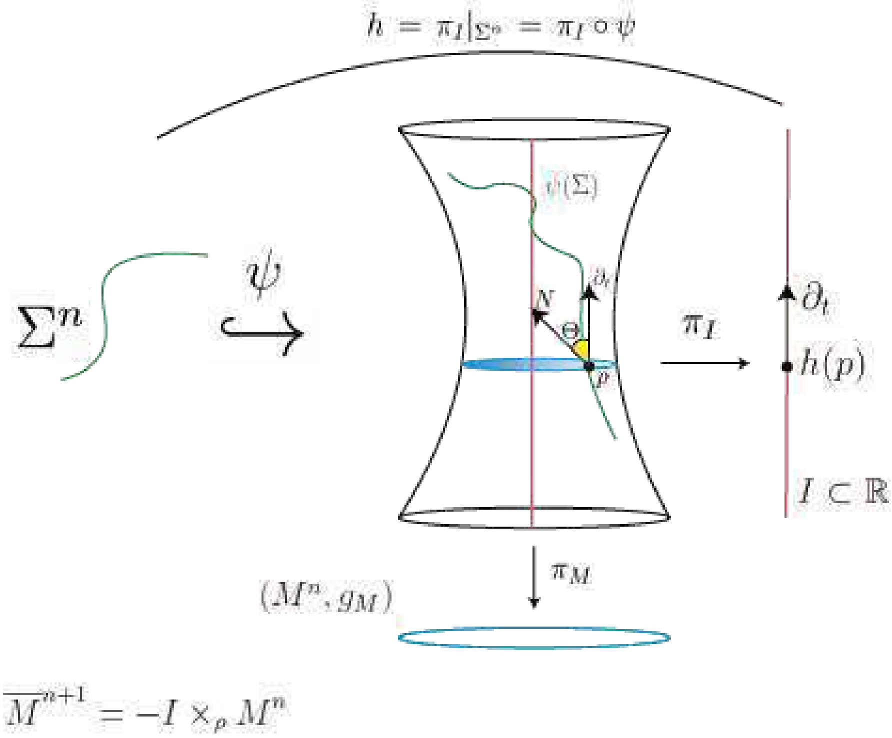

Figure 1.2: GRW representation with angle and height functions

Moreover, the equality $\Theta = -1$ holds if and only if $N = \partial_t$, that is, Σ^n is an open portion of a slice. A simple computation shows that

$$\overline{\nabla}\pi_I = -\overline{g}(\overline{\nabla}\pi_I, \partial_t)\partial_t = -\partial_t. \tag{1.5}$$

So, from (1.5) we have

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \Theta N. \tag{1.6}$$

Thus, (1.6) gives the following relation

$$|\nabla h|^2 = \Theta^2 - 1, \tag{1.7}$$

where $|\ .\ |$ stands for the norm of a tangent vector field on Σ^n in the metric g. Concerning relation (1.7), we have that h is constant if and only if Σ^n is an open portion of a slice.

On the other hand, from (1.2) we have that

$$\overline{\nabla}_V \partial_t = \frac{\rho'(\pi_I)}{\rho(\pi_I)}\{V + \bar{g}(V, \partial_t)\partial_t\}. \tag{1.8}$$

Hence, from (1.6) and (1.8) we deduce that, for any $X \in \mathfrak{X}(\Sigma)$, the Hessian of h in the metric g is given by

$$\begin{aligned} \nabla^2 h(X, X) &= g(\nabla_X \nabla h, X) \\ &= -\frac{\rho'(h)}{\rho(h)}\{|X|^2 + g(X, \nabla h)^2\} + g(AX, X)\Theta. \end{aligned} \tag{1.9}$$

Hence, from (1.9) we obtain that the Laplacian of h in the metric g is (see, for instance, [12, Lemma 4.1] or [62, Proposition 3.2])

$$\Delta h = -\frac{\rho'(h)}{\rho(h)}\{n + |\nabla h|^2\} - nH\Theta. \tag{1.10}$$

We conclude this section by recalling the convergence condition of a GRW spacetime which was introduced by Alías and Colares [12]. We say that a GRW spacetime $\overline{M}^n$ obeys the *strong null convergence condition* (SNCC) if the sectional curvature K_M of the Riemannian fiber M^n obeys the relation

$$K_M \geq \sup_I(\rho\rho'' - \rho'^2). \tag{1.11}$$

1.2 Spacelike mean curvature flow solitons in GRW spacetimes and examples

Spacelike mean curvature flow solitons are solutions to the mean curvature flow equation, which describes the evolution of a hypersurface in spacetime under its mean curvature. In GRW spacetimes, which are a class of spacetimes with a warped product structure, there exist certain spacelike hypersurfaces that are invariant under the flow and are known as solitons.

These solitons have been studied in the context of cosmology and general relativity, and their properties have been investigated in terms of their geometry and physical significance. They are important because they provide insight into the dynamics of spacetime in the presence of matter and energy, and their study can lead to a better understanding of the nature of the universe. Overall, the study of Spacelike mean curvature flow solitons in GRW spacetimes is an active area of research in cosmology and general relativity.

We recall that the spacelike mean curvature flow $\Psi : [0,T) \times \Sigma^n \to \overline{M}^{n+1}$ of a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ in a $(n+1)$-dimensional Lorentzian manifold $\overline{M}^{n+1}$, satisfying $\Psi(0,\cdot) = \psi(\cdot)$, looks for solutions of the equation

$$\frac{\partial \Psi}{\partial t} = n\vec{H},$$

where $\vec{H}(t,\cdot)$ is the (non-normalized) mean curvature vector of $\Sigma_t^n = \Psi(t, \Sigma^n)$ (see, for instance, [111]). In our context, according to [27, Definition 1.1] and [69, Definition 1.1], a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ is said a *spacelike mean curvature flow soliton* with respect to $\mathcal{K} = \rho(t)\partial_t$ and with *soliton constant* $c \in \mathbb{R}$ if its (non-normalized) future mean curvature function satisfies

$$H = c\rho(h)\Theta. \tag{1.12}$$

In fact, considering that Ψ is a self-similar mean curvature flow with respect to some vector field X, we can reason as in [27, Proposition 2.1] to deduce that the corresponding mean curvature vector satisfies

$$\vec{H} = cX^{\perp},$$

for some constant $c \in \mathbb{R}$. In our setting, X is equal to $\mathcal{K} = \rho(t)\partial_t$ and, hence, assuming that $\psi : \Sigma^n \to \overline{M}^{n+1}$ satisfies equation (1.12) means that it is a solution of the mean curvature flow evolution equation.

Adopting the terminology introduced in [27] and [69], we will also consider the *soliton function*

$$\zeta_c(t) = n\rho'(t) + c\rho^2(t). \tag{1.13}$$

So, each slice $M_{t_*} = \{t_*\} \times M^n$ is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = \rho(t)\partial_t$ and with soliton constant c given by

$$c = -n\frac{\rho'(t_*)}{\rho(t_*)^2}. \tag{1.14}$$

Moreover, t_* is implicitly given by the condition $\zeta_c(t_*) = 0$.

We finished this section quoting important examples which will be addressed along the next sections.

Example 1.2.1. In a similar way of [79], for the Lorentzian product space $-I \times M^n$, from (1.14) we get that the slices $\{t\} \times M^n$ are spacelike mean curvature flow solitons with soliton constant $c = 0$ with respect to vector field $\mathcal{K} = \partial_t$. Similarly to what happens in the Minkowski space $\mathbb{R}_1^{n+1} = -\mathbb{R} \times \mathbb{R}^n$, such solitons are called spacelike translating solitons.

Example 1.2.2. As in [13, Section 4], the *future temporal cone* Λ^+ of the Minkowski space $\mathbb{R}_1^{n+1}$

is defined as being the following set

$$\Lambda^+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle < 0 \ \text{ and } \ \langle x, e_1\rangle < 0\},$$

where $e_1 = (1, 0, \cdots, 0)$. We observe that Λ^+ can be regarded as the following GRW spacetime

$$-\mathbb{R}^+ \times_t \mathbb{H}^n,$$

where $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = -1, \ x_1 > 0\}$ denotes the n-dimensional hyperbolic space. Indeed, it is not difficult to verify that the map $\Phi : -\mathbb{R}^+ \times_t \mathbb{H}^n \to \Lambda^+$, given by $\Phi(t, x) = tx$, is an isometry. In this setting, we have that the slices $\{\sqrt{-\frac{n}{c}}\} \times \mathbb{H}^n$ are spacelike mean curvature flow solitons with soliton constant $c < 0$ with respect to vector field $\mathcal{K} = t\partial_t$.

Example 1.2.3. *The 4-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^3$, where $\mathbb{R}^3$ stands for the 3-dimensional Euclidean space endowed with its canonical metric, is a classical exact solution to the Einstein field equation without cosmological constant. It is an open Friedmann-Robertson-Walker model, which incorporates homogeneity and isotropy (the cosmological principle) and permitted expansion (for more details, see* [123, Chapter 12]*). Here, we consider the $(n+1)$-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$. From* (1.14) *we conclude that the slice $\{(-\frac{2n}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$ is the only one that is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ and with soliton constant $c < 0$.*

Example 1.2.4. *According to the terminology introduced by Albujer and Alías* [4]*, a GRW spacetime $-\mathbb{R} \times_{e^t} M^n$ is called a* steady state type *spacetime. This terminology is due to the fact that the steady state model of the universe $\mathcal{H}^4$, proposed by Bondi-Gold* [54] *and Hoyle* [100] *when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous) but also at all times, is isometric to the RW spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^3$ (for more details, see* [96]*). From* (1.14) *we conclude that the slice $\{\log(-\frac{n}{c})\} \times M^n$ is the only one that is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = e^t\partial_t$ and with soliton constant $c < 0$.*

Example 1.2.5. *From* [115, Example 4.2]*, the $(n+1)$-dimensional de Sitter space $\mathbb{S}^{n+1}_1$ is isometric to the RW spacetime $-\mathbb{R} \times_{\cosh t} \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the n-dimensional unit Euclidean sphere endowed with its standard metric. Taking into account the terminology introduced in* [17]*, the open half-space $\mathbb{R}^+ \times \mathbb{S}^n \subset \mathbb{S}^{n+1}_1$ (respect. $\mathbb{R}^- \times \mathbb{S}^n \subset \mathbb{S}^{n+1}_1$) is called the* chronological future *(respect.* past*) of $\mathbb{S}^{n+1}_1$ with respect to the totally geodesic* equator *$\{0\} \times \mathbb{S}^n$. From* (1.14) *we see that the equator is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = \cosh t\,\partial_t$ and constant soliton $c = 0$ and the slices $\{\sinh^{-1}(\frac{-n\pm\sqrt{n^2-4c^2}}{2c})\} \times \mathbb{S}^n$ are spacelike mean curvature flow soliton with respect to $\mathcal{K} = \cosh t\,\partial_t$ and with soliton constant $0 < |c| \leq \frac{n}{2}$.*

Example 1.2.6. *Taking into account once more* [115, Example 4.2]*, we consider the open region of $\mathbb{S}^{n+1}_1$ which is isometric to the RW spacetime $-\mathbb{R}^+ \times_{\sinh t} \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space endowed with its standard metric. From* (1.14) *we have that* *the slices* $\{\cosh^{-1}(\frac{-n-\sqrt{n^2+4c^2}}{2c})\} \times \mathbb{H}^n$ *are spacelike mean curvature flow soliton with respect to* $\mathcal{K} = \sinh t\, \partial_t$ *and with soliton constant* $c < 0$*.*

Example 1.2.7. *Motivated by* [115, Example 4.3]*, we will consider the open subset of the* $(n+1)$*-dimensional anti-de Sitter space* $\mathbb{H}_1^{n+1}$ *which is isometric to the RW spacetime* $-(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n$*. In analogy with the nomenclature of the de Sitter space, the open half-space* $(0, \frac{\pi}{2}) \times \mathbb{H}^n \subset \mathbb{H}_1^{n+1}$ *(respect.* $(-\frac{\pi}{2}, 0) \times \mathbb{H}^n \subset \mathbb{H}_1^{n+1}$*) will be called the* chronological future *(respect.* past*) of* $\mathbb{H}_1^{n+1}$ *with respect to the totally geodesic* equator $\{0\} \times \mathbb{H}^n$*. From* (1.14) *we see that the equator is a spacelike mean curvature flow soliton with respect to* $\mathcal{K} = \cos t\, \partial_t$ *and constant soliton* $c = 0$ *and the slices* $\{\sin^{-1}(\frac{-n\pm\sqrt{n^2+4c^2}}{2c})\} \times \mathbb{H}^n$ *are spacelike mean curvature flow soliton with respect to* $\mathcal{K} = \cos t\, \partial_t$ *and with soliton constant* $c \neq 0$*.*

1.3 Standard static spacetimes

When we set out to describe a generic spacetime, the Alice in Wonderland quality of the experience is partly because coordinate invariance allows our time and distance scales to be arbitrarily rescaled, but also partly because the landscape can change from one moment to the next. The situation is drastically simplified when the spacetime has a timelike Killing vector. Such a spacetime is said to be stationary. Two examples are flat spacetime and the spacetime surrounding the rotating earth (in which there is a frame-dragging effect). Non-examples include the solar system, cosmological models, gravitational waves, and a cloud of matter undergoing gravitational collapse.

Although the standard static space is an environment whose metric has Lorentzian properties, we prefer to introduce the reader to the correct preliminaries only now, in the next section, instead of reporting on them together in section 1.1.

In this context, we cite some basic concepts, properties and examples concerning standard static spacetimes, which will be used and addressed in the next sections.

Let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional Lorentz manifold endowed with a timelike Killing vector field K. Suppose that the distribution $\mathcal{D}$ orthogonal to K has constant rank and it is integrable. We denote by $\Psi : M^n \times \mathbb{I} \to \overline{M}^{n+1}$ the flow generated by K, where M^n is an arbitrarily fixed spacelike integral leaf of $\mathcal{D}$ labeled as $t = 0$, which we will suppose to be connected, and $\mathbb{I}$ is the maximal interval of definition. Without lost of generality, in what follows we will also consider $\mathbb{I} = \mathbb{R}$.

In this setting, $\overline{M}^{n+1}$ can be regard as the standard static spacetime $M^n \times_\rho \mathbb{R}_1$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the standard static metric

$$\overline{g} = \pi_M^\star(g_M) - (\rho \circ \pi_M)^2 \pi_{\mathbb{R}}^\star(dt^2), \tag{1.15}$$

where π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M^n \times \mathbb{R}$ onto each factor, g_M is the induced Riemannian metric on the Riemannian base M^n, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and $\rho \in C^\infty(M)$ is the warping function, which is given by $\rho = |K| = \sqrt{-\overline{g}(K, K)}$, where $|\ |$ denotes the norm of a vector field on $\overline{M}^{n+1}$. In particular, when $\rho \equiv 1$, the resulting standard static spacetime $(\overline{M}^{n+1}, \overline{g})$ is just a Lorentzian product space with factors (M^n, g_M) and $(\mathbb{R}, -dt^2)$.

In what follows, we quote some classical examples of standard static spacetimes, where our results obtained in the next sections can be applied.

Example 1.3.1. Our first example is given by the Lorentz-Minkowski spacetime $\mathbb{L}^{n+1}$, which is isometric to the warped product $(\, \mathbb{R}^n \times \mathbb{R}_1\,,\ \pi^*_{\mathbb{R}^n}(g_{\mathbb{R}^n}) + \pi^*_{\mathbb{R}}(-dt^2)\,)$.

Example 1.3.2. The Einstein static universe $(\, \mathbb{S}^n \times \mathbb{R}_1\,,\ \pi^*_{\mathbb{S}^n}(g_{\mathbb{S}^n}) + \pi^*_{\mathbb{R}}(-dt^2)\,)$ is also a standard static space (see Example 5.11 of [48]).

Example 1.3.3. The exterior Schwarzschild spacetime is defined as follows: Let $\mathbb{R}^4$ be given coordinates (t, r, θ, φ), where (r, θ, φ) are the usual spherical coordinates on $\mathbb{R}^3$. Given a positive constant m, the exterior Schwarzschild spacetime is defined on the subset $r > 2m$ of $\mathbb{R}^4$, a subset which is topologically $\mathbb{R}^2 \times \mathbb{S}^2$. The Schwarzschild metric for the region $r > 2m$ is given in (t, r, θ, φ) coordinates by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Since the metric for this spacetime is invariant under time translations $t \to t + a$, the coordinate vector field $\partial/\partial t$ is a (globally defined) timelike Killing vector field (see Section 5.2 of [48] and Chapter 13 of [123]). Consequently, the exterior Schwarzschild spacetime is a standard static spacetime.

Example 1.3.4. A model that also presents static regions (which appeared shortly after the Schwarzschild spacetime) is the Reissner-Nordström spacetime, whose metric in (t, r, θ, φ) coordinates admits the representation

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

This metric has singularities in $r = 0$, $r = r_+$ and $r = r_-$, where $r_\pm = m \pm (m^2 - e^2)^{1/2}$, and in regions corresponding to $+\infty > r > r_+$ and $r_- > r > 0$ we have that the Reissner-Nordström spacetime is static (see Section 5.5 of [95]).

1.4 Entire spacelike graphs

Before we start on the results properly, we need to recall some basic facts related to these spacelike graphs.

Let $\Omega \subseteq M^n$ be a connected domain and let $u \in C^\infty(\Omega)$ be a smooth function such that $u(\Omega) \subseteq I$, then $\Sigma^n(u)$ will denote the (vertical) graph over Ω determined by u, that is,

$$\Sigma(u) = \{(u(p), p) : p \in \Omega\} \subset \overline{M}^{n+1} = -I \times_\rho M^n.$$

The graph is said to be entire if $\Omega = M^n$. Observe that $h(u(p), p) = u(p), p \in \Omega$. Hence, h and u can be identified in a natural way. The metric induced on Ω from the Lorentzian metric $\overline{g}$ defined in (1.1) via $\Sigma(u)$ is

$$g_u = -du^2 + \rho^2(u) g_M. \tag{1.16}$$

It can be easily seen that a graph $\Sigma(u)$ is a spacelike hypersurface if and only if $|Du|_M < \rho(u)$, where Du stands for the gradient of u in M and $|Du|_M$ its norm, both with respect to the metric g_M. On the other hand, in the case where M^n is a simply connected manifold, from [9, Lemma 3.1] we have that every complete spacelike hypersurface $\psi : \Sigma^n \to -I \times_\rho M^n$ such that the warping function ρ is bounded on Σ^n is an entire spacelike graph over M^n. In particular, this happens for complete spacelike hypersurfaces lying in a closed solid cylinder over M^n.

Remark 1.4.1. *Also pointing out that, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph $\Sigma(u)$ in a GRW spacetime is not necessarily complete, in the sense that the induced Riemannian metric (1.16) is not necessarily complete on M^n. For instance, Albujer constructed explicit examples of noncomplete entire maximal spacelike graphs (that is, whose mean curvature is identically zero) in the Lorentzian product space $-\mathbb{R} \times \mathbb{H}^2$ (see [3, Section 3]).*

The future-pointing Gauss map of a spacelike graph $\Sigma(u)$ over Ω is given by the vector field

$$N(p) = \frac{\rho(u(p))}{\sqrt{\rho^2(u(p)) - |Du(p)|_M^2}} \left(\partial_t|_{(u(p),p)} + \frac{Du(p)}{\rho^2(u(p))} \right), \quad p \in \Omega. \tag{1.17}$$

The Weingarten operator related to the future-pointing Gauss map (1.17) is given by

$$\begin{aligned} AX = &- \frac{1}{\rho(u)\sqrt{\rho^2(u) - |Du|_M^2}} D_X Du - \frac{\rho'(u)}{\sqrt{\rho^2(u) - |Du|_M^2}} X \\ &+ \left(\frac{-g_M(D_X Du, Du)}{\rho(u) \left(\rho^2(u) - |Du|_M^2\right)^{3/2}} + \frac{\rho'(u) g_M(Du, X)}{\left(\rho^2(u) - |Du|_M^2\right)^{3/2}} \right) Du, \end{aligned} \tag{1.18}$$

for any vector field X tangent to Ω, where D denotes the Levi-Civita connection of (M^n, g_M). Consequently, if $\Sigma(u)$ is a spacelike graph defined over a domain $\Omega \subseteq M^n$, it is not difficult to verify from (1.18) that the future mean curvature function $H(u)$ of $\Sigma(u)$ is given by the following nonlinear differential equation:

$$H(u) = \operatorname{div}_M \left(\frac{Du}{n\rho(u)\sqrt{\rho^2(u) - |Du|_M^2}} \right) + \frac{\rho'(u)}{n\sqrt{\rho^2(u) - |Du|_M^2}} \left(n + \frac{|Du|_M^2}{\rho^2(u)} \right), \tag{1.19}$$

where div_M stands for the divergence operator computed in the metric g_M.

1.5 Omori-Yau maximum principle, Liouville type results and maximum principle for complete non compact manifolds

We initiate quoting the generalized maximum principle of Omori [122] and Yau [147] (see also [16] for a modern and accessible reference to the generalized maximum principle of Omori-Yau).

Lemma 1.5.1. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u \in C^\infty(\Sigma)$ be a smooth function which is bounded from above on Σ^n. Then there exists a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k) = \sup_\Sigma u, \qquad \lim_k |\nabla u(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta u(p_k) \leq 0.$$

It is not difficult to see that Lemma 1.5.1 is equivalent to the following one.

Lemma 1.5.2. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u \in C^\infty(\Sigma)$ be a smooth function which is bounded from below on Σ^n. Then there exists a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k) = \inf_\Sigma u, \qquad \lim_k |\nabla u(p_k)| = 0 \qquad \text{and} \qquad \liminf_k \Delta u(p_k) \geq 0.$$

We will continue this subsection by citing an extension of Hopf's theorem on a complete Riemannian manifold (Σ^n, g) due to Yau in [148]. For this, we will adopt the following notation

$$\mathcal{L}^p_g(\Sigma) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u|^p d\Sigma < +\infty\},$$

where $d\Sigma$ stands for the measure defined from the metric g.

Lemma 1.5.3. *Let u be a smooth function defined on a complete Riemannian manifold (Σ^n, g), such that Δu does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1_g(\Sigma)$, then Δu vanishes identically on Σ^n.*

Lemma 1.5.4. *If u is a nonnegative smooth subharmonic function defined on (Σ^n, g), with $u \in \mathcal{L}^p_g(\Sigma)$ for some $p > 1$, then u must be constant.*

Lemma 1.5.5. *All complete noncompact Riemannian manifolds with nonnegative Ricci curvature have at least linear volume growth.*

Next we shall devote ourselves to presenting the analytical tool that will be used to establish our rigidity results in the next ones. For this, let (Σ^n, g) be a complete noncompact Riemannian manifold and let $d(\,\cdot\,, o) : \Sigma^n \to [0, +\infty)$ denote the Riemannian distance of (Σ^n, g), measured from a fixed point $o \in \Sigma^n$. We say that a smooth function $u \in C^\infty(\Sigma)$ *converges to zero at infinity* when it satisfies the following condition

$$\lim_{d(x,o)\to+\infty} u(x) = 0. \tag{1.20}$$

Keeping in mind this concept, the following lemma corresponds to item (a) of [21, Theorem 2.2].

We also need the following definition which is inspired in (1.20): Given a complete noncompact Riemannian immersion $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ and $t_* \in I$, we say that a function u defined on Σ^n *converges from below (above) to* t_* *at infinity* when $u \leq t_*$ ($u \geq t_*$) and the function $\tilde{u} := u - t_*$ converges to zero at infinity.

Lemma 1.5.6. *Let (Σ^n, g) be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma)$ be a vector field on Σ^n. Assume that there exists a nonnegative, non-identically vanishing function $u \in C^\infty(\Sigma)$ which converges to zero at infinity and such that $g(\nabla u, X) \geq 0$. If $\mathrm{div}_g X \geq 0$ on Σ^n, then $g(\nabla u, X) \equiv 0$ on Σ^n.*

For our purpose, we will also need to quote a suitable maximum principle that will be used to prove our nonexistence results. For this, let (Σ^n, g) be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p,t)$ the geodesic ball centered at p and with radius t. Given a polynomial function $\sigma : (0, +\infty) \to (0, +\infty)$, we say that Σ^n has *polynomial volume growth* like $\sigma(t)$ if there exists $p \in \Sigma^n$ such that

$$\mathrm{vol}(B(p,t)) = \mathcal{O}(\sigma(t)),$$

as $t \to +\infty$, where vol denotes the standard Riemannian volume related to the metric g. As it was already observed in the beginning of Section 2 in [11], if $p, q \in \Sigma^n$ are at distance d from each other, we can verify that

$$\frac{\mathrm{vol}(B(p,t))}{\sigma(t)} \geq \frac{\mathrm{vol}(B(q,t-d))}{\sigma(t-d)} . \frac{\sigma(t-d)}{\sigma(t)}.$$

So, the choice of p in the notion of volume growth is immaterial. For this reason, we will just say that Σ^n has polynomial volume growth.

Keeping in mind this previous digression, we close this section quoting the following key lemma which corresponds to a particular case of a new maximum principle due to Alías, Caminha and do Nascimento (see [11, Theorem 2.1]).

Lemma 1.5.7. *Let (Σ^n, g) be a connected, oriented, complete noncompact Riemannian manifold, and let $u \in \mathcal{C}^\infty(\Sigma)$ be a nonnegative smooth function such that $\Delta u \geq au$ on Σ^n, for some positive constant $a \in \mathbb{R}$. If Σ^n has polynomial volume growth and $|\nabla u|$ is bounded on Σ^n, then u vanishes identically on Σ^n.*

Chapter 2

Uniqueness and nonexistence results in GRW spacetimes

In this chapter, we obtain a apply the generalized maximum principle of Omori-Yau [122,147], as well as other maximum principles due to Yau in [148], in order to obtain new uniqueness and nonexistence results concerning complete spacelike hypersurfaces in a GRW spacetime. This is made through the assumption of the *strong null convergence condition* (SNCC) and appropriate constraints on the warping function ρ and on the future mean curvature of the spacelike hypersurface. The results presented in this chapter make part of [31].

2.1 A computational lemma

Considering a spacelike hypersurface Σ^n in a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ obeying (1.11), the next lemma gives sufficient conditions to the Ricci curvature of Σ^n with respect to the conformal metric $\hat{g} = \frac{1}{\rho(h)^2} g$ be bounded from below. Besides technical reasons to use this conformal metric $\hat{g}$, we point out the following geometric meaning of $\hat{g}$: We can write $g^* = \frac{1}{\rho(t)^2}\overline{g}$, where $g^* = -\frac{1}{\rho(t)^2}dt^2 + g_M$ is the product Lorentzian metric $-ds^2 + g_M$ in $J \times M^n$, being J the open interval obtained from the change $ds = \frac{1}{\rho(t)}dt$, and considering the Riemannian metric on Σ^n induced from g^*.

In what follows, we will suppose that Σ^n is contained in a *timelike bounded region* M^n, that is,

$$\mathcal{B}_{t_1,t_2} := \{(t,p) \in -I \times_\rho M^n : t_1 \leq t \leq t_2 \quad \text{and} \quad p \in M^n\}.$$

We also recall that a spacelike hypersurface has *bounded second fundamental form* when the Hilbert-Schmidt norm of its Weingarten operator is bounded.

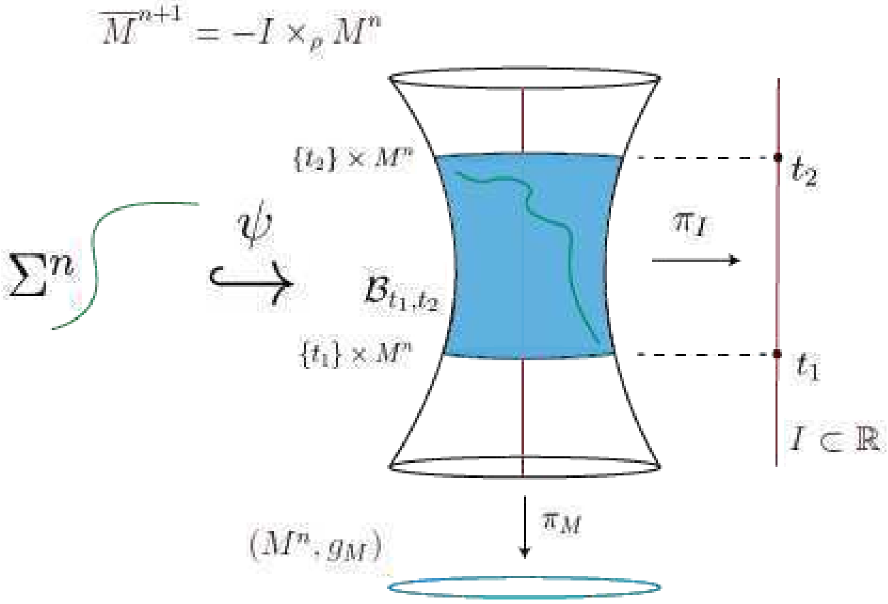

Figure 2.1: Representation of the timelike bounded region

Lemma 2.1.1. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying the SNCC (1.11) and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a spacelike hypersurface contained in $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$. If the second fundamental form and the angle function Θ are bounded, then the Ricci curvature $\widehat{\mathrm{Ric}}$ of Σ^n with respect to the conformal metric $\hat{g} := \dfrac{1}{\rho^2(h)} g$ is bounded from below.*

Proof. We recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ and the curvature tensor $\overline{R}$ of the ambient $-I \times_\rho M^n$ by the so-called Gauss' equation given by

$$g(R(X,Y)Z,W) = \bar{g}(\overline{R}(X,Y)Z,W) - g(AX,Z)g(AY,W) + g(AX,W)g(AY,Z), \tag{2.1}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$. Here, as in [123], the curvature tensor R is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [,] denotes the Lie blanket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Let us consider $X \in \mathfrak{X}(\Sigma)$ and take a (local) orthonormal frame $\{E_1, \cdots, E_n\}$. It follows from Gauss equation (2.1) that the Ricci curvature Ric of Σ^n with respect to the induced metric g satisfies

$$\mathrm{Ric}(X,X) \geq \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) - \frac{n^2H^2}{4}|X|^2. \tag{2.2}$$

To estimate the first summand on the right-hand side of (2.2), let us consider $X^* = (\pi_M)_*(X)$

and $E_i^* = (\pi_M)_*(E_i)$. So, from [123, Proposition 7.42] and (1.6) we have

$$\begin{aligned}\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) &= \sum_i g(R_M(X^*,E_i^*)X^*,E_i^*) + (n-1)((\log\rho)'(h))^2|X|^2 \\ &\quad -(n-2)(\log\rho)''(h)g(X,\nabla h)^2 - (\log\rho)''(h)|\nabla h|^2|X|^2,\end{aligned} \tag{2.3}$$

where R_M denotes the curvature tensor of M^n. By writing $X^* = X + \bar{g}(X,\partial_t)\partial_t$, we can easily estimate the first summand on the right-hand side of (2.3) to get

$$\begin{aligned}\sum_i g(R_M(X^*,E_i^*)X^*,E_i^*) &= \rho^2(h)(|X^*|_M^2|E^*|_M^2 - g(X^*,E^*)_M^2)K_M(X^*,E^*) \\ &\geq \frac{1}{\rho^2(h)}((n-1)|X|^2 + |\nabla h^2||X|^2 \\ &\quad +(n-2)g(X,\nabla h)^2)\min_i K_M(X^*,E_i^*).\end{aligned} \tag{2.4}$$

Consequently, since our ambient space obeys (1.11), from (2.4) we have that

$$\begin{aligned}\sum_i g(R_M(X^*,E_i^*)X^*,E_i^*) &\geq ((n-1)|X|^2 + |\nabla h|^2 \\ &\quad +(n-2)g(X,\nabla h)^2)(\log\rho)''(h)\end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.3), we get

$$\begin{aligned}\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) &\geq ((n-1)|X|^2 + |\nabla h|^2 + (n-2)g(X,\nabla h)^2)(\log\rho)''(h) \\ &\quad +(n-1)((\log\rho)'(h))^2|X|^2 \\ &\quad -(n-2)(\log\rho)''(h)g(X,\nabla h)^2 - (\log\rho)''(h)|\nabla h|^2|X|^2 \\ &= (n-1)\frac{\rho''(h)}{\rho(h)}|X|^2.\end{aligned} \tag{2.6}$$

Then, taking into account that $|A|^2 \geq nH^2$, from (2.2) and (2.6) we reach at

$$\mathrm{Ric}(X,X) \geq -\left((n-1)\frac{|\rho''(h)|}{\rho(h)} + \frac{n|A|^2}{4}\right)|X|^2. \tag{2.7}$$

On the other hand, we have the following equation (see, for instance, [53, Section 1J], [110, Section A] or [140, page 168])

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &= \mathrm{Ric}(X,X) + \frac{1}{\rho(h)^2}\Big\{(n-2)\rho(h)\nabla^2\rho(h)(X,X) \\ &\quad +(\rho(h)\Delta\rho(h) - (n-1)|\nabla\rho(h)|^2)|X|^2\Big\}.\end{aligned} \tag{2.8}$$

Consequently, from equation (2.8) we get

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &= \mathrm{Ric}(X,X)+\frac{1}{\rho^2(h)}\{(n-2)\rho(h)(\rho''(h)g(\nabla h,X)^2+\rho'(h)\nabla^2h(X,X)) \\ &\quad +(\rho(h)(\rho''(h)|\nabla h|^2+\rho'(h)\Delta h)-(n-1)(\rho'(h))^2|\nabla h|^2)|X|^2\}.\end{aligned} \tag{2.9}$$

Hence, considering (1.7), (1.9), (1.10) and (2.7) into (2.9), we obtain after a straightforward computation the following lower estimate

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &\geq \Big\{(n-1)\frac{(\rho'(h))^2}{\rho(h)}-(n-1)\left(\frac{|\rho''(h)|}{\rho(h)}+(n+1)\frac{(\rho'(h))^2}{\rho^2(h)}\right)\Theta^2 \\ &\quad -(n-\sqrt{n}-2)\frac{|\rho'(h)|}{\rho(h)}|A||\Theta|-\frac{n|A|^2}{4}\Big\}|X|^2.\end{aligned} \tag{2.10}$$

Therefore, taking into account that $|A|$ and $|\Theta|$ are bounded and that Σ^n lies in $\mathcal{B}_{t_1,t_2}$, from (2.10) we conclude that $\widehat{\mathrm{Ric}}$ is bounded from below. □

Remark 2.1.2. *We note that if we change the assumption SNCC to the NCC in Lemma 2.1.1, then the conclusion on $\hat{g}$ does not remain true.*

2.2 Statements and proofs of the main results

2.2.1 Uniqueness under Omori-Yau's generalized maximum principle

So, we are in position to present our first uniqueness result.

Theorem 2.2.1. *Let $\overline{M}^{n+1}=-I\times_\rho M^n$ be a GRW spacetime satisfying the SNCC (1.11) and let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a complete spacelike hypersurface lying in $\mathcal{B}_{t_1,t_2}\subset\overline{M}^{n+1}$, with $\rho'(t)>0$ for all $t_1\leq t\leq t_2$. Suppose that the second fundamental form is bounded, and that the future mean curvature H and the angle function Θ satisfy*

$$H\leq-\frac{\rho'(h)}{\rho(h)}\Theta. \tag{2.11}$$

If the height function h is such that

$$|\nabla h|\leq\inf_{\Sigma}\left|\frac{\rho'(h)}{\rho(h)}-H\right|, \tag{2.12}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. As before, let us consider on Σ^n the metric $\hat{g}=\dfrac{1}{\rho^2(h)}g$, which is conformal to its induced metric g. Denoting by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (1.7) and (1.10) we have

$$\hat{\Delta}h=-\rho(h)\rho'(h)\{n+(n-1)|\nabla h|^2\}-nH\rho(h)^2\Theta. \tag{2.13}$$

Thus, from (2.13) we get

$$\begin{aligned}\hat{\Delta}\rho(h) &= -n\rho(h)(\rho'(h))^2 - nH\rho'(h)\rho^2(h)\Theta \\ &\quad + \rho^3(h)\{(\log\rho)''(h) - (n-2)\frac{(\rho'(h))^2}{\rho^2(h)}\}|\nabla h|^2.\end{aligned} \tag{2.14}$$

For any positive real number α, with a straightforward computation from (2.14) we get obtain

$$\begin{aligned}\hat{\Delta}\rho^{-\alpha}(h) &= -\alpha\rho^{-\alpha-1}(h)\Big\{-n\rho(h)(\rho'(h))^2 - nH\rho'(h)\rho^2(h)\Theta \\ &\quad + \rho^3(h)\left((\log\rho)''(h) - (n+\alpha-3)\frac{(\rho'(h))^2}{\rho^2(h)}\right)|\nabla h|^2\Big\} \\ &= -\alpha\rho^{-\alpha-1}(h)\Big\{-n\rho(h)(\rho'(h))^2\Theta^2 - nH\rho'(h)\rho^2(h)\Theta \\ &\quad + \rho^3(h)\left((\log\rho)''(h) - (\alpha-3)\frac{(\rho'(h))^2}{\rho^2(h)}\right)|\nabla h|^2\Big\}.\end{aligned} \tag{2.15}$$

On the other hand, since we are assuming that $|A|$ is bounded (which implies that H is also bounded), and that $\Sigma^n \subset \mathcal{B}_{t_1,t_2}$, from (1.7) and (2.12) we get that Θ is bounded. So, Lemmas 1.5.1 and 2.1.1 guarantee the existence of a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n such that

$$\lim_k \rho^{-\alpha}(h)(p_k) = \sup_\Sigma \rho^{-\alpha}(h), \quad \lim_k |\hat{\nabla}\rho^{-\alpha}(h)(p_k)|_{\hat{g}} = 0 \quad \text{and} \quad \limsup_k \hat{\Delta}\rho^{-\alpha}(h)(p_k) \leq 0, \tag{2.16}$$

where $|\cdot|_{\hat{g}}$ and $\hat{\nabla}$ denote, respectively, the norm and gradient with respect to the metric $\hat{g}$.

But, it is not difficult to verify that

$$|\hat{\nabla}\rho^{-\alpha}(h)|_{\hat{g}} = \alpha\rho^{-\alpha}(h)|\rho'(h)||\nabla h|. \tag{2.17}$$

So, since $\Sigma^n \subset \mathcal{B}_{t_1,t_2}$, with $\rho'(t) > 0$ for all $t_1 \leq t \leq t_2$, from (2.16) and (2.17) we get that

$$\lim_k |\nabla h(p_k)| = 0. \tag{2.18}$$

Consequently, from (1.7) and (2.18) we have that

$$\lim_k \Theta(p_k) = -1. \tag{2.19}$$

Moreover, from (2.11), (2.15) and (2.18) we obtain

$$\begin{aligned}0 \geq \limsup_k \hat{\Delta}\rho^{-\alpha}(h)(p_k) &\geq n\alpha \limsup_k \Big\{\rho^{-\alpha-1}(h)\Big(\rho(h)(\rho'(h))^2\Theta^2 + H\rho'(h)\rho^2(h)\Theta\Big)\Big\}(p_k) \\ &\quad -\alpha\limsup_k \left\{\rho^{2-\alpha}(h)\left|(\log\rho)''(h) - (\alpha-3)\frac{(\rho'(h))^2}{\rho^2(h)}\right||\nabla h|^2\right\}(p_k) \\ &= n\alpha \sup_\Sigma \rho^{1-\alpha}(h) \limsup_k \Big\{\rho'(h)|\Theta|\Big(-\frac{\rho'(h)}{\rho(h)}\Theta - H\Big)\Big\}(p_k). \\ &\geq 0.\end{aligned} \tag{2.20}$$

Thus, from (2.20) and (2.19) we infer that

$$\lim_k \left(\frac{\rho'(h)}{\rho(h)} - H \right)(p_k) = 0. \tag{2.21}$$

Consequently, from (2.21) we get

$$\inf_\Sigma \left| \frac{\rho'(h)}{\rho(h)} - H \right| = 0. \tag{2.22}$$

Therefore, from (2.12) and (2.22) we conclude that Σ^n must be a slice of $\overline{M}^{n+1}$. □

Remark 2.2.2. *Related to the assumptions made in Theorem 2.2.1, we point out the following:*

(a) As it was observed in [18, Subsection 3.1], *the assumption $\rho'(t) > 0$ in $t_1 \leq t \leq t_2$ has a physical interpretation: If for each $p \in M^n$ we parameterize $I \times \{p\}$ by $\xi_p(t) = (t, p)$, since ∂_t is the velocity of each* galaxy *ξ_p, they are its integral curves. In particular, the function t is the common proper time of all the galaxies. By taking t constant, we get the slice $M_t = \{t\} \times M^n$. Then, the distance between two galaxies ξ_p and ξ_q in M_t is $\rho(t)\mathrm{d}_M(p, q)$, where d_M is the Riemannian distance in (M^n, g_M). In particular, when ρ has positive derivative in $t_1 \leq t \leq t_2$, we conclude that the universe is expanding in $t_1 \leq t \leq t_2$ for the comoving observers in M^n.*

(b) The differential inequality (2.11) and others similar to it have been used before in several other papers to get nonexistence and rigidity results concerning complete spacelike hypersurfaces (see, for instance, [5, 13, 15, 18, 28, 58–60, 72, 132]*).*

(c) Hypothesis (2.12) can be regarded as a control on the growth of the height function of the spacelike hypersurface through the difference between the mean curvature of the spacelike hypersurface and the mean curvature of the slices which are contained in $\mathcal{B}_{t_1,t_2}$ and intersect the spacelike hypersurface.

From the proof of Theorem 2.2.1 we obtain a nonexistence result concerning complete spacelike hypersurfaces with nonpositive future mean curvature.

Corollary 2.2.3. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying the SNCC (1.11). There does not exist a complete spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ lying in $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\rho'(t) > 0$ for all $t_1 \leq t \leq t_2$, having nonpositive future mean curvature and such its second fundamental form and angle function are bounded.*

Proof. Indeed, if such a complete spacelike hypersurface Σ^n existed, then (2.22) resulted in the following absurd

$$0 < \min_{t_1 \leq t \leq t_2} \frac{\rho'(t)}{\rho(t)} \leq \inf_\Sigma \frac{\rho'(h)}{\rho(h)} \leq \inf_\Sigma \left(\frac{\rho'(h)}{\rho(h)} - H \right) = 0.$$

□

2.2.2 Uniqueness under integrability properties

In what follows, we will assume that the warping function f of the ambient GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ satisfies the following inequality

$$(\log \rho)'' \leq \gamma((\log \rho)')^2, \tag{2.23}$$

for some nonnegative constant γ. Moreover, in the results that follow, inequality (2.23) is only needed at the points of a spacelike hypersurface.

As it was observed in [18], the inequality (2.23) is a mild hypothesis due to the fact that, for instance, when $\overline{M}^{n+1}$ obeys the SNCC (respect. NCC) and its Riemannian fiber M^n is flat (respect. Ricci-flat), we have that (2.23) is automatically satisfied.

In this setting, we obtain the following result.

Theorem 2.2.4. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (2.23), occurring the equality only at isolated points of I and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface lying in $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\rho'(t) \geq 0$ for all $t_1 \leq t \leq t_2$. If the future mean curvature function H and the angle function Θ satisfy (2.11), and the height function h is such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. We will consider again the conformal metric $\hat{g} := \dfrac{1}{\rho^2(h)} g$ and we will take $\alpha = \gamma + 3$. Since we are assuming that Σ^n lies in $\mathcal{B}_{t_1,t_2}$ and that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, from (2.17) we get that $|\hat{\nabla}\rho^{-\alpha}(h)|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$.

Moreover, since H satisfies (2.11) and $\rho'(t) \geq 0$ for all $t_1 \leq t \leq t_2$, from (2.15) and (2.23) we obtain that $\hat{\Delta}\rho^{-\alpha}(h) \geq 0$. Consequently, we can apply Lemma 1.5.3 to infer that $\hat{\Delta}\rho^{-\alpha}(h) = 0$ on Σ^n.

Therefore, since we are also assuming that the equality occurs in (2.23) just only at isolated points of I, returning to (2.15) we conclude that $|\nabla h|$ must vanishes identically on Σ^n and, hence, Σ^n must be a slice of $\overline{M}^{n+1}$. □

We also get a slight different version of Theorem 2.2.4.

Theorem 2.2.5. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (2.23) and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface lying in $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\rho'(t) > 0$ for all $t_1 \leq t \leq t_2$. If the future mean curvature function H and the angle function Θ satisfy (2.11), and the height function h is such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. As in the proof of Theorem 2.2.4, we get that $\hat{\Delta}\rho^{-\alpha}(h) = 0$ on Σ^n, for $\alpha = \gamma + 3$. Moreover, since Σ^n lies in $\mathcal{B}_{t_1,t_2}$, we can also verify that $|\hat{\nabla}\rho^{-2\alpha}(h)|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. But, we note that

$$\hat{\Delta}\rho^{-2\alpha}(h) = 2\rho^{-\alpha}(h)\hat{\Delta}\rho^{-\alpha}(h) + 2|\widehat{\nabla}\rho^{-\alpha}(h)|^2_{\hat{g}} = 2|\widehat{\nabla}\rho^{-\alpha}(h)|^2_{\hat{g}} \geq 0. \tag{2.24}$$

Thus, we can apply again Lemma 1.5.3 to obtain that $\hat{\Delta}\rho^{-2\alpha}(h) = 0$ on Σ^n. Hence, since we are assuming that $\rho'(t) > 0$ for $t_1 \leq t \leq t_2$, from (2.17) and (2.24) we obtain that $|\nabla h| = 0$ on Σ^n. Therefore, Σ^n must be a slice of $\overline{M}^{n+1}$. □

These previous lemmas 1.5.4 and 1.5.5 us to prove the following nonexistence result.

Theorem 2.2.6. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (2.23), occurring the equality only at isolated points of I, and whose fiber M^n is complete noncompact with nonnegative Ricci curvature. There does not exist a complete spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ lying in $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\rho'(t) \geq 0$ for all $t_1 \leq t \leq t_2$, having the future mean curvature function H and angle function Θ satisfying (2.11), and the height function h is such that $(\rho(h))^{-1} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > \gamma + 3$.*

Proof. Supposing by contradiction the existence of such a spacelike hypersurface Σ^n and taking once more $\alpha = \gamma + 3$, from (2.15) and (2.23) we get that $\hat{\Delta}\rho^{-\gamma-3}(h) \geq 0$ on Σ^n. Moreover, since we are assuming that $\Sigma^n \subset \mathcal{B}_{t_1,t_2}$, with $\rho'(t) \geq 0$ for all $t_1 \leq t \leq t_2$, and that $(\rho(h))^{-1} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > \gamma + 3$, it is not difficult to verify that $\rho^{-\gamma-3}(h) \in \mathcal{L}^p_{\hat{g}}(\Sigma)$ for $p = \frac{q}{\gamma+3} > 1$. Thus, we can apply Lemma 1.5.4 to get that $\rho(h)$ is constant on Σ^n. Hence, since we are also supposing that the equality occurs in (2.23) just only at isolated points of I, returning to (2.15) we conclude that $|\nabla h|$ must vanishes identically on Σ^n. Consequently, Σ^n is isometric (up to scaling) to M^n. So, since $\rho(h)$ is a positive constant, our assumption that $\rho(h) \in \mathcal{L}^q_g(\Sigma)$ also implies that M^n has finite volume. But, since M^n is assumed to be complete noncompact with nonnegative Ricci curvature, Lemma 1.5.5 leads us to a contradiction. □

2.2.3 Applications to Einstein-de Sitter spacetimes

The 4-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^3$ is a classical exact solution to the Einstein field equation without cosmological constant. It is an open Friedmann-Robertson-Walker model, which incorporates spatial homogeneity and isotropy (the cosmological principle) and permitted expansion (for more details, see [123, Chapter 12]). In [135], Rubio showed that the only complete constant mean curvature spacelike hypersurfaces in the 4-dimensional Einstein-de Sitter spacetime lying in a closed solid cylinder are spacelike slices and, in particular, there is no complete maximal hypersurface in such a region.

Here, observing that the $(n+1)$-dimensional Einstein-de Sitter spacetime

$$-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$$

satisfies (1.11), from Theorem 2.2.1 we obtain the following consequence.

Corollary 2.2.7. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface lying in a closed solid cylinder of the $(n+1)$-dimensional Einstein-de Sitter spacetime $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$, such that its second fundamental form is bounded. If the future mean curvature H and the angle function Θ satisfy $H \leq -\frac{2}{3t}\Theta$, and the height function h is such that $|\nabla h| \leq \inf_\Sigma \left|\frac{2}{3t} - H\right|$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When the ambient spacetime is the Einstein-de Sitter spacetime, Theorem 2.2.5 reads as follows.

Corollary 2.2.8. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface lying in a closed solid cylinder of the $(n+1)$-dimensional Einstein-de Sitter spacetime $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$. If the future mean curvature H and the angle function Θ satisfy $H \leq -\frac{2}{3t}\Theta$, and the height function h is such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

From Theorem 2.2.6 we obtain the following consequences.

Corollary 2.2.9. *There does not exist a complete spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ lying in a timelike bounded of the $(n+1)$-dimensional Einstein-de Sitter spacetime $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$, having future mean curvature H and the angle function Θ satisfying $H \leq -\frac{2}{3t}\Theta$ and the height function is such that $h^{-\frac{2}{3}} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > 3$.*

2.2.4 Applications to steady state type spacetimes

According to the terminology introduced by Albujer and Alías [4], a GRW spacetime $-I \times_\rho M^n$ such that $I = \mathbb{R}$ and whose warping function is just the exponential function $\rho(t) = e^t$ is called a *steady state type* spacetime. This terminology is due to the fact that the steady state model of the universe $\mathcal{H}^4$, proposed by Bondi-Gold [54] and Hoyle [100] when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous) but also at all times, is isometric to the GRW spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^3$ (for more details, see [96]).

Taking into account that a steady state type spacetime whose Riemannian fiber has nonnegative sectional curvature satisfies (1.11), from Theorem 2.2.1 we obtain the following application.

Corollary 2.2.10. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface lying in a closed solid cylinder of a steady state type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ whose Riemannian fiber M^n has nonnegative sectional curvature, such that its second fundamental form is bounded. If the future mean curvature H and the angle function Θ satisfy $H \leq -\Theta$, and the height function h is such that $|\nabla h| \leq \inf_\Sigma (1 - H)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

When the ambient spacetime is the steady state type spacetime, Theorem 2.2.5 reads as follows.

Corollary 2.2.11. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface lying in a closed solid cylinder of a steady state type spacetime $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$. If the future mean curvature H and the angle function Θ satisfy $H \leq -\Theta$, and the height function h is such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

From Theorem 2.2.6 we obtain the following consequences.

Corollary 2.2.12. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady state type spacetime, whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. There does not exist a complete spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ lying in a timelike bounded of $\overline{M}^{n+1}$, having future mean curvature H and the angle function Θ satisfying $H \leq -\Theta$ and the height function is such that $e^{-h} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > 3$.*

2.3 Calabi-Bernstein type results

Using the context developed in Section 1.4, we obtain a non-parametric version of Theorem 2.2.1.

Theorem 2.3.1. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying the SNCC (1.11) and let $\Sigma(u) \subset \overline{M}^{n+1}$ be an entire spacelike graph determined by a function $u \in C^\infty(M)$ with finite C^2 norm and whose future mean curvature satisfies*

$$H(u) \leq \frac{\rho'(u)}{\sqrt{\rho^2(u) - |Du|_M^2}}. \tag{2.25}$$

If, for some constant $0 < \beta < 1$,

$$|Du|_M \leq \min\left\{\beta\rho(u), \inf_M \left|\frac{\rho'(u)}{\rho(u)} - H(u)\right|\right\}, \tag{2.26}$$

then $u \equiv t_0$ for some $t_0 \in I$.

Proof. Observe first that, under the assumptions of the theorem, $\Sigma(u)$ is a complete spacelike hypersurface. Indeed, from (1.16) and the Cauchy-Schwarz inequality we get

$$g_u(X, X) = -g_M(Du, X^*)^2 + \rho^2(u)g_M(X^*, X^*) \geq (\rho^2(u) - |Du|_M^2)g_M(X^*, X^*), \tag{2.27}$$

for every tangent vector field X on $\Sigma(u)$, where (as before) X^* denotes the projection of X onto the Riemannian fiber M^n. Hence, from (2.26) and (2.27) we get, in particular, that

$$g_u(X, X) \geq \delta g_M(X^*, X^*), \tag{2.28}$$

where $\delta = (1 - \beta^2)\inf_M \rho^2(u)$. So, (2.28) implies that $L = \sqrt{\delta}L_M$, where L and L_M denote the length of a curve on $\Sigma(u)$ with respect to the Riemannian metrics g_u and g_M, respectively. As a consequence, since we are always assuming that M^n is complete, the induced metric g_u must be also complete.

Moreover, from (1.18) we conclude that (2.25) implies (2.11). Moreover, since we have that $N = N^* - \Theta\partial_t$, from (1.6) we get

$$|\nabla h|^2 = \rho^2(u)|N^*|_M^2. \tag{2.29}$$

Thus, from (1.17) and (2.29) we obtain

$$|\nabla h|^2 = \frac{|Du|_M^2}{\rho^2(u) - |Du|_M^2}. \tag{2.30}$$

Hence, from (2.30) we can also verify that (2.26) implies (2.12). Therefore, the result follows by applying Theorem 2.2.1. □

Remark 2.3.2. *We observe that through hypothesis (2.26) we are guaranteeing that the entire* *spacelike graph $\Sigma(u)$ is, indeed, complete, as well as, we are controlling the growth of the function u by a measure of how far $H(u)$ is different of the mean curvature of each slice that intersects $\Sigma(u)$.*

Remark 2.3.3. *According to* [76, Example 4.4], *we consider the following entire spacelike graph*

$$\Sigma(u) = \{(a \ln y, x, y) : y > 0\} \subset -\mathbb{R} \times \mathbb{H}^2,$$

where $0 < |a| < 1$. With a straightforward computation we can check that $|Du(x,y)|_{\mathbb{H}^2} = |a|$, $|A| = \frac{|a|}{\sqrt{1-a^2}}$ and $H(u) = -\frac{a}{2\sqrt{1-a^2}}$. Taking for instance $a = \frac{\sqrt{3}}{2}$, we see that hypotheses (2.25) *and* (2.26) *are satisfied by $\Sigma(u)$. Consequently, we conclude that the SNCC* (1.11) *is really necessary to get the desired conclusion in Theorem 2.3.1.*

On the other hand, following [57, Example 6.1], *let us consider the quadric model of $\mathbb{H}^2$ into the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ and let $\rho : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $\rho(t) = t$. If $\Omega = \{(x,y,z) \in \mathbb{L}^3 : x^2 + y^2 - z^2 < 0,\ z > 0\}$, then it is not difficult to check that the map $\phi : \mathbb{R}^+ \times_\rho \mathbb{H}^2 \to \Omega$, given by $\phi(t,(x,y,z)) = (tx, ty, tz)$, is an isometry. Hence, for each $z_0 > 0$,*

$$\Sigma_{z_0} := \phi^{-1}(\Omega \cap \{z = z_0\}) \subset \mathbb{R}^+ \times_\rho \mathbb{H}^2$$

is a maximal surface (that is, its mean curvature is identically zero), which is just the entire spacelike graph of the function $u_{z_0} : \mathbb{H}^2 \to \mathbb{R}^+$, defined by $u_{z_0}(x,y,z) = \frac{z_0}{z}$. Besides, we have that $\mathbb{R}^+ \times_f \mathbb{H}^2$ satisfies the SNCC (1.11)*. Thus, since $\Sigma_{z_0} = \Sigma(u_{z_0})$ verifies* (2.25)*, we conclude that the hypothesis* (2.26) *in Theorem 2.3.1 is necessary to infer that the function u is constant.*

Furthermore, from [118, Theorem 11 and Corollary 12] *we obtain the existence of a nontrivial entire spacelike graph $\Sigma(u) \subset -\mathbb{R} \times_{e^t} \mathbb{R}^n = \mathcal{H}^{n+1}$, with $u \in C^\infty(\mathbb{R}^n)$ being nonnegative and such that the mean curvature $H(u)$ is constant satisfying $H^2(u) \geq \frac{1}{1-|Du|^2_{\mathbb{R}^n}}$. Hence, we see that hypothesis* (2.25) *in Theorem 2.3.1 is also necessary to conclude the constancy of the function u.*

Reasoning as in the proof of Theorem 2.3.1, from Corollary 2.2.3 and the future mean curvature equation (1.19) we obtain the following nonexistence result.

Corollary 2.3.4. *Let M^n be a complete Riemannian manifold and $\rho : I \to \mathbb{R}$ a positive smooth function satisfying the SNCC* (1.11) *and such that ρ' is also positive on the open interval $I \subset \mathbb{R}$. For any constant $0 < \beta < 1$, there does not exist a smooth function $u : M^n \to I$ with finite C^2 norm, which is a solution of the following system of differential inequalities*

$$\begin{cases} \operatorname{div}_M\left(\dfrac{Du}{n\rho(u)\sqrt{\rho^2(u) - |Du|^2_M}}\right) + \dfrac{\rho'(u)}{n\sqrt{\rho^2(u) - |Du|^2_M}}\left(n + \dfrac{|Du|^2_M}{\rho^2(u)}\right) \leq 0 \\ \\ |Du|_M \leq \beta\rho(u) \end{cases}$$

Theorem 2.3.1 gives us the following applications in the context of the Einstein-de Sitter and steady state type spacetimes.

Corollary 2.3.5. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime and let $\Sigma(u) \subset \overline{M}^{n+1}$ be an entire spacelike graph determined by a function $u \in C^\infty(\mathbb{R}^n)$ with finite C^2 norm and whose future mean curvature satisfies $H(u) \leq \dfrac{2}{3u^{\frac{1}{3}}\sqrt{u^{\frac{4}{3}} - |Du|^2_{\mathbb{R}^n}}}$. If, for some constant $0 < \beta < 1$, $|Du|_{\mathbb{R}^n} \leq \min\{\beta u^{\frac{2}{3}}, \inf_{\mathbb{R}^n} \left|\frac{2}{3u} - H(u)\right|\}$, then $u \equiv t_0$ for some $t_0 > 0$.*

Corollary 2.3.6. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n has nonnegative sectional curvature and let $\Sigma(u) \subset \overline{M}^{n+1}$ be an entire spacelike graph determined by a function $u \in C^\infty(M)$ with finite C^2 norm and whose future mean curvature satisfies $H(u) \leq \dfrac{e^u}{\sqrt{e^{2u} - |Du|^2_M}}$. If, for some constant $0 < \beta < 1$, $|Du|_M \leq \min\{\beta e^u, \inf_M |1 - H(u)|\}$, then $u \equiv t_0$ for some $t_0 \in I$.*

Proceeding, from Theorem 2.2.4 we obtain the following Calabi-Bernstein type result.

Theorem 2.3.7. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (2.23), occurring the equality only at isolated points of I, and with $\rho'(t) \geq 0$ for all $t \in I$. Let $\Sigma(u) \subset \overline{M}^{n+1}$ be an entire spacelike graph determined by a bounded function $u \in C^\infty(M)$ whose future mean curvature satisfies (2.25). If, for some constant $0 < \beta < 1$, $|Du|_M \leq \beta\rho(u)$ and $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_0$ for some $t_0 \in I$.*

Proof. Since we are supposing that $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, it follows from the proof of Theorem 2.3.1 that $\Sigma(u)$ is a complete spacelike hypersurface.

On the other hand, it follows from (1.16) that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma$ stand for the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, and $G = det(g_{ij})$ with

$$g_{ij} = g_u(E_i, E_j) = \rho^2(u)\delta_{ij} - E_i(u)E_j(u).$$

Here, $\{E_1, \ldots, E^n\}$ denotes a local orthonormal frame with respect to the metric g_M. So, it is not difficult to verify that

$$|G| = \rho^{2(n-1)}(u)(\rho^2(u) - |Du|^2_M).$$

Consequently,

$$d\Sigma = \rho^{n-1}(u)\sqrt{\rho^2(u) - |Du|^2_M}dM. \tag{2.31}$$

Thus, from (2.30) and (2.31) we get

$$|\nabla h| d\Sigma = \rho^{n-1}(u)|Du|_M dM. \tag{2.32}$$

Hence, since we are assuming that u is bounded with $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, relation (2.32) guarantees that $|\nabla h| \in \mathcal{L}^1_g(\Sigma(u))$. Therefore, the result follows by applying Theorem 2.2.4. □

It is not difficult to see that from Theorem 2.2.5 we also get the following result.

Theorem 2.3.8. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (2.23) and with $\rho'(t) > 0$ for all $t \in I$. Let $\Sigma(u) \subset \overline{M}^{n+1}$ be an entire spacelike graph determined by a bounded function $u \in C^\infty(M)$ whose future mean curvature satisfies (2.25). If, for some constant $0 < \beta < 1$, $|Du|_M \le \beta\rho(u)$ and $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_0$ for some $t_0 \in I$.*

When the ambient space is either the Einstein-de Sitter spacetime or a steady state type spacetime, Theorem 2.3.8 reads as follows.

Corollary 2.3.9. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime and let $\Sigma(u) \subset \overline{M}^{n+1}$ be an entire spacelike graph determined by a bounded function $u \in C^\infty(\mathbb{R}^n)$ and whose future mean curvature satisfies $H(u) \le \dfrac{2}{3u^{\frac{1}{3}}\sqrt{u^{\frac{4}{3}} - |Du|^2_{\mathbb{R}^n}}}$. If, for some constant $0 < \beta < 1$, $|Du|_{\mathbb{R}^n} \le \beta u^{\frac{2}{3}}$ and $|Du|_{\mathbb{R}^n} \in \mathcal{L}^1_{g_{\mathbb{R}^n}}(\mathbb{R}^n)$, then $u \equiv t_0$ for some $t_0 > 0$.*

Corollary 2.3.10. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady state type spacetime and let $\Sigma(u)$ be an entire spacelike graph determined by a bounded function $u \in C^\infty(M)$ and whose future mean curvature satisfies $H(u) \le \dfrac{e^u}{\sqrt{e^{2u} - |Du|^2_M}}$. If, for some constant $0 < \beta < 1$, $|Du|_M \le \beta e^u$ and $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_0$ for some $t_0 \in I$.*

Taking into account once more relation (2.31), it is not difficult to see that from Theorem 2.2.6 we obtain the following nonexistence result.

Theorem 2.3.11. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (2.23), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. There does not exist a bounded entire solution $u \in C^\infty(M)$ of the future mean curvature equation (1.19), satisfying (2.25), with $|Du|_M \le \beta\rho(u)$, for some constant $0 < \beta < 1$, and such that $(\rho(u))^{-1} \in \mathcal{L}^q_{g_M}(M)$ for some q with $q > \gamma + 3$.*

We close this paper quoting the following applications of Theorem 2.3.11.

Corollary 2.3.12. *For any constant $0 < \beta < 1$, there does not exist a bounded smooth function $u : \mathbb{R}^n \to \mathbb{R}^+$ such that $u^{-\frac{2}{3}} \in \mathcal{L}^q_{g_{\mathbb{R}^n}}(\mathbb{R}^n)$, for some $q > 3$, and which is a solution of the following system of differential inequalities*

$$\begin{cases} \operatorname{div}_{\mathbb{R}^n}\left(\dfrac{Du}{nu^{\frac{2}{3}}\sqrt{u^{\frac{4}{3}} - |Du|^2_{\mathbb{R}^n}}}\right) + \dfrac{2|Du|^2_{\mathbb{R}^n}}{3nu^{\frac{5}{3}}\sqrt{u^{\frac{4}{3}} - |Du|^2_{\mathbb{R}^n}}} \le 0 \\ \\ |Du|_{\mathbb{R}^n} \le \beta u^{\frac{2}{3}} \end{cases}$$

Corollary 2.3.13. *Let M^n be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any constant $0 < \beta < 1$, there does not exist a bounded smooth function*

$u : M^n \to I$ such that $e^{-u} \in \mathcal{L}^q_{g_M}(M)$, for some $q > 3$, and which is a solution of the following system of differential inequalities

$$\begin{cases} \operatorname{div}_M \left(\dfrac{Du}{ne^u\sqrt{e^{2u} - |Du|^2_M}} \right) + \dfrac{|Du|^2_M}{ne^u\sqrt{e^{2u} - |Du|^2_M}} \leq 0 \\ \\ |Du|_M \leq \beta e^u \end{cases}$$

Chapter 3

Solitons of the spacelike mean curvature flow in GRW spacetimes

In the following results, we extend the techniques developed in [18, 27, 31, 40, 69] to study complete spacelike mean curvature flow solitons immersed in a generalized Robertson-Walker (GRW) spacetime, that is, a Lorentzian warped product $-I \times_\rho M^n$ with 1-dimensional negative definite base I and n-dimensional Riemannian fiber M^n. Under suitable constraints on the warping function ρ and on the curvatures of M^n, we apply suitable maximum principles in order to obtain nonexistence and uniqueness results concerning these solitons. we investigate geometric aspects of complete spacelike mean curvature flow solitons of codimension 1 in a generalized Robertson-Walker (GRW) spacetime $-I \times_\rho M^n$, with base $I \subset \mathbb{R}$, Riemannian fiber M^n and warping function $\rho \in C^\infty(I)$. For this, we apply suitable maximum principles to guarantee that such a mean curvature flow soliton is a slice of the ambient space, as well as to obtain nonexistence results concerning these solitons. In particular, we deal with entire graphs constructed over the Riemannian fiber M^n which are spacelike mean curvature flow solitons and we also explore the geometry of a conformal vector field in order to establish topological and further rigidity results for compact (without boundary) mean curvature flow solitons in a GRW spacetime. Applications to standard GRW spacetimes as, for instance, the Einstein-de Sitter and steady state type spacetimes, are given. Furthermore, we establish new Calabi-Bernstein type results related to entire spacelike mean curvature flow graphs constructed over the Riemannian fiber of the ambient spacetime. The results presented in this chapter make part of [41–43, 78].

3.1 Statements and proofs of the main results

Aiming to simplify the notation, along our main results we will consider the *modified soliton function* as being the function

$$\bar{\zeta}_c(t) := \rho'(t)\zeta_c(t), \tag{3.1}$$

where ζ_c is the soliton function defined in (1.13). So, we are in a position to state and prove our first nonexistence result concerning spacelike mean curvature flow solitons immersed in a GRW spacetime.

3.1.1 Nonexistence of spacelike mean curvature flow solitons in GRW spacetimes

Theorem 3.1.1. *Let* $\overline{M}^{n+1} = -I \times_\rho M^n$ *be a GRW spacetime satisfying* (1.11). *There does not exist a complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *with respect to* $\mathcal{K} = \rho(t)\partial_t$ *with soliton constant* c*, whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region* $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ *such that* $\bar{\zeta}_c(t)$ *has strict sign for all* $t \in [t_1, t_2]$.

Proof. By contradiction, let us assume the existence of a complete spacelike mean curvature flow soliton ψ satisfying the assumptions of Theorem 3.1.1. As before, consider on Σ^n the metric $\hat{g} = \frac{1}{\rho^2(h)} g$, which is conformal to its induced metric g. Denoting by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (1.7) and (1.10) we have

$$\hat{\Delta} h = -\rho(h)\rho'(h)\{n + (n-1)|\nabla h|^2\} - H\rho^2(h)\Theta. \tag{3.2}$$

Thus, from (2.13) we get

$$\begin{aligned}\hat{\Delta}\rho(h) \;=\; & -n\rho(h)(\rho'(h))^2 - H\rho'(h)\rho^2(h)\Theta \\ & +\rho^3(h)\{(\log\rho)''(h) - (n-2)((\log\rho)'(h))^2\}|\nabla h|^2.\end{aligned} \tag{3.3}$$

For any positive real number α, with a straightforward computation from (3.3) we get

$$\begin{aligned}\hat{\Delta}\rho^{-\alpha}(h) \;=\; & -\alpha\rho^{-\alpha-1}(h)\Big\{ -n\rho(h)(\rho'(h))^2 - H\rho'(h)\rho^2(h)\Theta \\ & +\rho^3(h)\left((\log\rho)''(h) - (n+\alpha-3)((\log\rho)'(h))^2\right)|\nabla h|^2\Big\} \\ =\; & -\alpha\rho^{-\alpha-1}(h)\Big\{ -n\rho(h)(\rho'(h))^2\Theta^2 - H\rho'(h)\rho^2(h)\Theta \\ & +\rho^3(h)\left((\log\rho)''(h) - (\alpha-3)((\log\rho)'(h))^2\right)|\nabla h|^2\Big\}.\end{aligned} \tag{3.4}$$

Hence, from (1.12), (3.1) and (3.15) we obtain

$$\hat{\Delta}\rho^{-\alpha}(h) = \alpha\rho^{-\alpha}(h)\Theta^2\bar{\zeta}_c(h) - \alpha\rho^{2-\alpha}(h)\left\{(\log\rho)''(h) - (\alpha-3)((\log\rho)'(h))^2\right\}|\nabla h|^2. \tag{3.5}$$

At this point, let us assume that $\bar{\zeta}_c(t) > 0$ for all $t_1 \le t \le t_2$. Since we are assuming that $|A|$ and Θ are bounded, we can apply Lemmas 1.5.1 and 2.1.1 to guarantee the existence of a sequence of points $\{p_k\}_{k\ge 1}$ in Σ^n such that

$$\lim_k \rho^{-\alpha}(h)(p_k) = \sup_\Sigma \rho^{-\alpha}(h), \quad \lim_k |\hat{\nabla}\rho^{-\alpha}(h)(p_k)|_{\hat{g}} = 0 \;\text{ and }\; \limsup_k \hat{\Delta}\rho^{-\alpha}(h)(p_k) \le 0, \tag{3.6}$$

where $|\cdot|_{\hat{g}}$ and $\hat{\nabla}$ denote, respectively, the norm and gradient with respect to the metric $\hat{g}$.

But, it is not difficult to verify that

$$|\hat{\nabla}\rho^{-\alpha}(h)|_{\hat{g}} = \alpha\rho^{-\alpha}(h)|\rho'(h)||\nabla h|. \tag{3.7}$$

So, since $\Sigma^n \subset \mathcal{B}_{t_1,t_2}$, with $|\rho'(t)| > 0$ for all $t_1 \le t \le t_2$, from (3.6) and (3.7) we get that

$$\lim_k |\nabla h(p_k)| = 0. \tag{3.8}$$

Consequently, from (1.7) and (3.8) we have that

$$\lim_k \Theta^2(p_k) = 1. \tag{3.9}$$

Moreover, from (3.16), (3.8) and (3.9) we obtain

$$\begin{aligned} 0 \ge \limsup_k \hat{\Delta}\rho^{-\alpha}(h)(p_k) \ge \alpha \limsup_k &\left\{\rho^{-\alpha}(h)\Theta^2\bar{\zeta}_c(h)\right\}(p_k) \\ &-\alpha \limsup_k \left\{\rho^{2-\alpha}(h)\Big|(\log\rho)''(h) - (\alpha-3)((\log\rho)'(h))^2\Big||\nabla h|^2\right\}(p_k) \\ &= \alpha \sup_\Sigma \rho^{-\alpha}(h) \limsup_k \bar{\zeta}_c(p_k) \ge 0. \end{aligned} \tag{3.10}$$

Hence, since $\sup_\Sigma \rho^{-\alpha}(h) > 0$ and $\bar{\zeta}_c(t) > 0$ for all $t_1 \le t \le t_2$, (3.10) gives us a contradiction.

Finally, in the case $\bar{\zeta}_c(t) < 0$ for all $t_1 \le t \le t_2$, using Lemma 1.5.2 instead of Lemma 1.5.1 we get

$$\begin{aligned} 0 \le \liminf_k \hat{\Delta}\rho^{-\alpha}(h)(p_k) \le \alpha \liminf_k &\left\{\rho^{-\alpha}(h)\Theta^2\bar{\zeta}_c(h)\right\}(p_k) \\ &+\alpha \liminf_k \left\{\rho^{2-\alpha}(h)\left|(\log\rho)''(h) - (\alpha-3)((\log\rho)'(h))^2\right||\nabla h|^2\right\}(p_k) \\ &= \alpha \inf_\Sigma \rho^{-\alpha}(h) \liminf_k \bar{\zeta}_c(p_k) \le 0. \end{aligned} \tag{3.11}$$

Therefore, since $\inf_\Sigma \rho^{-\alpha}(h) > 0$ and $\bar{\zeta}_c(t) < 0$ for all $t_1 \le t \le t_2$, (3.11) also leads us to a contradiction. $\square$

In what follows, we will assume that the warping function ρ of the ambient GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ satisfies the following inequality

$$(\log\rho)'' \le \gamma((\log\rho)')^2, \tag{3.12}$$

for some nonnegative constant γ. As it was observed in [18], the inequality (3.12) is a mild hypothesis due to the fact that, for instance, when $\overline{M}^{n+1}$ obeys the SNCC (respect. NCC) and its Riemannian fiber M^n is flat (respect. Ricci-flat), we have that (3.12) is automatically satisfied.

In the scenario that was discussed at the end of section 1.5 on the maximum principle for complete noncompact Riemannian manifolds, we will obtain some results in the following.

Theorem 3.1.2. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime whose warping function ρ satisfies inequality (3.12). There does not exist complete noncompact spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = \rho(t)\partial_t$ with soliton constant $c \neq 0$, bounded mean curvature and polynomial volume growth, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\bar{\zeta}_c(t) > 0$ for all $t \in [t_1, t_2]$.*

Proof. As in the proof of Theorem 3.1.2, let us consider on Σ^n the conformal metric $\hat{g} = \dfrac{1}{\rho^2(h)} g$. Denoting by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (1.7) and (1.10) we have

$$\hat{\Delta} h = -\rho(h)\rho'(h)\{n + (n-1)|\nabla h|^2\} - H\rho^2(h)\Theta. \tag{3.13}$$

Thus, from (3.13) we get

$$\begin{aligned} \hat{\Delta}\rho(h) &= -n\rho(h)(\rho'(h))^2 - H\rho'(h)\rho^2(h)\Theta \\ &\quad + \rho^3(h)\{(\log\rho)''(h) - (n-2)((\log\rho)'(h))^2\}|\nabla h|^2. \end{aligned} \tag{3.14}$$

For any positive real number α, with a straightforward computation from (3.14) we get

$$\begin{aligned} \hat{\Delta}\rho^{-\alpha}(h) &= -\alpha\rho^{-\alpha-1}(h)\Big\{ - n\rho(h)(\rho'(h))^2 - H\rho'(h)\rho^2(h)\Theta \\ &\quad + \rho^3(h)\left((\log\rho)''(h) - (n+\alpha-3)((\log\rho)'(h))^2\right)|\nabla h|^2 \Big\} \\ &= -\alpha\rho^{-\alpha-1}(h)\Big\{ - n\rho(h)(\rho'(h))^2\Theta^2 - H\rho'(h)\rho^2(h)\Theta \\ &\quad + \rho^3(h)\left((\log\rho)''(h) - (\alpha-3)((\log\rho)'(h))^2\right)|\nabla h|^2 \Big\}. \end{aligned} \tag{3.15}$$

Hence, from (1.12), (3.1) and (3.15) we reach at

$$\hat{\Delta}\rho^{-\alpha}(h) = \alpha\rho^{-\alpha}(h)\Theta^2\bar{\zeta}_c(h) - \alpha\rho^{2-\alpha}(h)\left\{(\log\rho)''(h) - (\alpha-3)((\log\rho)'(h))^2\right\}|\nabla h|^2. \tag{3.16}$$

So, observing that $\Theta^2 \geq 1$ and choosing $\alpha = 3 + \gamma$, we can use (3.12) and the assumption that $\bar{\zeta}_c(h) > 0$ on Σ^n to obtain from (3.16) the following estimate

$$\hat{\Delta}\rho(h)^{-\alpha} \;\geq\; \alpha\bar{\zeta}_c(h)\rho(h)^{-\alpha}. \tag{3.17}$$

Consequently, since we are assuming that $\Sigma^n \subset [t_1, t_2] \times M^n$, from (3.17) we get

$$\hat{\Delta}\rho(h)^{-\alpha} \;\geq\; a\rho(h)^{-\alpha}, \tag{3.18}$$

where $a = \alpha \inf_\Sigma \bar{\zeta}_c(h) > 0$.

Moreover, we have that

$$|\hat{\nabla}\rho(h)^{-\alpha}|_{\hat{g}} = \alpha\rho(h)^{-\alpha}|\rho'(h)||\nabla h| \leq \alpha\rho(h)^{-\alpha}|\rho'(h)||\Theta| = |c^{-1}|\alpha\rho(h)^{-\alpha-1}|\rho'(h)||H|. \tag{3.19}$$

So, since $\Sigma^n \subset [t_1, t_2] \times M^n$ and H is bounded on Σ^n, from (3.19) we conclude that $|\hat{\nabla}\rho(h)^{-\alpha}|_{\hat{g}}$

is also bounded on Σ^n.

On the other hand, considering the coefficients of conformal metric $\hat{g}_{ij} = \frac{1}{\rho(h)^2} g_{ij}$, where g_{ij} stands for the coefficients of the induced metric g, we have that

$$\hat{G} = \sqrt{\det(\hat{g}_{ij})} = \sqrt{\rho(h)^{-2n}\det(g_{ij})} = \rho(h)^{-n} G. \tag{3.20}$$

In particular, using once more that $\Sigma^n \subset [t_1, t_2] \times M^n$, from (3.20) jointly with the hypothesis that Σ^n has polynomial volume growth with respect to g, we guarantee that the same holds with respect to the conformal metric $\hat{g}$.

Therefore, we are in position to apply Lemma 1.5.7 to infer that $\rho(h)^{-\alpha}$ vanishes identically on Σ^n, which contradicts the fact that ρ is a positive function. □

3.1.2 Uniqueness and nonexistence results under integrability properties

Returning to the study of spacelike mean curvature flow solitons, we get the following result.

Theorem 3.1.3. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (3.12), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = \rho(t)\partial_t$ and with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ such that $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$. If the height function h is such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. We will consider again the conformal metric $\hat{g} := \dfrac{1}{\rho^2(h)} g$ and we will take $\alpha = \gamma + 3$. Since we are assuming that Σ^n lies in $\mathcal{B}_{t_1,t_2}$ and that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, from (3.7) we get that $|\hat{\nabla} \rho^{-\alpha}(h)|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$.

Moreover, since $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$, from (3.16) and (3.12) we obtain that $\hat{\Delta}\rho^{-\alpha}(h) \geq 0$. Consequently, we can apply Lemma 1.5.3 to infer that $\hat{\Delta}\rho^{-\alpha}(h) = 0$ on Σ^n.

Therefore, since we are assuming that the equality occurs in (3.12) just only at isolated points of I, returning to (3.16) we conclude that $|\nabla h|$ must vanishes identically on Σ^n. Therefore, Σ^n must be a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$. □

From Theorem 3.1.3 we also get the following nonexistence result.

Corollary 3.1.4. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (3.12), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = \rho(t)\partial_t$ and with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $\bar{\zeta}_c(t) > 0$ for all $t_1 \leq t \leq t_2$, and such that its height function h satisfies $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$.*

We are also able to present a slight different version of Theorem 3.1.3.

Theorem 3.1.5. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying* (3.12) *whose Riemannian fiber M^n is complete. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = \rho(t)\partial_t$ with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $\bar{\zeta}_c(t) \geq 0$ and $\rho'(t)$ vanishing only in isolated points of $[t_1, t_2]$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. As in the proof of Theorem 3.1.3, we get that $\hat{\Delta}\rho^{-\alpha}(h) = 0$ on Σ^n, for $\alpha = \gamma + 3$. Moreover, since Σ^n lies in $\mathcal{B}_{t_1,t_2}$, we can also verify that $|\widehat{\nabla}\rho^{-2\alpha}(h)|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. But, we note that

$$\hat{\Delta}\rho^{-2\alpha}(h) = 2\rho^{-\alpha}(h)\hat{\Delta}\rho^{-\alpha}(h) + 2|\widehat{\nabla}\rho^{-\alpha}(h)|^2_{\hat{g}} = 2|\widehat{\nabla}\rho^{-\alpha}(h)|^2_{\hat{g}} \geq 0. \tag{3.21}$$

Thus, we can apply again Lemma 1.5.3 to obtain that $\hat{\Delta}\rho^{-2\alpha}(h) = 0$ on Σ^n. Hence, since we are assuming that $\rho'(t) > 0$ for $t_1 \leq t \leq t_2$, from (3.7) and (3.21) we obtain that $|\nabla h| = 0$ on Σ^n. Therefore, Σ^n must be a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$. ☐

These previous lemmas enable us to prove the following nonexistence result.

Theorem 3.1.6. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying* (3.12)*, occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = \rho(t)\partial_t$ and soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$, and whose height function h is such that $(\rho(h))^{-1} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > \gamma + 3$.*

Proof. Supposing by contradiction the existence of such a spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ and taking once more $\alpha = \gamma + 3$, from (3.16) we obtain that $\hat{\Delta}\rho^{-\alpha}(h) \geq 0$ on Σ^n. Moreover, since Σ^n is contained in a timelike bounded region and $(\rho(h))^{-1} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > \alpha$, it is not difficult to verify that $\rho^{-\alpha}(h) \in \mathcal{L}^p_{\hat{g}}(\Sigma)$ for $p = \frac{q}{\alpha} > 1$. Thus, we can apply Lemma 1.5.4 to get that $\rho(h)$ is constant on Σ^n. Hence, since we are also supposing that the equality occurs in (3.12) just only at isolated points of I, returning to (3.16) we conclude that $|\nabla h|$ must vanish identically on Σ^n. Consequently, Σ^n is isometric (up to scaling) to M^n. So, since $f(h)$ is a positive constant, our assumption that $\rho(h) \in \mathcal{L}^q_g(\Sigma)$ also implies that M^n has finite volume. But, since M^n is a complete non-compact with nonnegative Ricci curvature, Lemma 1.5.5 leads us to a contradiction. ☐

In the results that follow in this section, we use a different analytical technique to that used at the beginning of this section: instead of making a conformal change in the metric, we decide to make a perturbation in the Laplacian.

In what follows, we will also consider the function

$$u = g(h) \in C^\infty(\Sigma^n), \tag{3.22}$$

where $g : I \to \mathbb{R}$ is an arbitrary primitive of ρ. Since $g' = \rho > 0$, then $u = g(h)$ can be thought as a reparametrization of the height function. In particular, from (1.6) we have that the gradient of u on Σ^n is given by

$$\nabla u \; = \; \rho(h)\nabla h \; = - \, \rho(h)\partial_t^\top \; = - \, \mathcal{K}^\top, \tag{3.23}$$

where $\mathcal{K}^\top$ denotes the tangential component of the closed conformal vector field $\mathcal{K} = \rho(t)\partial_t$. Taking into account this previous digression, we obtain the following auxiliary result.

Lemma 3.1.7. *Let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a spacelike mean curvature flow soliton with respect to $\mathcal{K} = \rho(t)\partial_t$ and with soliton constant $c \neq 0$. Then,*

$$H\langle AX, Y\rangle - c\nabla^2 u(X, Y) = c\rho'(h)\langle X, Y\rangle, \tag{3.24}$$

for all $X, Y \in \mathfrak{X}(\Sigma)$. Furthermore,

$$\nabla H = cA(\nabla u).$$

Proof. Firstly, we note that:

$$\begin{aligned} \nabla^2 u(X, X) &= \langle \nabla_X \nabla u, X\rangle \\ &= \langle \nabla_X(\rho(h)\nabla h), X\rangle \\ &= \rho(h)\langle \nabla_X \nabla h, X\rangle + \langle \nabla_X \rho(h)\nabla h, X\rangle \\ &= \rho(h)\nabla^2 h(X, X) + \rho'(h)\langle X, \nabla h\rangle^2. \end{aligned} \tag{3.25}$$

Thus, from (1.9) we get that

$$\begin{aligned} \nabla^2 u(X, X) &= \rho(h)\left(-\frac{\rho'(h)}{\rho(h)}\{|X|^2 + \langle X, \nabla h\rangle^2\} + \langle AX, X\rangle\Theta\right) + \rho'(h)\langle X, \nabla h\rangle^2 \\ &= -\rho'(h)|X|^2 - \rho'(h)\langle X, \nabla h\rangle^2 + \rho(h)\langle AX, X\rangle\Theta + \rho'(h)\langle X, \nabla h\rangle^2 \\ &= -\rho'(h)|X|^2 + \rho(h)\langle AX, X\rangle\Theta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{1}{c}\langle \nabla H, X\rangle &= \langle \overline{\nabla}_X \mathcal{K}, N\rangle + \langle \mathcal{K}, \overline{\nabla}_X N\rangle \\ &= -\langle A(X), \mathcal{K}\rangle = \langle X, A(\nabla u)\rangle, \end{aligned} \tag{3.26}$$

for every vector field $X \in \mathfrak{X}(\Sigma^n)$, so that from (3.23) we conclude the desired result. □

Remark 3.1.8. We point out that (3.24) is close to the that defining Ricci solitons and, therefore, it is interesting to make a study of mean curvature flow soliton under this point of view.

Naturally attached to $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ we can consider the support function

$$\begin{aligned} \varphi_{\mathcal{K}} : \Sigma^n &\to \mathbb{R} \\ q &\mapsto \varphi_{\mathcal{K}}(q) = \langle \mathcal{K}(q), N(q)\rangle. \end{aligned} \tag{3.27}$$

Hence, from (1.4) we have that

$$\varphi_{\mathcal{K}} \;=\; \rho(h)\langle N, \partial_t\rangle \;=\; \rho(h)\Theta \;\leq\; -\rho(h) \;<\; 0. \tag{3.28}$$

Furthermore, from [62, Proposition 2.1] and (1.7) we have

$$\Delta(\varphi_{\mathcal{K}}) \;=\; \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} - \{nN(\rho') - H\rho'\} + \langle \mathcal{K}, \nabla H\rangle, \tag{3.29}$$

where ∇H is the gradient of H in the metric of Σ^n, $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\overline{M}^{n+1}$ and $|A|$ is the Hilbert-Schmidt norm of A.

Besides, we get that

$$N(\rho') = -\rho''\Theta = -\frac{\rho''}{\rho}\varphi_{\mathcal{K}}. \tag{3.30}$$

On the other hand, since $N = N^* - \Theta\partial_t$, where $N^* = \pi_M(N)$ is the orthogonal projection of N onto M^n, it follows from [123, Corollary 7.43] that

$$\begin{aligned}
\overline{\mathrm{Ric}}(N,N) \;&=\; \overline{\mathrm{Ric}}(N^*,N^*) + \Theta^2\overline{\mathrm{Ric}}(\partial_t,\partial_t)\\
&=\; \mathrm{Ric}_M(N^*,N^*) + \langle N^*,N^*\rangle\left\{\frac{\rho''}{\rho} + (n-1)\frac{(\rho')^2}{\rho^2}\right\} - \frac{n\rho''}{\rho}\Theta^2\\
&=\; \mathrm{Ric}_M(N^*,N^*) - \left\{\frac{\rho''}{\rho} + (n-1)\frac{(\rho')^2}{\rho^2}\right\} - (n-1)\left(\frac{\rho'}{\rho}\right)'\Theta^2,
\end{aligned} \tag{3.31}$$

where Ric_M denotes the Ricci tensor of M^n. We note that it was used the relation $\langle N^*,N^*\rangle = \Theta^2 - 1$ in the last equality above.

Thus, inserting (3.30) and (3.31) into (3.29), we obtain

$$\begin{aligned}
\Delta(\varphi_{\mathcal{K}}) \;&=\; \left\{\mathrm{Ric}_M(N^*,N^*) + |A|^2 - \left\{\frac{\rho''}{\rho} + (n-1)\frac{(\rho')^2}{\rho^2}\right\} - (n-1)\left(\frac{\rho'}{\rho}\right)'\Theta^2\right\}\varphi_{\mathcal{K}}\\
&\quad + \left\{n\frac{\rho''}{\rho}\varphi_{\mathcal{K}} + H\rho'\right\} + \langle \mathcal{K}, \nabla H\rangle\\
&=\; \left\{\mathrm{Ric}_M(N^*,N^*) + |A|^2 + \frac{\rho''\rho - (\rho')^2}{f^2} - (n-1)\left(\frac{\rho'}{\rho}\right)'\Theta^2\right\}\varphi_{\mathcal{K}} + H\rho' + \langle \mathcal{K}, \nabla H\rangle\\
&=\; \{\mathrm{Ric}_M(N^*,N^*) + (n-1)(\ln\rho)''(1-\Theta^2) + |A|^2\}\varphi_{\mathcal{K}} + H\rho' + \langle \mathcal{K}, \nabla H\rangle\\
&=\; \{\mathrm{Ric}_M(N^*,N^*) - (n-1)(\ln\rho)''|\nabla h|^2 + |A|^2\}\varphi_{\mathcal{K}} + H\rho' + \langle \mathcal{K}, \nabla H\rangle.
\end{aligned} \tag{3.32}$$

From equations (1.12) and (3.27), we have that $H = c\,\varphi_{\mathcal{K}}$, and from (3.23) we get $\nabla u = -\mathcal{K}^\top$, where u is the reparametrization of the height function h given in (3.22). Consequently, we can rewrite (3.32) in the following way

$$\Delta(\varphi_{\mathcal{K}}) = \{c\rho'(h) + \mathrm{Ric}_M(N^*,N^*) - (n-1)(\ln\rho)''(h)|\nabla h|^2 + |A|^2\}\varphi_{\mathcal{K}} + \langle \nabla(cu), \nabla(\varphi_{\mathcal{K}})\rangle. \tag{3.33}$$

We recall that the *drift Laplacian* on Σ^n is defined by

$$\Delta_{cu}(\varphi) = \Delta(\varphi) - \langle \nabla(cu), \nabla\varphi \rangle \tag{3.34}$$

for all $\varphi \in C^\infty(\Sigma^n)$. So, from (3.33) and (3.34), we conclude that the drift Laplacian Δ_{cu} acting on $\varphi_{\mathcal{K}}$ is given by

$$\Delta_{cu}(\varphi_{\mathcal{K}}) = \{\tilde{\zeta}_c + \mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln\rho)''(h)|\nabla h|^2\}\varphi_{\mathcal{K}}, \tag{3.35}$$

where $\tilde{\zeta}_c \in C^\infty(\Sigma^n)$ is the function defined by

$$\tilde{\zeta}_c(q) = c\rho'(h(q)) + |A(q)|^2, \tag{3.36}$$

for every $q \in \Sigma^n$, which will be called the *second soliton function* associated to the spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$. Such nomenclature for $\tilde{\zeta}_c$ is motivated by [27, Equation (6.11)].

In what follows, we will assume that the GRW spacetime $-I \times_\rho M^n$ satisfies the *null convergence condition* (NCC)

$$\mathrm{Ric}_M \geq (n-1)(\rho\rho'' - \rho'^2)\langle\, ,\rangle_M, \tag{3.37}$$

which was originally established by Montiel [115], where Ric_M denotes the Ricci tensor of the Riemannian fiber M^n. It is not difficult to verify that all the GRW spacetimes described in Subsection 1.2 satisfy the NCC. For this, in the case of a steady state type spacetime (see Example 1.2.4), it is necessary to assume that its Riemannian fiber has nonnegative Ricci curvature.

Before we prove the first result, we will start by quoting an extension of Hopf's theorem on a complete Riemannian manifold Σ^n due to Yau in [148]. For this, we will adopt the following notation

$$\mathcal{L}^1(\Sigma^n) = \left\{\varphi \in C^\infty(\Sigma^n) \ : \ \int_{\Sigma^n} |\varphi|\, d\Sigma < +\infty\right\}$$

be the space of Lebesgue integrable functions on Σ^n, where $d\Sigma$ stands for the volume element induced by the metric of Σ^n and denote by $\mathcal{L}^1_{cu}(\Sigma^n)$ the set of Lebesgue integrable functions on Σ^n with respect to the modified volume element

$$d\mu = e^{cu} d\Sigma. \tag{3.38}$$

We also recall that a smooth function φ on Σ^n is said to be (cu)-*subharmonic* (respectively, (cu)-*superharmonic*) if $\Delta_{cu}(\varphi) \geq 0$ (respectively, $\Delta_{cu}(\varphi) \leq 0$) on Σ^n. So, it is not difficult to verify that from [63, Proposition 2.1] we obtain the following auxiliary lemma.

Lemma 3.1.9. *Let Σ^n be an n-dimensional complete oriented Riemannian manifold. If $\varphi \in C^\infty(\Sigma^n)$ is a (cu)-subharmonic function (or a (cu)-superharmonic function) on Σ^n and $|\nabla\varphi| \in \mathcal{L}^1_{cu}(\Sigma^n)$, then $\Delta_{cu}(\varphi) = 0$ on Σ^n.*

Given a GRW spacetime $\overline{M}^{n+1} = -I \times_\rho M^n$ obeying the NCC (3.37), we will require a suitable behavior of the second soliton function associated to a spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ and on the norm of the gradient of the its mean curvature function, in order to establish our first uniqueness result. For this, we will consider a *timelike bounded region* of $\overline{M}^{n+1}$ defined by

$$\mathcal{B}_{t_1,t_2} := \{(t,p) \in -I \times_\rho M^n : t_1 \leq t \leq t_2 \text{ and } p \in M^n\}.$$

Taking into account that all spacelike mean curvature solitons which appear in this paper are considered with respect to the closed conformal vector field $\mathcal{K} = \rho(t)\partial_t$, we are in position to present our first main result.

Theorem 3.1.10. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime which obeys the NCC (3.37), with equality holding only in isolated points of I. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant $c \neq 0$ and lying in a timelike bounded region $\mathcal{B}_{t_1,t_2}$. If its second soliton function $\tilde{\zeta}_c = |A|^2 + c\rho'(h)$ is nonnegative and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. From (3.37), we obtain that

$$\begin{aligned}
&\mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln \rho)''(h)|\nabla h|^2 \geq \qquad (3.39)\\
&\geq (n-1)(\rho(h)\rho''(h) - \rho'(h)^2)|N^*|_M^2 - (n-1)(\ln \rho)''(h)|\nabla h|^2\\
&= (n-1)(\rho(h)\rho''(h) - \rho'(h)^2)|N + \Theta\partial_t|_M^2 - (n-1)\left(\frac{\rho'}{\rho}\right)'(h)|\nabla h|^2\\
&= (n-1)\left\{(\rho(h)\rho''(h) - \rho'(h)^2)\frac{|\nabla h|^2}{\rho(h)^2} - \left(\frac{\rho(h)\rho''(h) - \rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2\right\} = 0.
\end{aligned}$$

Thus, since $\tilde{\zeta}_c \geq 0$ on Σ^n, from (3.35) and (3.39) we get that the support function φ_K defined in (3.27) satisfies

$$\Delta_{cu}(\varphi_\mathcal{K}) \leq \rho(h)\tilde{\zeta}_c\Theta \leq 0. \qquad (3.40)$$

On the other hand, since $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is contained in a timelike bounded region $\mathcal{B}_{t_1,t_2}$ of $-I \times_\rho M^n$, h is bounded on Σ^n and, consequently, the same happens with $u = g(h)$ and e^{cu}. So, since $c \neq 0$, from (3.38), (1.12), (3.27) and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$ we get $|\nabla(\varphi_\mathcal{K})| \in \mathcal{L}^1_{cu}(\Sigma^n)$. Next, from Lemma 3.1.9 we obtain $\Delta_{cu}(\varphi_\mathcal{K}) = 0$ on Σ^n. Since $\rho(h) > 0$ and $\Theta < 0$ on Σ^n, from (3.39) and (3.40) we must have on Σ^n that

$$\tilde{\zeta}_c = 0 \quad \text{and} \quad \mathrm{Ric}_M(N^*, N^*) - (n-1)(\ln \rho)''(h)|\nabla h|^2 = 0.$$

But, taking into account that the equality in (3.37) occurs only in isolated points of I, we can conclude that $|\nabla h| = 0$ on Σ^n and, consequently, h is constant on Σ^n. Therefore, $\psi(\Sigma^n)$ is a slice. □

Remark 3.1.11. From (1.8) we have that the slice M_t^n is a spacelike hypersurface whose shape operator (with respect to the orientation ∂_t) A_t is given by

$$\begin{aligned} \mathcal{A}_{t_*} \ : \ \mathfrak{X}(M^n_{t_*}) \ &\to \ \mathfrak{X}(M^n_{t_*}) \\ V \qquad &\mapsto \ \mathcal{A}_{t_*}(V) \ = \ -\overline{\nabla}_V(\partial_{t_*}) \ = \ -\frac{\rho'(t_*)}{\rho(t_*)}\, V. \end{aligned} \tag{3.41}$$

Thus from (3.41) we obtain that the principal curvatures $\kappa_i^{t_*}$ of the shape operator $\mathcal{A}_{t_*}$ of a slice $M^n_{t_*} = \{t_*\} \times M^n$, $t_* \in I$, are given by $\kappa_i^{t_*} = -\frac{\rho'(t_*)}{\rho(t_*)}$ for all $i \in \{1, \ldots, n\}$. So, from (1.14) and (3.36) we have

$$\tilde{\zeta}_c \ = \ c\,\rho'(t_*) + |\mathcal{A}_{t_*}|^2 \ = \ \sum_{i=1}^{n} \left(\kappa_i^{t_*}\right)^2 + \left(-\frac{n\,\rho'(t_*)}{\rho^2(t_*)}\right) \rho'(t_*) \ = \ 0$$

on $M^n_{t_*}$. Hence, our restriction on the values of the second soliton function $\tilde{\zeta}_c$ in Theorem 3.1.10 constitutes a mild hypothesis in the sense that it is natural to detect slices of $-I \times_\rho M^n$.

From Theorem 3.1.10 we also get the following nonexistence results.

Corollary 3.1.12. *There is no complete spacelike translating soliton lying in a timelike bounded region of a Lorentzian product space $-I \times M^n$, whose Riemannian fiber M^n has positive Ricci curvature, having soliton constant $c \neq 0$ and such that $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$.*

According to the classical terminology in linear potential theory, a Riemannian manifold Σ^n is called (cu)-*parabolic* if the constant functions are the only functions $\varphi \in C^2(\Sigma)$ which are bounded from below and satisfying $\Delta_{cu}(\varphi) \leq 0$. Inspired by the ideas of Romero et al. [130, 131], Albujer et al. established in [7, Theorem 1] the following parabolicity criterion which provides conditions for a complete spacelike hypersurface immersed in GRW spacetime $-I \times_\rho M^n$ to be (cu)-parabolic. For this, we will consider the function $\tilde{u} := g(\pi_I) \circ \tilde{\pi}$, where $\tilde{\pi} : \widetilde{M}^n \to M^n$ is the universal covering map of the Riemannian fiber M^n.

Lemma 3.1.13. *Let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a complete spacelike hypersurface immersed in a GRW spacetime $-I \times_\rho M^n$, whose Riemannian fiber M^n has $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. If the hyperbolic angle Θ is bounded from below and the warping function f and the height function h are such that $\sup_{\Sigma^n} \rho(h) < +\infty$ and $\inf_{\Sigma^n} \rho(h) > 0$, then Σ^n is (cu)-parabolic.*

We can state the following rigidity result for spacelike mean curvature flow solitons in GRW spacetimes.

Theorem 3.1.14. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the NCC (3.37), with equality holding only in isolated points of I, and such that the Riemannian fiber M^n has $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2}$. If the hyperbolic angle Θ is bounded from below and the second soliton function $\tilde{\zeta}_c = |A|^2 + c\rho'(h)$ is nonnegative, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. From (3.40) we get $\Delta_{cu}(\varphi_K) \leq 0$ on Σ^n. Thus, since we are assuming that $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is contained in a timelike bounded region, Lemma 3.1.13 guarantees that Σ^n is (cu)-parabolic and consequently φ_K is constant on Σ^n. At this point, we can reason as in the last part of the proof of Theorem 3.1.10 to conclude that there is $t \in I$ such that Σ^n is a slice of $\overline{M}^{n+1}$. $\square$

From Theorem 3.1.14 we also get the following nonexistence results.

Corollary 3.1.15. *Let $\overline{M}^{n+1} = -I \times M^n$ be a Lorentzian product space, whose Riemannian fiber M^n has positive Ricci curvature and $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. There is no complete spacelike translating soliton in $\overline{M}^{n+1}$, having soliton constant c and such that Θ is bounded from below.*

Considering the *strong null convergence condition* (SNCC)

$$K_M \geq \sup_I(\rho\rho'' - \rho'^2), \tag{3.42}$$

which was introduced by Alías and Colares [12], where K_M denotes the sectional curvature of the Riemannian fiber M^n and adding a suitable control to the growing of the height function through the second soliton function of a spacelike mean curvature flow soliton, we get the following version of the Omori-Yau's maximum principle:

Proposition 3.1.16. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the SNCC (3.42), and let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a complete spacelike mean curvature flow soliton having soliton constant $c \neq 0$. If the function $\frac{(n-1)\rho''(h)+c\rho(h)\rho'(h)}{\rho(h)}$ is bounded from below on Σ^n, then the Omori-Yau's maximum principle holds for the drift Laplacian Δ_{cu}, that is, for $\varphi \in C^2(\Sigma^n)$ with $\sup_\Sigma \varphi < +\infty$, there exists a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n, such that*

$$\lim_k \varphi(p_k) = \sup_\Sigma \varphi, \qquad \lim_k |\nabla\varphi(p_k)| = 0 \qquad \textit{and} \qquad \lim_k \Delta_{cu}\varphi(p_k) \leq 0.$$

Proof. We recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator A and the curvature tensor $\overline{R}$ of the ambient $-I \times_\rho M^n$ by the so-called Gauss equation, which is given by

$$\langle R(X,Y)Z,W\rangle = \langle \overline{R}(X,Y)Z,W\rangle - \langle AX,Z\rangle\langle AY,W\rangle + \langle AX,W\rangle\langle AY,Z\rangle, \tag{3.43}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma^n)$. Here, as in [123], the curvature tensor R is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\ ,\ ]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma^n)$.

Let us consider $X \in \mathfrak{X}(\Sigma^n)$ and take a (local) orthonormal frame $\{E_1, \cdots, E_n\}$. It follows from Gauss equation (3.43) that the Ricci curvature Ric of Σ^n satisfies

$$\mathrm{Ric}(X,X)=\sum_i\langle\overline{R}(X,E_i)X,E_i\rangle+|AX|^2+H\langle AX,X\rangle. \tag{3.44}$$

Thus, from (3.24) and (3.44) we get

$$\mathrm{Ric}(X,X)-c\nabla^2u(X,X)\geq\sum_i\langle\overline{R}(X,E_i)X,E_i\rangle+c\rho'(h)|X|^2. \tag{3.45}$$

To estimate the first summand on the right-hand side of inequality (3.45), let us consider $X^*=(\pi_M)_*(X)$ and $E_i^*=(\pi_M)_*(E_i)$. So, from [123, Proposition 7.42] and (1.6) we have

$$\begin{aligned}\sum_i\langle\overline{R}(X,E_i)X,E_i\rangle &= \sum_i\langle R_M(X^*,E_i^*)X^*,E_i^*\rangle+(n-1)((\ln\rho)'(h))^2|X|^2\\ &\quad -(n-2)(\ln\rho)''(h)\langle X,\nabla h\rangle^2-(\ln\rho)''(h)|\nabla h|^2|X|^2,\end{aligned} \tag{3.46}$$

where R_M denotes the curvature tensor of the Riemannian fiber M^n. By writing $X^*=X+\langle X,\partial_t\rangle\partial_t$, we can estimate the first summand on the right-hand side of (3.46) to get

$$\begin{aligned}\sum_i\langle R_M(X^*,E_i^*)X^*,E_i^*\rangle &= \rho^2(h)(|X^*|^2_M|E^*|^2_M-\langle X^*,E^*\rangle^2_M\rangle K_M(X^*,E^*)\\ &\geq \frac{1}{\rho^2(h)}((n-1)|X|^2+|\nabla h^2||X|^2\\ &\quad +(n-2)\langle X,\nabla h\rangle^2)\min_i K_M(X^*,E_i^*).\end{aligned} \tag{3.47}$$

Consequently, since our ambient space obeys (3.42), from (3.47) we have that

$$\sum_i\langle R_M(X^*,E_i^*)X^*,E_i^*\rangle\geq((n-1)|X|^2+|\nabla h|^2|X|^2+(n-2)\langle X,\nabla h\rangle^2)(\ln\rho)''(h). \tag{3.48}$$

Substituting (3.48) into (3.46), we get

$$\begin{aligned}\sum_i\langle\overline{R}(X,E_i)X,E_i\rangle &\geq ((n-1)|X|^2+|\nabla h|^2|X|^2+(n-2)\langle X,\nabla h\rangle^2)(\ln\rho)''(h)\\ &\quad +(n-1)((\ln\rho)'(h))^2|X|^2-(n-2)(\ln\rho)''(h)\langle X,\nabla h\rangle^2\\ &\quad -(\ln\rho)''(h)|\nabla h|^2|X|^2\\ &= (n-1)\frac{\rho''(h)}{\rho(h)}|X|^2.\end{aligned} \tag{3.49}$$

Hence, from (3.45) and (3.49) we obtain

$$\mathrm{Ric}-c\nabla^2u\geq((n-1)\frac{\rho''(h)}{\rho(h)}+c\rho'(h))\langle\,,\rangle.$$

Therefore, since the right-hand side of the above inequality is bounded from bellow, we conclude our proof by applying [68, Theorem 1]. □

Proceeding, we use Proposition 3.1.16 to establish the following result.

Theorem 3.1.17. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the SNCC (3.42), and let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a complete spacelike mean curvature flow soliton with soliton constant $c \neq 0$, such that $\frac{(n-1)\rho''(h)+c\rho(h)\rho'(h)}{\rho(h)}$ is bounded from below. If $\inf_\Sigma \rho(h) > 0$, the second soliton function $\tilde{\zeta}_c = |A|^2 + c\rho'(h)$ is nonnegative and the height function h satisfies*

$$|\nabla h| \leq \inf_{\Sigma^n} \tilde{\zeta}_c \quad \text{on} \quad \Sigma^n, \tag{3.50}$$

then Σ^n is a slice of $\overline{M}^{n+1}$.

Proof. Since $\varphi_\mathcal{K} < 0$ on Σ^n, Proposition 3.1.16 assures the existence of a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma^n$ such that

$$\lim_{k\to+\infty} \varphi_\mathcal{K}(p_j) = \sup_{\Sigma^n} \varphi_\mathcal{K} \qquad \text{and} \qquad \lim_{k\to+\infty} \Delta_{cu}\, \varphi_\mathcal{K}(p_k) \leq 0.$$

Hence, from (3.40) we get

$$0 \geq \lim_{k\to+\infty} \Delta_{cu}(\varphi_\mathcal{K})(p_j) = \sup_{\Sigma^n} \varphi_\mathcal{K} \lim_{k\to+\infty} \tilde{\zeta}_c(p_k) \geq 0. \tag{3.51}$$

But, since we are assuming that $\inf_\Sigma \rho(h) > 0$, we have that $\sup_{\Sigma^n} \varphi_\mathcal{K} < 0$. Consequently, from (3.51) we must have $\lim_{j\to+\infty} \tilde{\zeta}_c(p_k) = 0$ and, hence, $\inf_{\Sigma^n} \tilde{\zeta}_c = 0$. Therefore, the result follows from hypothesis (3.50). □

Remark 3.1.18. *We note that in Theorem 3.1.17 the hypotheses that $\frac{(n-1)\rho''(h)+c\rho(h)\rho'(h)}{\rho(h)}$ is bounded from below and $\inf_\Sigma \rho(h) > 0$ are automatically satisfied if we assume that the spacelike mean curvature flow soliton lies in a timelike bounded region of the ambient spacetime.*

Our next result can be regarded as a sort of extension of [68, Theorem 3].

Theorem 3.1.19. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying the SNCC (3.42). There is no complete spacelike mean curvature flow soliton immersed in $\overline{M}^{n+1}$, with soliton constant $c \neq 0$ such that $c\rho'(h) \geq 0$ and $\frac{(n-1)\rho''(h)+c\rho(h)\rho'(h)}{\rho(h)}$ is bounded from below.*

Proof. Let us suppose by contradiction the existence of such a complete spacelike mean curvature flow soliton Σ^n immersed in $\overline{M}^{n+1}$. Since we are supposing that $c\rho'(h) \geq 0$ and that $\overline{M}^{n+1}$ satisfies the SNCC (3.42), we conclude from (3.35) and (3.39) that

$$\Delta_{cu}\varphi_\mathcal{K} \leq |A|^2\varphi_\mathcal{K}.$$

From above equation, we get

$$\Delta_{cu}\varphi_\mathcal{K}^2 \geq 2\varphi_\mathcal{K}\Delta_{cu}\varphi_\mathcal{K} \geq 2|A|^2\varphi_\mathcal{K}^2.$$

Since $\varphi_{\mathcal{K}} = \frac{H}{c}$ we have

$$\Delta_{cu} H^2 \geq 2H^2|A|^2 \geq 2\frac{H^4}{n}. \tag{3.52}$$

With a straightforward computation, we can verify that

$$\Delta_{cu}\left(\frac{-1}{\sqrt{1+H^2}}\right) = \frac{\Delta_{cu}H^2}{2(1+H^2)^{3/2}} - \frac{3}{4}\frac{|\nabla H^2|^2}{(1+H^2)^{5/2}}. \tag{3.53}$$

Hence, from (3.52) and (3.53) we obtain

$$\Delta_{cu}\left(\frac{-1}{\sqrt{1+H^2}}\right) \geq \frac{H^4}{n(1+H^2)^{3/2}} - \frac{3}{4}\frac{|\nabla H^2|^2}{(1+H^2)^{5/2}}.$$

Therefore, since $\frac{(n-1)\rho''(h)+c\rho(h)\rho'(h)}{\rho(h)}$ is bounded from below, from Proposition 3.1.16 we can apply the Omori-Yau's maximum principle and reason as in the proof of [68, Theorem 3] to conclude that $H \equiv 0$, which corresponds to an absurd. □

From Theorem 3.1.19 we get the following nonexistence results.

Corollary 3.1.20. *There is no complete spacelike translating soliton with soliton constant $c \neq 0$ immersed in $-I \times M^n$, whose Riemannian fiber M^n has nonnegative sectional curvature.*

Remark 3.1.21. Fixing a constant $c \in \mathbb{R}$ with $0 < |c| < 1$, from [73, Example 4.4] we have that

$$\Sigma^n = \{(c \ln x_n, x_1, \ldots, x_n) : x_n > 0\} \subset -\mathbb{R} \times \mathbb{H}^n$$

is a complete spacelike translating soliton of the mean curvature flow with respect to ∂_t, having soliton constant c and constant future mean curvature

$$H = \frac{c}{\sqrt{1-c^2}} = c\,\Theta.$$

Moreover, we also get that

$$|\nabla h| = \frac{|c|}{\sqrt{1-c^2}} = |A|.$$

Hence, since the static GRW spacetime $-\mathbb{R} \times \mathbb{H}^n$ obeys neither the NCC (3.37) nor the SNCC (3.42), we can verify that it works as a counterexample related to our previous theorems. Consequently, we conclude that their hypothesis are, indeed, necessary.

Now, we deal with compact (without boundary) mean curvature flow solitons.

Theorem 3.1.22. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact mean curvature flow soliton with soliton constant $c \neq 0$. If $c > 0$, then*

$$\min_{\Sigma} H^2 \leq -cn\rho'(h_*) \quad \text{and} \quad \max_{\Sigma} H^2 \geq -cn\rho'(h^*),$$

where h_* and h^* are the minimum and maximum of the height function on Σ^n. Similarly, if $c < 0$, then

$$\min_\Sigma H^2 \leq -cn\rho'(h^*) \quad \text{and} \quad \max_\Sigma H^2 \geq -cn\rho'(h_*).$$

Proof. From (3.24) we have that

$$c\Delta u = -nc\rho'(h) - H^2. \tag{3.54}$$

Let us consider $c > 0$ and let p_0 be a point of minimum of the height function h. Since a primitive g of ρ is a increasing function, we have that $h(p_0) = h_*$ is minimum point of the function $u = g(h)$ and, hence, $\Delta u(p_0) \geq 0$. Thus, from (3.54) we get that

$$\min_\Sigma H^2 \leq H^2(p_0) \leq -nc\rho'(h_*).$$

Analogously, taking a point of maximum of h, we are able to conclude that

$$\max_\Sigma H^2 \geq -cn\rho'(h^*).$$

The proof of the case $c < 0$ follows the same steps of the case $c > 0$. □

From above result we conclude directly the following nonexistence result.

Corollary 3.1.23. *There exist no compact spacelike translating soliton with soliton constant $c \neq 0$ immersed in $-I \times M^n$.*

We finish this subsection establishing a rigidity result derived from Theorem 3.1.22.

Corollary 3.1.24. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a compact mean curvature flow soliton with soliton constant $c \neq 0$. Assume that $\rho''(t) \leq 0$ for $h_* \leq t \leq h^*$, where h_* and h^* are the minimum and maximum on Σ^n of its height function h. If H is constant, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. Indeed, since $\rho''(t) \leq 0$, we have that ρ' is non-decreasing. Besides, since H is constant, from Theorem 3.1.22 we conclude that

$$-nc\rho'(t) = H^2,$$

for $h_* \leq t \leq h^*$. Thus, from above equation jointly with (3.54), we have that $\Delta u = 0$. Therefore, since Σ^n is compact, we conclude that u is constant, which means that Σ^n is a slice of $\overline{M}^{n+1}$. □

3.1.3 Uniqueness under a parabolicity criterion

According to [133], a GRW spacetime $\overline{M}^{n+1} = I \times_\rho M^n$ is said to be *spatially parabolic* when its Riemannian fiber M^n is parabolic, that is, (M^n, g_M) is a noncompact complete Riemannian manifold such that the only superharmonic functions on it that are bounded from below are the constants. Analogously, $\overline{M}^{n+1}$ is said to be *spatially parabolic covered* when its universal Lorentzian covering is spatially parabolic. For our next uniqueness result, we need of the following parabolicity criterion due to Aledo, Rubio and Salamanca (see [18, Theorem 2.2])

Lemma 3.1.25. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed in a spatially parabolic covered GRW spacetime $\overline{M}^{n+1} = I \times_\rho M^n$. If $\sup_\Sigma \rho(h) < +\infty$ and the hyperbolic angle function Θ is bounded, then $(\Sigma^n, \hat{g})$, endowed with the conformal metric $\hat{g} = \frac{1}{\rho^2(h)} g$, is parabolic.*

Using Lemma 3.1.25 we obtain the following result.

Theorem 3.1.26. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a spatially parabolic covered GRW spacetime satisfying (3.12), holding the equality only at isolated points of I. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = \rho(t)\partial_t$ with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$. If the hyperbolic angle function Θ is bounded, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. First, we note that Lemma 3.1.25 guarantees that $(\Sigma^n, \hat{g})$ is parabolic. Moreover, it follows from (3.16) that $\rho(h)^{-\alpha}$ (where $\alpha = \gamma + 3$) is subharmonic on Σ^n. Thus, since the hypothesis that $\Sigma^n \subset \mathcal{B}_{t_1,t_2}$ implies in particular that $\rho(h)^{-\alpha}$ is bounded from above, it follows from the parabolicity of $(\Sigma^n, \hat{g})$ that $\rho(h)$ is constant on Σ^n. Consequently, since we are assuming that the equality holds in (3.12) only at isolated points of I, returning to (3.16) we conclude that $|\nabla h| = 0$ on Σ^n, which means that Σ^n is a slice. □

3.1.4 Rigidity of mean curvature flow solitons

Using the concepts reported in equations 3.1 and 3.12 above, we will now obtain the results for the stiffness in the GRW spacetimes.

Theorem 3.1.27. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime with complete noncompact Riemannian fiber M^n and whose warping function ρ satisfies inequality (3.12). The only complete noncompact spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = \rho(t)\partial_t$ with soliton constant c such that $\bar{\zeta}_c(h) \geq 0$, $\rho(h)$ is increasing (decreasing) and, for some $t_* \in I$, h converges from below (above) to t_* at infinity, is the slice M_{t_*}.*

Proof. As in the proof of Theorem 3.1.3, let us suppose by contradiction that such a mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ is not the slice M_{t_*} and let us consider on Σ^n the conformal metric $\hat{g} = \dfrac{1}{\rho^2(h)} g$. Denoting by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (1.7), (1.10) and (2.13) we get

$$\begin{aligned} \hat{\Delta}\rho(h) &= -n\rho(h)(\rho'(h))^2 - H\rho'(h)\rho^2(h)\Theta \\ &\quad + \rho^3(h)\{(\log\rho)''(h) - (n-2)((\log\rho)'(h))^2\}|\nabla h|^2. \end{aligned} \tag{3.55}$$

For any positive real number α, with a straightforward computation from (3.55), (1.12), (3.1) and (3.15), observing that $\Theta^2 \geq 1$ and choosing $\alpha = 3+\gamma$, we can use (3.12) and the assumption that $\bar{\zeta}_c(h) \geq 0$ on Σ^n to obtain from (3.16) the following estimate

$$\hat{\Delta}\rho(h)^{-\alpha} \quad \geq \quad \alpha\bar{\zeta}_c(h)\rho(h)^{-\alpha} \geq 0. \tag{3.56}$$

Moreover, we have that

$$|\hat{\nabla}\rho(h)^{-\alpha}|_{\hat{g}} = \alpha\rho(h)^{-\alpha}|\rho'(h)||\nabla h|. \tag{3.57}$$

At this point, taking into account (3.56) and (3.57), we can apply Lemma 1.5.6 the same choices of the smooth function $u = \rho(h)^{-\alpha} - \rho(t_*)^{-\alpha}$ and the vector field $X = \hat{\nabla}u$ to get that $\hat{g}(\hat{\nabla}u, X)$ is identically zero on Σ^n. Thus, returning to (3.57) we conclude that ∇h vanishes identically on Σ^n, which means that h is constant and (since it converges to t_* at infinity) Σ^n must be the slice M_{t_*}. Therefore, we reach at a contradiction. □

3.1.5 Applications to Einstein-de Sitter spacetimes

Observing that the $(n+1)$-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ (see Example 5.2.1) satisfies (1.11), from Theorem 3.1.1 we obtain the following consequence.

Corollary 3.1.28. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c \geq 0$, whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region of $\overline{M}^{n+1}$.*

Applying Theorem 3.1.3 to the Einstein-de Sitter spacetime, we obtain the following result.

Corollary 3.1.29. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. The only complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $t_2 = (-\frac{2n}{3c})^{\frac{3}{5}}$, and such that its height function h satisfies $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, is the slice $\{(-\frac{2n}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$.*

When the ambient spacetime is the Einstein-de Sitter spacetime, Corollary 3.1.4 reads as follows.

Corollary 3.1.30. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c \geq 0$, lying in a timelike bounded region of $\overline{M}^{n+1}$ and such that its height function h satisfies $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$.*

From Theorem 3.1.6 we obtain the following consequences.

Corollary 3.1.31. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c \geq 0$, lying in a timelike bounded region of $\overline{M}^{n+1}$ and such that its height function h satisfies $h^{-\frac{2}{3}} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > 3$.*

We close this section quoting the following applications of Theorem 3.1.26.

Corollary 3.1.32. *Let $\overline{M}^3 = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^2$ be the 3-dimensional Einstein-de Sitter spacetime. The only complete spacelike mean curvature flow soliton $\psi : \Sigma^2 \to \overline{M}^3$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^3$ with $t_2 = (-\frac{4}{3c})^{\frac{3}{5}}$, and such that its hyperbolic angle function Θ is bounded, is the slice $\{(-\frac{4}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$.*

From Theorem 3.1.2 we obtain the following consequence.

Corollary 3.1.33. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. There does not exist a complete noncompact spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c \geq 0$, whose mean curvature is bounded bounded, having polynomial volume growth and lying in a slab of $\overline{M}^{n+1}$.*

From Theorem 3.1.10 we derive the following consequence.

Corollary 3.1.34. *Let $\psi : \Sigma^n \looparrowright -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be a complete spacelike mean curvature flow soliton with soliton constant $c < 0$, lying in a timelike bounded region of the Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$. If $|A|$ does not vanish and $h \geq -\frac{8c^3}{27|A|^6}$ and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$, then Σ^n is the slice $\{(-\frac{2n}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$.*

From Theorem 3.1.19 we get the following nonexistence result.

Corollary 3.1.35. *There is no complete spacelike mean curvature flow soliton with soliton constant $c > 0$, lying in a timelike bounded region of the Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$.*

3.1.6 Applications to steady state type spacetimes

Since a steady state type spacetime (see Example 1.2.4) whose Riemannian fiber has nonnegative sectional curvature satisfies (1.11), from Theorem 3.1.1 we obtain the following application.

Corollary 3.1.36. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n has nonnegative sectional curvature. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t\partial_t$ with soliton constant $c \geq 0$, whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region of $\overline{M}^{n+1}$.*

When the ambient space is a steady state type spacetime, Theorem 3.1.3 gives us the following consequence.

Corollary 3.1.37. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n is complete. The only complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t \partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $t_2 = \log(-\frac{n}{c})$, and such that its height function h satisfies $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, is the slice $\{\log(-\frac{n}{c})\} \times M^n$.*

When the ambient spacetime is the steady state type spacetime, Corollary **??** reads as follows.

Corollary 3.1.38. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n is complete. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t \partial_t$ with soliton constant $c \geq 0$, lying in a timelike bounded region of $\overline{M}^{n+1}$ and such that its height function h satisfies $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$.*

From Theorem 3.1.6 we obtain the following consequences.

Corollary 3.1.39. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t \partial_t$ with soliton constant $c \geq 0$, lying in a timelike bounded region of $\overline{M}^{n+1}$ and such that its height function h satisfies $e^{-h} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > 3$.*

We close this section quoting the following applications of Theorem 3.1.26.

Corollary 3.1.40. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a spatially parabolic covered steady state type spacetime. The only complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t \partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $t_2 = \log(-\frac{n}{c})$, and such that its hyperbolic angle function Θ is bounded, is the slice $\{\log(-\frac{n}{c})\} \times M^n$.*

Remark 3.1.41. *Related to Corollary 3.1.40 in the case $n = 2$, when the Riemannian fiber M^2 is a complete Riemannian surface having nonnegative Gaussian curvature, a classical result due to Ahlfors [2] and Blanc-Fiala-Huber [101] guarantees that M^2 has parabolic universal covering.*

In this context, Theorem 3.1.2 gives the following:

Corollary 3.1.42. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime. There does not exist a complete noncompact spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t \partial_t$ with soliton constant $c \geq 0$, whose mean curvature is bounded, having polynomial volume growth and lying in a slab of $\overline{M}^{n+1}$.*

When the ambient space is a steady state type spacetime, Theorem 3.1.10 gives us the following rigidity result.

Corollary 3.1.43. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n has positive Ricci curvature. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant $c < 0$ and lying in a timelike bounded region. If $|A|$ does not vanish and $h \geq \ln\left(-\frac{|A|^2}{c}\right)$ and $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$, then Σ^n is the slice $\{\ln(-\frac{n}{c})\} \times M^n$.*

Corollary 3.1.44. *There is no complete spacelike mean curvature flow soliton lying in a timelike bounded region of a steady state type spacetime $-I \times_{e^t} M^n$, whose Riemannian fiber M^n has positive Ricci curvature, having soliton constant $c > 0$ and such that $|\nabla H| \in \mathcal{L}^1(\Sigma^n)$.*

From Theorem 3.1.14 we obtain the following applications.

Corollary 3.1.45. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n has positive Ricci curvature and $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c < 0$. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with soliton constant c and lying in a timelike bounded region $\mathcal{B}_{t_1,t_2}$. If Θ is bounded from below and $h \geq \ln\left(-\frac{|A|^2}{c}\right)$, then Σ^n is the slice $\{\ln(-\frac{n}{c})\} \times M^n$.*

Corollary 3.1.46. *Let $\overline{M}^{n+1} = -I \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n has positive Ricci curvature and $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c > 0$. There is no complete spacelike mean curvature flow soliton lying in a timelike bounded region of $\overline{M}^{n+1}$, having soliton constant c and such that Θ is bounded from below.*

From Theorem 3.1.19 we get the following nonexistence result.

Corollary 3.1.47. *There is no complete mean curvature flow soliton with soliton constant $c > 0$, lying in a timelike bounded region of the steady state type spacetime $-\mathbb{R} \times_{e^t} M^n$, whose Riemannian fiber M^n has nonnegative sectional curvature.*

3.1.7 Applications to de Sitter spaces

From [117, Example 4.2], the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ is isometric to the RW spacetime $-\mathbb{R} \times_{\cosh t} \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the n-dimensional unit Euclidean sphere endowed with its standard metric. Taking into account the terminology introduced in [17], the open half-space $\mathbb{R}^+ \times \mathbb{S}^n \subset \mathbb{S}_1^{n+1}$ (respect. $\mathbb{R}^- \times \mathbb{S}^n \subset \mathbb{S}_1^{n+1}$) is called the *chronological future* (respect. *past*) of $\mathbb{S}_1^{n+1}$ with respect to the totally geodesic *equator* $\{0\} \times \mathbb{S}^n$. From (1.14) we see that the equator is a spacelike mean curvature flow soliton with respect to $K = \cosh t\, \partial_t$ and constant soliton $c = 0$ and the slices $\{\sinh^{-1}(\frac{-n \pm \sqrt{n^2-4c^2}}{2c})\} \times \mathbb{S}^n$ are spacelike mean curvature flow soliton with respect to $K = \cosh t\, \partial_t$ and with soliton constant $0 < |c| \leq \frac{n}{2}$.

Considering the context of Example 1.2.5, from Theorem 3.1.1 we also get.

Corollary 3.1.48. *There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \mathbb{S}_1^{n+1}$ with respect to $\mathcal{K} = \cosh t\, \partial_t$ having soliton constant $c \geq 0$ (respect. $c \leq 0$), whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region contained in the chronological future (respect. past) of $\mathbb{S}_1^{n+1}$ with respect to the equator $\{0\} \times \mathbb{S}^n$.*

From Example 1.2.6 and Theorem 3.1.1 we obtain.

Corollary 3.1.49. *There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to -\mathbb{R}^+ \times_{\sinh t} \mathbb{H}^n \subset \mathbb{S}_1^{n+1}$ with respect to $\mathcal{K} = \sinh t\, \partial_t$ having soliton constant $c \geq 0$, whose* *second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region of* $-\mathbb{R}^+ \times_{\sinh t} \mathbb{H}^n \subset \mathbb{S}^{n+1}_1$.

Finally, in the setting of Example 1.2.7, Theorem 3.1.1 reads as follows.

Corollary 3.1.50. *There does not exist a complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \to -(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n \subset \mathbb{H}^{n+1}_1$ *with respect to* $\mathcal{K} = \cos t\, \partial_t$ *having soliton constant* $c \leq 0$ *(respect.* $c \geq 0$*), whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region contained in the chronological future (respect. past) of* $\mathbb{H}^{n+1}_1$ *with respect to the equator* $\{0\} \times \mathbb{H}^n$.

We close this section with the following consequence of Theorem 3.1.2:

Corollary 3.1.51. *There does not exist a complete noncompact spacelike mean curvature flow soliton* $\psi : \Sigma^n \to \mathbb{S}^{n+1}_1$ *with respect to* $\mathcal{K} = \cosh t\, \partial_t$ *having soliton constant* $c \geq 0$ *(respect.* $c \leq 0$*), whose mean curvature is bounded, having polynomial volume growth and lying in a slab contained in the chronological future (respect. past) of* $\mathbb{S}^{n+1}_1$ *with respect to the equator* $\{0\} \times \mathbb{S}^n$.

3.1.8 Applications to the Lorentz-Minkowski space

From Theorem 3.1.17 we obtain the following applications.

Corollary 3.1.52. *Let* $\psi : \Sigma^n \looparrowright -\mathbb{R}^+ \times_t \mathbb{H}^n$ *be a complete spacelike mean curvature flow soliton with soliton constant* $c < 0$. *If* $\inf_\Sigma h > 0$, *the second soliton function* $\tilde{\zeta}_c = |A|^2 + c$ *is nonnegative and* $|\nabla h| \leq \inf_{\Sigma^n} \tilde{\zeta}_c$, *then* Σ^n *is a slice* $\{\sqrt{-\frac{n}{c}}\} \times \mathbb{H}^n$.

Corollary 3.1.53. *There is no complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \looparrowright -\mathbb{R}^+ \times_t \mathbb{H}^n$ *with soliton constant* $c > 0$, *such that* $\inf_\Sigma h > 0$ *and* $|\nabla h| \leq \inf_{\Sigma^n} |A|^2 + c$.

3.1.9 Application to the anti-de Sitter space

From Theorem 3.1.19 we get the following nonexistence result.

Corollary 3.1.54. *There is no complete mean curvature flow soliton with soliton constant* $c \neq 0$ *immersed in* $-(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n \subset \mathbb{H}^{n+1}_1$, *such that* $c \sin(h) \leq 0$.

3.2 Entire spacelike mean curvature flow graphs

In this last section this chapter, we will use the theorems of the previous sections in order to establish new Calabi-Bernstein type results concerning entire spacelike graphs constructed over the Riemannian fiber of a GRW spacetime. If the reader needs a refresher on the preliminary concepts of entire graphs, we recommend a re-reading of Section 1.4.

Hence, from (1.12) and (1.19) we have that $\Sigma^n(u)$ is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = \rho(t)\partial_t$ and with soliton constant c if, and only if, $|Du|_M < \rho(u)$ and u is a solution of the following nonlinear differential equation:

$$\mathrm{div}_M\left(\frac{Du}{\rho(u)\sqrt{\rho(u)^2-|Du|_M^2}}\right)=-\frac{1}{\sqrt{\rho(u)^2-|Du|_M^2}}\left\{cf(u)^2+\rho'(u)\left(n+\frac{|Du|_M^2}{\rho(u)^2}\right)\right\}. \quad (3.58)$$

We say that $u\in C^\infty(M)$ has finite C^2 norm when

$$||u||_{\mathcal{C}^2(M)}:=\sup_{|k|\le 2}|D^k u|_{L^\infty(M)}<+\infty.$$

When an entire spacelike graph $\Sigma(u)$ is such that u has finite C^2 norm, it follows from (1.18) that $|A|$ is bounded and, consequently, $\Sigma(u)$ has bounded second fundamental form. We also note that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $0<\inf_M\rho(u)\le\sup_M\rho(u)<+\infty$.

In this setting, we obtain a nonparametric version of Theorem 3.1.1.

Theorem 3.2.1. *Let $\overline{M}^{n+1}=-I\times_\rho M^n$ be a GRW spacetime satisfying (1.11) and whose Riemannian fiber M^n is complete. Suppose that c is a constant such that the modified soliton function $\bar{\zeta}_c(t)$ has strict sign in I. There does not exist a smooth function $u:M^n\to I$ with finite C^2 norm which is solution of the spacelike mean curvature flow soliton equation (3.58) and such that $|Du|_M\le\beta\rho(u)$, for some constant $0<\beta<1$.*

Proof. Let us assume the existence of such a smooth function $u:M^n\to I$. It follows from (1.18) that the shape operator A of an entire spacelike graph $\Sigma(u)$ is bounded provided that u has finite C^2 norm. Note also that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $\Sigma(u)$ is contained in a bounded timelike region of $\overline{M}^{n+1}$.

On the other hand, under the assumptions of the theorem, $\Sigma(u)$ is a complete spacelike hypersurface. Indeed, proceeding as in [13, Corollary 5.1], from (1.16) and the Cauchy-Schwarz inequality we get

$$g_u(X,X)=-g_M(Du,X^*)^2+\rho^2(u)g_M(X^*,X^*)\ge(\rho^2(u)-|Du|_M^2)g_M(X^*,X^*), \quad (3.59)$$

for every tangent vector field X on $\Sigma(u)$, where (as before) X^* denotes the projection of X onto the Riemannian fiber M^n. Thus, since $|Du|_M\le\beta\rho(u)$, for some constant $0<\beta<1$, from (3.59) we get that

$$g_u(X,X)\ge\delta g_M(X^*,X^*), \quad (3.60)$$

where $\delta=(1-\beta^2)\inf_M\rho^2(u)$. So, (3.60) implies that $L=\sqrt{\delta}L_M$, where L and L_M denote the length of a curve on $\Sigma(u)$ with respect to the Riemannian metrics g_u and g_M, respectively. As a consequence, since we are always assuming that M^n is complete, the induced metric g_u must be also complete.

Moreover, from (1.17) we obtain that the hyperbolic angle function Θ of $\Sigma(u)$ is given by

$$\Theta = -\frac{\rho(u)}{\sqrt{\rho^2(u) - |Du|_M^2}}. \tag{3.61}$$

Hence, using once more that hypothesis that $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, from (3.61) we get that Θ is bounded. But, by applying Theorem 3.1.1 we have that $\Sigma(u)$ cannot exist. $\square$

From Theorem 3.2.1 we obtain the following applications.

Corollary 3.2.2. *For any constants $c \geq 0$ and $0 < \beta < 1$, there does not exist a smooth function $u : \mathbb{R}^n \to \mathbb{R}^+$ with finite C^2 norm which is a solution of the following system*

$$\begin{cases} \operatorname{div}_{\mathbb{R}^n}\left(\dfrac{Du}{u^{\frac{2}{3}}\sqrt{u^{\frac{4}{3}} - |Du|_{\mathbb{R}^n}^2}}\right) = -\dfrac{1}{\sqrt{u^{\frac{4}{3}} - |Du|_{\mathbb{R}^n}^2}}\left(cu^{\frac{4}{3}} + \frac{2n}{3u^{\frac{1}{3}}} + \dfrac{2|Du|_{\mathbb{R}^n}^2}{3u^{\frac{5}{3}}}\right) \\ \\ |Du|_{\mathbb{R}^n} \leq \beta u^{\frac{2}{3}} \end{cases} \tag{3.62}$$

Corollary 3.2.3. *Let M^n be a complete Riemannian manifold with nonnegative sectional curvature. For any constants $c \geq 0$ and $0 < \beta < 1$, there does not exist a smooth function $u : M^n \to I$ with finite C^2 norm which is a solution of the following system*

$$\begin{cases} \operatorname{div}_M\left(\dfrac{Du}{e^u\sqrt{e^{2u} - |Du|_M^2}}\right) = -\dfrac{1}{\sqrt{e^{2u} - |Du|_M^2}}\left(ce^{2u} + ne^u + \dfrac{|Du|_M^2}{e^u}\right) \\ \\ |Du|_M \leq \beta e^u \end{cases} \tag{3.63}$$

Remark 3.2.4. From Examples 1.2.5, 1.2.6 and 1.2.7, it is not difficult to see that we can also obtain applications of Theorem 3.2.1 to the de Sitter and anti-de Sitter spaces similar to Corollaries 3.2.2 and 3.2.3.

Proceeding, from Theorem 3.1.3 we obtain the following Calabi-Bernstein type result.

Theorem 3.2.5. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (3.12), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete. Suppose that c is a constant such that $\bar{\zeta}_c(t) \geq 0$ for all $t \in I$. If $\Sigma(u) \subset \overline{M}^{n+1}$ is an entire spacelike graph determined by a bounded function $u \in C^\infty(M)$ which is solution of the spacelike mean curvature flow soliton equation (3.58) with $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, and $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Since we are supposing that $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, it follows from the proof of Theorem 3.2.1 that $\Sigma(u)$ is a complete spacelike hypersurface.

On the other hand, reasoning once more as in [13, Corollary 5.1], we deduce from the induced metric (1.16) that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma$ stand for the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, and $G = det(g_{ij})$ with

$$g_{ij} = g_u(E_i, E_j) = \rho^2(u)\delta_{ij} - E_i(u)E_j(u). \tag{3.64}$$

Here, $\{E_1, \ldots, E^n\}$ denotes a local orthonormal frame with respect to the metric g_M. So, it is not difficult to verify that

$$|G| = \rho^{2(n-1)}(u)(\rho^2(u) - |Du|_M^2). \tag{3.65}$$

Hence, from (3.64) and (3.65) we obtain

$$d\Sigma = \rho^{n-1}(u)\sqrt{\rho^2(u) - |Du|_M^2}dM. \tag{3.66}$$

Moreover, since we have that $N = N^* - \Theta\partial_t$, from (5.3) we get

$$|\nabla h|^2 = \rho^2(u)|N^*|_M^2. \tag{3.67}$$

Thus, from (5.25) and (3.67) we obtain

$$|\nabla h|^2 = \frac{|Du|_M^2}{\rho^2(u) - |Du|_M^2}. \tag{3.68}$$

Consequently, from (3.68) and (3.66) we get

$$|\nabla h|d\Sigma = \rho^{n-1}(u)|Du|_M dM. \tag{3.69}$$

Hence, since we are assuming that u is bounded with $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, relation (3.69) guarantees that $|\nabla h| \in \mathcal{L}^1_g(\Sigma(u))$. Therefore, the result follows by applying Theorem 3.1.3. □

From Theorem 3.2.5 we obtain the following applications.

Corollary 3.2.6. *For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : \mathbb{R}^n \to \mathbb{R}^+$, with $u(x) \leq (-\frac{2n}{3c})^{\frac{3}{5}}$ for all $x \in \mathbb{R}^n$, $|Du|_{\mathbb{R}^n} \in \mathcal{L}^1_{g_{\mathbb{R}^n}}(\mathbb{R}^n)$ and which is solution of the system (3.62), is the constant $u = (-\frac{2n}{3c})^{\frac{3}{5}}$.*

Corollary 3.2.7. *Let M^n be a complete Riemannian manifold. For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : M^n \to \mathbb{R}$, with $u(x) \leq \log(-\frac{n}{c})$ for all $x \in M^n$, $|Du|_M \in \mathcal{L}^1_{g_M}(M)$ and which is solution of the system (3.63), is the constant $u = \log(-\frac{n}{c})$.*

Taking into account once more relation (3.66), it is not difficult to see that from Theorem 3.1.6 we obtain the following nonexistence result.

Theorem 3.2.8. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime satisfying (3.12), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. Suppose that c is a constant such that $\bar{\zeta}_c(t) \geq 0$ for all $t \in I$.*

There does not exist a bounded entire solution $u \in C^\infty(M)$ of the spacelike mean curvature flow soliton equation (3.58), with $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, and such that $(\rho(u))^{-1} \in \mathcal{L}^q_{g_M}(M)$ for some q with $q > \gamma + 3$.

We have the following applications of Theorem 3.2.8.

Corollary 3.2.9. *For any constants $c \geq 0$ and $0 < \beta < 1$, there does not exist a bounded smooth function $u : \mathbb{R}^n \to \mathbb{R}^+$ such that $u^{-\frac{2}{3}} \in \mathcal{L}^q_{g_{\mathbb{R}^n}}(\mathbb{R}^n)$, for some $q > 3$, and which is a solution of the system (3.62).*

Corollary 3.2.10. *Let M^n be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any constants $c \geq 0$ and $0 < \beta < 1$, there does not exist a bounded smooth function $u : M^n \to I$ such that $e^{-u} \in \mathcal{L}^q_{g_M}(M)$, for some $q > 3$, and which is a solution of the system (3.63).*

Observing once more that the assumption $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, implies that the hyperbolic angle function Θ given by (3.61) is bounded, Theorem 3.1.26 allows us to obtain the following result.

Theorem 3.2.11. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a spatially parabolic covered GRW spacetime satisfying (3.12), holding the equality only at isolated points of I. Suppose that c is a constant such that $\bar{\zeta}_c(t) \geq 0$ for all $t \in I$. If $\Sigma(u) \subset \overline{M}^{n+1}$ is an entire spacelike graph determined by a bounded function $u \in C^\infty(M)$ which is solution of the spacelike mean curvature flow soliton equation (3.58) with $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

We finish this manuscript with the following applications of Theorem 3.2.11.

Corollary 3.2.12. *For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : \mathbb{R}^2 \to \mathbb{R}^+$, with $u(x) \leq (-\frac{4}{3c})^{\frac{3}{5}}$ for all $x \in \mathbb{R}^2$, and which is solution of the system (3.62) for $n = 2$, is the constant $u = (-\frac{4}{3c})^{\frac{3}{5}}$.*

Corollary 3.2.13. *Let M^n be a complete Riemannian manifold with parabolic universal covering. For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : M^n \to \mathbb{R}$, with $u(x) \leq \log(-\frac{n}{c})$ for all $x \in M^n$, and which is solution of the system (3.63), is the constant $u = \log(-\frac{n}{c})$.*

Taking into account [8, Lemma 17], it is not difficult to see that we can reason to get the following nonparametric version of Theorem 3.1.27.

Corollary 3.2.14. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime with complete noncompact Riemannian fiber M^n and whose warping function ρ is increasing (decreasing) and satisfies inequality (3.12). Suppose in addition that c is a constant such that the modified soliton function $\bar{\zeta}_c(t) \geq 0$ for all $t \in I$. The only smooth function $u : M^n \to I$ which is solution of the mean curvature flow soliton equation (3.58), with $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$, and such that u converges from below (above) to some $t_* \in I$ at infinity is the constant function $u \equiv t_*$.*

Proof. Let $u \in C^\infty(M)$ be such a solution of equation (3.58). We start observing that, since M^n is complete and $|Du|_M \leq \beta\rho(u)$ (due to the boundedness of u), from (5.24) we conclude that the entire graph $\Sigma(u)$ must be complete. Therefore, we are in position to apply Theorem 3.1.27 to conclude that $u \equiv t_*$. □

Taking into account [8, Lemma 17] jointly with equation (5.9) in the proof of [13, Corollary 5.1], it is not difficult to see that we can reason to get the following nonparametric version of Theorem 3.1.2.

Theorem 3.2.15. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime whose Riemannian fiber M^n is complete noncompact, having polynomial volume growth and with its warping function ρ satisfying inequality (3.12). Suppose in addition that $c \neq 0$ is a constant such that the modified soliton function $\bar{\zeta}_c(t) > 0$ for all $t \in I$. There does not exist a bounded smooth function $u : M^n \to I$ which is solution of the mean curvature flow soliton equation (3.58) and such that $|Du|_M \leq \beta\rho(u)$, for some constant $0 < \beta < 1$.*

Proof. Let $u \in C^\infty(M)$ be such a solution of equation (3.58). From (5.25) we get

$$\Theta(u) = \frac{\rho(u)}{\sqrt{\rho(u)^2 - |Du|_M^2}}.$$

Thus, since we are assuming that u and $|Du|_M$ are bounded, from (1.12) and (3.61) we get that $H(u)$ is bounded away from zero.

On the other hand, reasoning as in the proof of [14, Theorem 1], we deduce from the induced metric (1.16) that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma$ stand for the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, and (as in the proof of Theorem 3.1.2) $G = det(g_{ij})$ with

$$g_{ij} = g_u(E_i, E_j) = \rho^2(u)\delta_{ij} - E_i(u)E_j(u). \tag{3.70}$$

Here, $\{E_1, \ldots, E^n\}$ denotes a local orthonormal frame with respect to the metric g_M. So, it is not difficult to verify that

$$|G| = \rho^{2(n-1)}(u)(\rho^2(u) - |Du|_M^2). \tag{3.71}$$

Then, from (3.70) and (3.71) we obtain

$$d\Sigma = \rho^{n-1}(u)\sqrt{\rho^2(u) - |Du|_M^2}dM. \tag{3.72}$$

Hence, since we are supposing that (M^n, g_M) has polynomial volume growth, we can use once more the hypotheses that u and $|Du|_M$ are bounded jointly with relation (3.72) to get that $(\Sigma(u), g_u)$ also has polynomial volume growth. Therefore, we are in position to apply Theorem 3.1.2 and conclude that $\Sigma(u)$ cannot exist. □

Theorem 3.2.16. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the SNCC (3.37), with equality occurring only in isolated points of I, and whose Riemannian fiber M^n is complete. Let $u \in C^\infty(M)$ be an entire solution of equation (3.58) for $c \neq 0$, with finite C^2 norm, such* *that $|Du|_M \leq \alpha\rho(u)$, for some constant $0 < \alpha < 1$, and the second soliton function $\tilde{\zeta}_c(u) = |A|^2 + c\rho'(u)$ is nonnegative. If $|Du|_M \in \mathcal{L}^1(M)$, then $u \equiv t_*$ for some $t_* \in I$, which is implicitly given by the condition $\zeta(t_*) = 0$.*

Proof. Let $z \in C^\infty(M)$ be such a solution of equation (3.58). It follows from (1.18) that the shape operator A of $\Sigma^n(z)$ is bounded, provided that u has finite C^2. We note also that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $\Sigma^n(u)$ is contained in a bounded timelike region of $\overline{M}^{n+1}$. Consequently, since we are also assuming that $|Du|_M \leq \alpha\rho(u)$, for some constant $0 < \alpha < 1$, we get that

$$|Du|_M^2 \leq \rho^2(u) - \beta,$$

for $\beta = (1 - \alpha^2)\inf_{\Sigma(u)} \rho^2(u)$. Thus, we can apply [6, Proposition 1] to conclude that $\Sigma^n(u)$ is complete.

We also have that $N = N^* - \Theta\partial_t$, where N^* denotes the projection of N onto the fiber M^n. Consequently, from (1.17), we get

$$|\nabla u|^2 = \langle N^*, N^*\rangle = \rho^2(u)\langle N^*, N^*\rangle_M. \tag{3.73}$$

Thus, from (1.17) and (3.73) we obtain

$$|\nabla u|^2 = \frac{|Du|_M^2}{\rho^2(u) - |Du|_M^2}. \tag{3.74}$$

On the other hand, it follows from (1.16) that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma^n$ stand for the Riemannian volume elements of (M^n, g_M) and $(\Sigma^n(u), g_u)$, respectively, and $G = det(g_{ij})$ with

$$g_{ij} = g_z(E_i, E_j) = \rho^2(u)\delta_{ij} - E_i(u)E_j(u).$$

Here, $\{E_1, \ldots, E^n\}$ denotes a local orthonormal frame with respect to the metric g_M. So, it is not difficult to verify that

$$|G| = \rho^{2(n-1)}(u)(\rho^2(u) - |Du|_M^2).$$

Consequently,

$$d\Sigma = \rho^{n-1}(u)\sqrt{\rho^2(u) - |Du|_M^2}dM. \tag{3.75}$$

Thus, from (3.74) and (3.75) we get

$$|\nabla u|d\Sigma = \rho(u)^{n-1}|Du|_M dM. \tag{3.76}$$

Hence, since z is bounded and $|Du|_M \in \mathcal{L}^1(M)$, from relation (3.76) we conclude that $|\nabla u| \in \mathcal{L}^1(\Sigma^n(u))$. Consequently, from (3.26) we get that $|\nabla(\varphi_\mathcal{K})| \in \mathcal{L}^1_{cu}(\Sigma^n(u))$. Therefore, we can reason as in the last part of the proof of Theorem 3.1.10 to conclude the result. □

From Theorem 3.1.14 we obtain the following consequence

Theorem 3.2.17. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the SNCC (3.37), with equality occurring only in isolated points of I, and whose Riemannian fiber M^n is complete with $(c\tilde{u})$-parabolic universal Riemannian covering for some constant $c \neq 0$. If $u \in C^\infty(M)$ is an entire solution of equation (3.58) for c, with finite C^1 norm, such that $|Du|_M \leq \alpha\rho(u)$, for some constant $0 < \alpha < 1$, and the second soliton function $\tilde{\zeta}_c(u) = |A|^2 + c\rho'(u)$ is nonnegative, then $u \equiv t_*$ for some $t_* \in I$, which is implicitly given by the condition $\zeta(t_*) = 0$.*

Proof. Observing that h satisfies (1.4) and (3.73), from (1.18) we obtain

$$|\nabla h|^2 = \frac{|Du|_M^2}{\rho(u)^2 - |Du|_M^2}. \tag{3.77}$$

Hence, since we are assuming that z has finite C^1 norm and taking into account once more that $\Theta^2 = |\nabla h|^2 + 1$, with aid of (3.77) we conclude that Θ is bounded. Therefore, the result follows by applying Theorem 3.1.14. □

From Theorem 3.1.17 we also obtain the following result

Theorem 3.2.18. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the SNCC (3.37) and whose Riemannian fiber M^n is complete. Let $u \in C^\infty(M)$ be a bounded entire solution of equation (3.58) for some constant $c \neq 0$, such that $|Du|_M \leq \alpha\rho(u)$, for some constant $0 < \alpha < 1$, and the second soliton function $\tilde{\zeta}_c(u) = |A|^2 + c\rho'(u)$ is nonnegative. If*

$$|Du|_M \leq \inf_M \tilde{\zeta}_c, \tag{3.78}$$

then $u \equiv t_*$ for some $t_* \in I$, which is implicitly given by the condition $\zeta(t_*) = 0$.

Proof. From (3.77) and (3.78), we see that hypothesis (3.50) is satisfied. Therefore, the result follows applying Theorem 3.1.17. □

We close this subsection with the following application of Theorem 3.1.19

Theorem 3.2.19. *Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime obeying the SNCC (3.42) and whose Riemannian fiber M^n is complete. For any constant $c \neq 0$, there is no bounded entire solution $u \in C^\infty(M)$ of equation (3.58), such that $|Du|_M \leq \alpha\rho(u)$, for some constant $0 < \alpha < 1$, and $c\rho'(u) \geq 0$.*

3.3 Stability of spacelike mean curvature flow solitons in GRW spacetimes

Let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a complete spacelike mean curvature flow soliton with with soliton constant c. We recall that a *variation with compact support and fixed boundary* of ψ :

$\Sigma^n \looparrowright -I \times_\rho M^n$ is a smooth mapping

$$F : (-\epsilon, \epsilon) \times \Sigma^n \to -I \times_\rho M^n \tag{3.79}$$

such that

(i) for $s \in (-\epsilon, \epsilon)$, the map $F_s : \Sigma^n \looparrowright -I \times_\rho M^n$ given by $F_s(q) = F(s, q)$ is an spacelike immersion with $F_0 = x$;

(ii) $F_s|_{\partial\Sigma} = \psi|_{\partial\Sigma}$ for all $s \in (-\epsilon, \epsilon)$.

In all that follows, we let dM_s denote the volume element of the metric induced on Σ^n by F_s and N_s the unit normal vector field along F_s. Moreover, we also consider in Σ^n the weighted volume form given by $d\mu_s = e^{-\rho} dM_s$. When $s = 0$ all these objects coincide with the ones defined in Σ^n, respectively.

The *variational field* associated to the variation F is the vector field $\frac{\partial F}{\partial s}|_{s=0}$. Letting

$$u_s = -\langle \frac{\partial F}{\partial s}, N_s \rangle, \tag{3.80}$$

we get

$$\frac{\partial F}{\partial s}|_{s=0} = u_0 N + (\frac{\partial F}{\partial s}|_{s=0})^\top,$$

where $(\cdot)^\top$ stands for tangential components.

Denoting the set of all smooth functions on Σ^n with compact support by $C_0^\infty(\Sigma^n)$, according to [46, Lemma 2.1] and [47, Lemma 2.1], every function $\varphi \in C_0^\infty(\Sigma^n)$ with

$$\int_{\Sigma^n} \varphi \, d\Sigma = 0 \tag{3.81}$$

induces a variation of $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ of the type (3.79), with variational normal field $\frac{\partial F}{\partial s}|_{s=0} = \varphi N$, and with *first variation* $\delta_\varphi \mathbb{A}$ of the *area functional*

$$\begin{aligned} \mathbb{A} : (-\epsilon, \epsilon) \to \; & \mathbb{R} \\ s \;\; & \mapsto \mathbb{A}(s) = \text{Area}(F_s(\Sigma^n)) = \int_{\Sigma^n} d\Sigma_s, \end{aligned}$$

given by

$$\delta_\varphi \mathbb{A} = \frac{d\mathbb{A}}{ds}(0) = \int_{\Sigma^n} \varphi H \, d\Sigma. \tag{3.82}$$

Here, N stands for a normal unit vector field globally defined on Σ^n, $d\Sigma_s$ denotes the volume element of Σ^n with respect to the metric induced by $F_s : \Sigma^n \looparrowright -I \times_\rho M^n$ and H is the mean curvature function of $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ with respect to N.

As a consequence of (3.82), *maximal* compact spacelike mean curvature flow solitons of $-I \times_\rho M^n$ (that is, with mean curvature identically zero) are characterized as critical points of the area functional $\mathbb{A}$ whereas any compact spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ with constant mean curvature H is a critical point of $\mathbb{A}$ restricted to functions $\varphi \in C^\infty(\Sigma^n)$ which satisfy the condition (3.81). Geometrically, this additional condition means that the variations under consideration preserve a certain volume functional (for more details, see [47]).

For these critical points, [46, Proposition 2.3] asserts that the stability of the corresponding variational problem is given by the second variation of the area functional $\mathbb{A}$, which is given by

$$\delta^2_\varphi \mathbb{A} \;=\; \frac{d^2\mathbb{A}}{ds^2}(0)(\varphi) \;=\; \int_{\Sigma^n} \Big\{\Delta(\varphi) - \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi\Big\}\varphi\, d\Sigma,$$

where Δ stands for the Laplacian operator on Σ^n, $\overline{\mathrm{Ric}}$ is the Ricci tensor of the GRW spacetime $-I \times_\rho M^n$ and $|A|$ denotes the length of the shape operator A of $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ with respect to N. In this setting, we establish the following

Definition 3.3.1. *A compact spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ with constant mean curvature H is said strongly stable if $\delta^2_\varphi \mathbb{A} \leq 0$, for every $\varphi \in C^\infty(\Sigma^n)$.*

In our next result, we impose a suitable behavior on the warping function ρ to obtain a nonexistent result of strongly stable spacelike mean curvature flow solitons immersed in $-I \times_\rho M^n$.

Theorem 3.3.2. *There is no strongly stable compact spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ with soliton constant $c \neq 0$, whose mean curvature H is constant and such that its height function h satisfies $c\rho'(h)\rho(h) + n\rho''(h) > 0$ on Σ^n.*

Proof. By contradiction, let us suppose the existence of such a soliton $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$. From the first equation in the proof of [62, Proposition 2.1] we have

$$\begin{aligned}\Delta(\varphi_{\mathcal{K}}) &= \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} - \{nN(\rho'(h)) - H\rho'(h)\} + \langle \mathcal{K}, \nabla H\rangle \\ &= \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} + c\,\varphi_{\mathcal{K}}\rho'(h) - nN(\rho'(h)),\end{aligned} \tag{3.83}$$

where $\varphi_{\mathcal{K}} \in C^\infty(\Sigma^n)$ is the support function defined in (3.27).

On the other hand, we also have

$$N(\rho'(h)) \;=\; -\langle \rho''(h)\partial_t, N\rangle \;=\; -\,\frac{\rho''(h)}{\rho(h)}\,\varphi_{\mathcal{K}}. \tag{3.84}$$

Thus, from (3.83) and (3.84) we get

$$\begin{aligned}\Delta(\varphi_{\mathcal{K}}) &= \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} + c\,\varphi_{\mathcal{K}}\rho'(h) - nN(\rho'(h)) \\ &= \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} + \Big\{c\,\rho'(h) + n\,\frac{\rho''(h)}{\rho(h)}\Big\}\,\varphi_{\mathcal{K}}.\end{aligned} \tag{3.85}$$

Moreover, since $H = c\,\varphi_{\mathcal{K}}$ is constant and $c \neq 0$, we have that $\varphi_{\mathcal{K}}$ is also constat on Σ^n. Hence, form (3.85), we obtain

$$-\{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} \;=\; \Big\{c\,\rho'(h) + n\,\frac{\rho''(h)}{\rho(h)}\Big\}\,\varphi_{\mathcal{K}}. \tag{3.86}$$

Now, from our hypothesis of strong stability and taking into account Definition 3.3.1, we currently have

$$\delta_\varphi^2 \mathbb{A} \;=\; \int_{\Sigma^n} \Big\{\Delta(\varphi) - \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi\Big\}\varphi \, d\Sigma \;\leq\; 0$$

for every $\varphi \in C^\infty(\Sigma^n)$.

Thus, making $\varphi = \varphi_{\mathcal{K}} < 0$ (see (3.28)), from hypothesis $c\rho'(h)\rho(h) + n\rho''(h) > 0$ jointly with (3.86) we get

$$0 < \int_{\Sigma^n} \left\{c\,\rho'(h) + n\,\frac{\rho''(h)}{\rho(h)}\right\}\varphi_{\mathcal{K}}^2 = \int_{\Sigma^n} \Big\{\underbrace{\Delta(\varphi_{\mathcal{K}})}_{0} - \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}}\Big\}\varphi_{\mathcal{K}}\, d\Sigma \leq 0,$$

and we reach at an absurd. □

From Theorem 3.3.2 we get the following application.

Corollary 3.3.3. *There is no strongly stable compact spacelike mean curvature flow soliton in a steady state type spacetime $-I \times_{e^t} M^n$ with soliton constant $c > 0$ and constant mean curvature.*

In what follows, we consider the function

$$\overline{u} \;=\; -g(\pi_I) \in C^\infty(-I \times_\rho M^n),$$

where $g : I \to \mathbb{R}$ is a primitive of the warping function f which was used for define the reparametrization $u = -g(h)$ of the height function h of the spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ (see (3.22)). From (1.3) we observe that $\overline{u} = u$ on Σ^n and, hence, $\bar{u}$ is a smooth extension of u. According to [45], we consider the *Bakry-Émery-Ricci tensor* tensor $\overline{\mathrm{Ric}}_{c\bar{u}}$ of $-I \times_\rho M^n$, which is given by

$$\overline{\mathrm{Ric}}_{c\bar{u}} \;=\; \overline{\mathrm{Ric}} + c\,\overline{\nabla}^2\,\bar{u} = \overline{\mathrm{Ric}} - c\rho'(h)\langle\,,\rangle, \tag{3.87}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\nabla}^2$ are the standard Ricci tensor and the Hessian in $-I \times_\rho M^n$, respectively. We will also consider the modified volume element

$$d\bar{\mu} \;=\; e^{c\bar{u}} dV, \tag{3.88}$$

where dV denotes the standard volume element of $-I \times_\rho M^n$. We note that on Σ^n, $d\bar{\mu}$ coincides with the modified volume element $d\mu$ previously defined in (3.38).

With all these considerations, we have that any function $\varphi \in C_0^\infty(\Sigma^n)$ with

$$\int_{\Sigma^n} \varphi \, d\mu \;=\; 0$$

induces a variation with compact support and fixed boundary of $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ with variational normal field $\frac{\partial F}{\partial s}|_{s=0} \;=\; \varphi N$ and with *first variation* $\delta_\varphi\,(\mathbb{A}_{cu})$ of the *modified area* *functional*

$$\begin{aligned} \mathbb{A}_{cu} : (-\epsilon, \epsilon) &\rightarrow \quad \mathbb{R} \\ s \quad &\mapsto \mathbb{A}_{cu}(s) = \int_{\Sigma^n} d\mu \end{aligned}$$

given by

$$\delta_\varphi\left(\mathbb{A}_{cu}\right) = \frac{d\mathbb{A}_{cu}}{ds}(0) = \int_{\Sigma^n} \varphi H_{c\bar{u}}\, d\mu \tag{3.89}$$

(see, for instance, [66, Lemma 3.2]), where $H_{c\bar{u}}$ is *the modified mean curvature* of $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ defined by

$$H_{c\bar{u}} = H - c\langle \overline{\nabla}(\bar{u}), N\rangle.$$

But, since $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is a spacelike mean curvature flow soliton with respect to the closed conformal vector field $\mathcal{K} = \rho(t)\partial_t$ and with soliton constant $c \neq 0$, from (1.12) and (3.23) we get that

$$\begin{aligned} H_{c\bar{u}} &= c\rho(h)\Theta - c\langle \overline{\nabla}(\bar{u}), N\rangle = c\rho(h)\Theta - c\langle (\overline{\nabla}(\bar{u})^\top + \overline{\nabla}(\bar{u})^\perp), N\rangle \\ &= c\rho(h)\Theta - c\langle \overline{\nabla}(\bar{u})^\perp, N\rangle = c\rho(h)\Theta - c\langle (-g'(h)\overline{\nabla}\pi_I))^\perp, N\rangle. \\ &= c\rho(h)\Theta - c\rho(h)\langle \partial_t, N\rangle = 0 \end{aligned} \tag{3.90}$$

Therefore, from (3.89) and (3.90) we obtain that any spacelike mean curvature flow soliton $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ with respect to the closed conformal vector field $\mathcal{K} = \rho(t)\partial_t$ and with soliton constant $c \neq 0$, is a critical point of the modified area functional $\mathbb{A}_{cu}$.

Furthermore, the *stability operator* $L_{cu} : C_0^\infty(\Sigma^n) \rightarrow C_0^\infty(\Sigma^n)$ for this variational problem is given by the second variation formula $\delta^2_\varphi\left(\mathbb{A}_{cu}\right)$ of $\mathbb{A}_{cu}$, which in our case is written as follows (see, for instance, [66, Proposition 3.5] for the case $H_{c\bar{u}} = 0$):

$$\delta^2_\varphi\left(\mathbb{A}_{cu}\right) = \frac{d^2\mathbb{A}_{cu}}{ds^2}(0)(\varphi) = \int_{\Sigma^n} \varphi L_{cu}(\varphi)\, d\mu,$$

with

$$L_{cu} = \Delta_{cu} - \{\overline{\mathrm{Ric}}_{c\bar{u}}(N, N) + |A|^2\},$$

where Δ_{cu} is the drift Laplacian operator on Σ^n given in (3.34). So, using (3.87) we can rewrite the stability operator L_{cu} as

$$L_{cu} = \Delta_{cu} - \{\overline{\mathrm{Ric}}(N, N) - cf'(h) + |A|^2\}. \tag{3.91}$$

The following notion of stability concerning spacelike mean curvature flow solitons in GRW spacetime now makes sense.

Definition 3.3.4. *Let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a spacelike mean curvature flow soliton with soliton constant $c \neq 0$. We say that $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is L_{cu}-stable if $\delta^2_\varphi\left(\mathbb{A}_{cu}\right) \leq 0$, for all $\varphi \in C_0^\infty(\Sigma^n)$.*

The next auxiliary result gives a sufficient condition to guarantee that a spacelike mean curvature flow soliton must be L_{cu}-stable (for its proof, see [70, Lemma 3.2]).

Lemma 3.3.5. *Let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a spacelike mean curvature flow soliton with soliton constant $c \neq 0$. If there exists a positive smooth function $\varphi \in C^\infty(\Sigma^n)$ such that $L_{cu}(\varphi) \leq 0$, then Σ^n is L_{cu}-stable.*

Now, we analyze the behavior of the warping function ρ along a spacelike mean curvature flow soliton in order to infer its L_{cu}-stability.

Theorem 3.3.6. *Let $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ be a spacelike mean curvature flow soliton with soliton constant $c \neq 0$.*

(a) If $\zeta_c'(t) \leq 0$ on Σ^n, then $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is L_{cu}-stable.

(b) If Σ^n is compact and $\zeta_c'(t) \geq 0$ on it, then $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is L_{cu}-stable if and only if $\zeta_c(t)$ is constant on Σ^n.

(c) If Σ^n is compact and $\zeta_c'(t) > 0$ on it, then $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ cannot be L_{cu}-stable.

Proof. From (3.85) we have

$$\Delta(\varphi_{\mathcal{K}}) \;=\; \{\overline{\mathrm{Ric}}(N,N) + |A|^2\}\varphi_{\mathcal{K}} + \left\{c\,\rho'(h) + n\,\frac{\rho''(h)}{\rho(h)}\right\}\varphi_{\mathcal{K}},$$

where $\varphi_{\mathcal{K}} \in C^\infty(\Sigma^n)$ is support function defined in (3.27). So, by applying $\varphi_{\mathcal{K}}$ to the stability operator L_{cu} and using the last equation we get

$$L_{cu}(\varphi_{\mathcal{K}}) \;=\; \Delta_{cu}(\varphi_{\mathcal{K}}) - \{\overline{\mathrm{Ric}}(N,N) - c\rho'(h) + |A|^2\}\varphi_{\mathcal{K}} \;=\; \left\{2c\rho'(h) + n\,\frac{\rho''(h)}{\rho(h)}\right\}\varphi_{\mathcal{K}}.$$

Hence,

$$L_{cu}(-\varphi_{\mathcal{K}}) \;=\; \{2c\rho'(h)\rho(h) + n\rho''(h)\}\,(-\varphi_{\mathcal{K}}), \tag{3.92}$$

with $-\varphi_{\mathcal{K}}$ being a positive smooth function on Σ^n and, with a direct application of Lemma 3.3.5, the result of item (a) is obtained directly.

Now, let us consider item (b). Note that in this case $C_0^\infty(\Sigma^n) = C^\infty(\Sigma^n)$. So, if $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is L_{cu}-stable, from Definition 3.3.4 and equation (3.92) we get

$$\begin{aligned} 0 \;\geq\; \delta^2_{(-\varphi_{\mathcal{K}})}\,(\mathbb{A}_{cu}) \;&=\; \int_{\Sigma^n} (-\varphi_{\mathcal{K}}) L_{cu}(-\varphi_{\mathcal{K}})\, d\mu \\ &=\; \int_{\Sigma^n} \{2c\rho'(h)f(h) + n\rho''(h)\}(-\varphi_{\mathcal{K}})^2\, d\mu \;\geq\; 0, \end{aligned} \tag{3.93}$$

which guarantees us $\zeta_c(t)$ is constant on Σ^n. The converse follows from item (a).

Finally, we prove item (c). Assuming the opposite, if we have $\psi : \Sigma^n \looparrowright -I \times_\rho M^n$ is L_{cu}-stable then, from the analysis of signs studied in (3.93),

$$0 \;\geq\; \int_{\Sigma^n} \{2c\rho'(h)\rho(h) + n\rho''(h)\}(-\varphi_{\mathcal{K}})^2\, d\mu \;>\; 0,$$

which constitutes a absurd. □

From Theorem 3.3.6 we obtain the following applications:

Corollary 3.3.7. *Every spacelike translating soliton immersed in the Lorentzian product space $-I \times M^n$, with soliton constant $c \neq 0$, is L_{cu}-stable.*

Corollary 3.3.8. *Every spacelike mean curvature flow soliton immersed in the future temporal cone $-\mathbb{R}^+ \times_t \mathbb{H}^n$, with soliton constant $c < 0$ and such that $h \geq \sqrt{-\frac{n}{c}}$, is L_{cu}-stable.*

Corollary 3.3.9. *There is no L_{cu}-stable compact spacelike mean curvature flow soliton immersed in a steady state type spacetime $-I \times_{e^t} M^n$ with soliton constant $c > 0$.*

Chapter 4

Rigidity of mean curvature flow solitons in standard static spacetime

In this chapter, we obtain rigidity results concerning complete noncompact solitons of the mean curvature flow related to a nonsingular Killing vector field K globally defined in standard static spacetime, which can be modeled as a warped product whose base corresponds to a fixed integral leaf of the distribution orthogonal to K and the warping function is equal to $|K|$. Our approach is based on a suitable maximum principle dealing with a notion of convergence to zero at infinity. As application, we study the uniqueness of solutions for the mean curvature flow soliton equation in these ambient spaces. The results presented in this chapter make part of [33, 34].

4.1 Spacelike hypersurfaces in a standard static spacetime

Let us consider a connected spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$, which means that the induced metric g on Σ^n via ψ is a Riemannian metric. Since K is a globally defined timelike vector field on $\overline{M}^{n+1}$, it follows that there exists a unique unitary timelike normal vector field N globally defined on Σ^n which is in same time-orientation as K. By using the inverse Cauchy-Schwarz inequality, we get

$$\overline{g}(N, K) \leq -\rho < 0 \quad \text{on} \quad \Sigma^n. \tag{4.1}$$

We will refer to that normal vector field N as to the *future-pointing Gauss map* of Σ^n. Throughout this work, N will always denote the future-pointing Gauss map of a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$. We also note that in a standard static spacetime $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ there exists a distinguished foliation whose leaves are given by the totally geodesic level hypersurfaces of the function $\pi_\mathbb{R}$. They are just the spacelike slices $M^n \times \{t_*\}$, $t_* \in \mathbb{R}$, whose future-pointing Gauss map N is given by the unit timelike vector field $\frac{1}{\rho}K$ restricted to $M^n \times \{t_*\}$.

The (vertical) height function of a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ is defined by

$h = \pi_{\mathbb{R}} \circ \psi$ and its angle function is given by $\Theta = g(N, K)$, where we recall that N denotes the future-pointing Gauss map of Σ^n. From (4.1), we note that Θ will be always a negative function. Moreover, from the decomposition $K = K^\top - \Theta N$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n, we obtain

$$\nabla h = -\frac{1}{\rho^2} K^\top \quad \text{and} \quad |\nabla h|^2 = \frac{\Theta^2 - \rho^2}{\rho^4}. \tag{4.2}$$

Here, for simplicity of notation, we are considering $\rho = \rho \circ \pi_M \circ \psi$ along Σ^n.

Let $\overline{\nabla}$, ∇ and D denote the Levi-Civita connections in $\overline{M}^{n+1}$, Σ^n and M^n, respectively. Then the Gauss and Weingarten formulas for the spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - g(AX, Y)N \tag{4.3}$$

and

$$AX = -\overline{\nabla}_X N, \tag{4.4}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$, where A stands for the Weingarten endomorphism of Σ^n with respect to its future-pointing Gauss map N.

Using once more the decomposition $K = K^\top - \Theta N$, from (4.3) and (4.4) we see that

$$\nabla_X K^\top = (\overline{\nabla}_X K)^\top - \Theta AX. \tag{4.5}$$

Consequently, from (4.2) and (4.5) we get the Hessian of the height function as follows

$$\begin{aligned} \nabla_X \nabla h &= \nabla_X \left(-\frac{1}{\rho^2} K^\top \right) \\ &= \frac{2}{\rho^3} g(\nabla \rho, X) K^\top - \frac{1}{\rho^2} (\overline{\nabla}_X K)^\top + \frac{1}{\rho^2} \Theta AX. \end{aligned} \tag{4.6}$$

So, taking a local orthonormal tangent frame $\{e_1, e_2, \ldots, e_n\}$ on Σ^n, from (4.6) we obtain

$$\begin{aligned} \Delta h &= \sum_{i=1}^n g\left(\frac{2}{\rho^3} g(\nabla \rho, e_i) K^\top - \frac{1}{\rho^2} (\overline{\nabla}_{e_i} K)^\top + \frac{1}{\rho^2} \Theta A e_i, e_i \right) \\ &= -\frac{2}{\rho} g(\nabla \rho, \nabla h) - \sum_{i=1}^n \frac{1}{\rho^2} \overline{g}(\overline{\nabla}_{e_i} K, e_i) - \frac{1}{\rho^2} \Theta H, \end{aligned} \tag{4.7}$$

where H stands for the mean curvature function of Σ^n related to N. But, since K is a Killing vector field on $\overline{M}^{n+1}$, it satisfies the following Killing equation

$$\overline{g}(\overline{\nabla}_X K, Y) + \overline{g}(X, \overline{\nabla}_Y K) = 0, \tag{4.8}$$

for every $X, Y \in \mathfrak{X}(\overline{M})$. Hence, from (4.7) and (4.8) we reach at the following suitable formula

$$\Delta h = -\frac{2}{\rho} g(\nabla \rho, \nabla h) - \frac{1}{\rho^2} \Theta H. \tag{4.9}$$

4.2 Rigidity of mean curvature flow solitons

Before we proceed an important observation for this section:

Remark 4.2.1. *When the ambient space* $\overline{M}^{n+1}$ *is a standard static spacetime of the type* $M^n \times_\rho \mathbb{R}_1$*, according to* [79, Definition 2] *(see also* [27, Definition 1.1] *and* [69, Definition 1.1]*), a Riemannian immersion*

$$\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$$

is called a mean curvature flow soliton *with respect to* K *and with* soliton constant $c \in \mathbb{R}$ *if its (non-normalized) mean curvature function satisfies*

$$H = c\,\Theta. \tag{4.10}$$

In the Lorentzian case, we will use the nomenclature spacelike mean curvature flow soliton*. We also observe that each slice* $M^n \times \{t\}$ *of* $\overline{M}^{n+1}$ *is a mean curvature flow soliton with respect to* K *and with soliton constant* $c = 0$*.*

Concerning the following rigidity result for spacelike mean curvature flow solitons in a standard static spacetime.

Theorem 4.2.2. *Let* $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ *be a standard static spacetime with complete noncompact Riemannian base* M^n*. The only complete noncompact spacelike mean curvature flow soliton* $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ *with respect to* K *and with soliton constant* $c \geq 0$ *(resp.* $c \leq 0$*), such that* ρ *is bounded on* Σ^n *and* h *converges from below (resp. above) to* t_* *at infinity, is the slice* M_{t_*}*.*

Proof. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be such a mean curvature flow soliton. Let us consider on Σ^n the metric $\hat{g} = \rho^{\frac{4}{n-2}} g$, which is conformal to its induced metric g.

It is well known that, in local coordinates $(x_1, \cdots, x_n)$ of Σ^n, the Laplacian of its height function on a metric $\hat{g}$ is given by

$$\hat{\Delta} h = \frac{1}{\hat{G}} \sum_{k,l=1}^{n} \partial_k \left(\hat{g}^{kl} \hat{G} \partial_l(h) \right), \tag{4.11}$$

where $\hat{g}_{kl} = \hat{g}(\partial_k, \partial_l)$, $\hat{G} = \sqrt{\det\left(\hat{g}_{kl}\right)}$ and $\left(\hat{g}^{kl}\right) = \left(\hat{g}_{kl}\right)^{-1}$.

Taking the conformal metric $\hat{g} = \rho^{\frac{4}{n-2}} g$, we have that $\hat{g}_{kl} = \rho^{\frac{4}{n-2}} g_{kl}$, $\hat{g}^{kl} = \frac{1}{\rho^{\frac{4}{n-2}}} g^{kl}$ and

$$\hat{G} = \sqrt{\det(\hat{g}_{kl})} = \sqrt{\rho^{\frac{4n}{n-2}} \det(g_{kl})} = \rho^{\frac{2n}{n-2}} G. \tag{4.12}$$

Thus, from (4.17) and (4.18) we obtain

$$\begin{aligned}\hat{\Delta}h &= \frac{1}{\rho^{\frac{2n}{n-2}}G}\sum_{k,l=1}^{n}\partial_k\left(\frac{1}{\rho^{\frac{4}{n-2}}}g^{kl}\rho^{\frac{2n}{n-2}}G\partial_l(h)\right)\\ &= \frac{\rho^{\frac{2n-4}{n-2}}}{\rho^{\frac{2n}{n-2}}}\sum_{k=1}^{n}g^{kl}\partial_k(\partial_l(h)) + \frac{1}{\rho^{\frac{2n}{n-2}}}\frac{2n-4}{n-2}\rho^{\frac{2n-4}{n-2}-1}\sum_{k=1}^{n}g^{kl}\partial_k(\rho)\partial_l(h)\\ &= \frac{1}{\rho^{\frac{4}{n-2}}}\Delta h + \frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}}g(\nabla\rho,\nabla h).\end{aligned} \tag{4.13}$$

Considering (4.9) and (4.10) into (4.19), we get

$$\begin{aligned}\hat{\Delta}h &= \frac{1}{\rho^{\frac{4}{n-2}}}\left(\frac{-2}{\rho}\overline{g}(\nabla\rho,\nabla h) - \frac{c}{\rho^2}\Theta^2\right) + \frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}}\overline{g}(\nabla\rho,\nabla h)\\ &= \left(\frac{-4}{2\rho^{\frac{n+2}{n-2}}} + \frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}}\right)\overline{g}(\nabla\rho,\nabla h) - \frac{c}{\rho^{\frac{2n}{n-2}}}\Theta^2\\ &= -\frac{c}{\rho^{\frac{2n}{n-2}}}\Theta^2.\end{aligned} \tag{4.14}$$

At this point, taking into account (4.14), we define the smooth function u over M^n by

$$u = \begin{cases} t_* - h & (\text{when } c \geq 0)\\ h - t_* & (\text{when } c \leq 0)\end{cases},$$

and the vector field $X = \hat{\nabla}u$, from (4.14) we get that

$$\operatorname{div}_{\hat{g}}X \geq 0. \tag{4.15}$$

Moreover, we have

$$\hat{g}(\hat{\nabla}u, X) = |\hat{\nabla}u|^2_{\hat{g}} \geq 0. \tag{4.16}$$

In addition, since h converges to t_* at infinity, we have that u is a nonnegative non-identically vanishing function which converges to zero (also related to the metric $\hat{g}$, since ρ is bounded on Σ^n). Thus, from (4.15) and (4.16) we can apply Lemma 1.5.6 to get that $\hat{g}(\hat{\nabla}u, X)$ is identically zero on Σ^n. Hence, returning to (4.16) we conclude that $\hat{\nabla}h$ vanishes identically on Σ^n, which means that h is constant and (since it converges to t_* at infinity) Σ^n must be the slice M_{t_*}. □

4.3 Uniqueness and nonexistence results under integrability properties

Now, we are in position to present our first uniqueness result concerning spacelike mean curvature flow solitons in a standard static spacetime.

Theorem 4.3.1. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a standard static spacetime with complete Riemannian base M^n and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with respect to K, with soliton constant $c \leq 0$ (resp. $c \geq 0$). Suppose that $h \geq 0$ (resp. $h \leq 0$) and that ρ is bounded along Σ^n. If $h \in \mathcal{L}^p_g(\Sigma)$ for some $p > 1$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. In local coordinates $(x_1, \cdots, x_n)$ of Σ^n, the Laplacian of its height function on the metric $\hat{g}$ is given by

$$\hat{\Delta} h = \frac{1}{\hat{G}} \sum_{k,l=1}^{n} \partial_k \left(\hat{g}^{kl} \hat{G} \partial_l(h) \right), \tag{4.17}$$

where $\hat{g}_{kl} = \hat{g}(\partial_k, \partial_l)$, $\hat{G} = \sqrt{\det\left(\hat{g}_{kl}\right)}$ and $\left(\hat{g}^{kl}\right) = \left(\hat{g}_{kl}\right)^{-1}$.

But, since $\hat{g} = \rho^{\frac{4}{n-2}} g$, we have that $\hat{g}_{kl} = \rho^{\frac{4}{n-2}} g_{kl}$, $\hat{g}^{kl} = \frac{1}{\rho^{\frac{4}{n-2}}} g^{kl}$ and

$$\hat{G} = \sqrt{\det(\hat{g}_{kl})} = \sqrt{\rho^{\frac{4n}{n-2}} \det(g_{kl})} = \rho^{\frac{2n}{n-2}} G. \tag{4.18}$$

Thus, from (4.17) and (4.18) we obtain

$$\begin{aligned} \hat{\Delta} h &= \frac{1}{\rho^{\frac{2n}{n-2}} G} \sum_{k,l=1}^{n} \partial_k \left(\frac{1}{\rho^{\frac{4}{n-2}}} g^{kl} \rho^{\frac{2n}{n-2}} G \partial_l(h) \right) \\ &= \frac{\rho^{\frac{2n-4}{n-2}}}{\rho^{\frac{2n}{n-2}}} \sum_{k=1}^{n} \partial_k(\partial_k(h)) + \frac{1}{\rho^{\frac{2n}{n-2}}} \frac{2n-4}{n-2} \rho^{\frac{2n-4}{n-2}-1} \sum_{k=1}^{n} \partial_k(\rho) \partial_k(h) \\ &= \frac{1}{\rho^{\frac{4}{n-2}}} \Delta h + \frac{2}{\rho^{\frac{n+2}{n-2}}} g(\nabla \rho, \nabla h). \end{aligned} \tag{4.19}$$

Inserting (4.9) and (4.10) into (4.19), we get

$$\hat{\Delta} h = \frac{1}{\rho^{\frac{4}{n-2}}} \left(-\frac{2}{\rho} g(\nabla \rho, \nabla h) - \frac{c}{\rho^2} \Theta^2 \right) + \frac{2}{\rho^{\frac{n+2}{n-2}}} g(\nabla \rho, \nabla h). \tag{4.20}$$

Hence, (4.20) allows us to the following formula

$$\hat{\Delta} h = -\frac{c}{\rho^{\frac{2n}{n-2}}} \Theta^2. \tag{4.21}$$

Consequently, since we are assuming that $h \geq 0$ (resp. $h \leq 0$) and $c \leq 0$ (resp. $c \geq 0$), from (4.21) we conclude that h (resp. $-h$) is a subharmonic function with respect to the metric $\widehat{g}$. Furthermore, since we are also supposing that ρ is bounded along Σ^n, from (4.18) we have that our hypothesis $h \in \mathcal{L}^p_g(\Sigma)$ implies that $h \in \mathcal{L}^p_{\hat{g}}(\Sigma)$. Therefore, we can apply Lemma 1.5.4 to guarantee that h is constant on Σ^n, that is, Σ^n must be a slice of $\overline{M}^{n+1}$. □

Theorem 4.3.1 jointly with Lemma 1.5.5 lead us to establish the following nonexistence result.

Theorem 4.3.2. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a standard static spacetime whose Riemannian base M^n is complete noncompact with nonnegative Ricci curvature, and having bounded warping* *function ρ. There is no complete spacelike mean curvature flow soliton with respect to K, with soliton constant $c \leq 0$ (resp. $c \geq 0$) and positive (resp. negative) height function satisfying $h \in \mathcal{L}^p_g(\Sigma)$ for some $p > 1$.*

Proof. Let us suppose the existence of such a spacelike mean curvature flow soliton, namely $\psi : \Sigma^n \to \overline{M}^{n+1}$. From Theorem 4.3.1, we get that Σ^n is a slice of $\overline{M}^{n+1}$. Consequently, $|h|$ must be equal to a positive constant α and, since we are assuming that $h \in \mathcal{L}^p_g(\Sigma)$, we obtain

$$\mathrm{vol}_{g_M}(M) = \mathrm{vol}_g(\Sigma) = \frac{1}{\alpha^p}\int_\Sigma |h|^p d_g\Sigma < +\infty. \tag{4.22}$$

On the other hand, taking into account that M^n is complete noncompact with nonnegative Ricci curvature, Lemma 1.5.5 assures that M^n has at least linear volume growth, which corresponds to a contradiction with (4.22). □

According to Definition 1 of [52], we say that a smooth Riemannian manifold (Σ^n, g) satisfies the $\mathcal{L}^1_g$*-Liouville property*, when every nonnegative superharmonic function $u \in \mathcal{L}^1_g(\Sigma)$ must be constant. Corollary 3 of [52] ensures that a stochastically complete manifold (and, in particular, a parabolic manifold) always satisfies the $\mathcal{L}^1_g$-Liouville property. However, in Section 2 of [52] the authors constructed nontrivial examples of stochastically incomplete (and, in particular, nonparabolic) manifolds satisfying the $\mathcal{L}^1_g$-Liouville property.

Motivated by these observations, it is not difficult to see that we can reason in a similar way as in the proof of Theorem 4.3.1 to get the following result

Theorem 4.3.3. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a standard static spacetime and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a spacelike mean curvature flow soliton with respect to K, with soliton constant $c \geq 0$ (resp. $c \leq 0$). Suppose that $h \geq 0$ (resp. $h \leq 0$) and that ρ is bounded along Σ^n. If Σ^n satisfies the $\mathcal{L}^1_g$-Liouville property and $h \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

In what follows, recall that a *timelike bounded region* $\mathcal{B}_{t_1,t_2}$ of $\overline{M}^{n+1}$ which is defined by

$$\mathcal{B}_{t_1,t_2} := \{(p,t) \in M^n \times_\rho \mathbb{R}_1 : t_1 \leq t \leq t_2 \text{ and } p \in M^n\}.$$

We close this section presenting our second uniqueness result concerning spacelike mean curvature flow solitons.

Theorem 4.3.4. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a standard static spacetime with complete Riemannian base M^n and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with respect to K, with soliton constant c and lying in a timelike bounded region $\mathcal{B}_{t_1,t_2}$ of $\overline{M}^{n+1}$. Suppose in addition that ρ is bounded along Σ^n. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. Taking local coordinates $(x_1, \cdots, x_n)$ in Σ^n and using that $\hat{g}^{kl} = \frac{1}{\rho^{\frac{4}{n-2}}} g^{kl}$, we get

$$\widehat{\nabla} h = \sum_{k,l=1}^n \widehat{g}^{kl}\partial_l(h)\partial_k = \frac{1}{\rho^{\frac{4}{n-2}}}\nabla h. \tag{4.23}$$

Consequently, since ρ is bounded along Σ^n, from (4.18) and (4.23) we obtain

$$\int_\Sigma |\widehat{\nabla} h|_{\hat{g}} d_{\hat{g}}\Sigma = \int_\Sigma \rho^{\frac{2(n-1)}{n-2}} |\nabla h| d_g \Sigma \leq \left(\sup_\Sigma \rho^{\frac{2(n-1)}{n-2}} \right) \int_\Sigma |\nabla h| d_g \Sigma. \tag{4.24}$$

Thus, since we are supposing that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, from (4.24) we conclude that $|\widehat{\nabla} h|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. So, taking into account (4.21), we can apply Lemma 1.5.3 to get that $\hat{\Delta} h$ vanishes identically on Σ^n.

On the other hand, we have that $|\widehat{\nabla} h^2|_{\hat{g}} = 2|h||\widehat{\nabla} h|_{\hat{g}}$. Thus, assuming that Σ^n lies in $\mathcal{B}_{t_1,t_2}$, we also obtain that $|\widehat{\nabla} h^2|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. Moreover, we have that

$$\hat{\Delta} h^2 = 2h\hat{\Delta} h + 2|\widehat{\nabla} h|^2_{\hat{g}} = 2|\widehat{\nabla} h|^2_{\hat{g}} \geq 0. \tag{4.25}$$

Hence, we can apply once more Lemma 1.5.3 to infer that $\hat{\Delta} h^2 = 0$ on Σ^n and, returning to (4.25), conclude that $|\widehat{\nabla} h|_{\hat{g}}$ is identically zero on Σ^n. Therefore, Σ^n must be a slice of $\overline{M}^{n+1}$. □

We recall that a spacetime is called *spatially closed* when it admits a closed (that is, compact without boundary) spacelike hypersurface. In this context, from Theorem 4.3.4 we get the following rigidity result

Corollary 4.3.5. *The only closed spacelike mean curvature flow soliton with respect to the timelike Killing vector field of a spatially closed standard static spacetime are the totally geodesic slices.*

4.4 New Calabi-Bernstein type results

In this last section of chapter we will use the theorems of the previous section in order to establish new Calabi-Bernstein type results concerning entire graphs constructed over the Riemannian base of a standard static spacetime $M^n \times_\rho \mathbb{R}_1$ and which are spacelike mean curvature flow solitons with respect to K. For this, we need to recall some basic facts related to these graphs.

According to [71], we define the *entire graph* $\Sigma(u)$ associated to a smooth function $u \in C^\infty(M)$ as the hypersurface given by

$$\Sigma(u) = \{\Psi(x, u(x)) : x \in M^n\} \subset M^n \times_\rho \mathbb{R}_1,$$

where $\Psi : M^n \times \mathbb{I} \to \overline{M}^{n+1}$ is the flow generated by the timelike Killing vector field K. The metric induced on M^n from the Lorentzian metric (1.15) via $\Sigma(u)$ is given by

$$g_u = g_M - \rho^2 du^2. \tag{4.26}$$

Remark 4.4.1. The entire graph $\Sigma(u)$ is spacelike if, and only if, $\rho^2|Du|^2_M < 1$, where Du denotes the gradient of a function u with respect to the metric g_M of M^n. Indeed, if $\Sigma(u)$ is spacelike, then from (4.26) we have

$$0 < g_u(Du, Du) = g_M(Du, Du) - \rho^2 g_M(Du, Du)^2.$$

Hence, we conclude that $\rho^2|Du|_M^2 < 1$. Conversely, if $\rho^2|Du|_M^2 < 1$ and X is a vector field tangent to $\Sigma(u)$, from (4.26) jointly with Cauchy-Schwarz inequality we obtain

$$g_u(X, X) = g_M(X^*, X^*) - \rho^2 g_M(Du, X^*)^2 \geq g_M(X^*, X^*)(1 - \rho^2|Du|_M^2), \tag{4.27}$$

where X^* is the orthogonal projection of X onto TM. Thus from (4.27) we get that $g_u(X, X) \geq 0$ and $g_u(X, X) = 0$ if, and only if, $X = 0$.

Now, let us consider the function $F : \overline{M}^{n+1} \to \mathbb{R}$ given by $F(x, t) = u(x) - t$. We have that $\Sigma(u) = \Psi(F^{-1}(0))$. Thus, for all vector field X tangent to $\overline{M}^{n+1}$, we get

$$X(F) = X^*(F) - \frac{1}{\rho^2}\bar{g}(X, \partial_t)\partial_t(F) = \bar{g}(\frac{1}{\rho^2}\partial_t + Du, X).$$

Then

$$\overline{\nabla} F = \frac{1}{\rho^2}\partial_t + Du$$

is a normal vector field on $F^{-1}(0)$. So, we claim that

$$N_0 = \Psi_*(\overline{\nabla} F) = \frac{1}{\rho^2}K + \Psi_*(Du) \tag{4.28}$$

is a normal vector field on $\Sigma(u)$. To show our claim, we define the map $\Psi^u : M^n \to \overline{M}^{n+1}$ by

$$\Psi^u(x) = (x, u(x)).$$

For $v \in TM$, we have that $\Psi^u_*(v)$ is tangent to $\Sigma(u)$ and, consequently,

$$\Psi^u_*(v) = \Psi_*(v) + v(u)\partial_t. \tag{4.29}$$

Now, we must verify that $\bar{g}(N_0, \Psi^u_*(v)) = 0$. Indeed, taking into account that $\bar{g}(\Psi_*(Du), \partial_t) = 0$, from (4.28) and (4.29) we get

$$\begin{aligned} \bar{g}(N_0, \Psi^u_*(v)) &= \frac{v(u)}{\rho^2}\bar{g}(\partial_t, \partial_t) + \bar{g}(\Psi_*(Du), \Psi_*(v)) \\ &= v(u) - v(u) = 0. \end{aligned}$$

Hence, since

$$|N_0| = \frac{(1 - \rho^2|Du|_M^2)^{1/2}}{\rho}, \tag{4.30}$$

it follows from (4.28) and (4.30) that the unit vector field

$$N = \frac{1}{\rho(1-\rho^2|Du|_M^2)^{1/2}}(K + \rho^2\Psi_*(Du)) \tag{4.31}$$

defines the future-pointing Gauss map of $\Sigma(u)$ and its corresponding angle function is given by

$$\Theta = g(N, K) = -\frac{\rho}{(1-\rho^2|Du|_M^2)^{1/2}}. \tag{4.32}$$

Moreover, for all vector field X tangent to M^n, the Weingarten endomorphism A of $\Sigma(u)$ with respect to N is given by

$$\begin{aligned} AX \;=\; & -\frac{\rho}{(1-\rho^2|Du|_M^2)^{1/2}} D_X Du - \frac{\rho^3 g(D_X Du, Du)}{(1-\rho^2|Du|_M^2)^{3/2}} Du - \frac{\rho^2 g(D\rho, X)|Du|_M^2}{(1-\rho^2|Du|_M^2)^{3/2}} Du \\ & -\frac{g(D\rho, X)}{(1-\rho^2|Du|_M^2)^{1/2}} Du - \frac{g(Du, X)}{(1-\rho^2|Du|_M^2)^{1/2}} D\rho. \end{aligned} \tag{4.33}$$

So, it follows from (4.33) that the mean curvature H_u of a spacelike entire graph $\Sigma(u)$ is given by

$$H_u = \mathrm{Div}_M\left(\frac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) + \frac{g(Du, D\rho)}{(1-\rho^2|Du|_M^2)^{1/2}}, \tag{4.34}$$

where Div_M stands for the divergence operator on M^n with respect to its metric g_M.

Hence, from (4.10) and (4.34) we have that $\Sigma(u)$ is a SMCFS with respect to K with soliton constant c if, and only if, $\rho|Du|_M < 1$ and u is a solution of the following nonlinear differential equation:

$$\mathrm{Div}_M\left(\frac{\rho Du}{(1-\rho^2|Du|_M^2)^{1/2}}\right) = -\frac{1}{(1-\rho^2|Du|_M^2)^{1/2}}\big(c\rho + g(Du, D\rho)\big). \tag{4.35}$$

Our next result corresponds to a nonparametric version of Theorem 4.2.2.

Corollary 4.4.2. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}_1$ be a standard static spacetime with complete noncompact Riemannian base M^n and whose warping function ρ is bounded. The only smooth function $u : M^n \to I$ which is solution of the mean curvature flow soliton equation (4.35) for some $c \geq 0$ (resp. $c \leq 0$) with $\rho|Du|_M \leq \lambda$, for some constant $0 < \lambda < 1$, and such that u converges from below (resp. above) to some $t_* \in I$ at infinity is the constant function $u \equiv t_*$.*

Proof. Since we are supposing that $\rho|Du|_M \leq \lambda$ for some constant $0 < \lambda < 1$, from (4.32) we get that Θ is bounded. Thus, [125, Lemma 19] assures that the entire spacelike graph $\Sigma(u)$ is, in fact, complete with respect to its induced metric from $\overline{M}^{n+1}$. Therefore, the result follows by applying Theorem 4.2.2 to conclude that $u \equiv t_*$. □

Remark 4.4.3. Consider the spacelike surface

$$\Sigma(u) = \{(x, y, u(x, y)) : y > 0\} \subset \mathbb{H}^2 \times \mathbb{R}_1,$$

where $u(x, y) = c \ln y$, $c \in \mathbb{R}$ is a constant such that $0 < |c| < 1$, $\mathbb{H}^2$ denotes the two-dimensional hyperbolic space equipped with its standard metric. From [76, Example 4.4], we have that $\Sigma(u)$

is a complete spacelike translating soliton with $|Du|_{\mathbb{H}^2} = |c|$ and constant mean curvature $H = \frac{c}{\sqrt{1-c^2}} = c\,\Theta$, with respect to orientation (4.31). Hence, we also conclude that in Corollary 6.5.2 the hypothesis that the function u converges to some $t_* \in I$ at infinity is necessary to infer that u must be constant.

Part II

Rigidity of hypersurfaces in certain warped products and results for submanifolds in weighted products

Chapter 5

Preliminaries for Part II

In this chapter we shall briefly introduce some basic facts and notations that will appear along Part II of this thesis.

5.1 Two-sided hypersurfaces in a warped product

Let (M, g_M) be an n-dimensional $(n \geq 2)$ connected Riemannian manifold and let $I \subset \mathbb{R}$ be an open interval in $\mathbb{R}$ endowed with the metric dt^2. The product manifold $\overline{M}^{n+1} = I \times M^n$ endowed with the Riemannian metric

$$\overline{g} = \pi_I^*(dt^2) + \rho(\pi_I)^2 \pi_M^*(g_M), \tag{5.1}$$

where ρ is a positive smooth function on I, the applications π_I and π_M denote the projections onto I and M, respectively, is called a warped product with fiber (M, g_M), base (I, dt^2) and warping function ρ. In such a case, we simply write $\overline{M}^{n+1} = I \times_\rho M^n$.

In this setting, we will consider the conformal closed vector field $K = \rho(\pi_I)\partial_t$ globally defined on $\overline{M}$, where $\partial_t = \frac{\partial}{\partial_t}$ stands for the unit coordinate vector field tangent to I. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of the base and the fiber (see Proposition 7.35 of [123]) , it follows that

$$\overline{\nabla}_X K = \rho'(\pi_I)X, \tag{5.2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$.

Throughout this work, we will deal with connected two-sided hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in $\overline{M}^{n+1} = I \times_\rho M^n$, which means that its normal bundle is trivial, that is, there is a globally defined unit normal vector field $N \in T\Sigma^\perp$ on it. We will also assume that Σ^n is transversal to K at every point and we will denote by g its induced metric. In this setting, we will consider the shape operator (or Weingarten endomorphism) of Σ^n, $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which is given by $A(X) = -\overline{\nabla}_X N$, and its mean curvature function $H = \frac{1}{n}\mathrm{tr}(A)$.

In the warped product $\overline{M}^{n+1} = I \times_\rho M^n$ there exists a remarkable family of two-sided hypersurfaces: its slices $M_{t_0} = \{t_0\} \times M$, with $t_0 \in I$. The shape operator and the mean curvature of M_{t_0} with respect to $N = -\partial_t$ are, respectively, $A_{t_0} = \frac{\rho'(t_0)}{\rho(t_0)} I$, where I denotes the identity operator, and $H_{t_0} = \frac{\rho'(t_0)}{\rho(t_0)}$.

We will deal with two particular functions naturally attached to a two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, the (vertical) height function $h = \pi_I \circ \psi$ and the angle function $\Theta = \overline{g}(N, \partial_t)$. The transversality condition above, together with the connectedness of Σ^n, gives that Θ does not change sign on Σ^n.

Let us denote by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics $\overline{g}$ and g, respectively. Then, a simple computation shows that the gradient of π_I on M^n is given by

$$\overline{\nabla}\pi_I = \overline{g}(\overline{\nabla}\pi_I, \partial_t)\partial_t = \partial_t$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \partial_t^\top, \tag{5.3}$$

where $\partial_t^\top = \partial_t - \Theta N$ is the tangential component of ∂_t along Σ^n. From (5.3) we deduce that

$$|\nabla h|^2 + \Theta^2 = 1, \tag{5.4}$$

where ∇h is the gradient of h in the metric g and $|X|^2 = g(X, X)$ for any $X \in \mathfrak{X}(\Sigma)$. Moreover, from (5.2) and (5.3) we deduce that the Hessian of h in the metric g is given by

$$\begin{aligned} \nabla^2 h(X, X) &= g(\nabla_X \partial_t^\top, X) \\ &= \overline{g}(\overline{\nabla}_X(\partial_t - \Theta N), X) \\ &= \frac{\rho'(h)}{\rho(h)}(|X|^2 - g(\nabla h, X)^2) + g(AX, X)\Theta, \end{aligned} \tag{5.5}$$

for any $X \in \mathfrak{X}(\Sigma)$. Hence, from (5.5) we obtain that the Laplacian of h in the metric g is

$$\Delta h = \frac{\rho'(h)}{\rho(h)}\left(n - |\nabla h|^2\right) + nH\Theta. \tag{5.6}$$

5.2 Mean curvature flow solitons

We recall that the mean curvature flow $\Psi : [0, T) \times \Sigma^n \to \overline{M}^{n+1}$ of an immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ in a $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$, satisfying $\Psi(0, \cdot) = \psi(\cdot)$, looks for solutions of the equation

$$\frac{\partial \Psi}{\partial t} = \vec{H},$$

where $\vec{H}(t, \cdot)$ is the (non-normalized) mean curvature vector of $\Sigma_t^n = \Psi(t, \Sigma^n)$. In our context, according to [27, Definition (1.1)], a two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a warped product $\overline{M}^{n+1} = I \times_\rho M^n$ is said a *mean curvature flow soliton* with respect to $K = \rho(t)\partial_t$

with *soliton constant* $c \in \mathbb{R}$ if its (non-normalized) mean curvature function satisfies

$$H = c\rho(h)\Theta. \tag{5.7}$$

Adopting the terminology introduced in [27], we will also consider the *soliton function*

$$\zeta_c(t) = n\rho'(t) + c\rho(t)^2. \tag{5.8}$$

As it was observed in [27], a slice $M_{t_*} = \{t_*\} \times M^n$ is a mean curvature flow soliton with respect to $K = f(t)\partial_t$ and with soliton constant c given by

$$c = -n\frac{\rho'(t_*)}{\rho(t_*)^2}. \tag{5.9}$$

Moreover, t_* is implicitly given by the condition $\zeta_c(t_*) = 0$.

The following cites important examples which will be addressed along the next two sections. In the first one, we consider a suitable warped product model for the Euclidean space minus a point.

Example 5.2.1. *Let $o = (0, \ldots, 0)$ be the origin of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We have that $\mathbb{R}^{n+1} \setminus \{o\}$ is isometric to $\mathbb{R}_+ \times_t \mathbb{S}^n$ (see* [115, Section 4, Example 1]*), whose slices $\{t\} \times \mathbb{S}^n$ are isometric to n-dimensional Euclidean spheres $\mathbb{S}^n(t)$ of radius $t \in \mathbb{R}_+$. In this setting, the mean curvature flow solitons with respect to $K = t\partial_t$ with soliton constant $c = -1$ are just the self-shrinkers. So, from* (5.9) *we conclude that $\mathbb{S}^n(\sqrt{n}) \equiv \{\sqrt{n}\} \times \mathbb{S}^n$ is the only slice which is a self-shrinker.*

In our next example, we consider a suitable warped product model for the real projective space.

Example 5.2.2. *We recall that the $(n+1)$-dimensional real projective space is given by the quotient $\mathbb{RP}^{n+1} = \mathbb{S}^{n+1}/\{\pm 1\}$, where $\{\pm 1\}$ is the group of diffeomorphisms of $(n+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{n+1}$ consisting of the identity map $q \mapsto q$ and the antipodal map $q \mapsto -q$. We consider the Riemannian metric in $\mathbb{RP}^{n+1}$ in such a way that the natural projection $\pi : \mathbb{S}^{n+1} \to \mathbb{RP}^{n+1}$ becomes a local isometry. If P stands for the north pole of $\mathbb{S}^{n+1}$, then we denote by Cut_P the cut locus of $\pi(P) \in \mathbb{RP}^{n+1}$. We have that Cut_P is the image of the equator of $\mathbb{S}^{n+1}$ orthogonal to P via the natural projection, namely, $Cut_P = \pi(\mathbb{S}^n) = \mathbb{RP}^n$. Moreover, as it was proved in* [53, Section 9.111]*, $\mathbb{RP}^{n+1} \setminus \{\pi(P) \cup Cut_P\}$ is isometric to the warped product $\left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$. From* (5.9) *we conclude that the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$ is the only one that is a mean curvature flow soliton with respect to $K = \sin t\partial_t$ with soliton constant $c < 0$.*

Proceeding, we consider the so-called pseudo-hyperbolic spaces.

Example 5.2.3. *According to* [144]*, warped products of the type $I \times_{e^t} M^n$ are called* pseudo-hyperbolic spaces*. This terminology is due to the fact that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is isometric to the warped product $\mathbb{R} \times_{e^t} \mathbb{R}^n$, where the slices constitute a family of* *horospheres sharing a same fixed point in the asymptotic boundary $\partial_\infty \mathbb{H}^{n+1}$ and giving a complete foliation of $\mathbb{H}^{n+1}$ (for more details about pseudo-hyperbolic spaces see, for instance, [24,115,144]). From (5.9) we conclude that the slice $\{\log(-\frac{n}{c})\} \times M^n$ is the only one that is a mean curvature flow soliton with respect to $K = e^t\partial_t$ with soliton constant $c < 0$.*

In our last examples, we deal with the Schwarzschild and Reissner-Nordström spaces.

Example 5.2.4. *Given a mass parameter $\mathfrak{m} > 0$, the* Schwarzschild space *is defined to be the product $\overline{M}^{n+1} = (r_0(\mathfrak{m}), +\infty) \times \mathbb{S}^n$ furnished with the metric $\bar{g} = V_{\mathfrak{m}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$, $V_{\mathfrak{m}}(r) = 1 - 2\mathfrak{m}r^{1-n}$ stands for its potential function and $r_0(\mathfrak{m}) = (2\mathfrak{m})^{1/(n-1)}$ is the unique positive root of $V_{\mathfrak{m}}(r) = 0$. Its importance lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m}}(r)dt^2 + \bar{g}$ is a solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance, [123,* Chapter 13] *for more details concerning Schwarzschild geometry).*

As it was observed in [69, Example 1.3], *$\overline{M}^{n+1}$ can be reduced in the form $I \times_f \mathbb{S}^n$ with metric (5.1) via the following change of variables:*

$$t = \int_{r_0(\mathfrak{m})}^{r} \frac{d\sigma}{\sqrt{V_{\mathfrak{m}}(\sigma)}}, \qquad f(t) = r(t), \qquad I = \mathbb{R}_+. \tag{5.10}$$

As it was noted in [69, Example 4.1], *since $V_{\mathfrak{m}}(r)$ is strictly increasing on $(r_0(\mathfrak{m}), +\infty)$, it follows from (5.10) that the warping function f satisfies:*

$$f'(t) = \frac{dr}{dt} = \sqrt{V_{\mathfrak{m}}(r(t))} > 0 \qquad \text{and} \qquad f''(t) = \frac{1}{2}\frac{dV_{\mathfrak{m}}}{dr}(r(t)) > 0. \tag{5.11}$$

Thus, from (5.9) and (5.11) we can verify that a slice $\{t_*\} \times \mathbb{S}^n$ is a mean curvature flow soliton with respect to $f(t)\partial_t = r\sqrt{V_{\mathfrak{m}}(r)}\partial_r$ with soliton constant $c < 0$ when $t_* = t(r_*)$ with $r_* > r_0(\mathfrak{m})$ solving the following equation

$$V_{\mathfrak{m}}(r) = \frac{c^2}{n^2}r^4. \tag{5.12}$$

We note that such a solution exists if and only if the function $\varphi_{\mathfrak{m}}(t) = \frac{c^2}{n^2}t^4 + \frac{2\mathfrak{m}}{t^{n-1}} - 1$ has a zero on $(r_0(\mathfrak{m}), +\infty)$. Notice that $\varphi_{\mathfrak{m}}$ is a convex function which goes to infinity if t goes to 0 or $+\infty$ and so $\varphi_{\mathfrak{m}}$ has a unique minimal point in $(0, \infty)$. Such value $\hat{r}$ is given implicitly by $\varphi'_{\mathfrak{m}}(\hat{r}) = 0$, that is,

$$\frac{4c^2}{n^2}\hat{r}^3 - \frac{2\mathfrak{m}(n-1)}{\hat{r}^n} = 0.$$

Therefore, the equation (5.12) has a solution if and only if $\hat{r} > r_0(\mathfrak{m})$ and $\varphi_{\mathfrak{m}}(\hat{r}) \leq 0$. The last condition can be rewritten in the following way:

$$\hat{r} = \left(\frac{\mathfrak{m}(n-1)n^2}{2c^2}\right)^{1/(n+3)} \geq \left(\frac{\mathfrak{m}(n+3)}{2}\right)^{1/(n-1)}. \tag{5.13}$$

In particular, there are two solutions $r_0(\mathfrak{m}) < r_{*,-} < \hat{r} < r_{*,+}$ if the strict inequality holds in (5.13), and a unique solution $r_* = \hat{r}$ if equality holds.

Example 5.2.5. *Given a mass parameter* $\mathfrak{m} > 0$ *and an electric charge* $\mathfrak{q} \in \mathbb{R}$*, with* $|\mathfrak{q}| \leq \mathfrak{m}$*, the* Reissner-Nordström space *is defined to be the product* $\overline{M}^{n+1} = (r_0(\mathfrak{m}, \mathfrak{q}), +\infty) \times \mathbb{S}^n$ *endowed with the metric* $\bar{g} = V_{\mathfrak{m},\mathfrak{q}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$*, where* $g_{\mathbb{S}^n}$ *is the standard metric of* $\mathbb{S}^n$*,* $V_{\mathfrak{m},\mathfrak{q}}(r) = 1 - 2\mathfrak{m}r^{1-n} + \mathfrak{q}^2 r^{2-2n}$ *stands for its potential function and* $r_0(\mathfrak{m}, \mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}}\right)^{1/(n-1)}$ *is the largest positive zero of* $V_{\mathfrak{m},\mathfrak{q}}(r)$*. The importance of this model lies in the fact that the manifold* $\mathbb{R} \times \overline{M}^{n+1}$ *equipped with the Lorentzian static metric* $-V_{\mathfrak{m},\mathfrak{q}}(r)dt^2 + \bar{g}$ *is a charged black-hole solution of the Einstein field equation in vacuum with zero cosmological constant.*

As in the previous example, $\overline{M}^{n+1}$ *can be reduced in the form* $I \times_f \mathbb{S}^n$ *with metric* (5.1) *via the same change of variables as in* (5.10)*. Furthermore, following the same previous steps, the warping function* f *has positive first and second derivatives. Moreover, we can verify that a slice* $\{t_*\} \times \mathbb{S}^n$ *is a mean curvature flow soliton with respect to* $f(t)\partial_t = r\sqrt{V_{\mathfrak{m},\mathfrak{q}}(r)}\partial_r$ *with soliton constant* $c < 0$ *when* $t_* = t(r_*)$ *with* $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ *solving the following equation*

$$V_{\mathfrak{m},\mathfrak{q}}(r) = \frac{c^2}{n^2} r^4. \tag{5.14}$$

We observe that such a case is more complicated to explicit all the values, but qualitatively we can say that such a solution of (5.14) *exists if and only if the function* $\varphi_{\mathfrak{m},\mathfrak{q}}(x) = \frac{c^2}{n^2}x^4 + \frac{2\mathfrak{m}}{x^{n-1}} - \frac{\mathfrak{q}^2}{x^{2n-2}} - 1$ *has a zero on* $(r_0(\mathfrak{m}), +\infty)$*. Note that* $\varphi_{\mathfrak{m},\mathfrak{q}}$ *goes to positive infinity if* x *goes to positive infinity and* $\varphi_{\mathfrak{m},\mathfrak{q}}$ *goes to negative infinity if* x *goes to zero. So,* $\varphi_{\mathfrak{m},\mathfrak{q}}$ *has at least one root in* $(0, +\infty)$ *and if such roots are greater than* $r_0(\mathfrak{m}, \mathfrak{q})$ *we get the desired solutions* r_**.*

5.3 Hypersurfaces immersed in Riemannian manifold endowed with a Killing vector field

Let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold endowed with a nowhere zero Killing vector field K. Suppose that the distribution $\mathcal{D}$ orthogonal to K is integrable. We denote by $\Psi : M^n \times \mathbb{I} \to \overline{M}^{n+1}$ the flow generated by K, where M^n is an arbitrarily fixed integral leaf of $\mathcal{D}$ labeled as $t = 0$, which we will suppose to be connected, and $\mathbb{I}$ is the maximal interval of definition. Without lost of generality, in what follows we will consider $\mathbb{I} = \mathbb{R}$.

In this setting, $\overline{M}^{n+1}$ can be regard as the warped product $M^n \times_\rho \mathbb{R}$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the warping metric

$$g = \pi_M^\star(g_M) + (\rho \circ \pi_M)^2 \pi_{\mathbb{R}}^\star(dt^2). \tag{5.15}$$

Here, π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M^n \times \mathbb{R}$ onto each factor, g_M is the induced Riemannian metric on the Riemannian base M^n, $\mathbb{R}$ is endowed with its usual metric dt^2 and the warping function $\rho \in C^\infty(M)$ is given by $\rho = |K| > 0$, where $|\ |$ denotes the norm of a vector field on $\overline{M}^{n+1}$.

Let us consider a connected *two-sided* hypersurface $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ immersed into such a warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$, which means that there exists a globally defined unit normal vector field N on Σ^n. In particular, for each $t \in \mathbb{R}$, the two-side hypersurface $M^n \times \{t\}$ oriented by $N = \frac{K}{\rho}$ is called a *slice* of $\overline{M}^{n+1}$, which is a totally geodesic.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections in $\overline{M}^{n+1}$ and Σ^n, respectively. Thus, also denoting by g the induced metric of Σ^n, its Gauss and Weingarten formulas are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N \tag{5.16}$$

and

$$AX = -\overline{\nabla}_X N, \tag{5.17}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma^n)$. Here $A : \mathfrak{X}(\Sigma^n) \to \mathfrak{X}(\Sigma^n)$ stands for the shape operator (or Weingarten endomorphism) of Σ^n with respect to N.

For our purposes, we will consider two particular smooth functions on Σ^n, namely, the (vertical) height function $h = \pi_{\mathbb{R}} \circ \psi$ and the angle function $\Theta = g(N, K)$. From the decomposition $K = K^\top + \Theta N$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along ψ, we obtain

$$\nabla h = \frac{1}{\rho^2} K^\top \quad \text{and} \quad |\nabla h|^2 = \frac{\rho^2 - \Theta^2}{\rho^4}. \tag{5.18}$$

Using once more the decomposition $K = K^\top + \Theta N$, from (5.16) and (5.17) we get that

$$\nabla_X K^\top = (\overline{\nabla}_X K)^\top + \Theta AX. \tag{5.19}$$

Consequently, from (5.18) and (5.16) we have that the Hessian of h is given by

$$\begin{aligned} \nabla_X \nabla h &= \nabla_X \left(\frac{1}{\rho^2} K^\top \right) \\ &= -\frac{2}{\rho^3} g(\nabla \rho, X) K^\top + \frac{1}{\rho^2} (\overline{\nabla}_X K)^\top + \frac{1}{\rho^2} \Theta AX. \end{aligned} \tag{5.20}$$

So, taking a local orthonormal tangent frame $\{e_1, \ldots, e_n\}$ on Σ^n, from (5.20) we obtain

$$\begin{aligned} \Delta h &= \sum_{i=1}^n g\left(-\frac{2}{\rho^3} g(\nabla \rho, e_i) K^\top + \frac{1}{\rho^2} (\overline{\nabla}_{e_i} K)^\top + \frac{1}{\rho^2} \Theta A e_i, e_i \right) \\ &= -\frac{2}{\rho} g(\nabla \rho, \nabla h) + \sum_{i=1}^n \frac{1}{\rho^2} g(\overline{\nabla}_{e_i} K, e_i) + \frac{1}{\rho^2} \Theta H, \end{aligned} \tag{5.21}$$

where $H = \mathrm{tr}(A)$ is the non-normalized mean curvature of Σ^n with respect to N.

But, since K is a Killing vector field on $\overline{M}^{n+1}$, it satisfies the following Killing equation

$$g(\overline{\nabla}_X K, Y) + g(X, \overline{\nabla}_Y K) = 0, \tag{5.22}$$

for every $X, Y \in \mathfrak{X}(\overline{M})$. Hence, from (5.21) and (5.22) we reach at the following suitable formula for the Laplacian of h

$$\Delta h = -\frac{2}{\rho} g(\nabla \rho, \nabla h) + \frac{1}{\rho^2}\Theta H. \tag{5.23}$$

5.4 Entire graphs

In the section of our paper, we will establish the basic foundations to later establish Moser-Bernstein type results concerning integer graphs constructed on the fibre of a deformed product. First of all, we need to recall some basic facts related to these graphs.

Let $\Omega \subseteq M^n$ be a domain. Then, each function $u \in C^\infty(\Omega)$ such that $u(\Omega) \subseteq I$ defines a vertical graph in the Riemannian warped product $\overline{M}^{n+1} = I \times_\rho M^n$. In such a case, $\Sigma(u)$ will denote the graph over Ω determined by u, that is,

$$\Sigma(u) = \{(u(p), p) : p \in \Omega\} \subset \overline{M}^{n+1}.$$

The graph is said to be entire if $\Omega = M^n$. Observe that $h(u(p), p) = u(p), p \in \Omega$. Hence, h and u can be identified in a natural way. The metric induced on Ω from the Riemannian metric of the ambient space via $\Sigma(u)$ is

$$g_u = du^2 + \rho(u)^2 g_M. \tag{5.24}$$

If M^n is complete and $\inf_M \rho(u) > 0$, then $\Sigma(u)$ furnished with the metric g_u is also complete. The unit vector field

$$N(p) = -\frac{\rho(u(p))}{\sqrt{\rho(u(p))^2 + |Du(p)|_M^2}} \left(\partial_t|_{(u(p),p)} - \frac{Du(p)}{\rho(u(p))^2} \right), \quad p \in \Omega, \tag{5.25}$$

where Du stands for the gradient of u in M and $|Du|_M = g_M(Du, Du)^{1/2}$, gives an orientation of $\Sigma(u)$ with respect to which we have $\Theta = \overline{g}(N, \partial_t) < 0$, so that the assumption of transversality to the vector field $K = \rho(t)\partial_t$ is not necessary here. The corresponding shape operator is given by

$$\begin{aligned} AX = &- \frac{1}{\rho(u)\sqrt{\rho(u)^2 + |Du|_M^2}} D_X Du + \frac{\rho'(u)}{\sqrt{\rho(u)^2 + |Du|_M^2}} X \\ &- \left(\frac{-g_M(D_X Du, Du)}{\rho(u)\left(\rho(u)^2 + |Du|_M^2\right)^{3/2}} - \frac{\rho'(u) g_M(Du, X)}{\left(\rho(u)^2 + |Du|_M^2\right)^{3/2}} \right) Du, \end{aligned} \tag{5.26}$$

for any vector field X tangent to Ω, where D denotes the Levi-Civita connection in M^n. Consequently, if $\Sigma(u)$ is a vertical graph over a domain $\Omega \subseteq M^n$, it is not difficult to verify from (5.26) that the mean curvature function $H(u)$ of $\Sigma(u)$ is given by the following nonlinear differential equation:

$$nH(u) = -\mathrm{div}_M \left(\frac{Du}{\rho(u)\sqrt{\rho(u)^2 + |Du|_M^2}} \right) + \frac{\rho'(u)}{\sqrt{\rho(u)^2 + |Du|_M^2}} \left(n - \frac{|Du|_M^2}{\rho(u)^2} \right), \tag{5.27}$$

where div_M stands for the divergence operator computed in the metric g_M.

5.5 Omori-Yau maximum principle and futter results

In order to obtain our main result, we initiate this section quoting the generalized maximum principle of Omori [122] and Yau [147] (see [16] for a modern and accessible reference to the generalized maximum principle of Omori-Yau).

Lemma 5.5.1. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u \in C^\infty(\Sigma)$ be a smooth function which is bounded from above on Σ^n. Then there exists a sequence $(p_k)_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k) = \sup_\Sigma u, \qquad \lim_k |\nabla u(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta u(p_k) \leq 0.$$

In our main result, we focus on Riemannian warped product spaces $I \times_\rho M^n$ satisfying the convergence condition

$$K_M \geq \sup_I (\rho'^2 - \rho\rho''), \tag{5.28}$$

where K_M stands for the sectional curvature of the fiber M^n. Warped products satisfying (5.28) have been studied, for instance, in [26, 75, 91]. The fact that this condition holds for the Ricci curvature instead of the sectional curvature is also well known (see, for instance, [23, 25, 115]).

we initiate this subsection considering an extension of Hopf's theorem on a complete Riemannian manifold (Σ^n, g) due to Yau in [148]. For this, let us also take

$$\mathcal{L}^p_g(\Sigma) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u|^p d\Sigma < +\infty\}, \tag{5.29}$$

where $d\Sigma$ stands for the measure related to the metric g.

Lemma 5.5.2. *Let u be a smooth function defined on a complete Riemannian manifold (Σ^n, g), such that Δu does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1_g(\Sigma)$, then Δu vanishes identically on Σ^n.*

Next, we quote two other auxiliary results also due to Yau in [148].

Lemma 5.5.3. *If u is a nonnegative smooth subharmonic function defined on (Σ^n, g), with $u \in \mathcal{L}^p_g(\Sigma)$ for some $p > 1$, then u must be constant.*

Lemma 5.5.4. *All noncompact complete Riemannian manifolds with nonnegative Ricci curvature have at least linear volume growth.*

Next we shall devote ourselves to presenting the analytical tool that will be used to establish our rigidity results in the next ones. For this, let (Σ^n, g) be a complete noncompact Riemannian manifold and let $d(\,\cdot\,, o) : \Sigma^n \to [0, +\infty)$ denote the Riemannian distance of (Σ^n, g), measured from a fixed point $o \in \Sigma^n$. We say that a smooth function $u \in C^\infty(\Sigma)$ *converges to zero at* *infinity* when it satisfies the following condition

$$\lim_{d(x,o)\to+\infty} u(x) = 0. \tag{5.30}$$

Keeping in mind this concept, the following lemma corresponds to item (a) of [21, Theorem 2.2].

Lemma 5.5.5. *Let (Σ^n, g) be a complete noncompact Riemannian manifold and let $X \in \mathfrak{X}(\Sigma)$ be a vector field on Σ^n. Assume that there exists a nonnegative, non-identically vanishing function $u \in C^\infty(\Sigma)$ which converges to zero at infinity and such that $g(\nabla u, X) \geq 0$. If $\operatorname{div}_g X \geq 0$ on Σ^n, then $g(\nabla u, X) \equiv 0$ on Σ^n.*

We also need the following definition which is inspired in (5.30): Given a complete noncompact Riemannian immersion $\psi : \Sigma^n \looparrowright I \times_\rho M^n$ and $t_* \in I$, we say that a function u defined on Σ^n *converges from below* (*above*) *to* t_* *at infinity* when $u \leq t_*$ $(u \geq t_*)$ and the function $\tilde{u} := u - t_*$ converges to zero at infinity.

For our purpose, we will also need to quote a suitable maximum principle that will be used to prove our nonexistence results. For this, let (Σ^n, g) be a connected, oriented, complete noncompact Riemannian manifold. We denote by $B(p,t)$ the geodesic ball centered at p and with radius t. Given a polynomial function $\sigma : (0, +\infty) \to (0, +\infty)$, we say that Σ^n has *polynomial volume growth* like $\sigma(t)$ if there exists $p \in \Sigma^n$ such that

$$\operatorname{vol}(B(p,t)) = \mathcal{O}(\sigma(t)),$$

as $t \to +\infty$, where vol denotes the standard Riemannian volume related to the metric g. As it was already observed in the beginning of Section 2 in [22], if $p, q \in \Sigma^n$ are at distance d from each other, we can verify that

$$\frac{\operatorname{vol}(B(p,t))}{\sigma(t)} \geq \frac{\operatorname{vol}(B(q,t-d))}{\sigma(t-d)} \cdot \frac{\sigma(t-d)}{\sigma(t)}.$$

So, the choice of p in the notion of volume growth is immaterial. For this reason, we will just say that Σ^n has polynomial volume growth.

Keeping in mind this previous digression, we close this section quoting the following key lemma which corresponds to a particular case of a new maximum principle due to Alías, Caminha and do Nascimento (see [22, Theorem 2.1]).

Lemma 5.5.6. *Let (Σ^n, g) be a connected, oriented, complete noncompact Riemannian manifold, and let $u \in \mathcal{C}^\infty(\Sigma)$ be a nonnegative smooth function such that $\Delta u \geq au$ on Σ^n, for some positive constant $a \in \mathbb{R}$. If Σ^n has polynomial volume growth and $|\nabla u|$ is bounded on Σ^n, then u vanishes identically on Σ^n.*

Chapter 6

Rigidity of hypersurfaces and Moser-Bernstein type results in certain warped products, with applications to pseudo-hyperbolic spaces

In this chapter we deal with complete two-sided hypersurfaces immersed in a warped product space of the type $I \times_\rho M^n$. Under suitable constraints on the warping function ρ, on the sectional curvature of the fiber M^n and on the mean curvature of such a hypersurface Σ^n, we apply some maximum principles in order to show that Σ^n must be a slice of $I \times_\rho M^n$. New Moser-Bernstein type results concerning entire graphs constructed over M^n are obtained, and applications to pseudo-hyperbolic spaces $I \times_{e^t} M^n$ are given. Here we present results of [32].

6.1 A computational lemma

Considering an immersed hypersurface Σ^n in a warped product space $I \times_\rho M^n$ satisfying (5.28), the next lemma gives sufficient conditions to its Ricci curvature with respect to the conformal metric $\hat{g} := \dfrac{1}{\rho(h)^2} g$. For this, we will suppose that Σ^n lies in a *slab* of $I \times_\rho M^n$, which means that Σ^n is contained in a bounded region of the type

$$[t_1, t_2] \times M^n = \{(t,p) \in I \times_\rho M^n : t_1 \leq t \leq t_2 \text{ and } p \in M^n\}.$$

Lemma 6.1.1. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product which satisfies the convergence condition (5.28) and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a hypersurface with bounded second fundamental form and lying in a slab of $\overline{M}^{n+1}$. Then, the Ricci curvature $\widehat{\mathrm{Ric}}$ of Σ^n with respect to the conformal metric $\hat{g} := \dfrac{1}{\rho(h)^2} g$ is bounded from below.*

Proof. First, we recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator A and the curvature tensor $\overline{R}$ of the ambient $I \times_f M^n$ by the so-called Gauss' equation given by[1]

$$g(R(X,Y)Z,W) = \bar{g}(\overline{R}(X,Y)Z,W) + g(A(X,Z),A(Y,W)) - g(A(X,W),A(Y,Z)), \quad (6.1)$$

for all tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Let us consider $X \in \mathfrak{X}(\Sigma)$ and take a local orthonormal frame $\{E_1, \cdots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss' equation (6.1) that the Ricci curvature Ric of Σ^n with respect to the induced metric g is given by

$$\begin{aligned} \mathrm{Ric}(X,X) &\geq \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) - (n|H||A| + |A|^2)|X|^2 \\ &\geq \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) - (\sqrt{n}+1)|A|^2|X|^2. \end{aligned} \quad (6.2)$$

Moreover, with a straightforward computation, we get

$$\begin{aligned} \overline{R}(X,E_i)X &= \overline{R}(X^*,E_i^*)X^* + \bar{g}(X,\partial_t)\overline{R}(X^*,E_i^*)\partial_t + \bar{g}(X,\partial_t)\bar{g}(E_i,\partial_t)\overline{R}(X^*,\partial_t)\partial_t \\ &+\bar{g}(E_i,\partial_t)\overline{R}(X^*,\partial_t)X^* + \bar{g}(X,\partial_t)\overline{R}(\partial_t,E_i^*)X^* + \bar{g}(X,\partial_t)^2\overline{R}(\partial_t,E_i^*)\partial_t, \end{aligned} \quad (6.3)$$

where $X^* = X - \bar{g}(X,\partial_t)\partial_t$ and $E_i^* = E_i - \bar{g}(E_i,\partial_t)\partial_t$ are the projections of the tangent vector fields X and E_i onto the fiber M^n, respectively. By repeated use of the formulas of Proposition 7.42 of [123] and using equation (5.3), from (6.3) we get

$$\begin{aligned} \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) &= \sum_i \bar{g}(R_M(X^*,E_i^*)X^*,E_i^*) - \frac{\rho(h)\rho''(h)}{\rho(h)^2}|X|^2 \\ &+\frac{\rho'(h)^2}{\rho(h)^2}\left(|\nabla h|^2 - (n-1)\right)|X|^2 \\ &+(n-2)\left(\frac{\rho'(h)^2 - \rho(h)\rho''(h)}{\rho(h)^2}\right)g(X,\nabla h)^2, \end{aligned} \quad (6.4)$$

where R_M denotes the curvature tensor of the fiber M^n. But, it is not difficult to verify that

$$\begin{aligned} \sum_i \bar{g}(R_M(X^*,E_i^*)X^*,E_i^*) = \ &\frac{1}{\rho^2}\sum_i K_M(X^*,E_i^*)(|X|^2 - g(\nabla h,E_i)^2|X|^2 \\ &-g(X,\nabla h)^2 - g(X,E_i)^2 + 2g(X,\nabla h)g(X,E_i)g(\nabla h,E_i)). \end{aligned}$$

Thus, by using the convergence condition (5.28) and with another straightforward computation,

[1] As in [123], the curvature tensor R of the hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

from (6.4) we obtain

$$\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) \geq -\frac{\rho''(h)}{\rho(h)}\left(n-|\nabla h|^2\right)|X|^2 \geq -n\frac{|\rho''(h)|}{\rho(h)}|X|^2. \quad (6.5)$$

On the other hand, we have the following equation (see, for instance, Section 1J of [53], Section A of [110] or page 168 of [140])

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &= \mathrm{Ric}(X,X) + \frac{1}{\rho(h)^2}\big\{(n-2)\rho(h)\nabla^2\rho(h)(X,X) \\ &\quad +(\rho(h)\Delta\rho(h)-(n-1)|\nabla\rho(h)|^2)|X|^2\big\}. \quad (6.6)\end{aligned}$$

So, inserting (5.4), (5.5) and (5.6) into (6.6) we get:

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &= \mathrm{Ric}(X,X) + \frac{1}{\rho(h)^2}\big\{(n-2)\rho(h)(\rho''(h)g(\nabla h,X)^2+\rho'(h)\nabla^2 h(X,X)) \\ &\quad +(\rho(h)(\rho''(h)|\nabla h|^2+\rho'(h)\Delta h)-(n-1)\rho'(h)^2|\nabla h|^2)|X|^2\big\}. \quad (6.7)\end{aligned}$$

Hence, considering (6.2) and (6.5), from (6.7) we obtain the following lower estimate:

$$\widehat{\mathrm{Ric}}(X,X) \geq -\frac{1}{\rho(h)}\big\{(2n-1)|\rho''(h)|+(n+\sqrt{n}-2)|\rho'(h)||A|+(\sqrt{n}+1)\rho(h)|A|^2\big\}|X|^2. \quad (6.8)$$

Therefore, taking into account once more that $|A|$ is bounded and that Σ^n lies in a slab of the ambient space, from (6.8) we conclude that $\widehat{\mathrm{Ric}}$ is bounded from below. □

6.2 Rigidity of two-sided hypersurfaces via Omori-Yau's maximum principle

From now on, we will orient the two-sided hypersurfaces in such a way that $\Theta \leq 0$. In this setting, extending the ideas of [18, 20, 62], we obtain the following result:

Theorem 6.2.1. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (5.28). Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface with bounded second fundamental form and lying in a slab $[t_1,t_2]\times M^n$, with $\rho'(t)>0$ for $t\in[t_1,t_2]$. If the height function h and the mean curvature function H satisfy*

$$0 < -\frac{\rho'(h)}{\rho(h)}\Theta \leq H \quad (6.9)$$

and

$$|\nabla h| \leq \inf_\Sigma \left| H - \frac{\rho'(h)}{\rho(h)}\right|, \quad (6.10)$$

then Σ^n is a slice.

Proof. As before, let us consider on Σ^n the metric $\hat{g} = \frac{1}{\rho(h)^2} g$, which is conformal to its induced metric g. If we denote by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (5.4) and (5.6) we get

$$\begin{aligned} \hat{\Delta} h &= \rho(h)^2 \Delta h - (n-2)\rho(h)\rho'(h)|\nabla h|^2 \\ &= n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + nH\rho(h)^2\Theta. \end{aligned} \tag{6.11}$$

With a straightforward computation, from (6.11) we obtain

$$\begin{aligned} \hat{\Delta}\rho(h) &= \rho''(h)\hat{g}(\widehat{\nabla} h, \widehat{\nabla} h) + \rho'(h)\hat{\Delta} h \\ &= \rho''(h)\rho(h)^2|\nabla h|^2 + \rho'(h)\left(n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + nH\rho(h)\Theta\right) \\ &= n\rho(h)\rho'(h)^2 + nH\rho'(h)\rho(h)^2\Theta + \rho(h)^3\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{6.12}$$

Given a positive real number α, we have that

$$\hat{\Delta}\rho(h)^{-\alpha} = \alpha(\alpha+1)\rho(h)^{-\alpha-2}\hat{g}(\widehat{\nabla}\rho(h), \widehat{\nabla}\rho(h)) - \alpha\rho(h)^{-\alpha-1}\hat{\Delta}\rho(h). \tag{6.13}$$

Using (6.12) in (6.13) we get

$$\begin{aligned} \hat{\Delta}\rho(h)^{-\alpha} &= -\alpha n\rho(h)^{-\alpha}\rho'(h)^2 - \alpha nH\rho'(h)\rho(h)^{-\alpha+1}\Theta + \alpha(\alpha+1)\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 \\ &- \alpha\rho(h)^{-\alpha+2}\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{6.14}$$

But, from (5.4) we have

$$-\alpha n\rho(h)^{-\alpha}\rho'(h)^2 = -\alpha n\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 - \alpha n\rho(h)^{-\alpha}\rho'(h)^2\Theta^2. \tag{6.15}$$

Thus, from (6.14) and (6.15) we obtain

$$\begin{aligned} \hat{\Delta}\rho(h)^{-\alpha} &= -n\alpha\rho(h)^{-\alpha}\{\rho'(h)^2\Theta^2 + H\rho(h)\rho'(h)\Theta\} \\ &\quad -\alpha\rho(h)^{-\alpha+2}\left[(\log\rho)''(h) - (\alpha-1)\frac{\rho'(h)^2}{\rho(h)^2}\right]|\nabla h|^2. \end{aligned} \tag{6.16}$$

On the other hand, since we assume that $|A|$ is bounded and that Σ^n lies in a slab of the ambient space (which obeys the convergence condition 5.28), Lemmas **??** and 6.1.1 guarantee the existence of a sequence of points $(p_k)_{k\geq 1}$ in Σ^n such that

$$\lim_k \rho(h)^{-\alpha}(p_k) = \sup_\Sigma \rho(h)^{-\alpha} \qquad \lim_k |\widehat{\nabla}\rho(h)^{-\alpha}(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \hat{\Delta}\rho(h)^{-\alpha}(p_k) \leq 0.$$

Since $\widehat{\nabla}\rho(h)^{-\alpha} = -\alpha\rho(h)^{1-\alpha}\rho'(h)\nabla h$ and taking into account once more that Σ^n lies in a slab and that $\rho'(h) > 0$ in this slab, we get that

$$\lim_k |\nabla h(p_k)| = 0. \tag{6.17}$$

Moreover, from (5.4) we also have that

$$\lim_k \Theta(p_k) = -1. \tag{6.18}$$

So, using (6.9), (6.16), (6.17) and (6.18), it is not difficult to verify that

$$0 \geq \limsup_k \hat{\Delta}\rho(h)^{-\alpha}(p_k) \geq n\alpha \sup_\Sigma \rho(h)^{-\alpha} \limsup_k -\{\rho'(h)^2 - Hf(h)\rho'(h)\}(p_k) \geq 0. \tag{6.19}$$

Thus, from (6.19) we infer that

$$\lim_k \left(H - \frac{\rho'(h)}{\rho(h)} \right)(p_k) = 0.$$

Hence,

$$\inf_\Sigma \left| H - \frac{\rho'(h)}{\rho(h)} \right| = 0.$$

Therefore, from our hypothesis (6.10) we conclude that Σ^n must be a slice of $I \times_\rho M^n$. □

6.3 Rigidity of two-sided hypersurfaces via integrability properties

In what follows, we will assume that the warping function ρ of the ambient space $\overline{M}^{n+1} = I \times_\rho M^n$ satisfies the following inequality

$$(\log \rho)'' \leq \gamma[(\log \rho)']^2, \tag{6.20}$$

for some constant $\gamma > -1$.

In order to obtain our next result, using the previous lemma 5.5.2, we get the following result:

Theorem 6.3.1. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose warping function satisfies (6.20), holding the equality only at isolated points of I, and with complete fiber M^n. Let $\psi : \Sigma^n \to \overline{M}$ be a complete two-sided hypersurface which lies in a slab of $\overline{M}^{n+1}$. If the height function h and the mean curvature function H satisfy (6.9) and $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice.*

Proof. Considering again the conformal metric $\hat{g} := \frac{1}{\rho(h)^2} g$, it is not difficult to verify that

$$|\hat{\nabla}\rho(h)^{-\alpha}|_{\hat{g}} = \alpha\rho(h)^{-\alpha}|\rho'(h)||\nabla h|, \tag{6.21}$$

for any positive constant α. Consequently, since we assume that Σ^n lies in a slab of $\overline{M}^{n+1}$ and $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, from (6.21) we get that $|\hat{\nabla}\rho(h)^{-\alpha}|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$.

Moreover, taking $\alpha = 1 + \gamma$, from (6.16) we also have that $\hat{\Delta}\rho^{-\alpha} \geq 0$. Thus, we can apply Lemma 1.5.3 to infer that $\hat{\Delta}\rho^{-\alpha} = 0$ on Σ^n. Hence, since we are assuming that equality occurs in (6.20) only at isolated points of I, returning to (6.16) we conclude that $|\nabla h|$ must vanish identically on Σ^n. Therefore, Σ^n is a slice. $\square$

We also get a slightly different version of Theorem 6.3.1:

Theorem 6.3.2. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose warping function satisfies (6.20), and has complete fiber M^n. Let $\psi : \Sigma^n \to \overline{M}$ be a complete two-sided hypersurface which lies in a slab $[t_1, t_2] \times M^n$, with $\rho'(t) > 0$ for $t \in [t_1, t_2]$. If the height function h and the mean curvature function H satisfy (6.9) and $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice.*

Proof. As in the proof of Theorem 6.3.1, taking $\alpha = 1 + \gamma$, we get that $\hat{\Delta}\rho^{-\alpha} = 0$ on Σ^n. Moreover, since Σ^n lies in a slab of $I \times_\rho M^n$, we can also verify that $|\widehat{\nabla}\rho^{-2\alpha}|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. But, we note that

$$\hat{\Delta}\rho^{-2\alpha} = 2\rho^{-\alpha}\hat{\Delta}\rho^{-\alpha} + 2|\widehat{\nabla}\rho^{-\alpha}|^2_{\hat{g}} = 2|\widehat{\nabla}\rho^{-\alpha}|^2_{\hat{g}} \geq 0.$$

Thus, we can apply Lemma 5.5.2 again to obtain that $\hat{\Delta}\rho^{-2\alpha} = 0$. Hence, since we assume that $\rho'(t) > 0$ for $t \in [t_1, t_2]$, from (6.21) we obtain that $|\nabla h| = 0$ on Σ^n. Therefore, Σ^n must be a slice. $\square$

The next result we will expound will make use of lemma 5.5.4 from section 5.5, also due to Yau in [148].

These previous lemmas enable us to prove the following nonexistence result.

Theorem 6.3.3. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product satisfying (6.20), holding equality only at isolated points of I, and whose fiber M^n is noncompact complete with nonnegative Ricci curvature. There do not exist complete two-sided hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ lying in a slab of $\overline{M}^{n+1}$, satisfying (6.9) and such that $f(h) \in \mathcal{L}^q_g(\Sigma)$ for some q with $q + \gamma < -1$.*

Proof. Supposing for contradiction the existence of such a hypersurface Σ^n, we get from (6.16) that $\hat{\Delta}\rho^{-1-\gamma}(h) \geq 0$ on Σ^n. Moreover, since we assume that Σ^n lies in a slab of $\overline{M}^{n+1}$ and $\rho(h) \in \mathcal{L}^q_g(\Sigma)$ for some q with $1 + \gamma + q < 0$, it is not difficult to verify that $\rho(h)^{-1-\gamma} \in \mathcal{L}^p_{\hat{g}}(\Sigma)$ for $p = -\frac{q}{1+\gamma} > 1$. Thus, we can apply Lemma 5.5.3 to get that $f(h)$ is constant on Σ^n. Hence, since we assume that the equality occurs in (6.20) only at isolated points of I, returning to (6.16) we conclude that $|\nabla h|$ must vanish identically on Σ^n. Consequently, Σ^n is isometric (up to scaling) to M^n. So, since $\rho(h)$ is a positive constant, our assumption that $\rho(h) \in \mathcal{L}^q_g(\Sigma)$ also implies that M^n has finite volume. But, since M^n is assumed to be noncompact complete with nonnegative Ricci curvature, Lemma 5.5.4 leads us to a contradiction. $\square$

6.4 Rigidity of two-sided hypersurfaces via a parabolicity criterion

We recall that a noncompact Riemannian manifold is said to be *parabolic* if the only subharmonic functions on it that are bounded from above are the constants. On the other hand, given two Riemannian manifolds (Σ, g) and (Σ', g'), a diffeomorphism ϕ from Σ onto Σ' is called a *quasi-isometry* if there exists a constant $c \geq 1$ such that

$$c^{-1}|v|_g \leq |d\phi(v)|_{g'} \leq c|v|_g,$$

for all $v \in T_p\Sigma$, $p \in \Sigma$. From Theorem 1 of [109] (see also Corollary 5.3 of [92]) we have the following:

Lemma 6.4.1. *Let (Σ, g) and (Σ', g') be two complete Riemannian manifolds. If Σ and Σ' are quasi-isometric, then Σ and Σ' are either both parabolic or neither is parabolic.*

We can use the previous lemma to get the following parabolicity criterion:

Lemma 6.4.2. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact hypersurface immersed in a warped product $\overline{M}^{n+1} = I \times_\rho M^n$, whose fiber (M^n, g_M) has parabolic universal covering. If Θ is bounded away from zero, then $(\Sigma^n, \hat{g})$, endowed with the conformal metric $\hat{g} = \frac{1}{\rho(h)^2} g$, is parabolic.*

Proof. Given $p \in \Sigma^n$ and $v \in T_p\Sigma^n$, from (5.1) and (5.4) we have

$$g(v, v) = g(v, \nabla h)^2 + \rho(h)^2 g_M(d\pi(v), d\pi(v)). \tag{6.22}$$

Thus, from (6.22) we get

$$\hat{g}(v, v) = \frac{1}{\rho(h)^2} g(v, v) \geq g_M(d\pi(v), d\pi(v)). \tag{6.23}$$

On the other hand, using (5.4) and the Cauchy-Schwarz inequality in (6.22) we also have

$$\Theta^2 g(v, v) \leq \rho(h)^2 g_M(d\pi(v), d\pi(v)). \tag{6.24}$$

Since Θ is bounded away from zero, there exists a positive constant β such that $\Theta^2 \geq \beta^2$. Consequently, from (6.24) we get

$$\beta^2 g(v, v) \leq \Theta^2 g(v, v) \leq \rho(h)^2 g_M(d\pi(v), d\pi(v)). \tag{6.25}$$

Thus, from (6.25) we have

$$\hat{g}(v, v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{6.26}$$

Hence, using inequalities (6.23) and (6.26) we get

$$g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v, v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{6.27}$$

So, taking $c = \frac{1}{\beta^2} \geq 1$, from (6.27) we obtain

$$\frac{1}{c} g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v, v) \leq c g_M(d\pi(v), d\pi(v)), \tag{6.28}$$

which means that π is a quasi-isometry between Σ and M.

Let Σ' be the universal Riemannian covering of Σ with projection $\pi_\Sigma : \Sigma' \to \Sigma$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \Sigma' \to M$ is a covering map. If M' is the universal Riemannian covering of M with projection $\pi' : M' \to M$, then there exists a diffeomorphism $\phi : \Sigma' \to M'$ such that $\pi' \circ \phi = \pi_0$. Moreover, from (6.28) it is not difficult to verify that ϕ is also a quasi-isometry. Therefore, since the universal Riemannian covering of M is parabolic, it follows from Lemma 6.4.1 that the universal Riemannian covering of Σ is parabolic and, hence, Σ must also be parabolic with respect to the metric $\hat{g}$. □

In order to state our next result, we recall that a function $\rho : I \to (0, +\infty)$ is said to be globally constant if $I = \mathbb{R}$ and ρ is constant.

Theorem 6.4.3. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering and such that its warping function ρ is not globally constant and satisfies (6.20). Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface with $\Theta \leq -\beta < 0$, for some positive constant β, and such that $\inf_\Sigma \rho(h) > 0$. If $\rho'(h)H \geq 0$ and $\dfrac{\rho'(h)^2}{\rho(h)^2}\Theta^2 \leq H^2$, then Σ^n is a slice.*

Proof. First, we note that Lemma 6.4.2 guarantees that $(\Sigma^n, \hat{g})$ is parabolic. Moreover, it follows from (6.16) that $\rho^{-\alpha}(h)$ (where $\alpha = 1 + \gamma$) is subharmonic on Σ^n. Thus, since the hypothesis $\inf_\Sigma \rho(h) > 0$ implies that $\rho(h)^{-\alpha}$ is bounded from above, it follows from the parabolicity of $(\Sigma^n, \hat{g})$ that $\rho(h)$ is constant on Σ^n. Consequently, returning to (6.16) we get that

$$H^2 = \frac{\rho'(h)^2}{\rho(h)^2}\Theta^2. \tag{6.29}$$

Let us suppose that h is not constant. So, $J = \operatorname{Im} h$ is a subinterval of I and $\rho|_J$ is constant, which implies that $\rho'(t) = 0$ for all $t \in J$. Hence, $\rho'(h)$ vanishes identically and, from (**??**) we conclude that Σ^n is minimal. Thus, from (6.11) it follows that h is a harmonic function on the parabolic Riemannian manifold $(\Sigma^n, \hat{g})$. Finally, since $\rho(h)$ is constant and taking into account that we assume that f is not globally constant, it is not difficult to see that h must be bounded either from below or from above. Consequently, h is constant on Σ^n, leading us to a contradiction. Therefore, Σ^n must be a slice. □

6.5 Applications to pseudo-hyperbolic spaces

According to the terminology introduced by Tashiro [144], when the warping function is exponential the corresponding warped product $I \times_{e^t} M^n$ is referred to as a *pseudo-hyperbolic space*. Tashiro's terminology is due to the fact that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$

is isometric to the warped product $\mathbb{R} \times_{e^t} \mathbb{R}^n$, where the slices constitute a family of horospheres sharing a fixed point in the asymptotic boundary $\partial_\infty \mathbb{H}^{n+1}$ and giving a complete foliation of $\mathbb{H}^{n+1}$. For more details about these spaces see, for instance, [24, 25, 91, 115].

We observe that a pseudo-hyperbolic space $I \times_{e^t} M^n$, whose fiber M^n has nonnegative sectional curvature, satisfies (5.28). So, from Theorem 6.2.1 we obtain the following consequence:

Corollary 6.5.1. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed in a slab of a pseudo-hyperbolic space $\overline{M}^{n+1} = I \times_{e^t} M^n$ whose fiber M^n is complete with nonnegative sectional curvature, and has bounded second fundamental form. If $H \geq 1$ and $|\nabla h| \leq \inf_\Sigma (H-1)$, then Σ^n is a slice.*

Taking into account that a pseudo-hyperbolic space satisfies (6.20) for $\gamma = 0$, in this case Theorem 6.3.2 reads as follows:

Corollary 6.5.2. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface immersed in a slab of a pseudo-hyperbolic space $\overline{M}^{n+1} = I \times_{e^t} M^n$ with complete fiber M^n. If $H \geq 1$ and $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice.*

From Theorem 6.3.3 we obtain the following consequence:

Corollary 6.5.3. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space, whose fiber M^n is noncompact complete with nonnegative Ricci curvature. There do not exist complete two-sided hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ lying in a slab of $\overline{M}^{n+1}$, with $H \geq 1$ and such that $e^h \in \mathcal{L}^q_g(\Sigma)$ for some q with $q < -1$.*

From Theorem 6.4.3 we get the following result:

Corollary 6.5.4. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete two-sided hypersurface with $\Theta \leq -\beta < 0$, for some positive constant β, and such that $\inf_\Sigma h > -\infty$. If $H \geq 1$, then Σ^n is a slice.*

6.6 Moser-Bernstein type results for entire graphs

We say that $u \in C^\infty(M)$ has finite C^k norm, for some $k \in \mathbb{N}$, when

$$||u||_{C^k(M)} := \sup_{|\gamma| \leq k} |D^\gamma u|_{L^\infty(M)} < +\infty.$$

It follows from (5.26) that the shape operator A of an entire graph $\Sigma(u)$ is bounded provided that u has finite C^2. Note also that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $\inf_M \rho(u) > 0$.

In this context, we obtain a nonparametric version of Theorem 6.2.1.

Theorem 6.6.1. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (5.28). Let $\Sigma(u)$ be an entire graph determined by a function $u \in C^\infty(M)$ with finite C^2 norm and satisfying*

$$0 < \frac{\rho'(u)}{\sqrt{\rho(u)^2 + |Du|_M^2}} \leq H(u). \tag{6.30}$$

If

$$|Du|_M \leq \inf_M \left| H(u) - \frac{\rho'(u)}{\rho(u)} \right|, \tag{6.31}$$

then $u \equiv t_0$ for some $t_0 \in I$.

Proof. From (5.26), we conclude that (6.30) implies (6.9). Moreover, since we have that $N = N^* + \Theta\partial_t$, where (as before) N^* denotes the projection of N onto the fiber M^n, from (5.3) we get

$$|\nabla h|^2 = \rho(u)^2 |N^*|_M^2. \tag{6.32}$$

Thus, from (5.26) and (6.32) we obtain

$$|\nabla h|^2 = \frac{|Du|_M^2}{\rho(u)^2 + |Du|_M^2}. \tag{6.33}$$

Hence, from (6.33) we can also verify that (6.31) implies (6.10). Therefore, the result follows by applying Theorem 6.2.1. □

Theorem 6.6.1 gives us the following application in the context of pseudo-hyperbolic spaces:

Corollary 6.6.2. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space such that its fiber M^n is complete with nonnegative sectional curvature. Let $\Sigma(u)$ be an entire graph determined by a function $u \in C^\infty(M)$ with finite C^2 norm and satisfying $e^u \leq H(u)$. If $|Du|_M \leq \inf_M |H(u) - 1|$, then $u \equiv t_0$ for some $t_0 \in I$.*

From Theorem 6.3.1 we obtain the following Moser-Bernstein type result:

Theorem 6.6.3. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose warping function satisfies (6.20), holding equality only at isolated points of I, and with complete fiber M^n. Let $\Sigma(u)$ be an entire graph determined by a bounded function $u \in C^\infty(M)$ satisfying (6.30). If $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_0$ for some $t_0 \in I$.*

Proof. Reasoning as in the proof of Theorem 1 of [23], from (5.24) we infer that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma$ stand for the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, and $G = \det(g_{ij})$ with

$$g_{ij} = g_u(E_i, E_j) = E_i(u)E_j(u) + \rho(u)^2\delta_{ij}.$$

Here, $\{E_1, \ldots, E^n\}$ denotes a local orthonormal frame with respect to the metric g_M. So, it is not difficult to verify that

$$|G| = \rho(u)^{2(n-1)}(\rho(u)^2 + |Du|_M^2).$$

Consequently,

$$d\Sigma = \rho(u)^{n-1}\sqrt{\rho(u)^2 + |Du|_M^2}dM. \tag{6.34}$$

Thus, from (6.33) and (6.34) we get

$$|\nabla h|d\Sigma = \rho(u)^{n-1}|Du|_M dM. \tag{6.35}$$

Hence, since we assume that u is bounded with $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, relation (6.35) guarantees that $|\nabla h| \in \mathcal{L}^1_g(\Sigma(u))$. Therefore, the result follows by applying Theorem 6.3.1. □

When the ambient space is a pseudo-hyperbolic space, Theorem 6.6.3 reads as follows:

Corollary 6.6.4. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space such that its fiber M^n is complete. Let $\Sigma(u)$ be an entire graph determined by a bounded function $u \in C^\infty(M)$ satisfying $e^u \leq H(u)$. If $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_0$ for some $t_0 \in I$.*

Taking relation (6.34) into account once more, it is not difficult to see that from Theorem 6.3.3 we obtain the following nonexistence result:

Theorem 6.6.5. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product satisfying (6.20), holding equality only at isolated points of I, and whose fiber M^n is noncompact complete with nonnegative Ricci curvature. There do not exist entire solutions $u \in C^\infty(M)$ of the mean curvature equation (5.27) with finite C^1 norm, satisfying (6.30) and such that $\rho(u) \in \mathcal{L}^q_{g_M}(M)$ for some q with $q+\gamma < -1$.*

From Theorem 6.6.5 we get the following application:

Corollary 6.6.6. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n is noncompact complete with nonnegative Ricci curvature. There do not exist entire solutions $u \in C^\infty(M)$ of the mean curvature equation*

$$nH(u) = -\operatorname{div}_M\left(\frac{Du}{e^u\sqrt{e^{2u} + |Du|_M^2}}\right) + \frac{e^u}{\sqrt{e^{2u} + |Du|_M^2}}\left(n - \frac{|Du|_M^2}{e^{2u}}\right),$$

with finite C^1 norm, satisfying $e^u \leq H(u)$ and such that $e^u \in \mathcal{L}^q_{g_M}(M)$ for some q with $q < -1$.

We can verify from (5.25) that the condition Θ bounded away from zero is equivalent to $|Du|_M \leq \alpha f(u)$ for some positive constant α. Using this fact, Theorem 6.4.3 allows us to obtain the last result of this paper:

Theorem 6.6.7. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n has parabolic universal covering and such that its warping function ρ is not globally constant and satisfies (6.20).*

Let $\Sigma(u)$ be an entire graph determined by a bounded function $u \in C^\infty(M)$ with $|Du|_M \leq \alpha\rho(u)$ for some positive constant α. If $\rho'(u)H(u) \geq 0$ and $\dfrac{\rho'(u)^2}{\rho(u)^2 + |Du|_M^2} \leq H(u)^2$, then $u \equiv t_0$ for some $t_0 \in I$.

We close our paper with applications of Theorem 6.6.7.

Corollary 6.6.8. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n has parabolic universal covering. Let $\Sigma(u)$ be an entire graph determined by a bounded function $u \in C^\infty(M)$ with $|Du|_M \leq \alpha e^u$ for some positive constant α. If $H(u) \geq 1$, then $u \equiv t_0$ for some $t_0 \in I$.*

In [24], Alías and Dajczer proved that the horospheres are the only complete surfaces properly immersed in $\mathbb{H}^3$ with constant mean curvature $-1 \leq H \leq 1$ and which are contained in a region between two horospheres which share a fixed point in $\partial_\infty \mathbb{H}^3$. Modelling $\mathbb{H}^3$ through the warped product space $\mathbb{R} \times_{e^t} \mathbb{R}^2$, we recall that these horospheres are just isometric to the slices $\{t\} \times \mathbb{R}^2$. So, from Corollary 6.6.8 we get our last result:

Corollary 6.6.9. *The only 2-dimensional entire graphs $\Sigma(u)$ lying between two horospheres $\{t_1\} \times \mathbb{R}^2$ and $\{t_2\} \times \mathbb{R}^2$ $(t_1 < t_2)$ of $\mathbb{H}^3 = \mathbb{R} \times_{e^t} \mathbb{R}^2$, with $|Du|_M \leq \alpha e^u$ for some positive constant α and such that $H(u) \geq 1$, are the horospheres $\{t\} \times \mathbb{R}^2$, with $t_1 \leq t \leq t_2$.*

Remark 6.6.10. *It is worth mentioning that López recently obtained in [114] gradient estimates for solutions to the Dirichlet problem for the constant mean curvature equation on a domain of a horosphere in three-dimensional hyperbolic spaces and, under suitable boundary conditions, he employed these estimates to solve the Dirichlet problem when the mean curvature H satisfies $H < 1$.*

Chapter 7

Mean curvature flow solitons in certain warped products: Nonexistence, rigidity and Moser-Bernstein type results

In the following results, our purpose is to apply suitable maximum principles in order to obtain nonexistence and rigidity results concerning complete n-dimensional mean curvature flow solitons with respect to the conformal vector field $K = \rho(t)\partial_t$ of a warped product space of the type $I \times_\rho M^n$. Applications to self-shrinkers in the Euclidean space, as well as to mean curvature flow solitons in the real projective, pseudo-hyperbolic, Schwarzschild and Reissner-Nordström spaces are also given. Furthermore, we study entire graphs constructed over the fiber M^n and which are mean curvature flow solitons with respect to K, obtaining new Moser-Bernstein type results (see Section 7.3) The results presented in this chapter make part of [42–44].

7.1 Auxiliary results

In order to investigate the nonexistence of complete mean curvature flow solitons, initially we introduce the following definition:

Definition 7.1.1. *The Laplacian operator Δ on a Riemannian manifold (Σ, g) satisfies the Omori-Yau maximum principle if for any $u \in C^2$ bounded from above, there exists a sequence $(p_k)_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k) = \sup_\Sigma u = u^*, \qquad \lim_k |\nabla u(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta u(p_k) \leq 0.$$

Now we recall the maximum principle due to Omori [122] and Yau [147]. Such concept gives us conditions to the validity of a maximum principle for the hessian or the Laplacian on a Riemannian manifold. Specifically, we quote the following result for the Laplacian:

Lemma 7.1.2 (Yau, [147]). *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u \in C^\infty(\Sigma)$ be a smooth function which is bounded from above on Σ^n. Then the Laplacian Δ satisfies the Omori-Yau maximum principle on Σ.*

Denoting by K_M the sectional curvature of the fiber M^n, we will consider warped product spaces $I \times_\rho M^n$ satisfying the convergence condition

$$K_M \geq \sup_I(\rho'^2 - \rho\rho''). \tag{7.1}$$

Warped products satisfying (7.1) have been studying, for instance, in [26, 27, 75, 91]. The case that this condition holds for the Ricci curvature instead of the sectional curvature is also well known (see, for instance, [23, 25, 115]). Furthermore, it is not difficult to verify that there exists a wide class of warped product satisfying (7.1), including, for instance, the Euclidean space minus a point $\mathbb{R}^{n+1} \setminus \{o\} = \mathbb{R}_+ \times_t \mathbb{S}^n$, the real projective space (minus a suitable point and its cut locus) $\left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$, the pseudo-hyperbolic spaces $I \times_{e^t} M^n$ with fiber having nonnegative sectional curvature and the Schwarzschild and Reissner-Nordström spaces $I \times_\rho \mathbb{S}^n$ (see Examples 5.2.1, 5.2.2, 5.2.3, 5.2.4 and 5.2.5).

Indeed, this verification for the Euclidean, the real projective, the pseudo-hyperbolic and the Schwarzschild spaces is quite simple. In the case of the Reissner-Nordström space, with a straightforward computation we get that

$$\rho'(t)^2 - \rho(t)\rho''(t) = 1 - \mathfrak{m}r(t)^{1-n} - n\left\{\mathfrak{m} - \mathfrak{q}^2 r(t)^{1-n}\right\} r(t)^{1-n}. \tag{7.2}$$

But, since $r(t) > r_0(\mathfrak{m}, \mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}}\right)^{1/(n-1)}$, it is not difficult to verify that we must have

$$\mathfrak{q}^2 r(t)^{1-n} < \mathfrak{m}. \tag{7.3}$$

Consequently, from (7.2) and (7.3) we conclude that the convergence condition (7.1) is also satisfied in the Reissner-Nordström space.

We recall that a hypersurface Σ^n lies in a *slab* of a warped product $I \times_\rho M^n$ when Σ^n is contained in a region of the type

$$[t_1, t_2] \times M^n = \{(t, p) \in I \times_\rho M^n : t_1 \leq t \leq t_2 \text{ and } p \in M^n\}.$$

Next, considering an immersed hypersurface Σ^n in a slab of a warped product space $I \times_\rho M^n$ satisfying (7.1), we will verify that the Omori-Yau maximum principle is satisfied.

Lemma 7.1.3. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product which satisfying the convergence condition (7.1) and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a hypersurface with bounded second fundamental form and lying in a slab of $\overline{M}^{n+1}$. Then, the Laplacian on Σ^n satisfies the Omori-Yau maximum principle.*

Proof. First, we recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator A and the curvature tensor $\overline{R}$ of the ambient $I \times_\rho M^n$ by the so-called Gauss' equation given by[1]

$$g(R(X,Y)Z,W) = \bar{g}(\overline{R}(X,Y)Z,W) + g(A(X,Z),A(Y,W)) - g(A(X,W),A(Y,Z)),$$

for every tangent vector fields $X, Y, Z, W \in \mathfrak{X}(\Sigma)$.

Let us consider $X \in \mathfrak{X}(\Sigma)$ and take a local orthonormal frame $\{E_1, \cdots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss equation (6.1) that the Ricci curvature Ric of Σ^n with respect to the induced metric g is given by

$$\begin{aligned} \mathrm{Ric}(X,X) &\geq \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) - (|H||A| + |A|^2)|X|^2 \\ &\geq \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) - (\sqrt{n}+1)|A|^2|X|^2. \end{aligned} \tag{7.4}$$

Moreover, with a straightforward computation, we get

$$\begin{aligned} \overline{R}(X,E_i)X &= \overline{R}(X^*,E_i^*)X^* + \bar{g}(X,\partial_t)\overline{R}(X^*,E_i^*)\partial_t + \bar{g}(X,\partial_t)\bar{g}(E_i,\partial_t)\overline{R}(X^*,\partial_t)\partial_t \\ &+ \bar{g}(E_i,\partial_t)\overline{R}(X^*,\partial_t)X^* + \bar{g}(X,\partial_t)\overline{R}(\partial_t,E_i^*)X^* + \bar{g}(X,\partial_t)^2\overline{R}(\partial_t,E_i^*)\partial_t, \end{aligned} \tag{7.5}$$

where $X^* = X - \bar{g}(X,\partial_t)\partial_t$ and $E_i^* = E_i - \bar{g}(E_i,\partial_t)\partial_t$ are the projections of the tangent vector fields X and E_i onto the fiber M^n, respectively.

Thus, by repeated use of the formulas of [123, Proposition 7.42] and using equation (5.3), from (7.20) we get

$$\begin{aligned} \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) &= \sum_i \bar{g}(R_M(X^*,E_i^*)X^*,E_i^*) - \frac{\rho(h)\rho''(h)}{\rho(h)^2}|X|^2 \\ &+ \frac{\rho'(h)^2}{\rho(h)^2}\left(|\nabla h|^2 - (n-1)\right)|X|^2 + (n-2)\left(\frac{\rho'(h)^2 - \rho(h)\rho''(h)}{\rho(h)^2}\right)g(X,\nabla h)^2, \end{aligned} \tag{7.6}$$

where R_M denotes the curvature tensor of the fiber M^n. But, it is not difficult to verify that

$$\begin{aligned} \sum_i \bar{g}(R_M(X^*,E_i^*)X^*,E_i^*) = &\ \frac{1}{\rho^2}\sum_i K_M(X^*,E_i^*)(|X|^2 - g(\nabla h,E_i)^2|X|^2 \\ &- g(X,\nabla h)^2 - g(X,E_i)^2 + 2g(X,\nabla h)g(X,E_i)g(\nabla h,E_i)). \end{aligned}$$

Thus, by using the convergence condition (7.1) and with another straightforward computation, from (7.6) we obtain

$$\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) \geq -\frac{\rho''(h)}{\rho(h)}\left(n - |\nabla h|^2\right)|X|^2 \geq -n\frac{|\rho''(h)|}{\rho(h)}|X|^2. \tag{7.7}$$

[1] As in [123], the curvature tensor R of the hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

On the other hand, we have the following equation (see, for instance, [53, Section 1J], [110, Section A] or [140, page 168])

$$\widehat{\mathrm{Ric}}(X,X)=\mathrm{Ric}(X,X)+\frac{1}{\rho(h)^2}\left\{(n-2)\rho(h)\nabla^2\rho(h)(X,X)+(\rho(h)\Delta\rho(h)-(n-1)|\nabla\rho(h)|^2)|X|^2\right\}.$$

Consequently, from this previous equation we get

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &= \mathrm{Ric}(X,X)+\frac{1}{\rho(h)^2}\big\{(n-2)\rho(h)(\rho''(h)g(\nabla h,X)^2+\rho'(h)\nabla^2h(X,X))\\ &\quad+(\rho(h)(\rho''(h)|\nabla h|^2+\rho'(h)\Delta h)-(n-1)\rho'(h)^2|\nabla h|^2)|X|^2\big\}. \qquad (7.8)\end{aligned}$$

Hence, considering (5.4), (5.5), (5.6), (7.4) and (7.7) into (7.8), we obtain the following lower estimate:

$$\widehat{\mathrm{Ric}}(X,X)\geq-\frac{1}{\rho(h)}\big\{(2n-1)|\rho''(h)|+(n+\sqrt{n}-2)|\rho'(h)||A|+(\sqrt{n}+1)\rho(h)|A|^2\big\}|X|^2. \quad (7.9)$$

Therefore, taking into account that $|A|$ is bounded and that Σ^n lies in a slab of the ambient space, from (7.4) and (7.7) we conclude that the Ricci curvature is bounded from below and by Lemma 7.1.2 the Laplacian satisfies the desired property. □

7.2 Statements and proofs of the main results

7.2.1 Nonexistence results via Omori-Yau maximum principle

Into the scope of a warped product $I\times_\rho M^n$ we are in position to state and prove our first nonexistence result concerning mean curvature flow solitons immersed in a slab of a warped product.

Theorem 7.2.1. *Let $\overline{M}^{n+1}=I\times_\rho M^n$ be a warped product whose fiber M^n satisfies hypothesis (7.1). There exists no complete mean curvature flow soliton $\psi:\Sigma^n\to\overline{M}^{n+1}$ with respect to $K=\rho(t)\partial_t$ with soliton constant $c\neq0$, having bounded second fundamental form, lying in a slab $[t_1,t_2]\times M^n$ and $\zeta_c(t)$ having a strict sign on $[t_1,t_2]$.*

Proof. Let us suppose by contradiction the existence of such a mean curvature flow soliton $\psi:\Sigma^n\to\overline{M}^{n+1}$. From (5.6) we have

$$\begin{aligned}\Delta h &= n\frac{\rho'(h)}{\rho(h)}-\frac{\rho'(h)}{\rho(h)}|\nabla h|^2+c\rho(h)\Theta^2 \qquad (7.10)\\ &= n\frac{\rho'(h)}{\rho(h)}\Theta^2+n\frac{\rho'(h)}{\rho(h)}|\nabla h|^2+c\rho\Theta^2\\ &= (n-1)\frac{\rho'(h)}{\rho(h)}|\nabla h|^2+\frac{n\rho'(h)+c\rho^2(h)}{\rho}\Theta^2\\ &= (n-1)\frac{\rho'(h)}{\rho(h)}|\nabla h|^2+\frac{\zeta_c(h)}{\rho(h)}\Theta^2,\end{aligned}$$

where we used (5.4) in the second equality. Since the second fundamental form has bounded norm and the hypersurface is contained in a slab, from Lemma 6.1.1 we are able to apply the Omori-Yau maximum principle. Indeed, there are sequences $\{x_k\}$ and $\{p_k\}$ such that

$$\lim_k h(p_k) = \sup_\Sigma h = h^*, \qquad \lim_k |\nabla h(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta h(p_k) \leq 0,$$

and

$$\lim_k h(x_k) = \inf_\Sigma h = h_*, \qquad \lim_k |\nabla h(x_k)| = 0 \qquad \text{and} \qquad \liminf_k \Delta h(x_k) \geq 0,$$

and thus, using that Θ goes to 1 along the sequences $\{p_k\}$ and $\{x_k\}$, we deduce from equation (7.10) that

$$\zeta_c(h^*) \leq 0 \leq \zeta_c(h_*),$$

which contradict our hypothesis on the function ζ_c.

□

Remark 7.2.2. It is worth to point out that complete mean curvature flow solitons immersed in a slab of a warped product $I \times_\rho M^n$ and having bounded second fundamental form constitute natural generalizations of the compact ones, and they have already been studied by Alías, de Lira and Rigoli in [27].

Taking into account Example 5.2.1, it is not difficult to verify that we get from the proof of Theorem 7.2.1 the following result concerning the nonexistence of complete self-shrinkers:

Corollary 7.2.3. *There exists no complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$ with bounded second fundamental form and lying in the closure of an n-dimensional annulus with either inner radius $r_{ir} > \sqrt{n}$ or outer radius $r_{or} < \sqrt{n}$.*

Remark 7.2.4. *We point out that the sphere of radius $\sqrt{n}$ satisfies all the hypotheses if we allow the inner radius r_{ir} (or outer radius r_{or}) equal to $\sqrt{n}$. We also notice that the self-shrinkers $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, for $1 \leq k \leq n-1$, of $\mathbb{R}^{n+1}$ have bounded second fundamental form but they do not belong to any n-dimensional annuli.*

Remark 7.2.5. *In Corollaries 7.2.17, 7.2.20, 7.2.25 and 7.2.29, if we assume $c > 0$ the condition ζ_c positive is immediate and so the nonexistence results follows directly.*

We will now present a non-existence result, whose fundamental technique for construction is the assumption of having polynomial volume growth. For the next result, let us establish one notation. Consider the *modified soliton function* as been the function

$$\bar{\zeta}_c(t) := \rho'(t)\zeta_c(t). \tag{7.11}$$

Theorem 7.2.6. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose warping function ρ satisfies inequality*

$$(\log \rho)'' \leq \gamma[(\log \rho)']^2. \tag{7.12}$$

There does not exist complete noncompact mean curvature flow solitons $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \rho(t)\partial_t$ with soliton constant $c \neq 0$ and mean curvature bounded away from zero, having polynomial volume growth and lying in a slab $[t_1, t_2] \times M^n$ with $\bar{\zeta}_c(t) < 0$ for all $t \in [t_1, t_2]$.

Proof. Let us suppose by contradiction the existence of such a mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ and let us consider on Σ^n the metric $\hat{g} = \frac{1}{\rho(h)^2} g$, which is conformal to its induced metric g. If we denote by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (5.4) and (5.6) we get

$$\begin{aligned}\hat{\Delta}h &= \rho(h)^2\Delta h - (n-2)\rho(h)\rho'(h)|\nabla h|^2 \\ &= n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + H\rho(h)^2\Theta. \end{aligned} \tag{7.13}$$

With a straightforward computation, from (7.13) we obtain

$$\begin{aligned}\hat{\Delta}\rho(h) &= \rho''(h)\hat{g}(\widehat{\nabla}h, \widehat{\nabla}h) + \rho'(h)\hat{\Delta}h \\ &= \rho''(h)f(h)^2|\nabla h|^2 + \rho'(h)\left(n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + H\rho(h)\Theta\right) \\ &= n\rho(h)\rho'(h)^2 + H\rho'(h)\rho(h)^2\Theta + \rho(h)^3\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{7.14}$$

Given a positive real number α, we have that

$$\hat{\Delta}\rho(h)^{-\alpha} = \alpha(\alpha+1)\rho(h)^{-\alpha-2}\hat{g}(\widehat{\nabla}\rho(h), \widehat{\nabla}\rho(h)) - \alpha\rho(h)^{-\alpha-1}\hat{\Delta}\rho(h). \tag{7.15}$$

Using (7.14) in (7.15) we get

$$\begin{aligned}\hat{\Delta}\rho(h)^{-\alpha} &= -\alpha n\rho(h)^{-\alpha}\rho'(h)^2 - \alpha H\rho'(h)\rho(h)^{-\alpha+1}\Theta + \alpha(\alpha+1)\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 \\ &- \alpha\rho(h)^{-\alpha+2}\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{7.16}$$

But, from (5.4) we have

$$-\alpha n\rho(h)^{-\alpha}\rho'(h)^2 = -\alpha n\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 - \alpha n\rho(h)^{-\alpha}\rho'(h)^2\Theta^2. \tag{7.17}$$

Thus, from (7.26), (7.44), (5.7) and (7.11) we obtain

$$\begin{aligned}\hat{\Delta}\rho(h)^{-\alpha} &= -\alpha\rho(h)^{-\alpha}\bar{\zeta}_c(h)\Theta^2 \\ &\quad -\alpha\rho(h)^{-\alpha+2}\left\{(\log\rho)''(h) - (\alpha-1)[(\log\rho)'(h)]^2\right\}|\nabla h|^2. \end{aligned} \tag{7.18}$$

Now, taking into account hypothesis (7.12) and choosing $\alpha = 1+\gamma > 0$, from (7.45) we get

$$\hat{\Delta}\rho(h)^{-\alpha} \geq -\bar{\zeta}_c(h)\alpha\rho(h)^{-\alpha}\Theta^2. \tag{7.19}$$

At this point we observe that, since $c \neq 0$, $\Sigma^n \subset [t_1, t_2] \times M^n$ and H is bounded away from zero, from (5.7) we see that Θ^2 is also bounded away from zero. So, since we are also assuming that $\bar{\zeta}_c(t) < 0$ for all $t \in [t_1, t_2]$, from (7.19) we reach at the following inequality

$$\hat{\Delta}\rho(h)^{-\alpha} \;\; \geq \;\; a\rho(h)^{-\alpha},$$

where $a = \alpha \inf_\Sigma |\bar{\zeta}_c(h)|$.

Moreover, it is not difficult to verify that

$$|\hat{\nabla}\rho(h)^{-\alpha}|_{\hat{g}} = \alpha\rho(h)^{-\alpha}|\rho'(h)||\nabla h| \leq \alpha\rho(h)^{-\alpha}|\rho'(h)|. \tag{7.20}$$

So, since $\Sigma^n \subset [t_1, t_2] \times M^n$, from (7.20) we conclude that $|\hat{\nabla}\rho(h)^{-\alpha}|_{\hat{g}}$ is bounded on Σ^n.

On the other hand, considering the coefficients of conformal metric $\hat{g}_{ij} = \frac{1}{\rho(h)^2} g_{ij}$, where g_{ij} stands for the coefficients of the induced metric g, we have that

$$\hat{G} = \sqrt{\det(\hat{g}_{ij})} = \sqrt{\rho(h)^{-2n}\det(g_{ij})} = \rho(h)^{-n}G. \tag{7.21}$$

In particular, using once more that $\Sigma^n \subset [t_1, t_2] \times M^n$, from (7.21) jointly with the hypothesis that Σ^n has polynomial volume growth with respect to g, we guarantee that the same holds with respect to the conformal metric $\hat{g}$.

Therefore, we are in position to apply Lemma 5.5.6 to infer that $\rho(h)^{-\alpha}$ vanishes identically on Σ^n, which contradicts the fact that ρ is a positive function. $\square$

Let $o = (0, \ldots, 0)$ be the origin of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We have that $\mathbb{R}^{n+1} \setminus \{o\}$ is isometric to $\mathbb{R}_+ \times_t \mathbb{S}^n$ (see [116, Section 4, Example 1]), whose slides $\{t\} \times \mathbb{S}^n$ are isometric to n-dimensional Euclidean spheres $\mathbb{S}^n(t)$ of radius $t \in \mathbb{R}_+$. In this setting, the mean curvature flow solitons with respect to $K = t\partial_t$ with soliton constant $c = -1$ are just the self-shrinkers. So, from (5.9) we conclude that $\mathbb{S}^n(\sqrt{n}) \equiv \{\sqrt{n}\} \times \mathbb{S}^n$ is the only slice which is a self-shrinker.

It is not difficult to verify that we get from Theorem 7.2.6 the following result concerning the nonexistence of complete self-shrinkers:

Corollary 7.2.7. *There does not exist complete noncompact n-dimensional self-shrinker immersed in $\mathbb{R}^{n+1}$ with mean curvature bounded away from zero, having polynomial volume growth and lying in the closure of an n-dimensional annulus with with inner radius $r_{ir} > \sqrt{n}$.*

Remark 7.2.8. *We note that, for each $1 \leq m \leq n-1$, the cylinder $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ is a self-shrinker immersed in $\mathbb{R}^{n+1}$ with mean curvature $|H| = \frac{\sqrt{m}}{n}$ and having polynomial volume growth, however it does not belong to any n-dimensional annuli. It is also worth to point out that Cao and Li [64] proved that an n-dimensional complete self-shrinker immersed in $\mathbb{R}^{n+p}$, with polynomial volume growth and whose second fundamental form satisfies $|A|^2 \leq 1$, must be isometric to one of the followings: a round sphere $\mathbb{S}^n(\sqrt{n})$, a cylinder $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, with $1 \leq m \leq n-1$, or a hyperplane $\mathbb{R}^n$, all of them immersed in $\mathbb{R}^{n+1}$.*

7.2.2 Rigidity results via an extension of Hopf's maximum principle

We initiate this section regarding an extension of Hopf's theorem on a complete Riemannian manifold (Σ^n, g) due to Yau in [148]. For this, let us also consider $\mathcal{L}^1_g(\Sigma) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u| d\Sigma < +\infty\}$, where $d\Sigma$ is the measure related to the metric g.

Using the previous lemma, we have the following result:

Theorem 7.2.9. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product. Let $\psi : \Sigma^n \to \overline{M}$ be a complete mean curvature flow soliton with respect to $K = \rho(t)\partial_t$ with soliton constant $c \neq 0$, lying in a slab $[t_1, t_2] \times M^n$, with $\zeta_c(t)$ does not changing the sign. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Considering $F(t) = \int_{t_0}^t \rho(v)^{1-n} dv$ and compute the Laplacian of $F(h)$ as follows:

$$\begin{aligned} \Delta F(h) &= F'(h)\Delta h + F''(h)|\nabla h|^2 \\ &= \frac{1}{\rho(h)^{n-1}}\Delta h + (1-n)\rho(h)^{-n}\rho'(h)|\nabla h|^2 \\ &= \frac{\zeta_c(h)}{\rho(h)^n}\Theta^2 + (n-1)\frac{\rho'(h)}{\rho(h)^n}|\nabla h|^2 + (1-n)\rho(h)^{-n}\rho'(h)|\nabla h|^2 \\ &= \rho(h)^{-n}\zeta_c(h)\Theta^2, \end{aligned} \tag{7.22}$$

where we used equation (5.6) in the third equality. Thus $F(h)$ is either subharmonic or superharmonic. Since Σ is contained is a slab and $|\nabla h| \in \mathcal{L}^1(\Sigma)$, we have that $|\nabla F(h)| = \rho(h)^{1-n}|\nabla h|$ belongs to the 1-Lebesgue space too.

Applying Lemma 5.5.2 we deduce that $\Delta F(h) = 0$ and thus $\zeta_c(h)\Theta^2 = 0$ along Σ. Next, note that

$$\Delta F(h)^2 = 2F(h)\Delta F(h) + 2|\nabla F(h)|^2 = 2\rho(h)^{1-n}|\nabla h|^2 \geq 0.$$

Applying Lemma 5.5.2 again, we deduce that $\nabla h = 0$ on Σ and from (5.4) we have $\Theta = 1$. Thus, $\zeta_c(h)$ vanishes on Σ, as we claimed.

□

From Theorem 7.2.9 we get the following rigidity result:

Corollary 7.2.10. *The only complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$, lying in the closure of an n-dimensional annulus with either inner radius $r_{ir} \geq \sqrt{n}$ or outer radius $r_{or} \leq \sqrt{n}$ and such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$ is $\mathbb{S}^n(\sqrt{n})$.*

Remark 7.2.11. *Related to Corollary 7.2.10, it is worth to mention that Pigola and Rimoldi [128] studied geometric properties of complete non-compact bounded self-shrinkers obtaining natural restrictions that force these hypersurfaces to be compact. In particular, they proved that the only complete bounded self-shrinker of $\mathbb{R}^3$ with $|A| \leq 1$ is $\mathbb{S}^2(\sqrt{2})$. Afterwards, Cavalcante and Espinar [67] showed that the only complete self-shrinker of $\mathbb{R}^{n+1}$ properly immersed in a closed cylinder $\overline{\mathbb{B}^{k+1}(r)} \times \mathbb{R}^{n-k}$, for some $k \in \{1, \cdots, n\}$ and radius $r \leq \sqrt{k}$, is the cylinder $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$.*

Next we will display some results from technique results using lemmas 5.5.5 and 5.5.6 as the main source.

Theorem 7.2.12. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product with complete noncompact fiber M^n and whose warping function ρ satisfies inequality (7.12). The only complete noncompact mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \rho(t)\partial_t$ with soliton constant c such that $\bar{\zeta}_c(h) \le 0$, $\rho(h)$ is increasing (decreasing) and, for some $t_* \in I$, h converges from below (above) to t_* at infinity, is the slice M_{t_*}.*

Proof. Let us suppose by contradiction that such a mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ is not the slice M_{t_*} and let us consider on Σ^n the metric $\hat{g} = \frac{1}{\rho(h)^2} g$, which is conformal to its induced metric g. If we denote by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (5.4) and (5.6) we get

$$\begin{aligned} \hat{\Delta} h &= \rho(h)^2 \Delta h - (n-2)\rho(h)\rho'(h)|\nabla h|^2 \\ &= n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + H\rho(h)^2\Theta. \end{aligned} \tag{7.23}$$

With a straightforward computation, from (7.23) we obtain

$$\begin{aligned} \hat{\Delta}\rho(h) &= \rho''(h)\hat{g}(\widehat{\nabla} h, \widehat{\nabla} h) + \rho'(h)\hat{\Delta} h \\ &= \rho''(h) f(h)^2 |\nabla h|^2 + \rho'(h)\left(n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + H\rho(h)\Theta\right) \\ &= n\rho(h)\rho'(h)^2 + H\rho'(h)\rho(h)^2\Theta + \rho(h)^3\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{7.24}$$

Given a positive real number α, we have that

$$\hat{\Delta}\rho(h)^{-\alpha} = \alpha(\alpha+1)\rho(h)^{-\alpha-2}\hat{g}(\widehat{\nabla}\rho(h), \widehat{\nabla}\rho(h)) - \alpha\rho(h)^{-\alpha-1}\hat{\Delta}\rho(h). \tag{7.25}$$

Using (7.24) in (7.25) we get

$$\begin{aligned} \hat{\Delta}\rho(h)^{-\alpha} &= -\alpha n\rho(h)^{-\alpha}\rho'(h)^2 - \alpha H\rho'(h)\rho(h)^{-\alpha+1}\Theta + \alpha(\alpha+1)\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 \\ &- \alpha\rho(h)^{-\alpha+2}\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{7.26}$$

But, from (5.4) we have

$$-\alpha n\rho(h)^{-\alpha}\rho'(h)^2 = -\alpha n\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 - \alpha n\rho(h)^{-\alpha}\rho'(h)^2\Theta^2. \tag{7.27}$$

Thus, from (7.26), (7.44), (5.7) and (7.11) we obtain

$$\begin{aligned} \hat{\Delta}\rho(h)^{-\alpha} &= -\alpha\rho(h)^{-\alpha}\bar{\zeta}_c(h)\Theta^2 \\ &\quad -\alpha\rho(h)^{-\alpha+2}\left\{(\log\rho)''(h) - (\alpha-1)[(\log\rho)'(h)]^2\right\}|\nabla h|^2. \end{aligned} \tag{7.28}$$

Now, taking into account hypothesis (7.12) and choosing $\alpha = 1 + \gamma > 0$, from (7.45) we get

$$\hat{\Delta}\rho(h)^{-\alpha} \quad \geq \quad -\bar{\zeta}_c(h)\alpha\rho(h)^{-\alpha}\Theta^2. \tag{7.29}$$

Since we are also assuming that $\bar{\zeta}_c(h) \leq 0$, choosing the smooth function $u = \rho(h)^{-\alpha} - \rho(t_*)^{-\alpha}$ and the vector field $X = \hat{\nabla}u$, from (7.29) we get that

$$\mathrm{div}_{\hat{g}} X = \hat{\Delta}\rho(h)^{-\alpha} \geq 0. \tag{7.30}$$

Moreover,

$$\hat{g}(\hat{\nabla}u, X) = |\hat{\nabla}\rho(h)^{-\alpha}|^2_{\hat{g}} = \alpha\rho(h)^{-\alpha}|\rho'(h)||\nabla h| \geq 0. \tag{7.31}$$

But, since we are supposing that $\rho(h)$ is increasing (decreasing) and that h converges from below (above) to t_* at infinity, we have that u is a nonnegative non-identically vanishing function which converges to zero at infinity (also related to the metric $\hat{g}$, since h is bounded).

Hence, we can apply Lemma 5.5.5 to get that $\hat{g}(\hat{\nabla}u, X)$ is identically zero on Σ^n. Thus, returning to (7.31) we conclude that ∇h vanishes identically on Σ^n, which means that h is constant and (since it converges to t_* at infinity) Σ^n must be the slice M_*. Therefore, we reach at a contradiction. □

7.2.3 Rigidity results via a parabolicity criterion

We recall that a Riemannian manifold is said to be *parabolic* if the only subharmonic functions on it that are bounded from above are the constants. On the other hand, given two Riemannian manifolds (Σ, g) and (Σ', g'), a diffeomorphism ϕ from Σ onto Σ' is called a *quasi-isometry* if there exists a constant $\kappa \geq 1$ such that

$$\kappa^{-1}|v|_g \leq |d\phi(v)|_{g'} \leq \kappa|v|_g,$$

for all $v \in T_p\Sigma$, $p \in \Sigma$. From [109, Theorem 1] (see also [92, Corollary 5.3]) we have the following:

Lemma 7.2.13. *Let (Σ, g) and (Σ', g') be two complete Riemannian manifolds. If Σ and Σ' are quasi-isometric, then Σ and Σ' are both parabolic or neither is parabolic.*

We can use the previous lemma to get the following parabolicity criterion:

Lemma 7.2.14. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete hypersurface immersed in a warped product $\overline{M}^{n+1} = I \times_\rho M^n$, whose fiber (M^n, g_M) is complete with parabolic universal covering. If Θ is bounded away from zero, then $(\Sigma^n, \hat{g})$, endowed with the conformal metric $\hat{g} = \frac{1}{\rho(h)^2} g$, is parabolic.*

Proof. Given $p \in \Sigma^n$ and $v \in T_p\Sigma^n$, from (5.1) and (5.4) we have

$$g(v, v) = g(v, \nabla h)^2 + \rho(h)^2 g_M(d\pi(v), d\pi(v)). \tag{7.32}$$

Thus, from (7.32) we get

$$\hat{g}(v,v) = \frac{1}{\rho(h)^2} g(v,v) \geq g_M(d\pi(v), d\pi(v)). \tag{7.33}$$

On the other hand, using (5.4) and the Cauchy-Schwarz inequality in (7.32) we also have

$$\Theta^2 g(v,v) \leq \rho(h)^2 g_M(d\pi(v), d\pi(v)). \tag{7.34}$$

Since Θ is bounded away from zero, there exists a positive constant β such that $\Theta^2 \geq \beta^2$. Consequently, from (7.34) we get

$$\beta^2 g(v,v) \leq \Theta^2 g(v,v) \leq \rho(h)^2 g_M(d\pi(v), d\pi(v)). \tag{7.35}$$

Thus, from (7.35) we have

$$\hat{g}(v,v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{7.36}$$

Hence, using inequalities (7.33) and (7.36) we get

$$g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v,v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{7.37}$$

So, taking the constant $\kappa = \frac{1}{\beta^2} \geq 1$, from (7.37) we obtain

$$\kappa^{-1} g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v,v) \leq \kappa g_M(d\pi(v), d\pi(v)), \tag{7.38}$$

which means that π is a quasi-isometry between Σ and M.

Let Σ' be the universal Riemannian covering of Σ with projection $\pi_\Sigma : \Sigma' \to \Sigma$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \Sigma' \to M$ is a covering map. If M' is the universal Riemannian covering of M with projection $\pi' : M' \to M$, then there exists a diffeomorphism $\phi : \Sigma' \to M'$ such that $\pi' \circ \phi = \pi_0$. Moreover, from (7.38) it is not difficult to verify that ϕ is also a quasi-isometry. Therefore, since the universal Riemannian covering of M is parabolic, it follows from Lemma 6.4.1 that the universal Riemannian covering of Σ is parabolic and, hence, Σ must be also parabolic with respect to the metric $\hat{g}$. □

Using Lemma 6.4.2 we obtain the following result:

Theorem 7.2.15. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering and such that its warping function f satisfies*

$$(\log \rho)'' \leq \gamma[(\log \rho)']^2, \tag{7.39}$$

for some constant $\gamma > -1$, holding the equality only at isolated points of I. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \rho(t)\partial_t$ with soliton constant $c \neq 0$, *such that* Θ *is bounded away from zero and* $\inf_\Sigma \rho(h) > 0$*. If* $\bar{\zeta}_c(h) \le 0$*, then* Σ^n *is a slice* M_{t_*} *for some* $t_* \in [t_1, t_2]$ *which is implicitly given by the condition* $\zeta_c(t_*) = 0$*.*

Proof. Let us consider on Σ^n the metric $\hat{g} = \frac{1}{\rho(h)^2} g$, which is conformal to its induced metric g. If we denote by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (5.4) and (5.6) we get

$$\begin{aligned}\hat{\Delta} h &= \rho(h)^2 \Delta h - (n-2)\rho(h)\rho'(h)|\nabla h|^2 \\ &= n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + H\rho(h)^2\Theta. \end{aligned} \tag{7.40}$$

With a straightforward computation, from (7.40) we obtain

$$\begin{aligned}\hat{\Delta}\rho(h) &= \rho''(h)\hat{g}(\widehat{\nabla} h, \widehat{\nabla} h) + \rho'(h)\hat{\Delta} h \\ &= \rho''(h)\rho(h)^2|\nabla h|^2 + \rho'(h)\left(n\rho(h)\rho'(h)\Theta^2 + \rho(h)\rho'(h)|\nabla h|^2 + H\rho(h)\Theta\right) \\ &= n\rho(h)\rho'(h)^2 + H\rho'(h)\rho(h)^2\Theta + \rho(h)^3\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{7.41}$$

Given a positive real number α, we have that

$$\hat{\Delta}\rho(h)^{-\alpha} = \alpha(\alpha+1)\rho(h)^{-\alpha-2}\hat{g}(\widehat{\nabla}\rho(h), \widehat{\nabla}\rho(h)) - \alpha\rho(h)^{-\alpha-1}\hat{\Delta}\rho(h). \tag{7.42}$$

Using (7.41) in (7.42) we get

$$\begin{aligned}\hat{\Delta}\rho(h)^{-\alpha} &= -\alpha n\rho(h)^{-\alpha}\rho'(h)^2 - \alpha H\rho'(h)\rho(h)^{-\alpha+1}\Theta + \alpha(\alpha+1)\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 \\ &- \alpha\rho(h)^{-\alpha+2}\left((\log\rho)''(h) - (n-2)\frac{\rho'(h)^2}{\rho(h)^2}\right)|\nabla h|^2. \end{aligned} \tag{7.43}$$

But, from (5.4) we have

$$-\alpha n\rho(h)^{-\alpha}\rho'(h)^2 = -\alpha n\rho(h)^{-\alpha}\rho'(h)^2|\nabla h|^2 - \alpha n\rho(h)^{-\alpha}\rho'(h)^2\Theta^2. \tag{7.44}$$

Thus, from (7.43), (7.44), (5.7) and (7.11) we obtain

$$\begin{aligned}\hat{\Delta}\rho(h)^{-\alpha} &= -\alpha\rho(h)^{-\alpha}\bar{\zeta}_c(h)\Theta^2 \\ &\quad -\alpha\rho(h)^{-\alpha+2}\left\{(\log\rho)''(h) - (\alpha-1)[(\log\rho)'(h)]^2\right\}|\nabla h|^2. \end{aligned} \tag{7.45}$$

First, we note that Lemma 6.4.2 guarantees that $(\Sigma^n, \hat{g})$ is parabolic. Moreover, it follows from (7.45) that $\rho(h)^{-\alpha}$ (where $\alpha = 1+\gamma$) is subharmonic on Σ^n. Thus, since the hypothesis $\inf_\Sigma \rho(h) > 0$ implies that $\rho(h)^{-\alpha}$ is bounded from above, it follows from the parabolicity of $(\Sigma^n, \hat{g})$ that $\rho(h)$ is constant on Σ^n. Consequently, since we are assuming that the equality holds in (7.12) only at isolated points of I, returning to (7.45) we conclude that $|\nabla h| = 0$ on Σ^n, which means that Σ^n is a slice. □

In the context of self-shrinkers, Theorem 7.2.15 reads as follows:

Corollary 7.2.16. *The only complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$, lying in the closure of the unbounded domain determined by $\mathbb{S}^n(\sqrt{n}) \subset \mathbb{R}^{n+1}$ and such that Θ is bounded away from zero, is $\mathbb{S}^n(\sqrt{n})$.*

7.2.4 Applications to real projective space

Considering the discussion made in Example 5.2.2, from Theorem 7.2.1 we have:

Corollary 7.2.17. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \backslash \{\pi(P) \cup Cut_P\}$. There exists no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \sin t \partial_t$ with soliton constant $c < 0$, having bounded second fundamental form and lying in a slab $[t_1, t_2] \times M^n$, with either $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) < t_1 < \frac{\pi}{2}$ or $0 < t_2 < \cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})$.*

Considering the setting of Example 5.2.2, from Theorem 7.2.9 we have:

Corollary 7.2.18. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \backslash \{\pi(P) \cup Cut_P\}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \sin t \partial_t$ with soliton constant $c < 0$, lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with either $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) \le t_1 < \frac{\pi}{2}$ or $0 < t_2 \le \cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$.*

Taking into account once more Example 5.2.2, from Theorem 7.2.1 we get:

Corollary 7.2.19. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \backslash \{\pi(P) \cup Cut_P\}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \sin t \partial_t$ with soliton constant $c < 0$, such that Θ is bounded away from zero. If $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) \le h < \frac{\pi}{2}$, then Σ^n is the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$.*

7.2.5 Applications to pseudo-hyperbolic spaces

When the ambient space is a pseudo-hyperbolic space (see Example 5.2.3), from Theorem 7.2.1 we also obtain the following consequence:

Corollary 7.2.20. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n has nonnegative sectional curvature. There exists no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = e^t \partial_t$ with soliton constant $c < 0$, having bounded second fundamental form and lying in a slab $[t_1, t_2] \times M^n$, with either $t_1 > \log\left(-\frac{n}{c}\right)$ or $t_2 < \log\left(-\frac{n}{c}\right)$.*

From Theorem 7.2.6 we also obtain the following consequence:

Corollary 7.2.21. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space. There does not exist complete noncompact mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = e^t \partial_t$ with soliton constant $c < 0$ and mean curvature bounded away from zero, having polynomial volume growth and lying in a slab $[t_1, t_2] \times M^n$ with $t_1 > \log\left(-\frac{n}{c}\right)$.*

When the ambient space is a pseudo-hyperbolic space, Theorem 7.2.9 reads as follows:

Corollary 7.2.22. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n is complete. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = e^t\partial_t$ with soliton constant $c < 0$, lying in a slab $[t_1, t_2] \times M^n$, with $t_1 \geq \log\left(-\frac{n}{c}\right)$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is the slice $\{\log\left(-\frac{n}{c}\right)\} \times M^n$.*

From Theorem 7.2.15 we obtain the following result:

Corollary 7.2.23. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n is complete with parabolic universal covering. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = e^t\partial_t$ with soliton constant $c < 0$, such that Θ is bounded away from zero. If $h \geq \log(-\frac{n}{c})$, then Σ^n is the slice $\{\log\left(-\frac{n}{c}\right)\} \times M^n$.*

7.2.6 Applications to Schwarzschild space

Given a mass parameter $\mathfrak{m} > 0$, the *Schwarzschild space* is defined to be the product

$$\overline{M}^{n+1} = (r_0(\mathfrak{m}), +\infty) \times \mathbb{S}^n$$

furnished with the metric $\bar{g} = V_{\mathfrak{m}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$, $V_{\mathfrak{m}}(r) = 1 - 2\mathfrak{m}r^{1-n}$ stands for its potential function and $r_0(\mathfrak{m}) = (2\mathfrak{m})^{1/(n-1)}$ is the unique positive root of $V_{\mathfrak{m}}(r) = 0$. Its importance lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m}}(r)dt^2 + \bar{g}$ is a solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance, [123, Chapter 13] for more details concerning Schwarzschild geometry).

As it was observed in [69, Example 1.3], $\overline{M}^{n+1}$ can be reduced in the form $I \times_\rho \mathbb{S}^n$ with metric (5.1) via the following change of variables:

$$t = \int_{r_0(\mathfrak{m})}^{r} \frac{d\sigma}{\sqrt{V_{\mathfrak{m}}(\sigma)}}, \qquad \rho(t) = r(t), \qquad I = \mathbb{R}_+. \tag{7.46}$$

As it was noted in [69, Example 4.1], since $V_{\mathfrak{m}}(r)$ is strictly increasing on $(r_0(\mathfrak{m}), +\infty)$, it follows from (7.46) that the warping function ρ satisfies:

$$\rho'(t) = \frac{dr}{dt} = \sqrt{V_{\mathfrak{m}}(r(t))} > 0 \qquad \text{and} \qquad \rho''(t) = \frac{1}{2}\frac{dV_{\mathfrak{m}}}{dr}(r(t)) > 0. \tag{7.47}$$

Thus, from (5.9) and (7.47) we can verify that a slice $\{t_*\} \times \mathbb{S}^n$ is a mean curvature flow soliton with respect to $\rho(t)\partial_t = r\sqrt{V_{\mathfrak{m}}(r)}\partial_r$ with soliton constant $c < 0$ when $t_* = t(r_*)$ with $r_* > r_0(\mathfrak{m})$ solving the following equation

$$V_{\mathfrak{m}}(r) = \frac{c^2}{n^2}r^4. \tag{7.48}$$

We note that such a solution exists if and only if the function $\varphi_{\mathfrak{m}}(t) = \frac{c^2}{n^2}t^4 + \frac{2\mathfrak{m}}{t^{n-1}} - 1$ has a zero on $(r_0(\mathfrak{m}), +\infty)$. Notice that $\varphi_{\mathfrak{m}}$ is a convex function which goes to infinity if t goes to 0 or $+\infty$ and so $\varphi_{\mathfrak{m}}$ has a unique minimal point in $(0, \infty)$. Such value $\hat{r}$ is given implicitly by $\varphi'_{\mathfrak{m}}(\hat{r}) = 0$, that is,

$$\frac{4c^2}{n^2}\hat{r}^3 - \frac{2\mathfrak{m}(n-1)}{\hat{r}^n} = 0.$$

Therefore, the equation (7.48) has a solution if and only if $\hat{r} > r_0(\mathfrak{m})$ and $\varphi_{\mathfrak{m}}(\hat{r}) \leq 0$. The last condition can be rewritten in the following way:

$$\hat{r} = \left(\frac{\mathfrak{m}(n-1)n^2}{2c^2}\right)^{1/(n+3)} \geq \left(\frac{\mathfrak{m}(n+3)}{2}\right)^{1/(n-1)}. \tag{7.49}$$

Taking into account the previous setting, from Theorem 7.2.6 we get:

Corollary 7.2.24. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Schwarzschild space, where the warping function ρ is obtained from (7.46). There does not exist complete noncompact mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$ and mean curvature bounded away from zero, having polynomial volume growth and lying in a slab $[t_1, t_2] \times \mathbb{S}^n$ with $\rho(t_1) \geq \sqrt{-\frac{n}{c}}$.*

Proof. First, using (7.47) and taking the positive constant $\gamma = \frac{n-1}{2V_{\mathfrak{m}}(r(t_1))}$, we can verify that inequality (7.12) is satisfied. Moreover, since $V_{\mathfrak{m}}(r(t)) < 1$ for all $t \in I$, $r(t_1) = \rho(t_1) \geq \sqrt{-\frac{n}{c}}$ implies

$$V_{\mathfrak{m}}(r(t)) < 1 \leq \frac{c^2}{n^2} r(t)^4$$

and, consequently, we have that

$$\bar{\zeta}_c(t) = nV_{\mathfrak{m}}(r(t)) + cr(t)^2\sqrt{V_{\mathfrak{m}}(r(t))} < 0$$

for all $t \geq t_1$. Therefore, we can apply Theorem 7.2.6 to conclude our result. □

In particular, there are two solutions $r_0(\mathfrak{m}) < r_{*,-} < \hat{r} < r_{*,+}$ if the strict inequality holds in (7.49), and a unique solution $r_* = \hat{r}$ if equality holds.

Considering the context of Example 5.2.4, from Theorem 7.2.1 we get:

Corollary 7.2.25. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Schwarzschild space. There exists no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$, having bounded second fundamental form and lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with $\rho(t_2) \geq \sqrt{-\frac{n}{c}}$.*

Proof. Using (7.47) and definition of ζ_c we have

$$n\sqrt{V_{\mathfrak{m}}(r(t_1))} + cr(t_1)^2 \leq \zeta_c(t) \leq n\sqrt{V_{\mathfrak{m}}(r(t_2))} + cr(t_2)^2.$$

Since $V_{\mathfrak{m}}(r(t)) < 1$ for all $t \in I$, $r(t_2) = \rho(t_2) \geq \sqrt{-\frac{n}{c}}$ implies

$$\zeta_c(t) = n\sqrt{V_{\mathfrak{m}}(r(t))} + cr(t)^2 < 0,$$

for all $t \geq t_1$.Moreover, since $V_{\mathfrak{m}}(r(t)) < 1$ for all $t \in I$, $r(t_1) = \rho(t_1) \geq \sqrt{-\frac{n}{c}}$ implies

$$V_{\mathfrak{m}}(r(t)) < 1 \leq \frac{c^2}{n^2} r(t)^4$$

and, consequently, we have that

$$\bar{\zeta}_c(t) = nV_{\mathfrak{m}}(r(t)) + cr(t)^2\sqrt{V_{\mathfrak{m}}(r(t))} < 0$$

for all $t \geq t_1$. Therefore, we can apply Theorem 7.2.1 to conclude our result. $\square$

Taking into account again the context of Example 5.2.4, from Theorem 7.2.9 we also obtain:

Corollary 7.2.26. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Schwarzschild space and suppose that inequality (7.49) is satisfied. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$, lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with $V_{\mathfrak{m}}(r(t)) \leq \frac{c^2}{n^2} r(t)^4$ for all $t \in [t_1, t_2]$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m})$ solves equation (7.48).*

In the setting of Example 5.2.4, we also have the following consequence of Theorem 7.2.15:

Corollary 7.2.27. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Schwarzschild space and suppose that inequality (7.49) is satisfied. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$, such that Θ is bounded away from zero. If $V_{\mathfrak{m}}(r(h)) \leq \frac{c^2}{n^2} r(h)^4$ on Σ^n, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m})$ solves equation (7.48).*

7.2.7 Applications to Reissner-Nordström space

Given a mass parameter $\mathfrak{m} > 0$ and an electric charge $\mathfrak{q} \in \mathbb{R}$, with $|\mathfrak{q}| \leq \mathfrak{m}$, the *Reissner-Nordström space* is defined to be the product

$$\overline{M}^{n+1} = (r_0(\mathfrak{m}, \mathfrak{q}), +\infty) \times \mathbb{S}^n$$

endowed with the metric $\bar{g} = V_{\mathfrak{m},\mathfrak{q}}(r)^{-1} dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$, $V_{\mathfrak{m},\mathfrak{q}}(r) = 1 - 2\mathfrak{m} r^{1-n} + \mathfrak{q}^2 r^{2-2n}$ stands for its potential function and $r_0(\mathfrak{m}, \mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}}\right)^{1/(n-1)}$ is the largest positive zero of $V_{\mathfrak{m},\mathfrak{q}}(r)$. The importance of this model lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m},\mathfrak{q}}(r) dt^2 + \bar{g}$ is a charged black-hole solution of the Einstein field equation in vacuum with zero cosmological constant.

As in the case of the Schwarzschild space, $\overline{M}^{n+1}$ can be reduced in the form $I \times_\rho \mathbb{S}^n$ with metric (??) via the same change of variables as in (7.46). Furthermore, following the same previous steps, the warping function ρ has positive first and second derivatives. Moreover, we can verify that a slice $\{t_*\} \times \mathbb{S}^n$ is a mean curvature flow soliton with respect to $\rho(t)\partial_t = r\sqrt{V_{\mathfrak{m},\mathfrak{q}}(r)}\partial_r$ with soliton constant $c < 0$ when $t_* = t(r_*)$ with $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ solving the following equation

$$V_{\mathfrak{m},\mathfrak{q}}(r) = \frac{c^2}{n^2} r^4. \tag{7.50}$$

We observe that such a case is more complicated to explicit all the values, but qualitatively we can say that such a solution of (7.50) exists if and only if the function

$$\varphi_{\mathfrak{m},\mathfrak{q}}(x) = \frac{c^2}{n^2}x^4 + \frac{2\mathfrak{m}}{x^{n-1}} - \frac{\mathfrak{q}^2}{x^{2n-2}} - 1$$

has a zero on $(r_0(\mathfrak{m}), +\infty)$. Note that $\varphi_{\mathfrak{m},\mathfrak{q}}$ goes to positive infinity if x goes to positive infinity and $\varphi_{\mathfrak{m},\mathfrak{q}}$ goes to negative infinity if x goes to zero. So, $\varphi_{\mathfrak{m},\mathfrak{q}}$ has at least one root in $(0, +\infty)$ and if such roots are greater than $r_0(\mathfrak{m}, \mathfrak{q})$ we get the desired solutions r_*.

We can reason as in the proof of Corollary 7.2.24 to obtain the following nonexistence result:

Corollary 7.2.28. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Reissner-Nordström space, where the warping function ρ is obtained from (7.46). There does not exist complete noncompact mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$ and mean curvature bounded away from zero, having polynomial volume growth and lying in a slab $[t_1, t_2] \times \mathbb{S}^n$ with $V_{\mathfrak{m},\mathfrak{q}}(r(t)) < \frac{c^2}{n^2}r(t)^4$ for all $t \in [t_1, t_2]$.*

In the setting of Example 5.2.5, we can reason as in the proof of Corollary 7.2.25 to obtain the following nonexistence result:

Corollary 7.2.29. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Reissner-Nordström space. There exists no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$, having bounded second fundamental form and lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with $V_{\mathfrak{m},\mathfrak{q}}(r(t)) < \frac{c^2}{n^2}r(t)^4$ for all $t \in [t_1, t_2]$.*

Taking into account again the context of Example 5.2.5, from Theorem 7.2.9 we also obtain:

Corollary 7.2.30. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Reissner-Nordström space and suppose that there is $r_* > r_0(\mathfrak{m}, \mathfrak{q})$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$, lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with $V_{\mathfrak{m},\mathfrak{q}}(r(t)) \leq \frac{c^2}{n^2}r(t)^4$ for all $t \in [t_1, t_2]$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ solves equation (7.50).*

In the setting of Example 5.2.5, we also have the following consequence of Theorem 7.2.15:

Corollary 7.2.31. *Let $\overline{M}^{n+1} = I \times_\rho \mathbb{S}^n$ be the Reissner-Nordström space and suppose that there is $r_* > r_0(\mathfrak{m}, \mathfrak{q})$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \rho(t)\partial_t$ with soliton constant $c < 0$, such that Θ is bounded away from zero. If $V_{\mathfrak{m},\mathfrak{q}}(r(h)) \leq \frac{c^2}{n^2}r(h)^4$ on Σ^n, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ solves equation (7.50).*

7.3 Entire mean curvature flow graphs

Ecker and Huisken [84] proved that if an entire graph with polynomial volume growth is a self-shrinker, then it is necessarily a hyperplane. Later on, Wang [145] removed the condition of polynomial volume growth in Ecker-Huisken's Theorem. More recently, Colombo, Mari and Rigoli [69] extended this study for the context of entire mean curvature flow graphs in warped products. Motivated by these works, the last section of this paper is devoted to establish new Moser-Bernstein type results concerning entire graphs constructed over the fiber M^n of a warped product $\overline{M}^{n+1} = I \times_\rho M^n$, which are mean curvature flow solitons with respect to $K = \rho(t)\partial_t$ with soliton constant $c \neq 0$.

Hence, from (5.7) and (5.27) we have that $\Sigma(u)$ is a mean curvature flow soliton with respect to $K = f(t)\partial_t$ with soliton constant c if, and only if, u is a solution of the following nonlinear differential equation:

$$\operatorname{div}_M\left(\frac{Du}{\rho(u)\sqrt{\rho(u)^2+|Du|_M^2}}\right) = \frac{1}{\sqrt{\rho(u)^2+|Du|_M^2}}\left\{c\rho(u)^2+\rho'(u)\left(n-\frac{|Du|_M^2}{\rho(u)^2}\right)\right\}. \quad (7.51)$$

We say that $u \in C^\infty(M)$ has finite C^2 norm when

$$||u||_{\mathcal{C}^2(M)} := \sup_{|\gamma|\leq 2} |D^\gamma u|_{L^\infty(M)} < +\infty.$$

In this context, we establish our first Moser-Bernstein type result:

Theorem 7.3.1. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (5.28). Suppose in addition that $c \neq 0$ and $\zeta_c(t) \geq 0$. If $u \in C^\infty(M)$ is an entire solution of equation (7.51), with finite C^2 norm and such that $|Du|_M \leq \mathcal{C} \inf_M |\zeta_c(u)|$ for some positive constant $\mathcal{C}$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Let $u \in C^\infty(M)$ be such a solution of equation (7.51). It follows from (5.26) that the shape operator A of $\Sigma(u)$ is bounded, provided that u has finite C^2 norm. We note also that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $\inf_M \rho(u) > 0$. Hence, since we are assuming that M^n is complete, we get that $(\Sigma(u), g_u)$ must be also complete.

Therefore, we can reason as in the proof of Theorem 7.2.1 obtaining that $\inf_M |\zeta_c(u)| = 0$ and, hence, the result follows from our constraint on $|Du|_M$. □

From the proof of Theorem 7.3.1 we also get the following nonexistence result:

Corollary 7.3.2. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (5.28). Suppose in addition that $c \neq 0$ and $\inf_I \zeta_c(t) > 0$. There exists no entire solution with finite C^2 norm of the equation (7.51).*

Proceeding, Theorem 7.2.9 allows us to obtain our next result.

Theorem 7.3.3. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n is complete. Suppose in addition that $c \neq 0$ and $\zeta_c(t)$ does not changing the sign. If $u \in C^\infty(M)$ is a bounded entire solution of equation (7.51) such that $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Let $u \in C^\infty(M)$ be such a bounded entire solution of equation (7.51). Denoting by dM and $d\Sigma$ the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, from [23, Equation (3.7)] we have that

$$|\nabla h| d\Sigma = \rho(u)^{n-1} |Du|_M dM. \tag{7.52}$$

Hence, since we are assuming that u is bounded with $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, from relation (7.52) we conclude that $|\nabla h| \in \mathcal{L}^1_g(\Sigma(u))$. Therefore, the result follows by applying Theorem 7.2.9. □

From (5.25) we see that the assumption Θ bounded away from zero is equivalent to $|Du|_M \leq \mathcal{C}\rho(u)$ for some positive constant $\mathcal{C}$. So, Theorem 4.3.4 allows us to obtain our last Moser-Bernstein type result:

Theorem 7.3.4. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering and such that its warping function f satisfies (7.12), holding the equality only at isolated points of I. Suppose in addition that $c \neq 0$ and $\bar{\zeta}_c(t) \leq 0$. If $u \in C^\infty(M)$ is a bounded entire solution of equation (7.51) such that $|Du|_M \leq \mathcal{C}\rho(u)$ for some positive constant $\mathcal{C}$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Our next result corresponds to a nonparametric version of Theorem 4.3.1.

Corollary 7.3.5. *Let $\overline{M}^{n+1} = I \times_\rho M^n$ be a warped product with complete noncompact fiber M^n and whose warping function ρ is increasing (decreasing) and satisfies inequality (7.12). Suppose in addition that c is a constant such that the modified soliton function $\bar{\zeta}_c(t) \leq 0$ for all $t \in I$. The only smooth function $u : M^n \to I$ which is solution of the mean curvature flow soliton equation (7.51), with $|Du|_M$ bounded on M^n and such that u converges from below (above) to some $t_* \in I$ at infinity is the constant function $u \equiv t_*$.*

Proof. Let $u \in C^\infty(M)$ be such a solution of equation (7.51). We start observing that, since M^n is complete and $\inf_M \rho(u) > 0$ (due to the boundedness of u), from (5.24) we conclude that the entire graph $\Sigma(u)$ must be complete. Therefore, we are in position to apply Theorem 4.3.1 to conclude that $u \equiv t_*$. □

Chapter 8

Hypersurface in Riemannian manifold endowed with a Killing vector field

In the following results, we study the uniqueness and nonexistence of mean curvature flow solitons (MCFS) with respect to a nowhere zero Killing vector field K globally defined in a Riemannian space, via suitable Liouville type results. For this, we consider the ambient space as been a warped product of the type $M^n \times_\rho \mathbb{R}$, where the base M^n is an arbitrarily fixed integral leaf of the distribution orthogonal to K and the warping function $\rho \in C^\infty(M)$ is given by $\rho = |K|$. In particular, assuming that M^n is closed (that is, compact without boundary), we conclude that the only closed MCFS with respect to K are the totally geodesic slices. Furthermore, we establish new Moser-Bernstein type results concerning entire Killing graphs constructed through the flow of K and which are complete MCFS with respect to it. The results presented in this chapter make part of [33, 35].

8.1 Main results

Remark 8.1.1. *In a similar way of the Euclidean context, when the ambient space $M^n \times_\rho \mathbb{R}$, according to Definition 2 of [79] (see also Definition 1.1 of [27] and Definition 1.1 of [69]), a two-sided hypersurface $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ immersed in a warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ is called a* mean curvature flow soliton *(MCFS) with respect to K and with* soliton constant *$c \in \mathbb{R}$ if its mean curvature function H satisfies*

$$H = c\,\Theta. \tag{8.1}$$

In particular, we observed that each slice $M^n \times \{t\}$ of $\overline{M}^{n+1}$ is a MCFS with respect to K with soliton constant $c = 0$.

Now, we are in position to present our first uniqueness result concerning MCFS in a warped product $M^n \times_\rho \mathbb{R}$.

Theorem 8.1.2. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with complete base M^n and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a complete MCFS with respect to K and with soliton constant $c \geq 0$ (resp.*

$c \leq 0$). *Suppose that* $h \geq 0$ (*resp.* $h \leq 0$) *and that* ρ *is bounded along* Σ^n. *If* $h \in \mathcal{L}^p_g(\Sigma)$ *for some* $p > 1$, *then* Σ^n *is a slice of* $\overline{M}^{n+1}$.

Proof. It is well known that, in local coordinates $(x_1, \cdots, x_n)$ of Σ^n, the Laplacian of its height function on a metric $\hat{g}$ is given by

$$\hat{\Delta} h = \frac{1}{\hat{G}} \sum_{k,l=1}^{n} \partial_k \left(\hat{g}^{kl} \hat{G} \partial_l(h) \right), \tag{8.2}$$

where $\hat{g}_{kl} = \hat{g}(\partial_k, \partial_l)$, $\hat{G} = \sqrt{\det\left(\hat{g}_{kl}\right)}$ and $\left(\hat{g}^{kl}\right) = \left(\hat{g}_{kl}\right)^{-1}$.

Taking the conformal metric $\hat{g} = \rho^{\frac{4}{n-2}} g$, we have that $\hat{g}_{kl} = \rho^{\frac{4}{n-2}} g_{kl}$, $\hat{g}^{kl} = \frac{1}{\rho^{\frac{4}{n-2}}} g^{kl}$ and

$$\hat{G} = \sqrt{\det(\hat{g}_{kl})} = \sqrt{\rho^{\frac{4n}{n-2}} \det(g_{kl})} = \rho^{\frac{2n}{n-2}} G. \tag{8.3}$$

Thus, from (8.2) and (8.3) we obtain

$$\begin{aligned}
\hat{\Delta} h &= \frac{1}{\rho^{\frac{2n}{n-2}} G} \sum_{k,l=1}^{n} \partial_k \left(\frac{1}{\rho^{\frac{4}{n-2}}} g^{kl} \rho^{\frac{2n}{n-2}} G \partial_l(h) \right) \\
&= \frac{\rho^{\frac{2n-4}{n-2}}}{\rho^{\frac{2n}{n-2}}} \sum_{k=1}^{n} \partial_k(\partial_k(h)) + \frac{1}{\rho^{\frac{2n}{n-2}}} \frac{2n-4}{n-2} \rho^{\frac{2n-4}{n-2}-1} \sum_{k=1}^{n} \partial_k(\rho) \partial_k(h) \\
&= \frac{1}{\rho^{\frac{4}{n-2}}} \Delta h + \frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}} g(\nabla \rho, \nabla h).
\end{aligned} \tag{8.4}$$

Considering (5.23) and (5.7) into (8.4), we get

$$\begin{aligned}
\hat{\Delta} h &= \frac{1}{\rho^{\frac{4}{n-2}}} \left(\frac{-2}{\rho} g(\nabla \rho, \nabla h) + \frac{c}{\rho^2} \Theta^2 \right) + \frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}} g(\nabla \rho, \nabla h) \\
&= \left(\frac{-4}{2\rho^{\frac{n+2}{n-2}}} + \frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}} \right) g(\nabla \rho, \nabla h) + \frac{c}{\rho^{\frac{2n}{n-2}}} \Theta^2 \\
&= \left(\frac{(-4n+8)\rho^{\frac{n+2}{n-2}} + (4n-8)\rho^{\frac{n+2}{n-2}}}{2(n-2)\rho^{\frac{2n+4}{n-2}}} \right) g(\nabla \rho, \nabla h) + \frac{c}{\rho^{\frac{2n}{n-2}}} \Theta^2.
\end{aligned} \tag{8.5}$$

Hence, (8.5) allows us to the following formula

$$\hat{\Delta} h = \frac{c}{\rho^{\frac{2n}{n-2}}} \Theta^2. \tag{8.6}$$

Consequently, since we are assuming that $h \geq 0$ (resp. $h \leq 0$) and $c \geq 0$ (resp. $c \leq 0$), from (8.6) we conclude that h (resp. $-h$) is a subharmonic function with respect to the metric $\widehat{g}$. On the other hand, since we are also supposing that ρ is bounded along Σ^n, from (8.3) we have that our hypothesis $h \in \mathcal{L}^p_g(\Sigma)$ implies that $h \in \mathcal{L}^p_{\hat{g}}(\Sigma)$. Therefore, we can apply Lemma 5.5.2 to guarantee that h is constant on Σ^n, that is, Σ^n must be a slice of $\overline{M}^{n+1}$. $\square$

From Theorem 8.1.2 we get a rigidity result related to closed (compact without boundary) MCFS.

Corollary 8.1.3. *The only closed MCFS with respect to the Killing vector field K of a warped product $M^n \times_\rho \mathbb{R}$ whose base M^n is closed, are the totally geodesic slices.*

Theorem 8.1.2 jointly with Lemma 5.5.3 lead us to get the following nonexistence result.

Theorem 8.1.4. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product whose base M^n is complete noncompact with nonnegative Ricci curvature, and having bounded warping function ρ. There is no complete MCFS with respect to K, with soliton constant $c \geq 0$ (resp. $c \leq 0$) and positive (resp. negative) height function satisfying $h \in \mathcal{L}^p_g(\Sigma)$ for some $p > 1$.*

Proof. Let us suppose the existence of such a MCFS, namely $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$. From Theorem 8.1.2, we get that Σ^n is a slice of $\overline{M}^{n+1}$. Consequently, $|h|$ must be equal to a positive constant α and, since we are assuming that $h \in \mathcal{L}^p_g(\Sigma)$, we obtain

$$\mathrm{vol}_{g_M}(M) = \mathrm{vol}_g(\Sigma) = \frac{1}{\alpha^p}\int_\Sigma |h|^p d_g\Sigma < +\infty. \tag{8.7}$$

On the other hand, taking into account that M^n is complete noncompact with nonnegative Ricci curvature, Lemma 5.5.3 assures that M^n has at least linear volume growth, which corresponds to a contradiction with (8.7). □

According to Definition 1 of [52], we say that a smooth Riemannian manifold (Σ^n, g) satisfies the *$\mathcal{L}^1_g$-Liouville property*, when every nonnegative superharmonic function $u \in \mathcal{L}^1_g(\Sigma)$ must be constant. Corollary 3 of [52] ensures that a stochastically complete manifold (and, in particular, a parabolic manifold) always satisfies the $\mathcal{L}^1_g$-Liouville property. However, in Section 2 of [52] the authors constructed examples of stochastically incomplete (and, in particular, nonparabolic) manifolds satisfying the $\mathcal{L}^1_g$-Liouville property.

It is not difficult to verify that we can reason as in the proof of Theorem 8.1.2 to obtain the following uniqueness result concerning MCFS satisfying the $\mathcal{L}^1_g$-Liouville property.

Theorem 8.1.5. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with base M^n and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a MCFS with respect to K and with soliton constant $c \geq 0$ (resp. $c \leq 0$). Suppose that $h \geq 0$ (resp. $h \leq 0$) and that ρ is bounded along Σ^n. If Σ^n satisfies the $\mathcal{L}^1_g$-Liouville property and $h \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is contained into a slice of $\overline{M}^{n+1}$.*

In our next uniqueness result, we will suppose that the MCFS Σ^n lies in *slab* of $\overline{M}^{n+1}$, which means that Σ^n is contained in a bounded region of the type

$$M^n \times [t_1, t_2] = \{(p,t) \in M^n \times_\rho \mathbb{R} : t_1 \leq t \leq t_2 \text{ and } p \in M^n\}.$$

Theorem 8.1.6. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with complete base M^n and let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be a MCFS with respect to K and with soliton constant c, lying in a slab of $\overline{M}^{n+1}$. Suppose in addition that ρ is bounded along Σ^n. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice of $\overline{M}^{n+1}$.*

Proof. Considering once more the conformal metric $\hat{g} = \rho^{\frac{4}{n-2}} g$, since $\hat{g}^{kl} = \frac{1}{\rho^{\frac{4}{n-2}}} g^{kl}$, we get

$$\widehat{\nabla} h = \sum_{k,l=1}^{n} \widehat{g}^{kl} \partial_l(h) \partial_k = \frac{1}{\rho^{\frac{4}{n-2}}} \nabla h. \tag{8.8}$$

Thus, assuming that ρ is bounded along Σ^n, from (8.3) and (8.8) we obtain

$$\int_{\Sigma} |\widehat{\nabla} h|_{\hat{g}} d_{\hat{g}} \Sigma = \int_{\Sigma} \rho^{\frac{2(n-1)}{n-2}} |\nabla h| d_g \Sigma \leq \left(\sup_{\Sigma} \rho^{\frac{2(n-1)}{n-2}} \right) \int_{\Sigma} |\nabla h| d_g \Sigma. \tag{8.9}$$

Consequently, since we are supposing that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, from (8.9) we have that $|\widehat{\nabla} h|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. So, taking into account (8.6), we can apply Lemma 5.5.4 to conclude that $\hat{\Delta} h$ vanishes identically on Σ^n.

On the other hand, we have that $|\widehat{\nabla} h^2|_{\hat{g}} = 2|h||\widehat{\nabla} h|_{\hat{g}}$. Thus, assuming that Σ^n lies in slab of $\overline{M}^{n+1}$, we also get that $|\widehat{\nabla} h^2|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. Moreover, we have that

$$\hat{\Delta} h^2 = 2h\hat{\Delta} h + 2|\widehat{\nabla} h|^2_{\hat{g}} = 2|\widehat{\nabla} h|^2_{\hat{g}} \geq 0. \tag{8.10}$$

Hence, we can apply once more Lemma 5.5.4 to infer that $\hat{\Delta} h^2 = 0$ on Σ^n and, returning to (8.10), we conclude that $|\widehat{\nabla} h|_{\hat{g}}$ is identically zero on Σ^n. Therefore, Σ^n must be a slice of $\overline{M}^{n+1}$. □

Now, we are in position to present our first rigidity result concerning mean curvature flow solitons in a Riemannian warped product.

Theorem 8.1.7. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with complete noncompact base M^n. The only complete noncompact mean curvature flow soliton $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ with respect to K and with soliton constant $c \geq 0$ (resp. $c \leq 0$), such that ρ is bounded on Σ^n and h converges from above (resp. below) to t_* at infinity, is the slice M_{t_*}.*

Proof. Let $\psi : \Sigma^n \looparrowright \overline{M}^{n+1}$ be such a mean curvature flow soliton. Let us consider on Σ^n the metric $\hat{g} = \rho^{\frac{4}{n-2}} g$, which is conformal to its induced metric g.

It is well known that, in local coordinates $(x_1, \cdots, x_n)$ of Σ^n, the Laplacian of its height function on a metric $\hat{g}$ is given by

$$\hat{\Delta} h = \frac{1}{\hat{G}} \sum_{k,l=1}^{n} \partial_k \left(\hat{g}^{kl} \hat{G} \partial_l(h) \right), \tag{8.11}$$

where $\hat{g}_{kl} = \hat{g}(\partial_k, \partial_l)$, $\hat{G} = \sqrt{\det\left(\hat{g}_{kl}\right)}$ and $\left(\hat{g}^{kl}\right) = \left(\hat{g}_{kl}\right)^{-1}$.

Taking the conformal metric $\hat{g} = \rho^{\frac{4}{n-2}} g$, we have that $\hat{g}_{kl} = \rho^{\frac{4}{n-2}} g_{kl}$, $\hat{g}^{kl} = \frac{1}{\rho^{\frac{4}{n-2}}} g^{kl}$ and

$$\hat{G} = \sqrt{\det(\hat{g}_{kl})} = \sqrt{\rho^{\frac{4n}{n-2}} \det(g_{kl})} = \rho^{\frac{2n}{n-2}} G. \tag{8.12}$$

Thus, from (8.11) and (8.12) we obtain

$$
\begin{aligned}
\hat{\Delta}h &= \frac{1}{\rho^{\frac{2n}{n-2}}G}\sum_{k,l=1}^{n}\partial_k\left(\frac{1}{\rho^{\frac{4}{n-2}}}g^{kl}\rho^{\frac{2n}{n-2}}G\partial_l(h)\right)\\
&= \frac{\rho^{\frac{2n-4}{n-2}}}{\rho^{\frac{2n}{n-2}}}\sum_{k=1}^{n}g^{kl}\partial_k(\partial_l(h)) + \frac{1}{\rho^{\frac{2n}{n-2}}}\frac{2n-4}{n-2}\rho^{\frac{2n-4}{n-2}-1}\sum_{k=1}^{n}g^{kl}\partial_k(\rho)\partial_l(h)\\
&= \frac{1}{\rho^{\frac{4}{n-2}}}\Delta h + \frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}}g(\nabla\rho,\nabla h).
\end{aligned}
\tag{8.13}
$$

Considering (5.23) and (5.7) into (8.13), we get

$$
\begin{aligned}
\hat{\Delta}h &= \frac{1}{\rho^{\frac{4}{n-2}}}\left(\frac{-2}{\rho}g(\nabla\rho,\nabla h)+\frac{c}{\rho^2}\Theta^2\right)+\frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}}g(\nabla\rho,\nabla h)\\
&= \left(\frac{-4}{2\rho^{\frac{n+2}{n-2}}}+\frac{2n-4}{(n-2)\rho^{\frac{n+2}{n-2}}}\right)g(\nabla\rho,\nabla h)+\frac{c}{\rho^{\frac{2n}{n-2}}}\Theta^2\\
&= \frac{c}{\rho^{\frac{2n}{n-2}}}\Theta^2.
\end{aligned}
\tag{8.14}
$$

Since we are also assuming that h converges from above (resp. below) to t_* and $c \geq 0$ (resp. $c \leq 0$), choosing the smooth function $u = h - t_*$ (resp. $u = t_* - h$) and the vector field $X = \hat{\nabla}u$, from (8.14) we get that

$$
\mathrm{div}_{\hat{g}}X \geq 0. \tag{8.15}
$$

Moreover, we have

$$
\hat{g}(\hat{\nabla}u, X) = |\hat{\nabla}u|^2_{\hat{g}} \geq 0. \tag{8.16}
$$

In addition, since h converges to t_* at infinity, we have that u is a nonnegative non-identically vanishing function which converges to zero (also related to the metric $\hat{g}$, since ρ is bounded on Σ^n). Thus, from (8.15) and (8.16) we can apply Lemma 5.5.5 to get that $\hat{g}(\hat{\nabla}u, X)$ is identically zero on Σ^n. Hence, returning to (8.16) we conclude that $\hat{\nabla}h$ vanishes identically on Σ^n, which means that h is constant and (since it converges to t_* at infinity) Σ^n must be the slice M_{t_*}. □

8.2 Moser-Bernstein type results for MCFS

In this last section, to establish new Moser-Bernstein type results concerning entire graphs constructed over the base M^n of a warped product $M^n \times_\rho \mathbb{R}$, which are MCFS. Before, we need to recall some basic facts related to these graphs.

According to [71], we define the *entire Killing graph* $\Sigma(u)$ associated to a smooth function $u \in C^\infty(M)$ as been the hypersurface given by

$$
\Sigma(u) = \{\Psi(x, u(x)) : x \in M^n\} \subset M^n \times_\rho \mathbb{R},
$$

where $\Psi : M^n \times \mathbb{I} \to \overline{M}^{n+1}$ is the flow generated by the Killing vector field K. The metric induced on M^n from (5.15) via $\Sigma(u)$ is given by

$$g_u = g_M + \rho^2 du^2. \tag{8.17}$$

From (8.17) and by a straightforward computation we can verify that

$$N = \frac{1}{\rho(1+\rho^2|Du|_M^2)^{1/2}}(K - \rho^2\Psi_*(Du))$$

describes an unit normal vector field over $\Sigma(u)$ such that its angle function Θ is given by

$$\Theta = g(N, K) = \frac{\rho}{(1+\rho^2|Du|_M^2)^{1/2}} > 0. \tag{8.18}$$

Moreover, for all vector field X tangent to M^n, the Weingarten endomorphism A of $\Sigma(u)$ with respect to N is given by

$$\begin{aligned} AX = {} & \frac{\rho}{(1+\rho^2|Du|_M^2)^{1/2}} D_X Du - \frac{\rho^3 g(D_X Du, Du)}{(1+\rho^2|Du|_M^2)^{3/2}} Du - \frac{\rho^2 g(D\rho, X)|Du|_M^2}{(1+\rho^2|Du|_M^2)^{3/2}} Du \\ & + \frac{g(D\rho, X)}{(1+\rho^2|Du|_M^2)^{1/2}} Du + \frac{g(Du, X)}{(1+\rho^2|Du|_M^2)^{1/2}} D\rho. \end{aligned} \tag{8.19}$$

So, it follows from (8.19) that the mean curvature H_u of an entire graph $\Sigma(u)$ is given by

$$H_u = \mathrm{Div}_M\left(\frac{\rho Du}{(1+\rho^2|Du|_M^2)^{1/2}}\right) + \frac{g(Du, D\rho)}{(1+\rho^2|Du|_M^2)^{1/2}}, \tag{8.20}$$

where Div_M stands for the divergence operator on M^n with respect to its metric g_M.

Hence, from (5.7) and (8.20) we have that an entire graph $\Sigma(u)$ is a MCFS with respect to K and with soliton constant c if, and only if, u is a solution of the following elliptic non-linear partial differential equation

$$\mathrm{Div}_M\left(\frac{\rho Du}{(1+\rho^2|Du|_M^2)^{1/2}}\right) = \frac{1}{(1+\rho^2|Du|_M^2)^{1/2}}\big(c\rho - g(Du, D\rho)\big). \tag{8.21}$$

Now, we are in position to state and prove our first Moser-Bernstein type result.

Theorem 8.2.1. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with complete base M^n and bounded warping function ρ. Let $\Sigma(u)$ be an entire Killing graph determined by a smooth function $u \in C^\infty(M)$, which is nonnegative (resp. nonpositive) solution of equation (8.21) with $c \geq 0$ (resp. $c \leq 0$). Suppose in addition that $|Du|_M$ is bounded on M^n. If $u \in \mathcal{L}^p_{g_M}(M)$ for some $p > 1$, then $u \equiv t_0$ for some nonnegative (resp. nonpositive) $t_0 \in \mathbb{R}$.*

Proof. For any vector field X tangent to $\Sigma(u)$, from (8.17) we get

$$g_u(X, X) = g_M(X^*, X^*) + \rho^2 g_M(Du, X^*)^2 \geq g_M(X^*, X^*). \tag{8.22}$$

Thus, (8.22) implies that

$$L_u(\gamma) \geq L_M(\gamma^*), \tag{8.23}$$

where $L_u(\gamma)$ stands for the length of a curve γ on $\Sigma(u)$ with respect to the induced metric (8.17) and $L_M(\gamma^*)$ denotes the length of the projection γ^* of γ onto M^n with respect to its metric g_M. Consequently, since projections onto M^n of divergent curves on $\Sigma(u)$ give divergent curves on M^n and as we are assuming that the metric g_M is complete, we can apply Hopf-Rinow Theorem to conclude that the induced metric (8.17) is also complete.

Moreover, it follows from (8.17) that $d_g\Sigma = \sqrt{|G|}d_{g_M}M$, where $d_{g_M}M$ and $d_g\Sigma$ stand for the volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, and $G = det(g_{ij})$ with

$$g_{ij} = g_u(E_i, E_j) = \rho^2 E_i E_j + \delta_{ij}, \tag{8.24}$$

for each $i, j \in \{1, \dots, n\}$. Here, $\{E_1, \dots, E_n\}$ denotes a local orthonormal frame with respect to the metric g_M.

Using the definition of determinant for a squared matrix, we obtain

$$\det(g_{ij}) = \sum_{\sigma \in S_n} (\text{sign}\sigma) g_{1\sigma(1)} g_{2\sigma(2)} \dots g_{n\sigma(n)}, \tag{8.25}$$

where S_n is the set of bijective functions $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ and signσ is the sign of the permutation σ. Considering (8.24) into (8.25), we get

$$\det(g_{ij}) = \sum_{\sigma} (\text{sign}\sigma)(\rho^2 E_1 E_{\sigma(1)} + \delta_{1\sigma(1)})(\rho^2 E_2 E_{\sigma(2)} + \delta_{2\sigma(2)}) \dots (\rho^2 E_n E_{\sigma(n)} + \delta_{n\sigma(n)}). \tag{8.26}$$

With a straightforward computation, from (8.26) we obtain

$$\begin{aligned} \det(g_{ij}) &= \sum_{\sigma,k} (\text{sign}\sigma) \rho^2 E_{i_1} E_{\sigma(i_1)} \cdot \rho^2 E_{i_2} E_{\sigma(i_2)} \dots \rho^2 E_{i_k} E_{\sigma(i_k)} \delta_{i_{k+1}\sigma(i_{k+1})} \dots \delta_{i_n\sigma(i_n)} \\ &+ \sum_{\sigma} (\text{sign}\sigma) \rho^{2n} E_1 E_{\sigma(1)} \dots E_n E_{\sigma(n)} + \sum_{\sigma} (\text{sign}(\sigma)) \delta_{1\sigma(1)} \dots \delta_{n\sigma(n)}. \end{aligned} \tag{8.27}$$

On the other hand, we note that

$$\sum_{\sigma} (\text{sign}\sigma) \rho^{2n} E_1 E_{\sigma(1)} \dots E_n E_{\sigma(n)} = 0 \quad \text{and} \quad \sum_{\sigma} (\text{sign}(\sigma)) \delta_{1\sigma(1)} \dots \delta_{n\sigma(n)} = 1.$$

Thus,

$$\begin{aligned} \det(g_{ij}) &= \sum_{\sigma,k} (\text{sign}\sigma) \rho^2 E_{i_1} E_{\sigma(i_1)} \cdot \rho^2 E_{i_2} E_{\sigma(i_2)} \dots \rho^2 E_{i_k} E_{\sigma(i_k)} \delta_{i_{k+1}\sigma(i_{k+1})} \dots \delta_{i_n\sigma(i_n)} \\ &= \sum_{\sigma} (\text{sign}\sigma) \rho^2 E_{i_1} E_{\sigma(i_1)} \delta_{i_2\sigma(i_2)} \dots \delta_{i_n\sigma(i_n)} \\ &+ \sum_{\sigma,k \geq 2} (\text{sign}\sigma) \rho^{2k} E_{i_1} E_{\sigma(i_1)} \cdot E_{i_2} E_{\sigma(i_2)} \dots E_{i_k} E_{\sigma(i_k)} \delta_{i_{k+1}\sigma(i_{k+1})} \dots \delta_{i_n\sigma(i_n)}. \end{aligned} \tag{8.28}$$

If σ is different from the identity, then

$$\rho^2 E_{i_1} E_{\sigma(i_1)} \delta_{i_2\sigma(i_2)} \dots \delta_{i_n\sigma(i_n)} = 0.$$

Otherwise, we have

$$\sum_{\sigma} (\text{sign}\sigma)\rho^2 E_{i_1} E_{\sigma(i_1)} \delta_{i_2\sigma(i_2)} \dots \delta_{i_n\sigma(i_n)} = \rho^2(E_1^2 + E_2^2 + \dots E_n^2).$$

Now, considering

$$\sum_{\sigma,k\geq 2} (\text{sign}\sigma)\rho^{2k} E_{i_1} E_{\sigma(i_1)} \cdot E_{i_2} E_{\sigma(i_2)} \dots E_{i_k} E_{\sigma(i_k)} \delta_{i_{k+1}\sigma(i_{k+1})} \dots \delta_{i_n\sigma(i_n)},$$

for each fixed $k \geq 2$ and each fixed index $i_1, i_2, \dots, i_n$, we have

$$\sum_{\sigma,k\geq 2} (\text{sign}\sigma)\rho^{2k} E_{i_1}^2 E_{i_2}^2 \dots E_{i_k}^2 = 0, \tag{8.29}$$

for sign$\sigma = 1$ if the permutation in σ is even and sign$\sigma = -1$ if the permutation in σ is odd. Thus,

$$\sum_{\sigma,k\geq 2} (\text{sign}\sigma)\rho^{2k} E_{i_1} E_{\sigma(i_1)} \cdot E_{i_2} E_{\sigma(i_2)} \dots E_{i_k} E_{\sigma(i_k)} \delta_{i_{k+1}\sigma(i_{k+1})} \dots \delta_{i_n\sigma(i_n)} = 0.$$

So, using expressions (8.28) and (8.29), we obtain

$$|G| = 1 + \rho^2 |Du|^2.$$

Consequently, we reach at the following relation

$$d_g\Sigma = (1 + \rho^2 |Du|_M^2)^{1/2} d_{g_M} M. \tag{8.30}$$

Hence, since we are assuming that $u \in \mathcal{L}^p_{g_M}(M)$ for some $p > 1$ and that ρ and $|Du|_M$ are bounded, relation (8.30) guarantees that $h \in \mathcal{L}^p_g(\Sigma)$ for some $p > 1$. Therefore, since the metric (8.17) is complete, the result follows by applying Theorem 8.1.2. □

Proceeding, from Theorem 8.1.4 we establish the following nonexistence result concerning equation (8.21).

Theorem 8.2.2. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with complete noncompact base M^n having nonnegative Ricci curvature and with bounded warping function ρ. There is no entire Killing graph $\Sigma(u)$ determined by a positive (resp. negative) smooth function $u \in C^\infty(M)$, which is solution of equation (8.21) with $c \geq 0$ (resp. $c \leq 0$), such that $|Du|_M$ is bounded on M^n and $u \in \mathcal{L}^p_{g_M}(M)$ for some $p > 1$.*

Proof. Let us suppose the existence of such an entire Killing graph $\Sigma(u)$, determined by a positive (resp. negative) smooth function $u \in C^\infty(M)$. From Theorem 8.2.1, we get that $u \equiv t_0$ for some positive (resp. negative) $t_0 \in \mathbb{R}$. Since we are assuming that $u \in \mathcal{L}^p_{g_M}(M)$ for some $p > 1$, we obtain

$$\mathrm{vol}_{g_M}(M) = \frac{1}{|t_0|^p} \int_M |u|^p d_{g_M} M < +\infty. \tag{8.31}$$

But, taking into account that M^n is complete noncompact with nonnegative Ricci curvature, Lemma 5.5.3 assures that M^n has at least linear volume growth, which corresponds to a contradiction with (8.31). □

Taking into account once more relation (8.30), it is not difficult to verify that we can also obtain the following nonparametric version of Theorem 8.1.5.

Theorem 8.2.3. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with base M^n satisfying the $\mathcal{L}^1_{g_M}$-Liouville property and with bounded warping function ρ. Let $\Sigma(u)$ be an entire Killing graph determined by a smooth function $u \in C^\infty(M)$, which is nonnegative (resp. nonpositive) solution of equation (8.21) with $c \geq 0$ (resp. $c \leq 0$). Suppose in addition that $|Du|_M$ is bounded on M^n. If $u \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_0$ for some nonnegative (resp. nonpositive) $t_0 \in \mathbb{R}$.*

We close our paper stating and proving other Moser-Bernstein type result, which is derived from Theorem 8.1.6.

Theorem 8.2.4. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a warped product with complete base M^n and with bounded warping function ρ. Let $\Sigma(u)$ be an entire Killing graph determined by a bounded smooth function $u \in C^\infty(M)$, which is solution of equation (8.21). If $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

Proof. As in the beginning of the proof of Theorem 8.2.1, we get that the entire graph $\Sigma(u)$ is complete with respect to its induced metric (8.17).

On the other hand, since

$$N^* = N - N^\perp = \frac{\rho \Psi_*(Du)}{(1 + \rho^2 |Du|_M^2)^{1/2}},$$

we have that

$$|N^*|_M^2 = \frac{\rho^2 |Du|_M^2}{1 + \rho^2 |Du|_M^2}. \tag{8.32}$$

Thus, from (8.32) we get

$$|\nabla h|^2 = \frac{1}{\rho^2} |N^*|_M^2 = \frac{|Du|_M^2}{1 + \rho^2 |Du|_M^2}. \tag{8.33}$$

Hence, from (8.30) and (8.33) we reach at the following relation

$$|\nabla h| d_g \Sigma = |Du|_M d_{g_M} M. \tag{8.34}$$

Consequently, since we are supposing that $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, relation (8.34) assures that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$. Therefore, we can apply Theorem 8.1.6 to conclude the proof. □

Our next result corresponds to a nonparametric version of Theorem 8.1.7.

Corollary 8.2.5. *Let* $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ *be a warped product with complete noncompact base* M^n *and whose warping function* ρ *is bounded. The only smooth function* $u : M^n \to I$ *which is solution of the mean curvature flow soliton equation* (8.21) *for some* $c \geq 0$ (*resp.* $c \leq 0$) *and such that* u *converges from above* (*resp. below*) *to some* $t_* \in I$ *at infinity is the constant function* $u \equiv t_*$.

Proof. For any vector field X tangent to $\Sigma(u)$, from (8.17) we get

$$g_u(X, X) = g_M(X^*, X^*) + \rho^2 g_M(Du, X^*)^2 \geq g_M(X^*, X^*). \tag{8.35}$$

Thus, (8.35) implies that

$$L_u(\gamma) \geq L_M(\gamma^*), \tag{8.36}$$

where $L_u(\gamma)$ stands for the length of a curve γ on $\Sigma(u)$ with respect to the induced metric (8.17) and $L_M(\gamma^*)$ denotes the length of the projection γ^* of γ onto M^n with respect to its metric g_M. Consequently, since projections onto M^n of divergent curves on $\Sigma(u)$ give divergent curves on M^n and as we are assuming that the metric g_M is complete, we can apply Hopf-Rinow Theorem to conclude that the induced metric (8.17) is also complete.

Hence, since we are $u \in C^\infty(M)$ be such a solution of equation (8.21), we are in position to apply Theorem 8.1.7 to conclude that $u \equiv t_*$. □

Remark 8.2.6. According to [77, Example 10], we have that

$$\Sigma(u) = \{(x, y, c \ln y) : y > 0\} \subset \mathbb{H}^2 \times \mathbb{R},$$

where $u(x, y) = c \ln y$, $c \in \mathbb{R}$ is a constant and $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ stands for the two-dimensional hyperbolic space endowed with the complete metric $\langle , \rangle_{\mathbb{H}^2} = \frac{1}{y^2}(dx^2 + dy^2)$, is an entire translating graph having constant mean curvature $H = \frac{c}{\sqrt{1+c^2}} = c\,\Theta$ with respect to the orientation (5.25). Therefore, we conclude that in Corollary 8.2.5 the hypothesis that the function u converges to some $t_* \in I$ at infinity is necessary to guarantee that u is constant.

Chapter 9

Rigidity and nonexistence of submanifolds in weighted warped products

The *weighted warped products* are an extension of *warped products* in differential geometry. In this construction, a manifold is deformed or twisted through a function called " warping function. The difference in weighted warped products lies in the inclusion of a weight or measure in the warping function. These weights or measures can be used to model and capture additional effects in the geometry and physics of spacetime. The introduction of weights allows for greater flexibility in constructing spaces with interesting geometric properties.

In this chapter of the thesis, we will explore the fundamental concepts of weighted warped products. We will discuss the definitions and properties of these products, as well as their applications in various areas of mathematics and physics, such as general relativity, cosmology, string theory, and other related fields. We will investigate the geometric and physical properties of the base spaces and weighted warping functions, examine concrete examples of these constructions, and discuss the implications of these weighted warped products for understanding the structure of spacetime and their applications to specific problems.

It is expected that this chapter will provide a solid foundation for understanding weighted warped products and serve as a starting point for further advanced investigations in this fascinating area of differential geometry. The results presented in this chapter make part of [30].

9.1 Submanifolds in weighted warped products

Throughout this last part of the thesis, we will pay attention to then $(n+p)$-dimensional Riemannian manifold $(M^{n+p}, \langle , \rangle_M)$, our ambient space $I \times_\rho M^{n+p}$ is the $(n+p+1)$-dimensional product manifold $I \times M^{n+p}$, where I is an open interval of $\mathbb{R}$, endowed with the Riemmanian warped metric

$$\langle , \rangle = dt^2 + \rho(t)^2 \langle , \rangle_M, \tag{9.1}$$

where $\rho : I \to \mathbb{R}$ is a positive smooth function on I. In other words, $I \times_\rho M^{n+p}$ is nothing but a Riemannian warped product with Riemannian base (I, dt^2), Riemannian fiber $(M^{n+p}, \langle , \rangle_M)$ and warping function ρ.

Let Σ^n be a codimension $p+1$ submanifold immersed into a $I \times_\rho M^{n+p}$. That is, Σ^n is an n-dimensional connected manifold for which there exists a smooth immersion $\psi : \Sigma^n \to I \times_\rho M^{n+p}$. As usual, we will denote this induced metric also by $\langle , \rangle$.

In this setting, we denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $I \times_\rho M^{n+p}$ and Σ^n, respectively. The Gauss formula of Σ^n in $I \times_\rho M^{n+p}$ is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \tag{9.2}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $\alpha : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^\perp(\Sigma)$ stands for the vector valued second fundamental form of Σ^n, which is defined by

$$\alpha(X, Y) = (\overline{\nabla}_X Y)^\perp, \tag{9.3}$$

where $(\overline{\nabla}_X Y)^\perp$ denotes the normal component of $\overline{\nabla}_X Y$ along Σ^n. Moreover, the Weingarten formula is given by

$$\overline{\nabla}_X \eta = -A_\eta X + \nabla^\perp_X \eta, \tag{9.4}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$ and normal vector field $\eta \in \mathfrak{X}^\perp(\Sigma)$, where $\nabla^\perp$ is just the normal connection of Σ^n and $A_\eta : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the shape operator with respect to η; that is, the self-adjoint operator on $\mathfrak{X}(\Sigma)$ defined by

$$\langle A_\eta X, Y \rangle = \langle \alpha(X, Y), \eta \rangle, \quad \forall X, Y \in \mathfrak{X}(\Sigma).$$

The mean curvature vector field $\vec{H}$ of Σ^n is defined by

$$\vec{H} = \frac{1}{n}\mathrm{tr}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \alpha(E_i, E_i), \tag{9.5}$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on Σ^n.

Now, let φ be a weight function defined in $I \times_\rho M^{n+p}$. The *φ-divergence operator* on Σ^n is defined by

$$\mathrm{div}_\varphi(X) = e^\varphi \mathrm{div}(e^{-\varphi} X), \tag{9.6}$$

where X is a tangent vector field on Σ^n. From this, we define the drift Laplacian by

$$\Delta_\varphi u = \mathrm{div}_\varphi(\nabla u) = \Delta u - \langle \nabla u, \nabla \varphi \rangle, \tag{9.7}$$

where u is a smooth function on Σ^n. We will also refer to such an operator as the *φ-Laplacian* of Σ^n.

According to Gromov [93], the *weighted mean curvature vector field*, or simply *φ-mean curvature vector field*, $\vec{H}_\varphi$ of Σ^n is defined by

$$\vec{H}_\varphi = \vec{H} + \frac{1}{n}(\overline{\nabla}\varphi)^\perp, \tag{9.8}$$

where $\vec{H}$ denotes the standard mean curvature vector field of Σ^n defined in (9.5) and $(\overline{\nabla}\varphi)^\perp \in \mathfrak{X}^\perp(\Sigma)$ stands for the normal component of $\overline{\nabla}\varphi$ along Σ^n.

At this point, we observe that a splitting theorem due to Fang, Li and Zhang (see Theorem 1.1 of [87]) guarantees that if a weighted warped product manifold $(I\times_\rho M^{n+p})_\varphi$ with bounded weight function φ is such that $\overline{\mathrm{Ric}}_\varphi$ is nonnegative, then φ must be constant along I. So, motivated by this result, along this work we will consider weighted warped products $(I \times_\rho M^{n+p})_\varphi$ whose weight function φ does not depend on the parameter $t \in I$, that is $\langle\overline{\nabla}\varphi, \partial_t\rangle = 0$ and, for sake of simplicity, we will denote them by $I \times_\rho M^{n+p}_\varphi$.

In particular,

$$\overline{\nabla}_v\partial_t = \frac{\rho'(\tau)}{\rho(\tau)}v,$$

for every tangent vector $v \in T_{(\tau,x)}M_\tau$. This means that M_τ is a totally umbilical hypersurface in $I \times_\rho M^{n+p}$ with shape operator (with respect to the orientation ∂_t) given by

$$A_\tau v = -\overline{\nabla}_v\partial_t = -\frac{\rho'(\tau)}{\rho(\tau)}v,$$

for every $v \in T_{(\tau,x)}M_\tau$. Therefore, $\tau \in I \to M_\tau \subset I \times_\rho M^{n+p}$ determines a foliation of $I \times_\rho M^{n+p}$ by totally umbilical hypersurface with constant mean curvature given by

$$\mathcal{H}(\tau) = \frac{1}{n+p}\mathrm{tr}(A_\tau) = -\frac{\rho'(\tau)}{\rho(\tau)}. \tag{9.9}$$

In this setting, we have that the φ-mean curvature of a slice M_τ is just equal to its standard mean curvature. Indeed, from (9.9) and (9.8) we obtain

$$\mathcal{H}_\varphi(\tau) = \mathcal{H}(\tau) + \frac{1}{n}\langle\overline{\nabla}\varphi, \partial_t\rangle = -\frac{\rho'(\tau)}{\rho(\tau)}.$$

9.2 Statement and proof of the main results

Let $\psi : \Sigma^n \to I \times_\rho M^{n+p}$ be an immersed submanifold of codimension $p+1$. The height function of Σ^n, denoted by h, is the restriction of the projection $\pi_I(t,x) = t$ to Σ^n, that is, $h : \Sigma^n \to I$ is given by $h = \pi_I|_\Sigma = \pi_I \circ \psi$. From (5.3), we have that the gradient of π_I on $I \times_\rho M^{n+p}$ is given by $\overline{\nabla}\pi_I = \partial_t$. Then, the gradient of h on Σ^n is given by

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \partial_t^\top,$$

where $\partial_t = \partial_t^\top + \partial_t^\perp$. Here $\partial_t^\top \in \mathfrak{X}(\Sigma)$ and $\partial_t^\perp \in \mathfrak{X}^\perp(\Sigma)$ denote, respectively, the tangential and normal components of ∂_t.

In what follows, we will also consider the function $u = g(h)$, where $g : I \to \mathbb{R}$ is an arbitrary primitive of ρ. Since $g' = \rho > 0$, then $u = g(h)$ can be thought as a reparametrization of the height function. In particular, the gradient of u on Σ^n is given by

$$\nabla u = \rho(h)\nabla h = \rho(h)\nabla \partial_t^\top = K^\top, \tag{9.10}$$

where $K^\top$ denotes the tangential component of the closed conformal vector field K defined

$$K(t,x) = \rho(t)\partial_t|_{(t,x)}, \quad (t,x) \in I \times_\rho M^{n+p}. \tag{9.11}$$

In fact,

$$\overline{\nabla}_V K = \rho'(t)V \tag{9.12}$$

for every vector field V on $I \times_\rho M^{n+p}$, where $\overline{\nabla}$ denotes the Levi-Civita connection of $I \times_\rho M^{n+p}$.

Using (9.2), (9.4) and taking into account that $K = K^\top + K^\perp$, we obtain

$$\overline{\nabla}_X K = \nabla_X K^\top + \alpha(X, K^\top) - A_{K^\perp}X + \nabla_X^\perp K^\perp$$

for every $X \in \mathfrak{X}(\Sigma)$. Hence,

$$(\overline{\nabla}_X K)^\top = \nabla_X K^\top - A_{K^\perp}X \tag{9.13}$$

and

$$(\overline{\nabla}_X K)^\perp = \alpha(X, K^\top) + \nabla_X^\perp K^\perp.$$

On the other hand, equation (9.12) implies $\overline{\nabla}_X K = f'(h)X$, so that

$$(\overline{\nabla}_X K)^\top = f'(h)X \tag{9.14}$$

and

$$(\overline{\nabla}_X K)^\perp = 0.$$

Thus, from (9.13) and (9.14), we see that

$$\nabla_X K^\top = \rho'(h)X + A_{K^\perp}X. \tag{9.15}$$

From (9.10) and (9.15) we get

$$\nabla_X \nabla u = \nabla_X K^\top = \rho'(h)X + A_{K^\perp}X,$$

and tracing this expression we have that

$$\begin{aligned}\Delta u &= n\rho'(h) + \mathrm{tr}(A_{K^\perp}) \\ &= n(\rho'(h) + \langle \vec{H}, K\rangle) \\ &= n(\rho'(h) + \rho(h)\langle \vec{H}, \partial_t\rangle).\end{aligned} \tag{9.16}$$

Since we are considering Σ^n immersed in $I \times_\rho M_\varphi^{n+p}$, from (9.7), (9.10) and (9.16) we get

$$\begin{aligned}\Delta_\varphi u &= \Delta u - \langle \nabla u, \overline{\nabla}\varphi\rangle \\ &= n(\rho'(h) + \rho(h)\langle \vec{H}, \partial_t\rangle) - \rho(h)\langle \partial_t^\top, \overline{\nabla}\varphi\rangle \\ &= n(\rho'(h) + \rho(h)\langle \vec{H}, \partial_t\rangle) - \rho(h)\langle \partial_t - \partial_t^\perp, \overline{\nabla}\varphi\rangle \\ &= n(\rho'(h) + \rho(h)\langle \vec{H}, \partial_t\rangle) + \rho(h)\langle \partial_t^\perp, (\overline{\nabla}\varphi)^\perp\rangle.\end{aligned} \tag{9.17}$$

Thus, from (9.8) and (9.17) we obtain

$$\begin{aligned}\Delta_\varphi u &= n(\rho'(h) + \rho(h)\langle \vec{H} + \frac{1}{n}(\overline{\nabla}\varphi)^\perp, \partial_t\rangle) \\ &= n(\rho'(h) + \rho(h)\langle \vec{H}_\varphi, \partial_t\rangle).\end{aligned} \tag{9.18}$$

Consequently, from (9.18) we have the following lemma:

Lemma 9.2.1. *Let Σ^n be a submanifold immersed in $I \times_\rho M_\varphi^{n+p}$. If $u = g(h)$, where $g : I \to \mathbb{R}$ is an arbitrary primitive of ρ and h is the height function of Σ^n, then*

$$\Delta_\varphi u = n(\rho'(h) + \rho(h)\langle \vec{H}_\varphi, \partial_t\rangle).$$

Taking into account Lemma 9.2.1, we can reason as in the proof of Lemma 1 of [29] in order to obtain the following result:

Lemma 9.2.2. *Let Σ^n be a closed submanifold immersed in $I \times_\rho M_\varphi^{n+p}$. Then*

(i) $\min_\Sigma \langle \vec{H}_\varphi, \partial_t\rangle \le \mathcal{H}_\varphi(h^*)$, *where* $h^* = \max_\Sigma h$, *and*

(ii) $\max_\Sigma \langle \vec{H}_\varphi, \partial_t\rangle \ge \mathcal{H}_\varphi(h_*)$, *where* $h_* = \min_\Sigma h$.

Now, we are in a position to present our first rigidity result.

Theorem 9.2.3. *Let $I \times_\rho M_\varphi^{n+p}$ be a weighted warped product such that $(\log \rho)'' \ge 0$, and let $\psi : \Sigma^n \to I \times_\rho M_\varphi^{n+p}$ be a closed submanifold with φ-mean curvature vector field $\vec{H}_\varphi$ such that the support function $\langle \vec{H}_\varphi, \partial_t\rangle$ is constant. Then, $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times M^{n+p}$, for some $\tau \in I$. Moreover, when $p = 1$, $\phi := \pi_M \circ \psi : \Sigma^n \to M^{n+1}$ is a hypersurface with φ-mean curvature $H_{\phi,\varphi}$ satisfying*

$$|\vec{H}_\varphi|^2 = \frac{H_{\phi,\varphi}^2 + \rho'(\tau)^2}{\rho(\tau)^2}. \tag{9.19}$$

Proof. We will proceed based on the proof of Theorem 1 of [29]. From Lemma 9.2.2 and using the fact that $(\log\rho)'' \geq 0$ we have

$$\min_{\Sigma}\langle \vec{H}_\varphi, \partial_t\rangle \leq \mathcal{H}_\varphi(h^*) \leq \mathcal{H}_\varphi(h_*) \leq \max_{\Sigma}\langle \vec{H}_\varphi, \partial_t\rangle. \tag{9.20}$$

Thus, since we are assuming that $\langle \vec{H}_\varphi, \partial_t\rangle$ is constant, from (9.20) we get

$$\mathcal{H}_\varphi(h_*) = \mathcal{H}_\varphi(h^*) = \langle \vec{H}_\varphi, \partial_t\rangle = constant. \tag{9.21}$$

Using once more that $(\log\rho)'' \geq 0$, it follows from (9.21) that $\mathcal{H}_\varphi(t) = \langle \vec{H}_\varphi, \partial_t\rangle = constant$ on $[h_*, h^*]$. That is, $\mathcal{H}_\varphi(h) = \langle \vec{H}_\varphi, \partial_t\rangle$ on Σ^n.

So, $\mathcal{H}_\varphi(h) = -\frac{\rho'(h)}{\rho(h)} = \langle \vec{H}_\varphi, \partial_t\rangle$ implies $\rho'(h) + \rho(h)\langle \vec{H}_\varphi, \partial_t\rangle = 0$ on Σ^n, which by (9.17) allows us to conclude that $\Delta_\varphi u = 0$ on Σ^n. That is, u is a φ-harmonic function on Σ^n, which is a closed manifold. Hence, from (9.6) and (9.7), we can apply the divergence theorem to infer that $u = g(h)$ is constant on Σ^n, and since $g(t)$ is an increasing function this means that h is itself constant on Σ^n. Hence, $\psi(\Sigma)$ is contained in a slice M_τ.

When $p = 1$, as in the proof of Theorem 1 of [29], we can consider the (locally defined) unit normal vector field N of the hypersurface $\phi : \Sigma^n \to M^{n+1}$, with $\langle N, N\rangle_M = 1$. Thus, from (9.8) jointly with equation (4.18) of [29] and using again the assumption that φ does not depend on the parameter $t \in I$, it is not difficult to verify that holds the following equation

$$\vec{H}_\varphi = \frac{H_{\phi,\varphi}}{\rho(\tau)^2}N + \frac{\rho'(\tau)}{\rho(\tau)}\partial_t. \tag{9.22}$$

It is worth to note that, since $\overline{\nabla}\varphi = \frac{1}{\rho(\tau)^2}\nabla\varphi$ and $\langle N, N\rangle = \rho(\tau)^2\langle N, N\rangle_M = \rho(\tau)^2$, it was used the relation

$$H_\phi + \frac{1}{n}\langle \overline{\nabla}\varphi, N\rangle = H_\phi + \frac{1}{n}\langle \nabla\varphi, N\rangle_M = H_{\phi,\varphi}$$

to get (9.22). Therefore, from (9.22) we deduce relation (9.19). □

When the ambient space is a weighted product space of the form $\mathbb{R}^p \times M^{n+1}_\varphi$, we can apply p times Theorem 9.2.3 in order to get the following codimension reduction result:

Corollary 9.2.4. *The only n-dimensional closed φ-minimal submanifolds immersed in a weighted product space $\mathbb{R}^p \times M^{n+1}_\varphi$ are the closed φ-minimal hypersurfaces immersed in M^{n+1}_φ.*

From relation (9.19) in Theorem 9.2.3 we also obtain the following nonexistence result:

Corollary 9.2.5. *There do not exist closed φ-minimal submanifolds Σ^n immersed in a weighted warped product $I \times_\rho M^{n+1}_\varphi$ such that $(\log\rho)'' \geq 0$ and ρ' does not vanish on I.*

The following key lemma is a weak Omori-Yau's generalized maximum principle for the drift Laplacian. A proof of it can be found in [129].

Lemma 9.2.6. *Let Σ^n_φ be a complete weighted manifold whose Bakry-Émery-Ricci curvature tensor is bounded from below and let $u : \Sigma^n \to \mathbb{R}$ be a smooth function satisfying $\sup_\Sigma u < +\infty$. Then, there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma^n$ such that*

$$\lim_k u(p_k) = \sup_\Sigma u \quad \text{and} \quad \limsup_k \Delta_\varphi u(p_k) \leq 0.$$

The previous lemma jointly with Lemma 9.2.1 enable us to obtain an extension of Lemma 9.2.2. For this, we just proceed in a similar way of the proof of Lemma 2 of [29].

Lemma 9.2.7. *Let Σ^n be a complete submanifold immersed in $I \times_\rho M^{n+p}_\varphi$, such that its Bakry-Émery-Ricci tensor is bounded from below.*

(i) If Σ^n lies above a slice of $I \times_\rho M^{n+p}_\varphi$, then $\sup_\Sigma \langle \vec{H}_\varphi, \partial_t\rangle \geq \mathcal{H}_\varphi(h_*)$, where $h_* = \inf_\Sigma h \in I$;

(ii) If Σ^n lies below a slice of $I \times_\rho M^{n+p}_\varphi$, then $\inf_\Sigma \langle \vec{H}_\varphi, \partial_t\rangle \leq \mathcal{H}_\varphi(h^*)$, where $h^* = \sup_\Sigma h \in I$.

In our next result, we will assume that the ambient space obeys a convergence condition which was established by Montiel [115]. Before, we recall that a *slab* of a weighted warped product $I \times_\rho M^{n+p}_\varphi$ is just a region between two slices M_{τ_1} and M_{τ_2}, for some $\tau_1 < \tau_2$.

Theorem 9.2.8. *Let $I \times_\rho M^{n+p}_\varphi$ be a weighted warped product such that $(\log\rho)'' \geq 0$, with the equality $(\log\rho)'' = 0$ holding only at isolated points of I, and which obeys the following convergence condition*

$$K_M \geq \sup_I(\rho'^2 - \rho\rho''), \tag{9.23}$$

where K_M stands for the sectional curvature of M^{n+p}. Suppose in addition that the Hessian of the weight function φ is bounded from below. Let $\psi : \Sigma^n \to I \times_\rho M^{n+p}_\varphi$ be a complete submanifold which lies in a slab of $I \times_\rho M^{n+p}_\varphi$, with bounded second fundamental form and such that the support function $\langle \vec{H}_\varphi, \partial_t\rangle$ is constant. Then, $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times M^{n+p}$, for some $\tau \in I$. Moreover, when $p = 1$, $\phi := \pi_M \circ \psi : \Sigma^n \to M^{n+1}$ is a hypersurface with φ-mean curvature $H_{\phi,\varphi}$ satisfying (9.19).

Proof. We start showing that the Bakry-Émery-Ricci tensor of Σ^n is bounded from below. For this, we recall that the curvature tensor R of Σ^n can be described in terms of its second fundamental form α and the curvature tensor $\overline{R}$ of the ambient $I \times_\rho M^{n+p}_\varphi$ by the so-called Gauss equation given by

$$\langle R(X,Y)Z,W\rangle = \langle \overline{R}(X,Y)Z,W\rangle + \langle \alpha(X,Z),\alpha(Y,W)\rangle - \langle \alpha(X,W),\alpha(Y,Z)\rangle,$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$. As in [123], here we are considering that the curvature tensor is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X,\nabla_Y]Z,$$

where [ ] denotes the Lie bracket.

Taking $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \cdots, E_n\}$ of $\mathfrak{X}(\Sigma)$, it follows from this previous Gauss equation that the Ricci curvature tensor of Σ^n is given by

$$\begin{aligned} \mathrm{Ric}(X,X) &= \sum_{i=1}^{n} \langle \overline{R}(X,E_i)X, E_i\rangle + n\langle \alpha(X,X), \vec{H}\rangle - \sum_{i=1}^{n} |\alpha(X,E_i)|^2 \\ &= \sum_{i=1}^{n} \langle \overline{R}(X,E_i)X, E_i\rangle + n\langle \sum_{k=1}^{p+1} A_k X, X\rangle H_k - \sum_{i=1}^{n} \left| \sum_{k=1}^{p+1} \langle A_k X, E_i\rangle \eta_k \right|^2, \end{aligned} \tag{9.24}$$

where $\alpha(X,Y) = \sum_{i=1}^{p+1} \langle A_i X, Y\rangle \eta_i$ and $\{\eta_1, \cdots, \eta_{p+1}\}$ is a local orthonomal frame of $\mathfrak{X}^{\perp}(\Sigma)$. Consequently, taking account that $\vec{H}$ can be expressed in the following way

$$\vec{H} = \sum_{i=1}^{p+1} H_i \eta_i, \tag{9.25}$$

for some smooth functions $H_1, H_2, \ldots, H_{p+1}$ defined on Σ^n, from (9.24) and (9.25) we get

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n} \langle \overline{R}(X,E_i)X, E_i\rangle - \sum_{i=1}^{p+1} \left| A_i X - \frac{nH_i}{2} X \right|^2 + \frac{n^2 |\vec{H}|^2}{4} |X|^2. \tag{9.26}$$

Moreover, since we are assuming that holds the convergence condition (9.23), from inequality (4.17) of [19] we have that

$$\sum_i \langle \overline{R}(X,E_i)X, E_i\rangle \geq -n \frac{|\rho''(h)|}{\rho(h)} |X|^2. \tag{9.27}$$

Thus, inserting (9.27) into (9.26) and considering again (9.25), we obtain

$$\begin{aligned} \mathrm{Ric(X,X)} &\geq -n \frac{|\rho''(h)|}{\rho(h)} |X|^2 - \sum_{i=1}^{p+1} \left| A_i X - \frac{nH_i}{2} X \right|^2 + \frac{n^2 |\vec{H}|^2}{4} |X|^2 \\ &\geq -\left(n \frac{|\rho''(h)|}{\rho(h)} + |\alpha|^2 \right) |X|^2. \end{aligned} \tag{9.28}$$

Hence, since we are assuming that Σ^n lies in a slab of $I \times_\rho M_\varphi^{n+p}$, $|\alpha|$ is bounded and $\operatorname{Hess} \varphi$ is bounded from below, from (6) and (9.28) we get that the Bakry-Émery-Ricci tensor[1] of Σ^n is bounded from below.

Consequently, we can reason as in the proof of Theorem 9.2.3 (but using now Lemma 9.2.7 instead of Lemma 9.2.2) in order to show that

$$\mathcal{H}_\varphi(h^*) = \mathcal{H}_\varphi(h_*) = \langle \vec{H}_\varphi, \partial_t \rangle = constant. \tag{9.29}$$

[1] *Bakry-Émery-Ricci tensor* Ric_φ as being the following extension of the standard Ricci tensor Ric of M^n:

$$\mathrm{Ric}_\varphi = \mathrm{Ric} + \operatorname{Hess} \varphi.$$

Our constraint on $\log f$ implies that the function $\mathcal{H}_\varphi(t)$ is strictly decreasing on I. Hence, from (9.29) we get that $h_* = h^*$ and, consequently, h is constant on Σ^n. Therefore, $\psi(\Sigma)$ must be contained in a slice $\{\tau\} \times M^{n+p}$. □

From Theorem 9.2.8 we obtain the next nonexistence result:

Corollary 9.2.9. *Let $I \times_\rho M_\varphi^{n+1}$ be a weighted warped product such that $(\log \rho)'' \geq 0$, with the equality $(\log \rho)'' = 0$ holding only at isolated points of I, and which obeys the convergence condition (9.23). Suppose in addition that ρ' does not vanish on I and $\operatorname{Hess}\varphi$ is bounded from below. There do not exist complete φ-minimal submanifolds $\psi : \Sigma^n \to I \times_\rho M_\varphi^{n+1}$ lying in a slab of $I \times_\rho M_\varphi^{n+1}$ and with bounded second fundamental form.*

9.3 Further results

The next key lemma is just an extension of a Liouville-type result due to Yau in [147], and its proof can be found in [56].

Lemma 9.3.1. *The only φ-harmonic bounded functions defined on an n-dimensional complete weighted Riemannian manifold Σ_φ^n, whose Bakry-Émery-Ricci tensor is nonnegative, are the constant ones.*

Using this previous lemma we can prove the following result:

Theorem 9.3.2. *Let $I \times_\rho M_\varphi^{n+p}$ be a weighted warped product such that $(\log \rho)'' \geq 0$ and let $\psi : \Sigma^n \to I \times_\rho M_\varphi^{n+p}$ be a complete submanifold which lies in a slab of $I \times_\rho M_\varphi^{n+p}$, having nonnegative Bakry-Émery-Ricci tensor and such that the support function $\langle \vec{H}_\varphi, \partial_t \rangle$ is constant. Then, $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times M^{n+p}$, for some $\tau \in I$. Moreover, when $p = 1$, $\phi := \pi_M \circ \psi : \Sigma^n \to M^{n+1}$ is a hypersurface with φ-mean curvature $H_{\phi,\varphi}$ satisfying (9.19).*

Proof. We can proceed as in the proof of Theorem 9.2.8 to infer that the function $u = g(h)$ is a φ-harmonic function on Σ^n. Hence, since $\psi(\Sigma)$ lies in a slab of $I \times_\rho M_\varphi^{n+p}$, we can apply Lemma 9.3.1 to conclude that u is constant and, consequently, h is constant on Σ^n. Therefore, $\psi(\Sigma)$ must be contained in a slice $\{\tau\} \times M^{n+p}$. □

Considering once more the ambient space being a weighted product space of the form $\mathbb{R}^p \times M_\varphi^{n+1}$, we obtain our second codimension reduction result by applying recursively Theorem 9.3.2. More precisely,

Corollary 9.3.3. *The only n-dimensional complete φ-minimal submanifolds having nonnegative Bakry-Émery-Ricci tensor and lying in a slab of a weighted product space $\mathbb{R}^p \times M_\varphi^{n+1}$ are the complete φ-minimal hypersurfaces immersed in M_φ^{n+1}.*

From Theorem 9.3.2 we also get the following nonexistence result:

Corollary 9.3.4. *There do not exist complete φ-minimal submanifolds $\psi : \Sigma^n \to I \times_\rho M_\varphi^{n+1}$ having nonnegative Bakry-Émery-Ricci tensor and lying in a slab of a weighted warped product $I \times_\rho M_\varphi^{n+1}$ such that $(\log \rho)'' \geq 0$ and ρ' does not vanish on I.*

An important example of weighted manifold is the so-called *Gaussian space* $\mathbb{G}^n$, which corresponds to the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian probability measure $d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx^2$. Concerned with the weighted product space $\mathbb{R} \times \mathbb{G}^n$, Hieu and Nam extended the classical Bernstein's theorem [49] showing that the only weighted minimal graphs $\Sigma^n(u)$ of functions $u(x_2, \cdots, x_{n+1}) = x_1$ over $\mathbb{G}^n$ are the hyperplanes $x_1 = constant$ (see Theorem 4 of [97]).

Taking into account Corollary 9.3.3, we can use Theorem 4 of [97] to obtain a new Bernstein-type result. In what follows a $(p+1)$-*graph* in $\mathbb{R}^{p+1} \times \mathbb{G}^n$ defined over $\mathbb{G}^n$ is a graph $u : \mathbb{G}^n \to \mathbb{R}^{p+1}$, with $(u(x), x) \in \mathbb{R}^{p+1} \times \mathbb{G}^n$.

Theorem 9.3.5. *The only complete φ-minimal bounded $(p+1)$-graphs in $\mathbb{R}^{p+1} \times \mathbb{G}^n$ defined over $\mathbb{G}^n$, having nonnegative Bakry-Émery-Ricci tensor, are the n-dimensional hyperplanes $\{q\} \times \mathbb{G}^n$ with $q \in \mathbb{R}^{p+1}$.*

Taking into account

$$\mathrm{Ric}_\varphi = \mathrm{Ric} + \mathrm{Hess}\,\varphi, \tag{9.30}$$

and (9.28), from Theorem 9.3.5 we obtain the following:

Corollary 9.3.6. *The only complete φ-minimal bounded $(p+1)$-graphs in $\mathbb{R}^{p+1} \times \mathbb{G}^n$ defined over $\mathbb{G}^n$, with the second fundamental form satisfying $|\alpha| \leq 1$, are the n-dimensional hyperplanes $\{q\} \times \mathbb{G}^n$ with $q \in \mathbb{R}^{p+1}$.*

In order to establish our last results, let us consider

$$\mathcal{L}_\varphi^k(\Sigma) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u|^k(x) e^{-\varphi(x)} d\Sigma < +\infty\}.$$

While Lebesgue integrable spaces (see equation (5.29)) are associated with the standard Lebesgue measure, weighted Lebesgue integrable spaces incorporate an additional weight measure to model specific effects. This difference in the definition of the spaces results in different properties and applications, allowing for a more refined and adaptable analysis of functions.

The following result is a consequence of Theorem 1.1 of [127].

Lemma 9.3.7. *Let u be a nonnegative smooth φ-subharmonic function on a complete Riemannian manifold Σ^n. If $u \in \mathcal{L}_\varphi^k(\Sigma)$, for some $k > 1$, then u is constant.*

It is not difficult verify that we can apply Lemmas 9.2.1 and 9.3.7 to obtain our last result:

Theorem 9.3.8. *Let $I \times_\rho M_\varphi^{n+p}$ be a weighted warped product and let $\psi : \Sigma^n \to I \times_\rho M_\varphi^{n+p}$ be a complete φ-minimal submanifold with $\rho'(h) \geq 0$. If $u = g(h) \in \mathcal{L}_\varphi^k(\Sigma)$, for some $k > 1$, then $\psi(\Sigma)$ is contained in a slice $\{\tau\} \times M^{n+p}$, for some $\tau \in I$. Moreover, when $p = 1$, $\phi := \pi_M \circ \psi : \Sigma^n \to M^{n+1}$ is a φ-minimal hypersurface.*

Remark 9.3.9. According to a result due to Wei and Wylie [146], all noncompact complete Riemannian manifolds with nonnegative Bakry-Émery-Ricci tensor for some bounded weight function φ have at least linear φ-volume growth (i.e., for any $x \in \Sigma^n$, $\mathrm{vol}_\varphi(B(x, R))$ has at least linear growth on R, where $B(x, R)$ is the geodesic ball in Σ^n centered at x with radius R). Consequently, if we assume in Theorem 9.3.8 that Σ^n has nonnegative Bakry-Émery-Ricci tensor and that $\varphi(h)$ is bounded, we also conclude that Σ^n must be compact.

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